Journal of Number Theory 137 (2014) 50–66

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

A conjecture of De Koninck regarding particular square values of the sum of divisors function

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A R T I C L E   I N F O

Article history:
Received 28 March 2013
Received in revised form 5 August 2013
Accepted 20 October 2013
Available online 24 December 2013
Communicated by David Goss

MSC:
11A25
11A41

Keywords:
Sum of divisors
Squarefree core
De Koninck's conjecture
Compactification

A B S T R A C T

We study integers $n > 1$ satisfying the relation $\sigma(n) = \gamma(n)^2$, where $\sigma(n)$ and $\gamma(n)$ are the sum of divisors and the product of distinct primes dividing n, respectively. If the prime dividing a solution n is congruent to 3 modulo 8 then it must be greater than 41, and every solution is divisible by at least the fourth power of an *odd* prime. Moreover at least 2/5 of the exponents a of the primes dividing any solution have the property that $a + 1$ is a prime power. Lastly we prove that the number of solutions up to $x > 1$ is at most $x^{1/6+\epsilon}$, for any $\epsilon > 0$ and all $x > x_\epsilon$.

1. Introduction

A decade ago, Jean-Marie De Koninck asked for all integer solutions n to the equation

$$\sigma(n) = \gamma(n)^2 \tag{1}$$

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0022-314X/$ – see front matter
http://dx.doi.org/10.1016/j.jnt.2013.10.011

where $\sigma(n)$ denotes the sum of all positive divisors of n, and $\gamma(n)$ is the product of the distinct prime divisors of n. The only known solutions with $1 \leqslant n \leqslant 10^{11}$ are $n = 1$ and $n = 1782$, and so De Koninck sensibly conjectured that there exist no other solutions. It is included in Richard Guy's compendium [4, Section B11] as an unsolved problem.

In [2] a number of restrictions on the form of Eq. (1) were developed: the two solutions $n = 1$ and $n = 1782$ are the only ones having $\omega(n) \leqslant 4$; furthermore, if an integer $n > 1$ is fourth power free (i.e. $p^4 \nmid n$ for all primes p), then it was shown that n cannot satisfy De Koninck's equation.

The aim of this work is to present further items of evidence in support of De Koninck's conjecture, and to indicate the necessary structure of a hypothetical counter-example. In fact, upon combining together the results of [2] and this article, then any non-trivial solution other than 1782 must be even, have one prime divisor to power 1 and possibly another prime divisor to a power congruent to 1 modulo 4, while all other odd prime divisors should occur only to even powers. Here we shall establish that if the prime to power 1 is congruent to 3 modulo 8, then it must be no less than 43 (Proposition 1). Moreover, we prove that at least one odd prime divisor must appear with an exponent no smaller than 4 (Theorem 1).

Applying an idea from [3], we show in Corollary 2 that more than 2/5 of the exponents a appearing in the prime factorization of any solution of Eq. (1) are such that $a + 1$ is a prime or a prime power. We then count the number of potential solutions n up to x, in the following manner: using results of Pollack and Pomerance [8], and by extending a method of [2, Thm. 1], we shall prove in Theorem 2 that the number of solutions $n \leqslant x$ to Eq. (1) can be at most $x^{1/6+\epsilon}$, for any $\epsilon > 0$ and every $x > x_\epsilon$.

Finally, by exploiting the properties of the product compactification of $\mathbb{N}$, we show there are only finitely many solutions to (1) supported on any given finite set of primes $\mathcal{P}$. Indeed we will prove a more general result for the equation

$$\sigma(n)^\alpha \times \phi(n)^\beta = \theta \times n^\mu \times \gamma(n)^\tau \tag{2}$$

where $\alpha, \beta, \mu, \tau \in \mathbb{Z}$ with $\theta > 0$ some fixed rational, and $\alpha + \beta > \mu$ (see Theorem 3). The argument itself has a rather different flavor from that in [5].

Notations. If p is prime then $v_p(n)$ is the highest power of p which divides n, $\omega(n)$ will denote the number of distinct prime divisors of n, and $\mathcal{K}$ is the set of all solutions to $\sigma(n) = \gamma(n)^2$. Lastly, the symbols p, q, p_i, q_i are reserved exclusively for odd primes.

2. Preliminary lemmas

We begin by recalling some basic structure theory concerning solutions to Eq. (1). The following two background results were proved in [2].

Lemma 1. *If $n > 1$ belongs to $\mathcal{K}$, then one has a decomposition*

$$n = 2^e \times p_1 \times \prod_{i=2}^{s} p_i^{a_i}$$

where $e \geqslant 1$*, and* a_i *is even for all* $i = 3, \dots, s$*. Furthermore, either* a_2 *is even in which case* $p_1 \equiv 3 \pmod 8$*, or instead* $a_2 \equiv 1 \pmod 4$ *and* $p_1 \equiv p_2 \equiv 1 \pmod 4$*.*

Lemma 2. *If* $n > 1$ *is an element of* $\mathcal{K}$ *and does not equal to* $1782 = 2 \cdot 3^4 \cdot 11$*, then* n *has at least* 5 *distinct prime factors, and there exists a prime* (*either even or odd*) *dividing* n *to at least a fourth power.*

The proof of the next result is due Pollack, and can be found in [6].

Lemma 3. *If* $\sigma(n)/n = N/D$ *with* $\gcd(N, D) = 1$*, then given* $x \geqslant 1$ *and* $d \geqslant 1$:

$$\#\{n \leqslant x \text{ such that } D = d\} = x^{o(1)} \quad \text{as } x \to \infty.$$

Lastly we will require Apéry's solution to the generalized Ramanujan–Nagel equations.

Lemma 4. *(See Apéry [1].) The Diophantine equation* $x^2 + D = 2^{n+2}$*, with given non-zero integer* $D \neq 7$*, has at most two solutions. In addition*:

(i) *if* $D = 23$ *then* $(x, n) \in \{(3, 5), (45, 11)\}$,
(ii) *if* D *has the form* $2^m - 1$ *with* $m \geqslant 4$*, then* $(x, n) \in \{(1, m), (2^m - 1, 2m - 1)\}$.

Hence, in both these cases, there are exactly two solutions.

3. Restrictions on primes dividing members of $\mathcal{K}$

In this section, we shall make a preliminary study of restrictions on the possible values of p_1 and p_2 associated to elements of $\mathcal{K}$, additional to those described in Lemma 1 above. Clearly $p_1 + 1$ cannot be divisible by any cube, otherwise Eq. (1) is violated. Hence for prime numbers congruent to 3 modulo 8, this excludes first 107 and secondly (in increasing order) 499 from occurring.

We will henceforth refer to these as **bad De Koninck primes**; indeed there are an infinite number of primes $p \equiv 3 \pmod 8$ such that $p + 1$ is divisible by a proper cube. In Proposition 1 below, we shall prove that 3, 11 and 19 are also bad. In the case $p_1 \equiv p_2 \equiv 1 \pmod 4$ and $a_2 = 1$, this same constraint applied to $(p_1 + 1)(p_2 + 1)$ excludes for those primes less than 100, the pairs

$$\{5, 17\},\ \{5, 53\},\ \{5, 89\},\ \{13, 53\},\ \{13, 97\},\ \{17, 29\},\ \{17, 41\},\ \{17, 53\},$$
$$\{17, 89\},\ \{29, 53\},\ \{29, 89\},\ \{37, 53\},\ \{41, 53\},\ \{41, 89\},\ \{41, 97\}$$

called here **bad De Koninck pairs**. Later in Corollary 1, we show $\{5, 13\}$ is also bad.

Proposition 1. *Under the same notations as Lemma 1, if a solution $n \in \mathcal{K}$ satisfies both $\omega(n) > 4$ and $p_1 \equiv 3 \pmod 8$, then the prime $p_1 \geqslant 43$.*

Proof. First suppose that $p_1 = 3$, in which case

$$\left(2^{e+1} - 1\right) \times 4 \times \prod_{i=2}^{m} \frac{\sigma(p_i^{a_i})}{p_i^2} = 4 \times 3^2.$$

As a direct consequence $2^{e+1} - 1 < 9$ so $e \in \{1, 2\}$, and by [2, Theorem 3] we can assume $a_2 \geqslant 4$. Therefore

$$3 \times 13 = 3\left(3^2 + 3 + 1\right) < \left(2^{e+1} - 1\right) \times \frac{\sigma(p_2^{a_2})}{p_2^2} < 3^2,$$

which is obviously false, and we conclude that $p_1 \neq 3$.

Next if one supposes that $p_1 = 11$, then

$$\left(2^{e+1} - 1\right) \times 12 \times \prod_{i=2}^{m} \sigma\left(p_i^{a_i}\right) = 4 \times 11^2 \times \prod_{i=2}^{m} p_i^2,$$

thus $3 \times (2^{e+1} - 1) < 11^2$ which implies that $1 \leqslant e \leqslant 4$. If all of the a_i were strictly less than 4, then by [2, Theorem 3] again we would have $e = 4$, in which case

$$\left(2^{e+1} - 1\right) \times (p_1 + 1) \times \prod_{i=2}^{m} \frac{\sigma(p_i^{a_i})}{p_i^2} = 4 \times 11^2.$$

The latter implies

$$31 \times 3 \times \prod_{i=2}^{m} \frac{\sigma(p_i^{a_i})}{p_i^2} = 11^2,$$

hence there exists an $i \geqslant 2$ with $11 \mid p_i^2 + p_i + 1$; this is impossible since $11 \not\equiv 1 \bmod 3$. It follows there is at least one $i \geqslant 2$ with $a_i \geqslant 4$, and without loss of generality suppose that it is a_2 say. One therefore obtains an inequality

$$\left(2^{e+1} - 1\right) \times \frac{p_1 + 1}{4} \times \frac{\sigma(p_2^{a_2})}{p_2^2} < 11^2$$

and consequently,

$$3^2 \times \frac{\sigma(3^4)}{3^2} = 11^2 < 11^2$$

which is clearly false. Therefore $p_1 \neq 11$.

Finally suppose $p_1 = 19$. Using Lemma 1 we can write $n = 2^e \times p_1 \times \prod_{i=2}^m p_i^{a_i}$, whence

$$\left(2^{e+1} - 1\right) \times (p_1 + 1) \times \prod_{i=2}^{m} \frac{\sigma(p_i^{a_i})}{p_i^2} = 4p_1^2.$$

Thus $(2^{e+1} - 1) \times 5F = 19^2$ where F is a positive rational value strictly greater than 1. As a consequence $(2^{e+1} - 1) < 19^2/5$, implying that $1 \leqslant e \leqslant 5$.

Case (1). If $e = 5$ then

$$9 \times 7 \times 5F = 19^2$$

and it follows that $F < 19^2/315 < 1.15$. If some exponent $a_i \geqslant 3$ then $F \geqslant \sigma(p_i^3)/p_i^2 > 3$ which cannot occur, and therefore one may assume that $a_i = 2$ for every $i \in \{2, \dots m\}$. Now by studying the left hand side, there must exist a prime p_i (which we will call p_2) that equals 3. Then $\sigma(p_2^2) = 3^2 + 3 + 1 = 13$ yields a new prime, denoted p_3, with $\sigma(p_3^2) = 13^2 + 13 + 1 = 3 \times 61$. One thereby obtains a left hand side with at least three 3's in the numerator but at most two 3's in the denominator, while the right hand side has none. This contradiction shows $e < 5$.

Case (2). If $e = 4$ then

$$31 \times 5 \times \prod_{i=2}^{m} \frac{\sigma(p_i^{a_i})}{p_i^2} = 19^2.$$

Arguing as in the previous case, without loss of generality assume $a_i = 2$ for $i \geqslant 2$. Examining the left hand side, one of the primes p_i must equal 31; let us call it p_2. Then we have $\sigma(p_2^2) = 31^2 + 31 + 1 = 3 \times 331$, thence the new prime $p_3 = 331$ gives $\sigma(p_3^2) = 331^2 + 331 + 1 = 3 \times 7 \times 5233$, and ultimately $p_4 = 7$ with $7^2 + 7 + 1 = 3 \times 19$. Hence there are at least three 3's in the numerator and exactly two in the denominator, with none occurring on the right hand side. This shows $e < 4$.

Case (3). If $e = 3$ then $15 \times 5F = 19^2$ implies $75 \times (3^2 + 3 + 1) < 19^2$, which is false.

Case (4). If $e = 2$ then we would get $7 \times 5 \times 13 < 19^2$, which again is false.

Case (5). Henceforth we consider the situation where $e = 1$. It follows that

$$3 \times 5 \times \prod_{i=2}^{m} \sigma\left(p_i^{a_i}\right) = 19^2 \times \prod_{i=2}^{m} p_i^2$$

implying $3 \mid n$. One can then take $p_2 = 3$, and (by Lemma 1) assume that a_2 is even. Suppose first that $a_2 \geqslant 6$. Then $\sigma(3^{a_2}) \geqslant \sigma(3^6) = 1093$, in which case

$$5 \times 1093 \times \prod_{i=3}^{m} \sigma\left(p_i^{a_i}\right) \leqslant 5 \times \sigma\left(3^{a_2}\right) \times \prod_{i=3}^{m} \sigma\left(p_i^{a_i}\right) = 19^2 \times 3 \times \prod_{i=3}^{m} p_i^2$$

which is false, whence $a_2 \in \{2, 4\}$. However if $a_2 = 2$, then

$$3 \times 5 \times \left(3^2 + 3 + 1\right) \times \prod_{i=3}^{m} \frac{\sigma(p_i^{a_i})}{p_i^2} = 19^2 \times 3^2$$

and there must exist an odd prime dividing n which is greater than 3, and which divides n to a power not less than 4. This eventuality in turn implies

$$5 \times 13 \times \left(5^2 + 5 + 1\right) < 19^2 \times 3$$

which again is impossible.

Hence the only remaining possibility is that $a_2 = 4$. Because $\sigma(19) = 2^2 \times 5$ and $\sigma(3^4) = 11^2$, one may then assume $p_3 = 5$ and $p_4 = 11$, which gives us the equality

$$\sigma\left(2^1\right)\sigma\left(19^1\right)\sigma\left(3^4\right)\sigma\left(5^{a_3}\right)\sigma\left(11^{a_4}\right) \times \prod_{i=5}^{m} \frac{\sigma(p_i^{a_i})}{p_i^2} = 2^2 \times 19^2 \times 3^2 \times 5^2 \times 11^2.$$

Canceling like terms yields

$$5^{a_3} \times 11^{a_4} < \sigma\left(5^{a_3}\right)\sigma\left(11^{a_4}\right) \leqslant 19^2 \times 3 \times 5$$

which is false if either $a_3 \geqslant 4$ or $a_4 \geqslant 4$; since both are even, clearly $a_3 = a_4 = 2$.

Now $2 \cdot 19 \cdot 3^4 \cdot 5^2 \cdot 11^2 \notin \mathcal{K}$ so there exists a prime $p_5 \geqslant 7$ such that $p_5^{a_5} \| n$ with a_5 even. If $a_5 \geqslant 4$ then one would have

$$\left(7^2 + 7\right) \times 5^2 \times 11^2 < 5^2 \times 11^2 \times \frac{\sigma(p_5^{a_5})}{p_5^2} < 19^2 \times 3 \times 5$$

which is certainly false; thus all primes other than $2, 3, 19$ which divide n must do so exactly to the power 2.

As a consequence $m \geqslant 5$, and we can write

$$n = 2 \times 19 \times 3^4 \times 5^2 \times 11^2 \times \prod_{i=5}^{m} p_i^2.$$

Substituting this form into the equation $\sigma(n) = \gamma(n)^2$ and then canceling, one deduces

$$31 \times 131 \times \prod_{i=5}^{m} \left(\frac{p_i^2 + p_i + 1}{p_i^2} \right) = 19^2 \times 3 \times 5.$$

Therefore the set of p_i with $5 \leqslant i \leqslant m$ includes $\{31, 131\}$ and none out of $\{3, 5, 19\}$. However $\sigma(31^2) = 3 \times 331$, $\sigma(131^2) = 17293$ and $\sigma(17293^2) = 3 \cdot 13 \cdot 7668337$, hence $3^2 = 9$ divides the numerator of the product on the left and does not cancel with any denominator. This circumstance is impossible, as 9 does not divide the right hand side.

The above contradiction completes the proof that $p_1 \neq 19$. $\square$

Proposition 2. *If $p_1 \equiv 1 \pmod 4$ and $a_2 \geqslant 5$, then $p_1 \geqslant 173$.*

Proof. Applying Lemma 1 one knows $p_2 \geqslant 5$, and we can write

$$\left(2^{e+1} - 1\right) \times \frac{\sigma(p_2^{a_2})}{p_2^2} \times \prod_{i=3}^{m} \frac{\sigma(p_i^{a_i})}{p_i^2} = \frac{4p_1^2}{p_1 + 1}.$$

However $\sigma(5^5)/5^2 = 2906/25 \leqslant \sigma(p_2^{a_2})/p_2^2$ in which case $(2^{e+1} - 1) \times \frac{2906}{25} < \frac{4p_1^2}{p_1+1} < 4p_1$; the latter inequality is only satisfied by primes $p_1 \geqslant 173$. $\square$

Proposition 3. *If $n \in \mathcal{K}$ is a solution with $p_1 \equiv 3 \pmod 8$ such that n is not divisible by the fourth power of any odd prime, then p_1 cannot divide $2^{e+1} - 1$.*

Proof. Using [2, Theorem 3], one can express

$$n = 2^e \times p_1 \times \prod_{i=2}^{m} p_2^2$$

and moreover $2^{e+1} - 1 \leqslant 4p_1^2/(p_1 + 1) < 4p_1$. Thus under the assumption that $p_1 \mid 2^{e+1} - 1$, either $p_1 = 2^{e+1} - 1$ or $3p_1 = 2^{e+1} - 1$.

First suppose that $p_1 = 2^{e+1} - 1$. From the proof of [2, Theorem 3], one has

$$\frac{1}{4} \times \prod_{i=2}^{m} \frac{p_i^2 + p_i + 1}{p_i^2} \leqslant 0.73$$

consequently $(p_1 - 1) \times 0.73 \geqslant p_1$. The latter inequality implies $p_1 < 3$, which is false.

Alternatively if $3p_1 = 2^{e+1} - 1$, because $9 \neq 2^{e+1} - 1$ for any value of e, clearly $3 \neq p_1$, so we can instead set $p_2 = 3$. Similarly $13 = 3^3 + 3 + 1 \neq p_1$, and $13^2 + 13 + 1 = 3 \times 61$ with $61 \neq p_1$. However $3 \mid 61^2 + 61 + 1$ giving at least three powers of 3 dividing the left hand side of $\sigma(n) = \gamma(n)^2$, which again yields a contradiction. $\square$

The following three technical lemmas are key ingredients in the proof of Theorem 1.

Lemma 5. *If $n \in \mathcal{K}$ is divisible by 3, there exists an odd prime p such that $p^4 \mid n$.*

Proof. Assume (hypothetically) n is not divisible by the fourth power of an odd prime. If $p_1 \equiv 3 \pmod 8$ then using Lemma 1, we can write

$$\left(2^{e+1}-1\right) \times (p_1+1) \times \left(p_2^2+p_2+1\right) \times \cdots \times \left(p_m^2+p_m+1\right) = 4p_1^2p_2^2\cdots p_m^2.$$

By Lemma 1 once more, we know $p_1 \neq 3$ so instead put $p_2 = 3$. Consider the system:

$$\begin{array}{llll} 3^2+3+1 = 13; & 13 \equiv 5 \bmod 8, & 13 \neq p_1, & 13 = p_3 \\ 13^2+13+1 = 3\times 61; & 61 \equiv 5 \bmod 8, & 61 \neq p_1, & p_4 = 61 \\ 61^2+61+1 = 3\times 13\times 97; & 97 \equiv 1 \bmod 8, & 97 \neq p_1, & p_5 = 97. \end{array}$$

We observe that the left hand side of the previous equation must be divisible by $3^3 = 27$ whilst the right hand side is only divisible by $3^2 = 9$, yielding a contradiction.

If $p_1 \equiv 1 \pmod 4$, one has the decomposition

$$\left(2^{e+1}-1\right) \times (p_1+1) \times (p_2+1) \times \left(p_3^2+p_3+1\right) \times \cdots \times \left(p_m^2+p_m+1\right) = 4p_1^2p_2^2\cdots p_m^2.$$

Neither p_1 nor p_2 can be 3, thus we may take $p_3 = 3$.

If $p_1 = 13$ then $p_1 + 1 = 2 \times 7$, and we set $p_4 = 7$; therefore $7^2 + 7 + 1 = 3 \times 19$ and $19^2 + 19 + 1 = 3 \times 127$, again giving too many 3's.

If neither p_1 nor p_2 is 13, we can choose $p_4 = 13$ and thereby obtain $13^2+13+1 = 3\times 61$.

If $61 = p_1$ or p_2 (let's say $p_1 = 61$), we can write $n = 2^e \cdot 61 \cdot p_2 \cdot p_3^2 \cdots p_m^2$ and so $p_1 + 1 = 2 \times 31$ with $31 \neq p_2$. Consequently we can choose $p_4 = 31$, leading to the equation $\sigma(31^2) = 31^2 + 31 + 1 = 3 \times 331$ and again too many 3's.

Finally if $61 \neq p_1, p_2$ then we still pick up an additional 3, since $3 \mid 61^2 + 61 + 1$. □

Lemma 6. *If a solution $n \in \mathcal{K}$ is not divisible by the fourth power of an odd prime and $p_1 \equiv 3 \pmod 8$, then $3 \mid n$.*

Proof. Suppose $n \in \mathcal{K}$ but $3 \nmid n$. In general, if a prime $q \mid p^2 + p + 1$ then either $q = 3$, or we must have $q \equiv 1 \pmod 3$ so that $3 \mid q^2 + q + 1$. Now from the expression

$$\left(2^{e+1}-1\right) \times (p_1+1) \times \left(p_2^2+p_2+1\right) \times \cdots \times \left(p_m^2+p_m+1\right) = 4p_1^2p_2^2\cdots p_m^2$$

we can define $Q := \prod_{i=2}^m (p_i^2 + p_i + 1)$. Because $3 \nmid n$, each prime number p_j with $1 \leqslant j \leqslant m$ which appears as a factor of Q does not appear in the form $p_i^2 + p_i + 1$; this means we must have $Q \mid p_1^2$. However by Lemma 2, the integer Q has at least three quadratic factors, giving rise to a contradiction. □

Lemma 7. *If $n \in \mathcal{K}$ satisfies $p_1 \equiv 1 \pmod 4$ and $3 \nmid n$, then n is divisible by the fourth power of an odd prime.*

Proof. Suppose $p_1 \equiv p_2 \equiv 1 \pmod 4$. Then in the notation of Lemma 6, it follows that there are two quadratic factors for $Q = \prod_{i=2}^{m}(p_i^2+p_i+1)$ and (following cancelation) three possible forms for the equation $\sigma(n) = \gamma(n)^2$. We shall treat each of these separately.

Case (1):

$$p_3^2 + p_3 + 1 = p_1$$
$$p_4^2 + p_4 + 1 = p_2$$
$$\left(2^{e+1} - 1\right)\left(\frac{p_1+1}{2}\right)\left(\frac{p_2+1}{2}\right) = p_1 p_2 p_3^2 p_4^2.$$

Note that $\frac{p_2+1}{2}$ has at least one prime divisor, and at most three prime divisors.

(1.1) If $\frac{p_2+1}{2}$ has only one prime divisor then $\frac{p_2+1}{2} = p_1$; under this scenario, there are seven possibilities for $\frac{p_1+1}{2}$.

(1.1.1) If $\frac{p_1+1}{2} = p_2$ then $2^{e+1} - 1 = p_3^2 p_4^2$, which is impossible.
(1.1.2) If $\frac{p_1+1}{2} = p_3^2$ then $p_3 \mid p_1 + 1$; however $p_3 \mid p_1 - 1$ so $p_3 \mid \gcd(p_1+1, p_1-1) = 2$, which is impossible.
(1.1.3) If $\frac{p_1+1}{2} = p_4^2$ then

$$p_3^2 + p_3 + 1 = \frac{p_2+1}{2} = \frac{p_4^2+p_4+2}{2}$$

implying both $p_3 \mid p_4 + 1$ and $p_4 \mid p_3 + 1$, which is clearly false.
(1.1.4) If $\frac{p_1+1}{2} = p_3 p_4$ then $p_3 \mid p_1 + 1$; however $p_3 \mid p_1 - 1$ hence $p_3 \mid (p_1+1, p_1-1) = 2$, which is again false.
(1.1.5) If $\frac{p_1+1}{2} = p_2 p_3^2$ then $2^{e+1} - 1 = p_4^2$, which is impossible.
(1.1.6) If $\frac{p_1+1}{2} = p_2 p_4^2$ then $2^{e+1} - 1 = p_3^2$, which is impossible.
(1.1.7) If $\frac{p_1+1}{2} = p_2 p_3 p_4$ then $p_3 \mid p_1 + 1$; now $p_3 \mid p_1 - 1$ thus $p_3 \mid \gcd(p_1+1, p_1-1) = 2$, which is false.

(1.2) If $\frac{p_2+1}{2}$ has two prime divisors, either $\frac{p_2+1}{2} = p_3^2$; or p_4^2; or $p_3 p_4$.

(1.2.1) If $\frac{p_2+1}{2} = p_3^2$, then either $\frac{p_1+1}{2} = p_2$ or $\frac{p_1+1}{2} = p_4^2$:
(1.2.1.1) If $\frac{p_1+1}{2} = p_2$ then one has $2p_4(p_4 + 1) = p_3(p_3 + 1)$, which implies $p_3 \mid p_4 + 1$ and $p_4 \mid p_3 + 1$; the last two conditions are incompatible.
(1.2.1.2) If $\frac{p_1+1}{2} = p_4^2$ then $p_4(p_4 + 1) = 2(p_3 + 1)(p_3 - 1)$, which implies that $p_4 < p_3$; further $p_3(p_3 + 1) = 2(p_4 + 1)(p_4 - 1)$ which implies $p_3 < p_4$, impossible!
(1.2.2) If $\frac{p_2+1}{2} = p_4^2$, then either $\frac{p_1+1}{2} = p_2$ or $\frac{p_1+1}{2} = p_3^2$:
(1.2.2.1) If $\frac{p_1+1}{2} = p_2$ then $p_4 = 2$, which is false.
(1.2.2.2) If $\frac{p_1+1}{2} = p_3^2$ then $p_3 = 2$, which is false.

(1.2.3) If $\frac{p_2+1}{2} = p_3p_4$, then either $\frac{p_1+1}{2} = p_2$ or $\frac{p_1+1}{2} = p_3p_4$:
(1.2.3.1) If $\frac{p_1+1}{2} = p_2$ then $p_3 \mid p_4 + 1$ and $p_4 \mid p_3 + 1$, which is impossible.
(1.2.3.2) If $\frac{p_1+1}{2} = p_3p_4$ then $p_1 = p_2$, which is false as they are distinct primes.

(1.3) If $\frac{p_2+1}{2}$ has three prime divisors, either $\frac{p_2+1}{2} = p_1p_3^2$; or $p_1p_3p_4$; or $p_1p_4^2$.

(1.3.1) If $\frac{p_2+1}{2} = p_1p_3^2$ then one deduces $2^{e+1} - 1 = p_4^2$, which is false.
(1.3.2) If $\frac{p_2+1}{2} = p_1p_3p_4$ then $\frac{p_1+1}{2} = p_2$, which implies that $p_4 \mid p_3 + 1$ and $p_3 \mid p_4 + 1$; the latter conditions are incompatible.
(1.3.3) If $\frac{p_2+1}{2} = p_1p_4^2$ then we find $2^{e+1} - 1 = p_3^2$, which is false.

Combining (1.1), (1.2), and (1.3) together, clearly Case (1) is impossible in its entirety.

Case (2):

$$p_3^2 + p_3 + 1 = p_1$$
$$p_4^2 + p_4 + 1 = p_1p_2^2$$
$$\left(2^{e+1} - 1\right)\left(\frac{p_1+1}{2}\right)\left(\frac{p_2+1}{2}\right) = p_3^2p_4^2.$$

Here $p_3 \equiv p_4 \equiv 2 \pmod 3$, $\frac{p_1+1}{2} \equiv \frac{p_2+1}{2} \equiv 1 \pmod 3$, and there are at least two prime factors in $2^{e+1} - 1$ (which being congruent to 3 modulo 4 cannot include p_2, and being congruent to 1 modulo 3 cannot include p_3 or p_4). It follows that there is at least one prime factor in $\frac{p_1+1}{2}$ and $\frac{p_2+1}{2}$ respectively, which leaves us only $\frac{p_1+1}{2} = p_3$ or $\frac{p_1+1}{2} = p_4$, and these are both impossible.

Case (3):

$$p_3^2 + p_3 + 1 = p_1$$
$$p_4^2 + p_4 + 1 = p_1p_2$$
$$\left(2^{e+1} - 1\right)\left(\frac{p_1+1}{2}\right)\left(\frac{p_2+1}{2}\right) = p_2p_3^2p_4^2.$$

Note that it cannot happen that one of p_2, p_3, p_4 is the only prime divisor of $\frac{p_2+1}{2}$. Furthermore $2^{e+1} - 1$ must have at least two prime divisors, and it cannot be a square; in addition $2^{e+1} - 1 \equiv p_2 \equiv \frac{p_1+1}{2} \equiv \frac{p_2+1}{2} \equiv 1 \pmod 3$ and $p_3 \equiv p_4 \equiv 2 \pmod 3$. One therefore deduces

$$\frac{p_1+1}{2} = p_2$$
$$2^{e+1} - 1 = p_3p_4$$
$$\frac{p_2+1}{2} = p_3p_4.$$

From these three equations, we obtain

$$p_3 = \frac{\sqrt{2^{e+5} - 31} - 1}{2}$$

and by the result of Apéry in Lemma 4, this is clearly an impossible occurrence. □

We are now ready to give the main result of this section.

Theorem 1. *If $n \in \mathcal{K}$ then n is divisible by the fourth power of an* **odd** *prime.*

Proof. Firstly applying Lemma 5, if $n \in \mathcal{K}$ and $3 \mid n$ then $p^4 \mid n$ for some odd prime p. Without loss of generality, we may therefore assume $n \in \mathcal{K}$ and $3 \nmid n$.

If $p_1 \equiv 1 \pmod 4$ then the result is covered by Lemma 7. Likewise if $p_1 \equiv 3 \pmod 8$ then the result is covered by Lemma 6. Finally the remaining case $p_1 \equiv 7 \pmod 8$ is already excluded courtesy of Lemma 1. □

Corollary 1. *If $\{p_1, p_2\} = \{5, 13\}$ then $a_2 \geqslant 5$.*

Proof. Assume that $a_2 = 1$. Since $\sigma(n) = \gamma(n)^2$, setting $p_3 = 3$ and $p_4 = 7$ implies

$$\left(2^{e+1} - 1\right) \cdot (2 \times 3) \cdot (2 \times 7) \times \prod_{i=3}^{m} \sigma\left(p_i^{a_i}\right) = 4 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 13^2 \times \prod_{i=5}^{m} p_i^2.$$

Using the divisibility of n by the fourth power of an odd prime (which minimally is 3):

$$\left(2^{e+1} - 1\right) \times 3 \cdot 7 \cdot \frac{121}{9} \cdot \frac{31}{5^2} \cdot \frac{57}{7^2} \cdot \frac{183}{13^2} < 1$$

so $(2^{e+1} - 1) \cdot 2 < 1$, which is false for $e \in \mathbb{N}$. Therefore $a_2 > 1$, in which case $a_2 \geqslant 5$. □

4. The exponents for members of $\mathcal{K}$

We now study the exponents a_i occurring in the decomposition of a De Koninck number. The first step is to adapt an idea of Chen and Chen [3], in order to relate $\omega(n)$ with $\sum_{i=0}^{m} d(a_i + 1)$, where $d(x)$ is defined to be the number of divisors of an integer $x \geqslant 1$. The second step is to apply the AM/GM inequality, then further analyse the exponents.

Lemma 8. *Let a solution $n \in \mathcal{K}$ be represented as the product $n = 2^e \times p_1 \times \prod_{i=2}^{m} p_i^{a_i}$. If we set $p_0 = 2,\ a_0 = e$ and $a_1 = 1$, then there are inequalities*

$$2\omega(n) \leqslant \sum_{i=0}^{m} d(a_i + 1) \leqslant 3\omega(n).$$

Proof. One need only derive the upper bound, since the lower bound follows from (1).

First consider the case where $i \geqslant 2$ and a_i is even, so p_i is odd. Put $w_i = d(a_i+1)-1$ and write $n_{i,1}, \dots, n_{i,w_i}$ to denote all the positive integer divisors of a_i+1 other than 1. Let $q_{i,j}$ be a primitive prime divisor of $(p_i^{n_{i,j}}-1)/(p_i-1)$ for $0 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant w_i$. In particular, there are divisibilities

$$q_{i,j} \,\Big|\, \frac{p_i^{n_{i,j}}-1}{p_i-1} \,\Big|\, \frac{p_i^{a_i+1}-1}{p_i-1}$$

and if $\Omega(x)$ counts the number of prime factors of x with multiplicity, then

$$w_i \leqslant \omega\big(\sigma\big(p_i^{a_i}\big)\big) \leqslant \Omega\big(\sigma\big(p_i^{a_i}\big)\big).$$

Alternatively, if $i=0$ then primitive divisors exist except for $e+1=6$, and in that case

$$w_0 = d(e+1)-1 = 3 = \Omega\big(2^6-1\big).$$

Lastly if $i=1$ or $a_2=1$, then we have $1 = d(a_i+1)-1 < 2 \leqslant \Omega(p_i+1) = \Omega(\sigma(p_i^{a_i}))$.

Therefore in all cases $d(a_i+1)-1 \leqslant \Omega(\sigma(p_i^{a_i}))$, hence there is an inequality

$$\begin{aligned}\sum_{i=0}^{m} d(a_i+1) - \omega(n) &= \sum_{i=0}^{m}\big(d(a_i+1)-1\big)\\ &\leqslant \sum_{i=0}^{m} \Omega\big(\sigma\big(p_i^{a_i}\big)\big) = \Omega\big(\sigma(n)\big) = \Omega\big(\gamma(n)^2\big) = 2\omega(n)\end{aligned}$$

thereby completing the derivation of the upper bound. □

Corollary 2. *If $n \in \mathcal{K}$ then in the notation of Lemma 8, a proportion of more than 2/5 of the numbers a_i+1 must be either prime or prime powers.*

Proof. Because $2^{\omega(a_i+1)} \leqslant d(a_i+1)$, using the arithmetic–geometric mean and Lemma 8:

$$\big(2^{\sum_{i=0}^{m}\omega(a_i+1)}\big)^{\frac{1}{\omega(n)}} \leqslant \frac{\sum_{i=0}^{m} 2^{\omega(a_i+1)}}{\omega(m)} \leqslant 3.$$

Moreover, taking the logarithm of both sides, one deduces

$$\sum_{i=0}^{m} \omega(a_i+1) \leqslant \left(\frac{\log 3}{\log 2}\right) \times \omega(n).$$

For an integer $i \geqslant 1$, let $n_i := \#\{j\colon\ \omega(a_j+1) = i\}$. Then the above inequality becomes

$$n_1 + 2n_2 + 3n_3 + \cdots \leqslant \left(\frac{\log 3}{\log 2}\right) \times (n_1 + n_2 + \cdots)$$

which implies

$$\left(2-\frac{\log 3}{\log 2}\right)(n_2+n_3+\cdots)\leqslant\left(2-\frac{\log 3}{\log 2}\right)n_2+\left(3-\frac{\log 3}{\log 2}\right)n_3+\cdots\leqslant\left(\frac{\log 3}{\log 2}-1\right)n_1.$$

Rearranging the n_i's yields

$$n_1+n_2+\cdots\leqslant\left(\frac{\frac{\log 3}{\log 2}-1}{2-\frac{\log 3}{\log 2}}+1\right)\times n_1$$

and as the bracketed term equals 2.41 to two decimal places, we conclude that

$$\frac{2}{5}<\frac{1}{2.41}\leqslant\frac{n_1}{n_1+n_2+\cdots+n_m}\leqslant n_1$$

as required. □

5. Counting the elements in $\mathcal{K}\cap[1,x]$

For every real $x>0$, we will from now on use the notation $\mathcal{K}(x):=\mathcal{K}\cap[1,x]$. In [2, Theorem 4], it was shown that the size of the solutions $\mathcal{K}(x)$ is asymptotically bounded by $x^{1/4+o(1)}$ as x tends to infinity (and this result was itself an improvement on the work of Pomerance and Pollack [8], which instead gave an upper bound of $x^{1/3+o(1)}$). In this section we will sharpen the bound still further, as described directly below.

Theorem 2. *The estimate*

$$\#\mathcal{K}(x)\leqslant x^{1/6+o(1)}$$

holds as $x\to\infty$.

Proof. Let $n>1$ be in $\mathcal{K}(x)$, so we may express it as $n=A\times B$ where $\gcd(A,B)=1$, with A squarefree and B squarefull. Exploiting Lemma 1, then $A\in\{p_1,2p_1,p_1p_2,2p_1p_2\}$ and B is divisible by at least one prime to the fourth power or greater.

Under the notation of Lemma 3, one can write

$$\frac{N}{D}=\frac{\sigma(n)}{n}=\frac{\gamma(n)^2}{n}=\frac{\gamma(A)^2}{A}\times\frac{\gamma(B)^2}{B}=\frac{A}{B/\gamma(B)^2}>1$$

with $\gcd(A,B/\gamma(B)^2)=1$. It follows that $B/\gamma(B)^2<A$, whence

$$\frac{B^2}{\gamma(B)^2}<AB=n\leqslant x\quad\Longrightarrow\quad\frac{B}{\gamma(B)}\leqslant\sqrt{x}.$$

Now by Lemma 1, we can always decompose $B = \delta \times C^2 \times D$ where $\delta \in \{1, 2^3\}$, C is a squarefree integer, D is a 4-full integer, and such that δ, C and D are pairwise coprime. As a consequence,

$$\frac{B}{\gamma(B)} = \frac{\delta}{\gamma(\delta)} \times C \times \frac{D}{\gamma(D)} \quad \Longrightarrow \quad \frac{D}{\gamma(D)} \leqslant \sqrt{x}.$$

In addition

$$\frac{B}{\gamma(B)^2} = \frac{\delta}{\gamma(\delta)^2} \times \frac{D}{\gamma(D)^2}$$

so that

$$\frac{B}{\gamma(B)^2} = \frac{D}{\gamma(D)^2} \quad \text{or} \quad \frac{B}{\gamma(B)^2} = 2 \times \frac{D}{\gamma(D)^2}.$$

Moreover one knows that $D/\gamma(D) \leqslant \sqrt{x}$ above, which means $D/\gamma(D)^2 \leqslant \sqrt{x}$.

Now if two 4-full numbers D_1 and D_2 satisfy $D_1/\gamma(D_1) = D_2/\gamma(D_2)$, then we must also have $D_1/\gamma(D_1)^2 = D_2/\gamma(D_2)^2$. Hence the number of choices for $D/\gamma(D)^2 \leqslant \sqrt{x}$ with $D/\gamma(D) \leqslant \sqrt{x}$ and D 4-full, is less than or equal to the number of choices for $D/\gamma(D) \leqslant \sqrt{x}$ which is of type $x^{\frac{1}{6}+o(1)}$.

Therefore the number of choices for $B/\gamma(B)^2$ is also $x^{\frac{1}{6}+o(1)}$, and the proof is completed upon applying Lemma 3. □

6. Applications of the product compactification

For each prime p, let $\mathbb{N}_p$ denote the one point compactification of $\mathbb{N}$; in particular, each finite point $n \in \mathbb{N}$ is itself an open set, and a basis for the neighborhoods of the point at infinity, p^∞ say, is given by the open sets $U_p^{(\epsilon)} = \{p^e \in \mathbb{N}\colon\ e \geqslant 1/\epsilon\} \cup \{p^\infty\}$ with $\epsilon > 0$. If $\mathbb{P}$ indicates the set of prime numbers, let us write

$$\hat{\mathbb{N}} := \prod_{p \in \mathbb{P}} \mathbb{N}_p$$

for the product of these indexed spaces, endowed with the standard product topology. Then $\hat{\mathbb{N}}$ is a compact metrizable space so it is sequentially compact, hence every sequence in $\hat{\mathbb{N}}$ has a convergent subsequence.

Remark. We shall call $\hat{\mathbb{N}}$ equipped with its topology the *product compactification of* $\mathbb{N}$. A nice account detailing properties of the so-called 'supernatural topology' in attacking the odd perfect number problem, is given by Pollack in [7].

Consider now the more general equation

$$\sigma(n)^\alpha \times \phi(n)^\beta = \theta \times n^\mu \times \gamma(n)^\tau \tag{3}$$

where $\alpha, \beta, \mu, \tau \in \mathbb{Z}$ and $\theta > 0$ is a rational number. Write $\mathcal{K} = \mathcal{K}_{\alpha,\beta,\mu,\tau}$ for the set of solutions

$$\mathcal{K}_{\alpha,\beta,\mu,\tau} = \left\{ n \in \mathbb{N}\colon\ \sigma(n)^\alpha \times \phi(n)^\beta = \theta \times n^\mu \times \gamma(n)^\tau \right\}$$

which clearly depends on the initial choice of quintuple $(\alpha, \beta, \mu, \tau, \theta)$.

Theorem 3. *Let $\mathcal{P} \subset \mathbb{P}$ denote a fixed finite set of primes, and assume that $\alpha + \beta > \mu$. Then there exist only finitely many $n \in \mathcal{K}$ with support in $\mathcal{P}$.*

Before we give the demonstration, we point out that choosing $\alpha = 1$, $\beta = 0$, $\mu = 0$, $\tau = 2$ and $\theta = 1$ implies there exist only finitely many solutions to De Koninck's equation (1), supported on any prescribed finite set of primes $\mathcal{P}$.

Proof. Given $A, B, M, T \in \mathbb{Z}$, define a multiplicative function $h = h_{A,B,M,T} : \mathbb{N} \to \mathbb{Q}_{>0}$ by the formula

$$h(n) := \frac{\sigma(n)^A \times \phi(n)^B}{n^M \times \gamma(n)^T}.$$

For every $r \geqslant 1$ and at each prime p, one calculates that

$$h\left(p^r\right) = p^{A-B-T}(p-1)^{B-A}\left(1 - p^{-r-1}\right)^A \times \left(p^r\right)^{A+B-M}$$

while $h(1) = 1$. This naturally leads us to the definition

$$\hat{h}\left(p^\infty\right) := \begin{cases} \infty & \text{if } A + B > M \\ 0 & \text{if } A + B < M \\ p^{A-B-T}(p-1)^{B-A} & \text{if } A + B = M, \end{cases}$$

and provides a unique extension $\hat{h} : \hat{\mathbb{N}} \to \mathbb{R} \cup \{\infty\}$ of the original arithmetic function h. In fact if $A + B = M$ and $T = 0$, one can then show $\hat{h}$ is continuous on the monoid $\hat{\mathbb{N}}$.

Fix a finite set of primes $\mathcal{L} = \{l_1, \ldots, l_k\}$, and put

$$\mathbb{N}_{\mathcal{L}} := \left\{ n \in \mathbb{N}\colon\ n = l_1^{e_1} \cdots l_k^{e_k},\ e_j \geqslant 1 \right\}.$$

Key claim. If $A + B \geqslant M$ then $h|_{\mathbb{N}_{\mathcal{L}}}$ is monotonic increasing with respect to divisibility.

To establish this claim suppose that $n = n' \times l_j^{e_j}$ with $n' \in \mathbb{N}_{\mathcal{L} \setminus \{l_j\}}$, and set $m = n' \times l_j^{e_j+1}$. Then $h(m) = h(n') \times h(l_j^{e_j+1})$ and

$$h\left(l_j^{e_j+1}\right) = \frac{\sigma(l_j^{e_j+1})^A \times \phi(l_j^{e_j+1})^B}{l_j^{(e_j+1)M} \times \gamma(l_j^{e_j+1})^T}$$

$$= \left(\frac{\sigma(l_j^{e_j+1})}{\sigma(l_j^{e_j})}\right)^A \times l_j^{B-M} \times \frac{\sigma(l_j^{e_j})^A \phi(l_j^{e_j})^B}{l_j^{e_j M} \gamma(l_j^{e_j})^T}$$
$$= l_j^{B-M} \times \left(\frac{l_j^{e_j+2}-1}{l_j^{e_j+1}-1}\right)^A \times h\bigl(l_j^{e_j}\bigr).$$

However

$$l_j^{B-M} \times \left(\frac{l_j^{e_j+2}-1}{l_j^{e_j+1}-1}\right)^A = l_j^{B-M} \times \left(\frac{l_j^{e_j+2}-l_j}{l_j^{e_j+1}-1} + \frac{l_j-1}{l_j^{e_j+1}-1}\right)^A$$
$$= l_j^{B-M} \times \left(l_j + \frac{l_j-1}{l_j^{e_j+1}-1}\right)^A > l_j^{A+B-M} \geqslant 1$$

since $A+B \geqslant M$. It follows that $h(l_j^{e_j+1}) > h(l_j^{e_j})$, in which case

$$h(m) = h\bigl(n'\bigr) \times h\bigl(l_j^{e_j+1}\bigr) > h\bigl(n'\bigr) \times h\bigl(l_j^{e_j}\bigr) = h(n).$$

The proof of the claim then follows by induction on the number of primes (with multiplicity) which divide the quotient of a general pair n and m, with $n \mid m$.

Now let us take $A = \alpha$, and choose $B, M \in \mathbb{Z}$ such that

$$\mu - \beta < M - B \leqslant \alpha.$$

Suppose there exists a sequence of elements in $\mathcal{K}$ supported on $\mathcal{P}$ which are all distinct. Under the supernatural topology, there exists a subsequence $(N_i)_{i\geqslant 1}$ and a limit $N_o \in \hat{\mathbb{N}}$ such that $N_i \to N_o$. The element N_o is supported on $\mathcal{P}$, otherwise at least one of the N_i would also not be supported on $\mathcal{P}$. We may therefore write $N_o = A \times B^\infty$ where $\operatorname{supp}(A) \subset \mathcal{P}$, $\operatorname{supp}(B) \subset \mathcal{P}$, and $\gcd(A,B) = 1$ with B squarefree. Furthermore

$$\operatorname{supp}(A) \cup \operatorname{supp}(B) = \mathcal{L} = \{l_1, \ldots, l_k\}, \text{say.}$$

Then there exists a subsequence $(N_{i_j})_{j\geqslant 1}$ of the sequence $(N_i)_{i\geqslant 1}$ satisfying for all $j \geqslant 1$:

(i) $\operatorname{supp}(N_{i_j}) = \mathcal{L}$,
(ii) N_{i_j} properly divides $N_{i_{j+1}}$, and
(iii) $A \| N_{i_j}$.

Each $N_{i_j} \in \mathcal{K}$ and h is monotonic on the monoid $(\mathbb{N}, \times)$, hence for all $j \geqslant 2$ one has

$$0 < h(N_{i_1}) < h(N_{i_j}) \overset{\text{by (3)}}{=} \frac{\theta \times \phi(N_{i_j})^{B-\beta}}{N_{i_j}^{M-\mu} \times \gamma(N_{i_j})^{T-\tau}}$$

$$
\begin{aligned}
&= \frac{\phi(N_{i_j})^{B-\beta}}{N_{i_j}^{M-\mu}} \times \frac{\theta}{(\prod_{s=1}^{k} l_s)^{T-\tau}} \\
&\leqslant N_{i_j}^{(B-\beta)-(M-\mu)} \times \frac{\theta}{(\prod_{s=1}^{k} l_s)^{T-\tau}}
\end{aligned}
$$

which tends to zero as $j \to \infty$ since $M - B > \mu - \beta$.

This immediately yields a contradiction, and completes the proof of the theorem. □

7. Final comments

In Theorem 2, we believe it should be possible to reduce the upper bound to $x^{o(1)}$. Moreover extending the list of bad De Koninck primes, for example by finding additional infinite sets, seems readily achievable.

In the fundamental Lemma 1, showing that the exponent e of the power of 2 equals 1 (or at least is odd) looks like a reasonable goal, but we have been unable to prove this.

Lastly, extending the method of Theorem 3 to include subsets of $\mathcal{K}$ with prime support of bounded size, seems altogether more challenging.

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