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TOWARD A RECONNECTION MODEL FOR SOLAR FLARE STATISTICS

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ABSTRACT

A model to account for observed solar flare statistics in terms of a superposition of independent random flaring elements (assumed to be sites of magnetic reconnection in the coronal magnetic field and hence termed "separators") is described. A separator of length l is assumed to flare as a Poisson process in time, with a rate $\nu(l)$ inversely proportional to the Alfvén transit time for the structure. It is shown that a relationship $\mathscr{E} \propto l^{\kappa}$ between the mean energy of events $\mathscr{E}$ at a separator and the separator length implies a relationship $E \propto \tau^{\kappa}$ between individual waiting times τ and energies E of events at the separator. The most plausible $\kappa = 2$ model is found to be compatible with simple pictures for magnetohydrodynamic energy storage prior to magnetic reconnection in a current sheet with anomalous (turbulent) resistivity. Formal inversion of the observed flare frequency-energy distribution is shown to imply a distribution $P(l) \propto l^{-1}$ of the separator lengths in active regions. A simulation confirms the basic results of the model. It is also demonstrated that a model comprising time-dependent separator numbers $N = N(t)$ can reproduce an observed power-law tail in the flare waiting-time distribution, for large waiting times.

Subject headings: MHD — Sun: activity — Sun: corona — Sun: flares — Sun: magnetic fields

1. INTRODUCTION

Solar flares are dynamic events involving the transfer of energy from the magnetic field to plasma in the solar corona. It is widely believed that magnetic reconnection is the physical mechanism underlying flares, but many of the details of the explosive energy release, for instance, the role of Hall currents and plasma turbulence, remain contentious (e.g., Priest & Forbes 2000). What is agreed, on the basis of three-dimensional kinematic (Lau & Finn 1990; Priest & Titov 1996) and dynamic studies (Craig et al. 1999), is that reconnection is associated with specialized sites in active regions defined by the distribution of magnetic null points and separators (null-null lines) in the coronal plasma. The classification of reconnection solutions, into "fan" and "separator" current sheet models and quasi-cylindrical "spine" current solutions, then follows directly from the eigenstructure of magnetic null points.

Observationally, there is a large body of evidence in favor of the idea that flares involve magnetic reconnection at large-scale topological structures in the coronal magnetic field (e.g., Bentley & Mariska 1996). For example, photospheric emission in flares appears to coincide with the calculated locations of topological structures in the field (e.g., Gorbachev & Somov 1988; Brown et al. 1994). In addition, the basic features of large, two-ribbon events are commonly explained in terms of reconnection at an extended X-point or separator above a magnetic arcade (for a recent example, see Somov et al. 2002). Satellite observations of compact flares showing X-ray emission high in the corona have also been interpreted in terms of large-scale reconnection (Masuda et al. 1994). It should also be noted that there is a variety of evidence, as well as theoretical arguments, for fine-scale fragmentation of flare energy release sites (e.g., Brown et al. 1994), suggesting that energy release in large-scale events may involve a structured collapse to small length scales.

Generally speaking, reconnection theory has paid little attention to understanding the statistical nature of solar flare bursts. Of central interest is the frequency-energy distribution $\mathscr{N}(E)$, which describes the number of flares per unit energy per unit time. Observations suggest that this is a featureless power law,

$$\mathscr{N}(E) = AE^{-\gamma} , \tag{1}$$

where $\gamma \approx 1.5$ (e.g., Crosby, Aschwanden, & Dennis 1993). Flares occurring in individual active regions appear to follow the same distribution (Wheatland 2000a). The power-law form implies that there is no average flare energy: since there is no bump in the distribution, the average flare output is defined by cutoffs imposed on the spectrum. However, detailed flare models (Craig 2001), as well as simple dimensional considerations (Litvinenko 1998), suggest that flare energies should be related to the overall size of the reconnecting structure.

Consider, for example, Somov et al. (2002), who interpreted observations of the large "Bastille Day 2000" flare in terms of reconnection at a separator of length $l \approx 10^8$ m in the solar corona. If such a structure persists in the corona above an active region and flares repeatedly, then the average flare energy should be related to l. Somov et al. suggest a linear dependence, but it seems clear that no single site reconnection model whose mean output is limited by a fixed parameter l can be expected to reproduce a frequency-energy distribution consistent with the form of equation (1). This suggests that the reconnection picture may require additional ingredients, for example, flaring at multiple sites, to explain observed solar flare statistics.

A further constraint is provided by flare waiting-time statistics and, in particular, by the distribution $P(\tau)$ of waiting times τ between flares. The waiting-time distribution (WTD) in individual active regions is consistent with a Poisson process in time, i.e., is consistent with flares occurring as independent events with a constant mean rate or with a piecewise-constant rate (Wheatland 2002), although there is some evidence for flare sympathy (Moon et al. 2002). For the simplest case of a constant-rate Poisson process, the WTD is a simple exponential, $P(\tau) = \lambda \exp(-\lambda\tau)$, where λ is the mean rate of flaring. Remarkably, there is no evidence that flare energies depend on flare waiting times (Wheatland 2000b). This is difficult to understand if flares occur repeatedly at a single magnetic structure in an active region, with the structure losing free energy accumulated during each waiting time. When the WTD for events from all active regions on the Sun is considered for long periods, another feature appears: the distribution exhibits a power-law tail, for times longer than a few hours (Boffetta et al. 1999). Wheatland & Litvinenko (2002) presented an explanation for this effect, based on flares occurring as a Poisson process with a time-varying rate, the rate being modulated, e.g., by the solar cycle.

A popular model for flare statistics emphasizing the fragmentation of flare energy release sites is the avalanche model (Lu & Hamilton 1991; Lu et al. 1993), which successfully accounts for a variety of flare statistics including the frequency-energy distribution, the frequency–peak flux distribution, and observed waiting-time statistics (e.g., Charbonneau et al. 2001). In this model the flare process is assumed to consist of a cascade of elementary energy release events that trigger one another, occurring in a system in a state of self-organized criticality (SOC). The avalanche picture is fundamentally different from the large-scale current sheet models of classical reconnection in its emphasis on small-scale reconnection processes and in the role of SOC. While the avalanche model provides an interesting approach to flare statistics, it is difficult to reconcile with developing ideas of three-dimensional reconnection at large-scale topological structures in a coronal magnetic field. Other approaches to flare statistics based on fragmentation of the energy release region have also been developed. For example, simplified MHD descriptions of externally driven magnetic loops exhibit intermittent episodes of energy release with power-law frequency distributions for energy, peak flux, and event duration (Dmitruk, Gómez, & DeLuca 1998; Georgoulis, Velli, & Einaudi 1998). Once again, the connection between this approach and large-scale three-dimensional reconnection is unclear.

The present paper presents a model for flare statistics based on independent events at a multiplicity of flaring elements. Each element, or separator, flares according to a minimal set of physically based assumptions consistent with the properties of coronal magnetic reconnection. Our motivation is to provide a physical description that accords with the latest generation of large-scale three-dimensional magnetic reconnection models and that is capable of reproducing observed flare statistics. The present approach is distinct from previous work along these lines (Craig 2001; Wheatland 2002; Craig & Wheatland 2002) in that it considers random (Poisson) flaring at each flaring element, subject to a physically sensible choice for the mean energy at each flare site. This is found to lead to a new, physically plausible result for energy buildup at each separator. We also address a further question: how to account for the long-term behavior of the flare WTD.

We begin in § 2 by summarizing the main ingredients of the physically based reconnection description. In § 3 the new model is developed analytically and investigated numerically in terms of a simulation based on typical active region parameters. In § 4 a modification of the model to account for the long-term behavior of the WTD is presented, and § 5 presents a discussion of the main results.

2. PREVIOUS WORK

In Wheatland (2002) and Craig & Wheatland (2002) an initial model to describe flare statistics based on independent flaring at a number of distinct sites (separators) was introduced. The goal of the model was to account for the observed power-law distribution of flare energies and the apparent Poisson statistics of flare occurrence in individual active regions, using basic assumptions appropriate for reconnection models of flares.

The elements of the underlying reconnection model are straightforward. There are a multiplicity of flaring sites, determined by the distribution of magnetic null points and separators within the active region field. Each site is characterized by its length scale l, the separator length. Since flares are assumed to be an MHD process, the separator scale is assumed to set both the mean frequency ν and the mean energy $\mathscr{E}$ of an event, specifically as follows:

1. The flaring " tick rates " are determined by the Alfvén transit time of the separator, with a large number of transits being required to accumulate energy: $\nu(l) = v_A/(ql)$, where v_A is the Alfvén speed and $q \sim 10^4$.
2. The average energy of each event at a separator is $\mathscr{E} = Ql^\kappa$, with $1 \leq \kappa \leq 3$.

These assumptions are discussed in Craig (2001), who suggests that $\kappa = 2$ should be regarded as the favored energy scaling, since it corresponds to an exact separator reconnection solution. More specifically, it represents a flux pile-up current sheet of area l^2, whose thickness is determined, independently of l, by the microphysics (turbulent resistivity) of the reconnection mechanism (Litvinenko & Craig 2000). Less favorable energetically, in terms of flare release, are " spine " models, whose tubular current distributions provide the scaling $\mathscr{E} \propto l$. By contrast, for purely classical current sheet mechanisms, for instance, models in which all dimensions of the current layer scale linearly with l, the scaling is $\mathscr{E} \propto l^3$. In what follows, we retain the general form $\mathscr{E} \propto \mathscr{L}^\kappa$ for analytic purposes but continue to regard $\kappa = 2$ as the most plausible physical model (see also the dimensional argument of Litvinenko 1998).

Suppose the probability distribution of events of energy E at the separator is denoted by $P(E|l)$. The frequency-energy distribution of events at the separator is then

$$\mathscr{N}(E,\ l) = \nu(l)P(E|l)\ . \tag{2}$$

Assuming separators flare independently and that there is a universal probability distribution $P(l)$ of separator lengths, the observed frequency-energy distribution of flares from the model " active region " consisting of N separators is

$$\mathscr{N}(E) = N \int_l \mathscr{N}(E,\ l)P(l)\,dl\ . \tag{3}$$

This is an integral equation whose kernel $\mathscr{N}(E,\ l)$ is determined by the flaring properties of the separator.

In the "clockwork" model of Wheatland (2002), each separator flares periodically and produces events of a specific output E. In this case $\mathcal{N}(E,\ l) = \nu(l)\delta(E - Ql^2)$, and Wheatland shows by direct construction that the flare frequency-energy distribution can be reproduced by the choice $P(l) \propto l^{-1}$. This model also produces roughly Poisson statistics for flare occurrence, at least provided that the number of flaring elements N is large enough.

The clockwork assumption is relaxed in Craig & Wheatland (2002), who point out, using a general inversion argument, that Wheatland's result $P(l) \propto l^{-1}$ can be expected for any model that associates a well-defined mean energy $\mathcal{E} = Ql^2$ with each separator. An analytic treatment also confirms that the WTD can be explained by a superposition of periodic processes provided that $N \gtrsim 10$ (see Craig & Wheatland 2002). Strictly speaking, however, there is always a slight departure from Poisson statistics because a Poisson process has no maximum waiting time, whereas for periodic separator flaring the maximum waiting time corresponds to the period of the shortest separator.

On the observational side, it is possible that the departure from Poisson statistics could pose a problem for the model. More theoretically, although it is encouraging that statistical flare data can be reasonably well approximated using the parallel flaring approach, the periodic nature of flaring at separators seems unnecessarily restrictive, as does the idealization of a fixed number N of flaring elements. In what follows we explore relaxing these assumptions.

3. POISSON FLARING AT EACH SEPARATOR

A natural modification is to assume Poisson flaring at each separator. Recall that a Poisson process is appropriate if flares are independent, i.e., there is a constant probability per unit time of an event occurring. If an individual separator produces flares as a Poisson process with a mean rate $\nu = \nu(l)$, then the WTD for flares at the separator is

$$P(\tau|l) = \nu e^{-\nu\tau}\ , \qquad (4)$$

where τ denotes a waiting time. As in the previous models, we assume $\nu(l) = v_{\rm A}/(ql)$.

We obtain the energy probability distribution at the separator by continuity:

$$P(E|l)\, dE = P(\tau|l)\, d\tau\ . \qquad (5)$$

The constraint that the mean energy satisfies

$$\mathcal{E} = Ql^{\kappa} = \int EP(E|l)\, dE \qquad (6)$$

implies a monotonic functional relationship between the waiting time τ and the flare energy E. To see this, write equation (6) in the form

$$Ql^{\kappa} = \int_0^{\infty} P(\tau|l)E(\tau)\, d\tau \qquad (7)$$

and substitute for $P(\tau|l)$ from equation (4). The result is

$$\int_0^{\infty} E(\tau)e^{-\nu\tau}\, d\tau = \mathcal{L}[E(\tau);\ \nu] = Q\left(\frac{v_{\rm A}}{q}\right)^{\kappa} \nu^{-(1+\kappa)}\ , \qquad (8)$$

where $\mathcal{L}$ denotes the Laplace transform. Inverting the Laplace transform gives

$$E(\tau) = B\tau^{\kappa},\ \text{where } B \equiv \left(\frac{v_{\rm A}}{q}\right)^{\kappa} \frac{Q}{\Gamma(1+\kappa)}\ , \qquad (9)$$

which specifies the explicit flare energy dependence on the waiting time at the separator.

3.1. *Energy Waiting Time Relation for $\mathcal{E} = Ql^2$*

It is interesting that the basic $\kappa = 2$ model provides the scaling $E \propto \tau^2$. The implication is that the energy accumulates, between flares, according to a linear increase in the flare $\boldsymbol{B}$-field with time. This behavior seems appropriate for a flux pile-up reconnection picture, in which flux accumulates in the vicinity of a current sheet prior to the onset of reconnection. It is also notable that other simple MHD energy supply pictures yield this scaling: for example, the braiding of multiple loops subject to random footpoint motions leads to a quadratic increase in coronal magnetic energy with time (Berger 1994). This contrasts with the linear $E(\tau)$ relation obtained by random twisting of individual flux tubes (Sturrock & Uchida 1981), which from the present viewpoint is energetically less plausible.

A further interesting result is provided by recalling that the mean flare waiting time at a given separator is ν^{-1}. According to equation (9), $E(\tau = \nu^{-1})$ is exactly one-half the mean separator energy $\mathcal{E}$ for the case $\kappa = 2$. The exact distribution of energies of events at a separator follows from equation (5) together with equation (4):

$$P(E|l) = \frac{1}{2B^{1/2}}\frac{\nu}{E^{1/2}}\exp\left[-\nu\left(\frac{E}{B}\right)^{1/2}\right]\ , \qquad (10)$$

where $B = (1/2)Q(v_{\rm A}/q)^2$.

According to equation (2), the frequency-energy distribution of flares at a separator is given by $\mathcal{N}(E,\ l) = \nu(l)P(E|l)$. From equation (10) the energy dependence in this relationship is $\mathcal{N}(E,\ l) \propto E^{-1/2}\exp[-\nu(E/B)^{1/2}]$. This shows that the distribution is a power law with index $-1/2$, with an exponential rollover around $E = B/\nu^2 = (1/2)Ql^2 \approx \mathcal{E}$. This demonstrates two points. First, the frequency-energy distribution at a single separator does not have the $\propto E^{-1.5}$ form observed for flares on the Sun. [Since $\mathcal{N}(E,\ l) \propto E^{1/\kappa - 1}$ is the general power-law form, neglecting the rollover, this conclusion holds good for all $1 \leq \kappa \leq 3$.] Second, the distribution reveals the mean energy of events at the separator, and hence the separator length, via a rollover around $E \approx \mathcal{E}(l)$. This is consistent with the arguments presented in § 1, that any macroscopic reconnection model will have a frequency-energy distribution that reveals a mean energy, related to the size of the reconnecting structure.

3.2. *The Superposition of Randomly Flaring Separators*

As noted above, the observed power-law form for the flare frequency-energy distribution is not reproduced by considering a single separator site along the lines discussed above. By superposing over an assemblage of flaring separators, however, a general inversion procedure can be developed, valid for any well-defined spectrum. We give here a formal inversion argument, along the lines of Craig & Wheatland (2002), before specializing to the favored $\kappa = 2$ model.

First note that according to equation (3), we must superpose over all separator lengths in the active region assemblage. Since $P(l)\,dl = P(\nu)\,d\nu$, this can also be accomplished by superposing over all mean frequencies ν according to

$$\mathcal{N}(E) = N\int_0^{\infty} \mathcal{N}(E,\ \nu)P(\nu)\,d\nu\ , \tag{11}$$

where, using equations (1) and (4),

$$\mathcal{N}(E,\ \nu) = \mathcal{N}[E,\ l(\nu)] = \nu P(\tau|l)\frac{d\tau}{dE}\ . \tag{12}$$

If we define

$$G(\tau) \equiv \mathcal{N}(E)\frac{dE}{d\tau}\ , \tag{13}$$

then we deduce that

$$N\int_0^{\infty} e^{-\tau\nu}\nu^2 P(\nu)\,d\nu = N\mathcal{L}\left[\nu^2 P(\nu);\ \tau\right] = G(\tau)\ , \tag{14}$$

which provides the formal inversion:

$$\nu^2 P(\nu) = \frac{1}{N}\mathcal{L}^{-1}[G(\tau);\ \nu]\ . \tag{15}$$

For the specific case of interest, we have that

$$\mathcal{N}(E) = AE^{-\gamma}\ , \quad E(\tau) = B\tau^{\kappa}\ ,$$
$$G(\tau) = \kappa AB^{1-\gamma}\tau^{\kappa-1-\gamma\kappa}\ , \tag{16}$$

where $\gamma \approx 1.5$ and B is defined by equation (8). The inversion formula then gives

$$\nu^2 P(\nu) = N^{-1}\kappa AB^{1-\gamma}\frac{\nu^{\gamma\kappa-\kappa}}{\Gamma(1+\gamma\kappa-\kappa)}\ . \tag{17}$$

Taking $\gamma = 3/2$ and $\kappa = 2$, we deduce that $P(\nu) \propto \nu^{-1}$. Since $P(l) = P(\nu)|d\nu/dl|$, we again recover $P(l) \propto l^{-1}$, consistent with Wheatland (2002) and Craig & Wheatland (2002). Therefore, given a well-defined mean energy $\mathcal{E} = Ql^2$ at each separator, the observations imply $P(l) \propto l^{-1}$ for the distribution of separator lengths.

Some care should be taken in interpreting the formal inversion results. Note that the observed power-law form $\mathcal{N} \propto E^{-\gamma}$ cannot hold for all values of E; there will always be upper and lower energy roll-offs in the spectrum corresponding to upper and lower limits for the flare energy. A related point is that any probability distribution $P(l)$ must be normalizable, and this cannot be achieved without invoking cutoffs in the separator-length distribution. These subtleties, ignored in the formal treatment given above, are now addressed in the context of the favored $\kappa = 2$ model.

3.3. *The $P(l) \propto l^{-1}$ Model*

Motivated by the previous inversion, we now consider the normalized form

$$P(l) = \begin{cases} \dfrac{l^{-1}}{\ln(l_2/l_1)}\ , & l_1 \le l_2\ , \\ 0\ , & \text{otherwise}\ , \end{cases} \tag{18}$$

where l_1 and l_2 are the lower and upper cutoffs in l (the smallest and largest separators, respectively). Using this expression in equation (3) together with equation (10) leads to

$$\mathcal{N}(E) = \frac{NB^{1/2}}{2\ln(l_2/l_1)}E^{-3/2}e^{-l_1x/l_2} \times\left[1 + \frac{l_1}{l_2}x - (1+x)e^{-x(1-l_1/l_2)}\right]\ , \tag{19}$$

where $x \equiv (2E/Q)^{1/2}/l_1$ and $B = (1/2)(v_{\rm A}/q)^2 Q$. It is easy to see that for $1 \ll x \ll l_2/l_1$, this expression evaluates to

$$\mathcal{N}(E) \approx CE^{-3/2} \text{ with } C = \frac{NB^{1/2}}{2\ln(l_2/l_1)}\ . \tag{20}$$

It follows that there is power-law behavior for the range of energies $\mathcal{E}_1 \ll E \ll \mathcal{E}_2$, where $\mathcal{E}_i \equiv Ql_i^2$ $(i = 1, 2)$. Outside this range the distribution $\mathcal{N}(E)$ rolls over.

The total rate of flaring λ in the model is obtained by integrating equation (19), which is rather complicated. An approximate expression for the rate of flaring λ_1 above $\mathcal{E}_1$ is obtained by integrating equation (20) between the approximate limits of power-law behavior $\mathcal{E}_1$ and $\mathcal{E}_2$, which leads to

$$\lambda_1 \approx \frac{Nv_{\rm A}}{ql_l\ln(l_2/l_1)}\ . \tag{21}$$

Note that this approximate result is exact in the clockwork model of Wheatland (2002) (his eq. [13]).

It is interesting to simulate the model, to confirm the analytic results. To do this we need to decide on values of the basic parameters. A characteristic Alfvén speed is $v_{\rm A} = 10^6$ m s^{-1}. The parameter Q determines the basic scaling between separator length and average flare energy at the separator. If we adopt a maximum separator length $l_2 = 3\times10^8$ m, corresponding to an average flare energy at that separator of $\mathcal{E}_2 = 10^{25}$ J, then we obtain the estimate $Q = \mathcal{E}_2/l_2^2 \approx 10^8$ J m^{-2}, which we assume for the simulation. The value l_2 corresponds to the approximate maximum physical extent of magnetic regions on the Sun, and $\mathcal{E}_2$ is consistent with the energy of a large flare. We consider a large, complex active region, which we assume to contain $N = 50$ separators. Wheatland (2001) found that large active regions produce up to 100 flares with soft X-ray peak flux greater than 10^{-6} W m^{-2} at the Earth during a transit of the disk. For the purposes of illustration, it is convenient to assume that the flare energy can be closely identified with the energy radiated in X-rays. In this case the threshold events correspond to an energy $E_{\rm th} \approx 10^{20}$ J. For simplicity (and to see the effect of a cutoff), we assume this energy is the mean energy for the shortest separator in the simulation, i.e., take $\mathcal{E}_1 = E_{\rm th}$, which then implies a lower cutoff in separator length $l_1 = (\mathcal{E}_1/Q)^{1/2} = 10^6$ m. If we assume 100 flares during a 14 day transit, we have a rate $\lambda_1 \approx 8.3\times10^{-5}$ s^{-1}. Using equation (21) with the adopted values of λ_1, N, $v_{\rm A}$, l_1, and l_2 then implies $q \approx 10^5$, which we take for this simulation.

Figure 1 shows the result of a simulation of 200 days of flaring from the model with these parameters. A set of 50 separators with lengths l_i $(i = 1, 2, \ldots, 50)$ was chosen according to the distribution in equation (18), and 400 days of flaring was simulated according to a Poisson process at each separator with the appropriate mean frequency. Event energies were determined by each waiting time τ at a separator, according to the $E = B\tau^2$ rule. The top panel shows the frequency-energy distribution of events for the simulation,

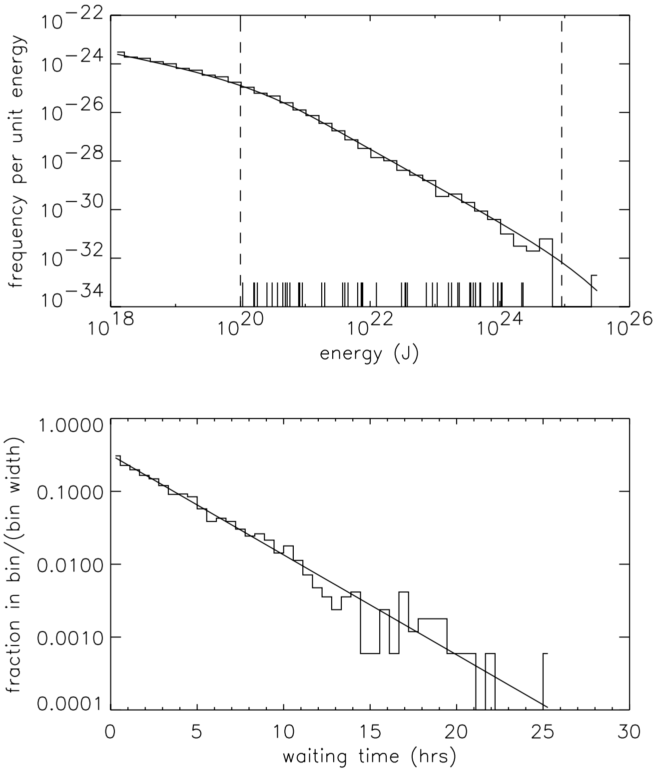

FIG. 1.—Simulation of 400 days of flaring for the $\kappa = 2$ model. *Top*: Energy distribution. *Bottom*: Waiting-time distribution.

as a solid histogram. The vertical dashed lines indicate $\mathscr{E}_1$ and $\mathscr{E}_2$, the average energies of events at the shortest and longest separators, respectively. The short solid vertical lines at the bottom again indicate the mean energies of the 50 separators chosen for this trial of the simulation. The solid curve is the analytic result in equation (19). This plot shows that the distribution is an approximate power law between the two limits $\mathscr{E}_1$ and $\mathscr{E}_2$, as expected, and more generally confirms the result of the analytic calculation. The bottom panel shows the WTD (*solid histogram*) as a loglinear plot and also shows the Poisson WTD (*solid line*) expected for the observed total rate.

A number of points about this simulation should be noted. First, the lower cutoff l_1 has been chosen for computational convenience and to illustrate the appearance of a low-energy rollover in the model. It is not intended to realistically model observed energy release events on the Sun, which appear to exhibit a power-law frequency-energy distribution down to the smallest observable events, either with the same index as that observed for large flares (e.g., Shimizu 1995) or with a steeper index (e.g., Benz & Krucker 1998). Based on equation (19), the model can reproduce a power law (with $\gamma = 3/2$) to arbitrarily low energies for a suitable choice of l_1. A related point to note is that the upper cutoff is intended to reflect a physical limit to the size of separators, and there is some observational evidence for a high-energy rollover in the frequency-energy distribution for flares (Kucera et al. 1997). The precise value of the model rollover can be adjusted by changing l_2. A final point to note is that the confirmation of equation (19) in the top panel of Figure 1 is nontrivial because the analytic result involves the approximation of a continuous distribution of separators, while the simulation involves the assumption of a finite number of separators with fixed lengths. However, the assumption of a static set of separators is unrealistic for active regions on the Sun, which are known to involve highly dynamic magnetic fields, and accordingly we do not attribute great significance to a departure from equation (19) for a static model with small numbers of separators. In § 4 we investigate a simple generalization to time-dependent separator numbers.

4. TIME-DEPENDENT SEPARATOR NUMBERS

Analysis of flare occurrence in individual active regions suggests that the flaring rate varies with time (Wheatland 2001). In addition, the WTD for flares from all active regions on the Sun observed for long periods of time exhibits a power-law tail for waiting times greater than about 10 hr (Boffetta et al. 1999). It should be noted that this behavior is distinctly different from the exponential WTD expected for a constant-rate Poisson process. However, the result can be explained in terms of flares occurring as a time-dependent Poisson process with a certain time distribution of rates (Wheatland & Litvinenko 2002).

These features of flare occurrence on the Sun can be included in the model developed in § 3 in a simple way by assuming that the number of separators varies with time: $N = N(t)$. Using equation (21) to relate the rate of events above energy $\mathscr{E}_1$ to the separator number leads to $\lambda_1(t) \propto N(t)$, assuming the other parameters are constant. Introducing time variation in the mean rate of flaring in this way changes the functional form of the WTD.

The form of the model WTD for the case of time-dependent separator numbers can be understood analytically as follows: Wheatland & Litvinenko (2002) pointed out that for a time-dependent Poisson process with a slowly varying rate $\lambda(t)$ (the variation must be slow with respect to a waiting time), the WTD at time t is

$$P_t(\tau) \approx \lambda(t) e^{-\lambda(t)\tau} . \qquad (22)$$

The observed WTD $P(\tau)$ for an observing interval $(0, T)$ is obtained by the average of $P_t(\tau)$ over the observing interval, weighted by the number of events $\lambda(t)\,dt$ in each interval $(t,\ t+dt)$:

$$P(\tau) = \frac{1}{N}\int_0^T \lambda(t) P_t(\tau)\,dt , \qquad (23)$$

where $N = \int_0^T \lambda(t)\,dt$ is the total number of events. Using equation (22), equation (23) can be rewritten as an integral over λ, leading to

$$P(\tau) = \frac{1}{\overline{\lambda}} \mathscr{L}\left[\lambda^2 f(\lambda);\ \tau\right] , \qquad (24)$$

where $\overline{\lambda} = N/T$ is the average rate, $\mathscr{L}$ denotes a Laplace transform, and $f(\lambda)$ is the time distribution of the rate. Using equation (21) to replace the rate in equation (24) by the separator number N gives

$$P_1(\tau) = \frac{v_A}{q_2 l_1 \ln(l_2/l_1)\overline{\lambda}} \mathscr{L}\left[N^2 F(N);\ \tau\right] , \qquad (25)$$

where $F(N)$ is the time distribution of N. The subscript 1 indicates that this is the WTD between events above energy $\mathscr{E}_1$.

It is instructive to interpret equation (25) using the Tauberian theorems for the Laplace transform. Specifically, we invoke the result that if the Laplace transform $F(s)$ of a function $f(t)$ has the form $F(s) \propto A/s^{r+1}$ as $s \to 0$, then the function has form $f(t) \propto At^r/\Gamma(r+1)$ as $t \to \infty$. From equation (25) it then follows that the appearance of a power law $P(\tau) \propto \tau^{-(3+\alpha)}$ in the WTD as $\tau \to \infty$ implies that the distribution of separator number $F(N)$ has the power-law form $F(N) \propto N^\alpha$ for $N \to 0$. Hence, the appearance of a power-law distribution in the WTD is attributable to a power law in the distribution of separator numbers, when the numbers are low. Note that even if the distribution of separator numbers is constant at low numbers ($\alpha = 0$), then a power law $\propto\tau^{-3}$ is predicted in the WTD.

To demonstrate the applicability of this model to the Sun, we have performed the following simulation: Wheatland & Litvinenko (2002) took the observed soft X-ray flares from the Sun for 1975–2001 above a peak flux 10^{-6} W m^{-2} (a "C1 class event") and determined a decomposition of the observed time history of flaring into a piecewise-constant Poisson process. Taking this time history of piecewise-constant rates and times and applying equation (21) leads to a time history of separator numbers for the Sun for 1975–2001 (these are the numbers of separators with mean flare energies greater than the energy of C1 flares: approximately 10^{20} J). The top panel of Figure 2 shows the resulting time history of separator numbers. In applying equation (21), the same values for v_A, q, l_1, and l_2 have been used as in the simulation in § 3.3. The figure reflects the variation of flare numbers with the solar cycle, and the last three solar cycles are clearly visible. Next, a time history of flaring was simulated for a model with this sequence of separator numbers N_i and intervals t_i. Specifically, for each time interval t_i the simulation procedure described in § 3.3 was performed, with N_i separators. The resulting WTD for the period 1975–2001 is shown by the solid histogram in the bottom panel of Figure 2. This log-log plot shows that the simulated WTD displays an extended power-law tail for waiting times greater than a few hours. The observed WTD is shown by the crosses in the figure (see Wheatland & Litvinenko 2002). The observed and simulated distributions are very similar and exhibit the same power-law tail (the power-law index of the tail of the observed WTD is -2.2 ± 0.1). There are differences in detail between the simulated and observed distribution. However, the piecewise-constant decomposition of the flare rate from which the separator numbers are determined is an approximation. This simulation is not intended to provide a completely realistic model of the flaring Sun over several solar cycles but is intended to show how a power-law tail to the WTD, such as that observed, can appear in the model when there is a time variation in separator numbers.

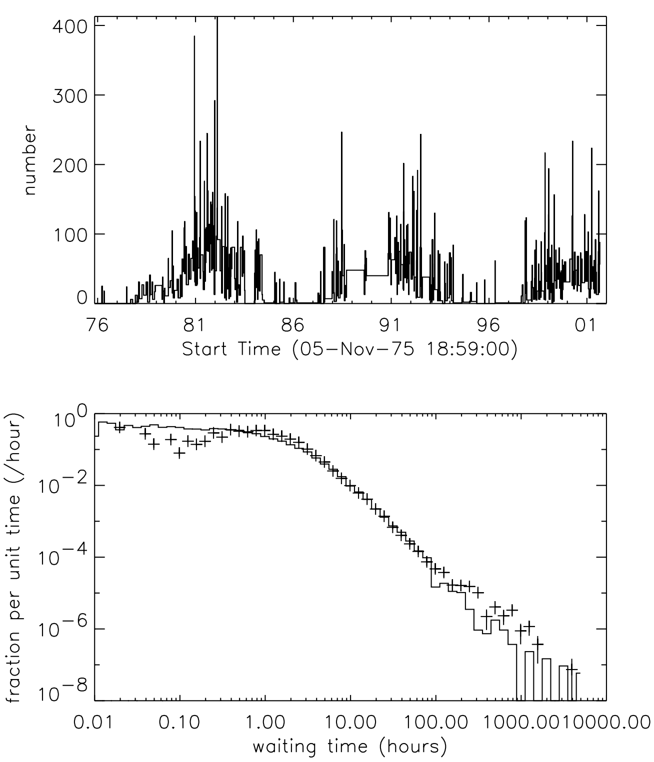

FIG. 2.—Simulation with time-dependent separator numbers, mimicking the variation of the total flare rate with the solar cycle. *Top*: Separator numbers. *Bottom*: Waiting-time distribution.

5. DISCUSSION AND CONCLUSIONS

In this paper a model for flare statistics is presented involving independent, random flaring at N distinct sites, each characterized by a single parameter, a length l. These sites are to be identified with topological features in the active region—null points and separators—at which magnetic reconnection can proceed. Accordingly, the sites are given basic physical properties consistent with MHD models of preflare reconnecting structures: the mean rate of flaring then reflects the large number of Alfvén transit times required to build up and orchestrate flare energy into a near-singular current sheet, while the average release energy reflects the geometrical extent of the site $\mathcal{E} \propto l^\kappa$, with $\kappa = 2$ the preferred value for current sheet reconnection. The assumption that each site produces flares as a Poisson process in time implies that there is a simple relationship $E \propto \tau^\kappa$ between the waiting time τ of an event and its release energy. For the preferred value $\kappa = 2$, this result is compatible with MHD energy storage in the vicinity of a current sheet.

In § 3 we considered the problem of accounting for statistical flare data, in particular the frequency-energy distribution. A formal inversion of the inferred distribution implies, for the choice $\kappa = 2$, a distribution $P(l) \propto l^{-1}$ of separator lengths. Consideration of the necessary cutoffs in the separator-length distribution implies that there will also be rollovers in the flare frequency-energy distribution. A simulation of the model confirms the analytic results. It was also shown in § 4 how an observed power-law tail in the flare waiting-time distribution, for long waiting times, arises as a result of time dependence in the mean rate of flaring. This can be incorporated in the model in a natural way by making the number of separators time-dependent, $N = N(t)$. A simulation confirms that this approach can reproduce the observed waiting-time distribution.

The model proposed here differs from the avalanche model for flare statistics in several basic ways. Note that in the avalanche model, flares comprise a multitude of microscopic, elementary energy release events that trigger one another. In the present model flares involve a single energy release event, at a given separator, and a given separator may produce flares with a wide range of possible energies, subject to the requirement of a well-defined mean energy. Flare statistics from an active region on the Sun are accounted for in terms of a sequence of flares at different, independently flaring separators. In the avalanche model the magnetic field (or more formally, the discrete field in the cellular automata) is in a state of self-organized criticality (SOC), which accounts for the appearance of a power law in the frequency-energy distribution. In the present model there is no SOC as such, and the power law in the frequency-energy distribution arises from the assumed scalings of event frequency and mean energy with separator length, together with the assumed universal distribution of separator lengths. It should be noted that the avalanche model also provides an explanation for the appearance of power laws in flare frequency–peak flux and frequency-duration distributions. In the present model the time history of energy release in individual flares is not resolved, and no attempt is made to explain the detailed energy budget of any one event. In principle this "coarse graining" could be remedied by modeling the detailed transition from primary energy release into the secondary phases of the flare, for example, adopting a hydrodynamic model incorporating conduction, radiation, and mass transfer (e.g., McClymont & Canfield 1983).

The basic requirement of the model is a $1/l$ distribution of separator lengths, which can be seen as a prediction of the model, and in principle is amenable to observational test (Wheatland 2002). The physical origin of this distribution remains an open question. Coronal magnetic fields, and presumably topological structures in those fields, are subject to complex processes of formation, evolution, and decay. The existence of a basic distribution of separator lengths arising from the common processes of field evolution seems plausible, but at the moment we have no detailed theory to explain the physical origin of the required $1/l$ dependence.

The present model is intended as a minimalist solution to the problem of reconciling large-scale three-dimensional reconnection theory with flare statistics, and there are many limitations to the model. In reality, active region magnetic fields are intrinsically dynamic and can be expected to evolve continually over the lifetime of an active region. The idealization of a fixed number of separators with fixed lengths is best regarded as a snapshot at some instant of the global field evolution. The generalization of the model to include a time-dependent separator number (§ 4) gives some idea of how time dependence can be introduced. It is also reasonable to assume that there would be some interaction between separators (leading to the possibility of flare sympathy), when in the present model the separators are strictly independent. This possibility will be explored in future work.

The predictions of the present model for flare occurrence at individual separators can be tested by high-resolution observations at short wavelengths. Recently, Aschwanden & Parnell (2002) found that the number N_e of EUV and soft X-ray brightenings in a quiet-Sun region was distributed with length L according to $N_e \propto L^{-a}$, with $a = 2.5 \pm 0.2$. Since observations in soft X-ray and EUV reveal secondary manifestations of flare energy release, it is difficult to compare this result directly with the model developed in this paper (see the comments above regarding coarse graining). Nevertheless, we note that the distribution obtained by Aschwanden & Parnell (2002) is comparable to $\nu(l)P(l)$ for our model. For our model we have $\nu(l)P(l) \propto l^{-2}$, so the power-law dependence on l in the model is not very different from Aschwanden & Parnell's result.

The approach developed in this paper demonstrates how global reconnection models might be reconciled with observed solar flare statistics. In its current form, the physical elements provided by reconnection theory are rudimentary, which in part is due to our incomplete understanding of energy storage and release in three-dimensional magnetic reconnection. As reconnection solutions are refined, it is expected that new results can be incorporated into this model to provide a more complete description of flare phenomena.

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