

Quadratics on complex Riemannian spaces of constant curvature, separation of variables, and the Gaudin magnet

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Integrable systems that are connected with orthogonal separation of variables in complex Riemannian spaces of constant curvature are considered herein. An isomorphism with the hyperbolic Gaudin magnet, previously pointed out by one of the authors, extends to coordinates of this type. The complete classification of these separable coordinate systems is provided by means of the corresponding L matrices for the Gaudin magnet. The limiting procedures (or ϵ calculus) which relate various degenerate orthogonal coordinate systems play a crucial role in the classification of all such systems.

I. CLASSICAL INTEGRABLE SYSTEMS ON COMPLEX CONSTANT CURVATURE SPACES AND THE COMPLEX GAUDIN MAGNET

Separation of variables in the Hamilton Jacobi equation

$$H(p_1, \dots, p_n; x_1, \dots, x_n) = \sum_{\alpha, \beta=1}^n g^{\alpha\beta} p_\alpha p_\beta = E, \quad p_\alpha = \frac{\partial W}{\partial x_\alpha}, \quad \alpha = 1, \dots, n \quad (1.1)$$

amounts to looking for a solution of the form

$$W = \sum_{\alpha=1}^n W_\alpha(x_\alpha; h_1, \dots, h_n), \quad h_n = E. \quad (1.2)$$

The solution is said to be a *complete integral* if $\det(\partial^2 W / \partial x_i \partial h_j)_{n \times n} \neq 0$. The solution then describes that of free motion on the corresponding Riemannian space with contravariant metric $g^{\alpha\beta}$. Indeed, if we require $b_j = (\partial W / \partial h_j) - h_n t \delta_{nj}$, $j = 1, \dots, n$ for parameters b_j , we find that the functions $\mathbf{x}(\mathbf{b}, \mathbf{h}), \mathbf{p}(\mathbf{b}, \mathbf{h})$ satisfy Hamilton's equations

$$\dot{x}_\alpha = \frac{\partial H}{\partial p_\alpha}, \quad \dot{p}_\alpha = -\frac{\partial H}{\partial x_\alpha}. \quad (1.3)$$

In this article we allow the Riemannian space to be complex and we consider variable separation of Eq. (1.1) for the following two classes of spaces:

- (1) The n dimensional complex sphere S_{nC} . This is commonly realized by the set of complex vectors $\mathbf{x} = (x_1, \dots, x_{n+1})$ which satisfy $\sum_{\alpha=1}^{n+1} x_\alpha^2 = 1$ and have infinitesimal distance $d\mathbf{x} \cdot d\mathbf{x} = \sum_{\alpha=1}^{n+1} dx_\alpha^2$.
- (2) The n dimensional complex Euclidean space E_{nC} . This is the set of complex vectors $\mathbf{x} = (x_1, \dots, x_n)$ with infinitesimal distance $d\mathbf{x} \cdot d\mathbf{x} = \sum_{\alpha=1}^n dx_\alpha^2$.

A fundamental problem from the point of view of separation of variables on these manifolds is to find all "inequivalent" coordinate systems. As yet, this is an unsolved problem, principally

because many such coordinate systems are intrinsically nonorthogonal. For orthogonal coordinate systems the problem is completely solved and in this case the constants h_i occurring in the complete integral can be chosen to be the values of an involutive set of constants of motion

$$A_j = \sum_{\alpha, \beta=1}^n a_{\alpha\beta}^{(j)} p_\alpha p_\beta, \quad j=1, \dots, n, \quad A_n = H, \quad \{A_j, A_k\} = 0, \quad (1.4)$$

where

$$\{F(x_\alpha, p_\beta), G(x_\alpha, p_\beta)\} = \sum_{\gamma=1}^n \left(\frac{\partial F}{\partial p_\gamma} \frac{\partial G}{\partial x_\gamma} - \frac{\partial F}{\partial x_\gamma} \frac{\partial G}{\partial p_\gamma} \right), \quad \alpha, \beta = 1, \dots, n$$

is the Poisson bracket. These constants of the motion are such that

- (1) each of the tensors $a_{\alpha\beta}^{(j)}$ is a Killing tensor and satisfies Killing's equations $\nabla_{(\alpha} a_{\beta\gamma)}^{(j)} = 0$, (Ref. 1), and
- (2) A_j can be represented as a sum of quadratic elements of the enveloping algebra of the Lie algebra of symmetries of each of these two considered spaces.

The Lie algebras of these spaces have, respectively, bases of the form

- (1) $SO(n+1): M_{\alpha\beta} = x_\alpha p_\beta - x_\beta p_\alpha, \quad \alpha, \beta = 1, \dots, n+1$. Here

$$\{M_{\alpha\beta}, M_{\gamma\delta}\} = \delta_{\alpha\gamma} M_{\beta\delta} + \delta_{\alpha\delta} M_{\beta\gamma} + \delta_{\beta\gamma} M_{\alpha\delta} + \delta_{\beta\delta} M_{\alpha\gamma}.$$

- (2) $E(n): M_{\alpha\beta}, P_\gamma = p_\gamma, \quad \alpha, \beta, \gamma = 1, \dots, n, \quad \alpha \neq \beta$. Here

$$\{M_{\alpha\beta}, P_\gamma\} = \delta_{\beta\gamma} P_\alpha - \delta_{\alpha\gamma} P_\beta, \quad \{P_\alpha, P_\beta\} = 0.$$

In this section the separable coordinate systems classified in Refs. 1–3 are given an algebraic interpretation. This is done using the complex analog of the isomorphism between all integrable systems connected with all possible separable systems and the m -site $SO(2,1)$ Gaudin magnet.^{4,5} The m -site complex Gaudin magnet can be realized as follows. Consider the direct sum of Lie algebras, each of rank 1

$$\mathcal{A} = \oplus_{\alpha=1}^m \mathfrak{so}_\alpha(3, \mathbb{C}). \quad (1.5)$$

The generators $\mathbf{s}_\alpha \in \mathbb{C}^3$, $\alpha = 1, \dots, m$ of \mathcal{A} satisfy the Poisson bracket relations

$$\{s_\alpha^i, s_\beta^j\} = -\delta_{\alpha\beta} \epsilon_{ijk} s_\alpha^k. \quad (1.6)$$

The following metric will be used subsequently when norms and scalar products are calculated:

$$\mathbf{s}_\alpha^2 = (\mathbf{s}_\alpha, \mathbf{s}_\alpha) = (s_\alpha^1)^2 + (s_\alpha^2)^2 + (s_\alpha^3)^2, \quad (\mathbf{s}_\alpha, \mathbf{s}_\beta) = s_\alpha^1 s_\beta^1 + s_\alpha^2 s_\beta^2 + s_\alpha^3 s_\beta^3.$$

If for each α , $\mathbf{s}_\alpha^2 = c_\alpha^2$ then the variables \mathbf{s}_α lie on the direct product of n complex spheres in \mathbb{C}^3 . The complex Gaudin magnet is the integrable Hamiltonian system described by the n integrals of motion H_α which are in involution with respect to the Poisson bracket

$$H_\alpha = 2 \sum_{\beta=1}^m \frac{(\mathbf{s}_\alpha, \mathbf{s}_\beta)}{e_\alpha - e_\beta}, \quad \{H_\alpha, H_\gamma\} = 0. \quad (1.7)$$

(We will give a simple proof of this involution property later.) Here e_α are taken to be pairwise distinct. This integrable Hamiltonian system is called an m -site $\text{SO}(3, \mathbb{C})$ -XXX Gaudin magnet. The H_α are all quadratic functions in the generators of the \mathcal{A} algebra and the following identities are satisfied:

$$\sum_{\alpha=1}^m H_\alpha = 0, \quad \sum_{\alpha=1}^m e_\alpha H_\alpha = \mathbf{J}^2 - \sum_{\alpha=1}^m c_\alpha^2 \quad (1.8)$$

where we have introduced the variables

$$\mathbf{J} = \sum_{\alpha=1}^m \mathbf{s}_\alpha, \quad \mathbf{J}^2 = (\mathbf{J}, \mathbf{J}) \quad (1.9)$$

the total sum of the momenta \mathbf{s}_α . Indeed

$$\{J^i, J^j\} = -\epsilon_{ijk} J^k, \quad \{J^k, H_\alpha\} = 0, \quad i, j, k = 1, 2, 3, \quad \alpha = 1, \dots, m. \quad (1.10)$$

The complete set of involutive integrals of motion is provided by H_α , \mathbf{J}^2 and, for example, J^3 . The integrals are generated by the 2×2 L matrix⁵⁻⁹

$$L(u) = \sum_{\alpha=1}^m \frac{1}{u - e_\alpha} \begin{pmatrix} s_\alpha^3 & -s_\alpha^1 - is_\alpha^2 \\ -s_\alpha^1 + is_\alpha^2 & -s_\alpha^3 \end{pmatrix} = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \quad (1.11)$$

where

$$\det L(u) = -(A^2 + BC) = - \sum_{\alpha=1}^m \frac{H_\alpha}{u - e_\alpha} - \sum_{\alpha=1}^m \frac{c_\alpha^2}{(u - e_\alpha)^2}. \quad (1.12)$$

Furthermore, $L(u)$ satisfies the linear r matrix algebra^{4,10}

$$\{ \overset{1}{L}(u), \overset{2}{L}(v) \} = \frac{i}{u-v} [P, \overset{1}{L}(u) + \overset{2}{L}(v)], \quad i = \sqrt{-1}, \quad (1.13)$$

where

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \overset{1}{L}(u) = L(u) \otimes I, \quad \overset{2}{L}(v) = I \otimes L(v). \quad (1.14)$$

The algebra specified by Eq. (1.13) contains all the necessary Hamiltonian structure of the problem in question. Note that Eq. (1.13) is equivalent to the easily proved relations

$$\begin{aligned} \{A(u), A(v)\} &= \{B(u), B(v)\} = \{C(u), C(v)\} = 0, \\ \{A(u), B(v)\} &= \frac{i}{u-v} (B(v) - B(u)), \quad \{A(u), C(v)\} = \frac{i}{u-v} (C(u) - C(v)), \\ \{B(u), C(v)\} &= \frac{2i}{u-v} (A(v) - A(u)). \end{aligned} \quad (1.15)$$

From Eq. (1.15) and the Leibnitz property of the Poisson bracket it is straightforward to deduce that

$$\{\det L(u), \det L(v)\} = 0.$$

In his article⁵ Kuznetsov has explicitly given the nature of the isomorphism between the XXX Gaudin magnet models and the separation of variables on the n dimensional real sphere S_n . The purpose of this article is to extend these ideas to complex orthogonal coordinate systems on the complex n sphere S_{nC} and, of course, as a consequence complex Euclidean space E_{nC} . Following Kuznetsov^{5,8} in the case of the sphere, we set $c_\alpha = 0$, $\alpha = 1, \dots, n+1$. The coordinates on the resulting cones are parametrized by

$$s_\alpha^1 = \frac{1}{4}(p_\alpha^2 + x_\alpha^2), \quad s_\alpha^2 = \frac{i}{4}(p_\alpha^2 - x_\alpha^2), \quad s_\alpha^3 = \frac{i}{2}p_\alpha x_\alpha. \quad (1.16)$$

It follows from Eq. (1.6) that $\{x_\alpha, x_\beta\} = \{p_\alpha, p_\beta\} = 0$, $\{p_\alpha, x_\beta\} = \delta_{\alpha\beta}$. Introducing the new variables $M_{\alpha\beta} = x_\alpha p_\beta - x_\beta p_\alpha$ which are the generators of rotations we have

$$(s_\alpha, s_\beta) = \frac{1}{8}M_{\alpha\beta}^2.$$

These generators satisfy the commutations relations given previously. This equality establishes the simple quadratic connection between the generators s_α of \mathcal{A} and the $M_{\alpha\beta}$ of $SO(n+1)$. Under this isomorphism the integrals given in Eq. (1.7) (and the subsequent discussion) transform into the following integrals for the free motion on the n sphere:

$$H = \sum_{\alpha=1}^{n+1} h_\alpha H_\alpha = \frac{1}{4} \sum_{\alpha < \beta} \frac{h_\alpha - h_\beta}{e_\alpha - e_\beta} M_{\alpha\beta}^2.$$

For $h_\gamma = e_\gamma$ we obtain the Casimir element of the $so(n+1)$ algebra $\sum_{\alpha < \beta} M_{\alpha\beta}^2$. The total momentum \mathbf{J} takes the form

$$J^1 = \frac{1}{4}(\mathbf{p} \cdot \mathbf{p} + \mathbf{x} \cdot \mathbf{x}), \quad J^2 = \frac{i}{4}(\mathbf{p} \cdot \mathbf{p} - \mathbf{x} \cdot \mathbf{x}), \quad J^3 = \frac{i}{2}\mathbf{p} \cdot \mathbf{x},$$

where the scalar product for the vectors \mathbf{x} and \mathbf{p} in C^{n+1} is Euclidean. The quantities \mathbf{M} and \mathbf{J} form the direct sum $so(n+1) \oplus so(3)$ as a result of the commutation relations

$$\{M_{\alpha\beta}, J^i\} = 0, \quad \{J^i, J^j\} = -\epsilon_{ijk} J^k.$$

Therefore, in addition to the involutive set of integrals H_α we can choose $J^2 = \frac{1}{4}\sum_{\alpha < \beta} M_{\alpha\beta}^2$ and $2(J^1 + iJ^2) = \mathbf{x} \cdot \mathbf{x} = c$, which gives the equation of the n sphere.

II. GENERIC ELLIPSOIDAL COORDINATES ON S_{nC} AND E_{nC}

Critical to the separation of variables on the n sphere S_{nC} is the system of ellipsoidal coordinates graphically pictured by the irreducible block

$$(S_{nC} | e_1 | e_2 | \dots | e_{n+1} |), \quad (2.1)$$

where in general $e_\alpha \neq e_\beta$ for $\alpha \neq \beta$. The separation variables are defined as zeros of the off diagonal element $B(u)$ of the L matrix, i.e., $B(u_j) = 0$, $j = 1, \dots, n$. It follows that

$$\sum_{\alpha=1}^{n+1} \frac{x_{\alpha}^2}{u-e_{\alpha}} = 0, \quad \text{for } u=u_j \quad \text{and} \quad x_{\alpha}^2 = c \frac{\prod_{j=1}^n (u_j - e_{\alpha})}{\prod_{\beta \neq \alpha} (e_{\beta} - e_{\alpha})}. \quad (2.2)$$

Each vector of momentum \mathbf{s}_{α} is associated with a cell e_{α} of the block. Note that $x_{\alpha}^2 = 2(s_{\alpha}^1 + is_{\alpha}^2)$. For each u_j the conjugate variable v_j is defined according to

$$v_j = -iA(u_j) = \frac{1}{2} \sum_{\alpha=1}^{n+1} \frac{x_{\alpha} p_{\alpha}}{u_j - e_{\alpha}}. \quad (2.3)$$

From Eq. (1.15) one can show that the u_i, v_i satisfy the canonical relations⁵

$$\{u_j, u_i\} = \{v_j, v_i\} = 0, \quad \{v_j, u_i\} = \delta_{ij}. \quad (2.4)$$

The change to the new variables v_j, u_i, c and J^3 is effectively the procedure of variable separation of Eq. (1.1) in ellipsoidal coordinates on the n sphere. Writing the L matrix in terms of the new variables we obtain

$$L(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & -A(u) \end{pmatrix}, \quad B(u) = -\frac{c}{2} \frac{\prod_{j=1}^n (u - u_j)}{2 \prod_{\alpha=1}^{n+1} (u - e_{\alpha})}, \quad (2.5)$$

$$A(u) = -\frac{2i}{c} B(u) \left(-iJ^3 + \sum_{j=1}^n \frac{v_j}{u - u_j} \frac{\prod_{\gamma=1}^{n+1} (u_j - e_{\gamma})}{\prod_{i \neq j} (u_j - u_i)} \right),$$

where $A(u_j) = iv_j$, $j=1, \dots, n$, and $A(u) \rightarrow (1/u)J^3 + \dots$ as $u \rightarrow \infty$.

To obtain $C(u)$ we first notice that equating residues at e_{α} on the right- and left-hand side of $A(u)$ gives

$$p_{\alpha} = \frac{2x_{\alpha}}{c} \left(-iJ^3 + \sum_{j=1}^n \frac{v_j}{e_{\alpha} - u_j} \frac{\prod_{\gamma=1}^{n+1} (u_j - e_{\gamma})}{\prod_{i \neq j} (u_j - u_i)} \right).$$

This together with the expression (2.2) for x_{α}^2 in terms of u_j gives $C(u)$ in the new variables. Three other useful formulas are

$$M_{\alpha\beta} = \frac{2x_{\alpha}x_{\beta}}{c} (e_{\alpha} - e_{\beta}) \sum_{j=1}^n v_j \frac{\hat{\prod}_{\gamma=1}^{n+1} (u_j - e_{\gamma})}{\prod_{i \neq j} (u_j - u_i)}, \quad (2.6)$$

where the hat in Eq. (2.6) means that the product terms with $\gamma=\alpha$ and $\gamma=\beta$ are omitted, and

$$J^2 = - \sum_{j=1}^n v_j^2 \frac{\prod_{\gamma=1}^{n+1} (u_j - e_{\gamma})}{\prod_{i \neq j} (u_j - u_i)} = \sum_{j=1}^n c v_j^2 \left(\sum_{\alpha=1}^{n+1} \frac{x_{\alpha}^2}{(u_j - e_{\alpha})^2} \right)^{-1}, \quad (2.7)$$

$$H_{\alpha} = \frac{x_{\alpha}^2}{c} \sum_{j=1}^n \frac{v_j^2}{u_j - e_{\alpha}} \frac{\prod_{\gamma=1}^{n+1} (u_j - e_{\gamma})}{\prod_{i \neq j} (u_j - u_i)}.$$

These relations together with Eq. (2.2) establish the explicit connection between the two sets of $2n+2$ variables p_{α}, x_{α} and u_i, v_j, c, J^3 . The equation for the eigenvalue curve $\Gamma: \det(L(u) - i\lambda I) = 0$ has the form

$$-\lambda^2 - A(u)^2 - B(u)C(u) = 0.$$

If we put $u=u_j$ into this equation then $\lambda = \pm v_j$. Thus variables u_j and v_j lie on the curve Γ

$$v_j^2 + \sum_{\alpha=1}^n \frac{H_\alpha}{u_j - e_\alpha} \equiv v_j^2 - \det L(u_j) = 0. \quad (2.8)$$

Equations (2.8) are the separation equations for each of the n degrees of freedom connected with the values of the integrals H_α . For the sphere $c_\alpha = 0$ these have the form

$$H_\alpha = \frac{1}{4} \sum_{\beta} \frac{M_{\alpha\beta}^2}{e_\alpha - e_\beta}, \quad \sum_{\alpha} H_\alpha = 0, \quad \sum_{\alpha} e_\alpha H_\alpha = \frac{1}{4} \sum_{\alpha < \beta} M_{\alpha\beta}^2 = J^2.$$

The Hamilton Jacobi equation (1.1) when parametrized by these variables has the form

$$\frac{1}{4} \sum_{\alpha < \beta} M_{\alpha\beta}^2 = J^2 = - \sum_{j=1}^n \left(\frac{\partial W}{\partial u_j} \right)^2 \frac{\prod_{\gamma=1}^{n+1} (u_j - e_\gamma)}{\prod_{i \neq j} (u_j - u_i)} = E, \quad v_j = \frac{\partial W}{\partial u_j}, \quad (2.9)$$

which can be solved by the separation of variables ansatz

$$W = \sum_{j=1}^n W_j(u_j, H_1, \dots, H_n) = \sum_{j=1}^n \int v_j du_j. \quad (2.10)$$

It will be convenient to employ an alternative form of the L matrix. If we use the vector

$$\mathbf{L}(u) = \sum_{\alpha=1}^{n+1} \frac{\mathbf{s}_\alpha}{u - e_\alpha}, \quad L(u) = (\mathbf{s}, \mathbf{L}(u)), \quad \det L(u) = -\mathbf{L}(u) \cdot \mathbf{L}(u), \quad (2.11)$$

where

$$s^1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad s^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad s^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

we see that \mathbf{L} satisfies

$$\{L^i(u), L^j(v)\} = \frac{\epsilon_{ijk}}{u-v} (L^k(u) - L^k(v)). \quad (2.12)$$

At this point we must consider a crucial difference between the real sphere and its complex counterpart. In the case of the complex sphere the generic ellipsoidal coordinates can admit multiply degenerate forms: the restriction $e_\alpha \neq e_\beta$, for $\alpha \neq \beta$ can be lifted. The resulting coordinates can be denoted by the block form

$$(S_{n\mathbb{C}} | e_1^{\lambda_1} | e_2^{\lambda_2} | \cdots | e_q^{\lambda_q} |, \quad \lambda_1 + \cdots + \lambda_q = n+1,$$

where the λ_α denote the multiplicities of e_α . To understand how the previous analysis applies to these types of coordinates we first illustrate with an example corresponding to the coordinates with diagram

$$(S_{n\mathbb{C}} | e_1^2 | e_3 | \cdots | e_{n+1} |).$$

In this case we write

$$\mathbf{L}(u) = \frac{a_1 \mathbf{s}_1}{u - e_1} + \frac{a_2 \mathbf{s}_2}{u - e_2} + \sum_{\alpha=3}^{n+1} \frac{\mathbf{s}_\alpha}{u - e_\alpha}. \quad (2.13)$$

Putting

$$\begin{aligned} x_2 \rightarrow x'_1 + \epsilon x'_2, \quad x_1 \rightarrow x'_1, \quad p_2 \rightarrow p'_1 + \epsilon p'_2, \quad p_1 \rightarrow p'_1, \\ a_1 + a_2 = 0, \quad a_2 = 1/\epsilon, \quad x_\alpha \rightarrow x'_\alpha, \quad p_\alpha \rightarrow p'_\alpha, \quad (\alpha \geq 3), \quad e_2 = e_1 + \epsilon \end{aligned} \quad (2.14)$$

then in the limit as $\epsilon \rightarrow 0$ we find

$$\mathbf{L}(u) = \frac{\mathbf{z}_1}{(u - e_1)^2} + \frac{\mathbf{z}_2}{u - e_1} + \sum_{\alpha=3}^{n+1} \frac{\mathbf{z}_\alpha}{u - e_\alpha},$$

where

$$\begin{aligned} \mathbf{z}_1 &= \left\langle \frac{1}{4} (p_1'^2 + x_1'^2), \frac{i}{4} (p_1'^2 - x_1'^2), \frac{i}{2} p_1' x_1' \right\rangle, \\ \mathbf{z}_2 &= \left\langle \frac{1}{2} (p_1' p_2' + x_1' x_2'), \frac{i}{2} (p_1' p_2' - x_1' x_2'), \frac{i}{2} (p_1' x_2' + p_2' x_1') \right\rangle, \\ \mathbf{z}_\alpha &= \mathbf{s}_\alpha, \quad \alpha = 3, \dots, n+1. \end{aligned} \quad (2.15)$$

The components of $\mathbf{z}_1, \mathbf{z}_2$ satisfy the $E(3, \mathbb{C})$ algebra relations

$$\{z_1^i, z_1^j\} = 0, \quad \{z_1^i, z_2^j\} = -\epsilon_{ijk} z_1^k, \quad \{z_2^i, z_2^j\} = -\epsilon_{ijk} z_2^k,$$

where the Poisson bracket (expressed in the primed coordinates) is

$$\{F, G\} = \sum_{j,k=1}^{n+1} B_{jk} \left(\frac{\partial F}{\partial p_j'} \frac{\partial G}{\partial x_k'} - \frac{\partial F}{\partial x_j'} \frac{\partial G}{\partial p_k'} \right)$$

and

$$B = (B_{jk}) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

It can easily be verified from these relations that relations (2.12) are again satisfied. Thus, $\{\mathbf{L} \cdot \mathbf{L}(u), \mathbf{L} \cdot \mathbf{L}(v)\} = 0$ so the coefficients of the various powers $(u - e_j)^{-k}$ in the expansion of $\mathbf{L} \cdot \mathbf{L}(u)$ form an involutive set of integrals of motion.

We obtain

$$\begin{aligned} \mathbf{L} \cdot \mathbf{L} &= \frac{1}{(u - e_1)^4} \mathbf{z}_1 \cdot \mathbf{z}_1 + \frac{2}{(u - e_1)^3} \mathbf{z}_1 \cdot \mathbf{z}_2 + \frac{1}{(u - e_1)^2} \mathbf{z}_2 \cdot \mathbf{z}_2 + \sum_{\alpha \geq 3} \frac{2\mathbf{z}_1 \cdot \mathbf{z}_\alpha}{(u - e_1)^2 (u - e_\alpha)} \\ &+ \sum_{\alpha \geq 3} \frac{2\mathbf{z}_2 \cdot \mathbf{z}_\alpha}{(u - e_1)(u - e_\alpha)} + \sum_{\alpha, \beta \geq 3, \alpha \neq \beta} \frac{1}{(u - e_\alpha)(u - e_\beta)} (\mathbf{z}_\alpha \cdot \mathbf{z}_\beta) \end{aligned}$$

$$\begin{aligned}
&= \frac{-1}{4(u-e_1)^2} (p'_1 x'_2 - p'_2 x'_1)^2 + \sum_{\alpha=3}^{n+1} \left[\frac{1}{4(u-e_1)^2(u-e_\alpha)} (p'_1 x'_\alpha - p'_\alpha x'_1)^2 \right. \\
&\quad \left. + \frac{1}{2(u-e_1)(u-e_\alpha)} (p'_1 x'_\alpha - p'_\alpha x'_1)(p'_2 x'_\alpha - p'_\alpha x'_2) \right] \\
&\quad + \sum_{\alpha, \beta=3, \alpha \neq \beta}^n \frac{1}{8(u-e_\alpha)(u-e_\beta)} (p'_\beta x'_\alpha - p'_\alpha x'_\beta)^2. \tag{2.16}
\end{aligned}$$

In particular

$$\begin{aligned}
\mathbf{z}_1 \cdot \mathbf{z}_1 &= 0, \quad \mathbf{z}_1 \cdot \mathbf{z}_2 = 0, \quad \mathbf{z}_\alpha \cdot \mathbf{z}_1 = (p'_1 x'_\alpha - p'_\alpha x'_1)^2 / 8, \quad \mathbf{z}_2 \cdot \mathbf{z}_2 = -(p'_1 x'_2 - p'_2 x'_1)^2 / 4, \\
\mathbf{z}_\alpha \cdot \mathbf{z}_2 &= (p'_1 x'_\alpha - p'_\alpha x'_1)(p'_2 x'_\alpha - p'_\alpha x'_2) / 4.
\end{aligned}$$

To relate this to the projective coordinates on the complex n sphere we recall that under the transformation (2.14) the fundamental quadratic forms $X = a_1 x_1^2 + a_2 x_2^2 + \sum_{\alpha=3}^{n+1} x_\alpha^2$, $P = a_1 p_1^2 + a_2 p_2^2 + \sum_{\alpha=3}^{n+1} p_\alpha^2$ transform to $X = 2x'_1 x'_2 + \sum_{\alpha=3}^{n+1} x_\alpha'^2$, $P = 2p'_1 p'_2 + \sum_{\alpha=3}^{n+1} p_\alpha'^2$. Therefore if we take the coordinates

$$\begin{aligned}
x'_1 &= \frac{1}{\sqrt{2}} (X_1 + iX_2), \quad x'_2 = \frac{1}{\sqrt{2}} (X_1 - iX_2), \quad x'_\alpha = X_\alpha, \quad \alpha = 3, \dots, n+1, \\
p'_1 &= \frac{1}{\sqrt{2}} (P_1 + iP_2), \quad p'_2 = \frac{1}{\sqrt{2}} (P_1 - iP_2), \quad p'_\alpha = P_\alpha, \quad \alpha = 3, \dots, n+1
\end{aligned}$$

we then can write

$$\begin{aligned}
p'_1 x'_\alpha - p'_\alpha x'_1 &= \frac{1}{\sqrt{2}} (M_{1\alpha} + iM_{2\alpha}), \quad p'_2 x'_\alpha - p'_\alpha x'_2 = \frac{1}{\sqrt{2}} (M_{1\alpha} - iM_{2\alpha}), \\
p'_1 x'_2 - p'_2 x'_1 &= i\sqrt{2}M_{12}, \quad X = \sum_{j=1}^{n+1} X_j^2, \quad P = \sum_{j=1}^{n+1} P_j^2,
\end{aligned}$$

where $M_{jk} = X_j P_k - X_k P_j$.

The integrals of motion H_α , Z in this case have, using partial fractions, the form

$$\begin{aligned}
\mathbf{L} \cdot \mathbf{L} &= \frac{1}{(u-e_1)^2} \left[-\frac{1}{4} (p'_1 x'_2 - p'_2 x'_1)^2 + \sum_{\alpha=3}^{n+1} \frac{1}{4(e_1-e_\alpha)} (p'_1 x'_\alpha - p'_\alpha x'_1)^2 \right] \\
&\quad + \sum_{\alpha=3}^{n+1} \frac{1}{u-e_\alpha} \left[\frac{1}{4(e_1-e_\alpha)^2} (p'_1 x'_\alpha - p'_\alpha x'_1)^2 - \frac{1}{2(e_1-e_\alpha)} (p'_1 x'_\alpha - p'_\alpha x'_1)(p'_2 x'_\alpha - p'_\alpha x'_2) \right. \\
&\quad \left. + \sum_{\beta} \frac{(p'_\alpha x'_\beta - p'_\beta x'_\alpha)^2}{4(e_\alpha - e_\beta)} \right] + \frac{1}{(u-e_1)} \sum_{\alpha=3}^{n+1} \left[\frac{-1}{4(e_1-e_\alpha)_2} (p'_1 x'_\alpha - p'_\alpha x'_1)^2 \right. \\
&\quad \left. + \frac{1}{2(e_1-e_\alpha)} (p'_1 x'_\alpha - p'_\alpha x'_1)(p'_2 x'_\alpha - p'_\alpha x'_2) \right]
\end{aligned}$$

$$= \sum_{\alpha=3}^{n+1} \frac{H_{\alpha}}{u-e_{\alpha}} + \frac{Z}{(u-e_1)^2} + \frac{Y}{u-e_1}, \quad (2.17)$$

where $Y = \sum_{\alpha=3}^{n+1} H_{\alpha}$.

The analysis presented so far could have been deduced from Kuznetsov⁵ where the double root is essentially contained in the s systems of type C on the real hyperboloid. Furthermore, the threefold root is contained in Kuznetsov's type D systems. The question we now answer is how to use these techniques on the case of ellipsoidal coordinates corresponding to multiply degenerate roots. For this we use the limiting procedures developed by Kalnins, Miller, and Reid.² We recall that the process of using these limiting procedures amounts to altering the elementary divisors of the two quadratic forms

$$\sum_{\alpha=1}^{n+1} \frac{x_{\alpha}^2}{u-e_{\alpha}} = 0, \quad \sum_{\alpha=1}^{n+1} x_{\alpha}^2 = c^2.$$

Theorem 1: Let u_i be the generic ellipsoidal coordinates on the n sphere, viz.,

$$a_{\alpha} x_{\alpha}^2 = \frac{c^2 \prod_{j=1}^n (u_j - e_{\alpha})}{\prod_{\beta \neq \alpha} (e_{\beta} - e_{\alpha})}, \quad \alpha = 1, \dots, n+1, \quad (2.18)$$

with corresponding infinitesimal distance

$$ds^2 = -\frac{1}{4} \sum_{i=1}^n \prod_{j \neq i} (u_i - u_j) \frac{(du_i)^2}{\prod_{j=1}^{n+1} (u_i - e_j)}$$

and coordinate curves

$$\sum_{\alpha=1}^{n+1} \frac{a_{\alpha} x_{\alpha}^2}{u-e_j} = 0, \quad \sum_{\alpha=1}^{n+1} a_{\alpha} x_{\alpha}^2 = c^2. \quad (2.19)$$

Then the degenerate ellipsoidal coordinates having the infinitesimal distance

$$ds^2 = -\frac{1}{4} \sum_{i=1}^n [\prod_{j \neq i} (u_i - u_j)] \frac{(du_i)^2}{\prod_{j=1}^p (u_i - e_j)^{N_j}} \quad (2.20)$$

can be obtained from generic ellipsoidal coordinates via the transformations

$$\begin{aligned} e_j^J &\rightarrow e_1^J + {}^J\epsilon_{j-1}^1, \quad j=1, \dots, N_J, \quad J=1, \dots, p, \\ p_j^J &\rightarrow p_1^J + \sum_{i=2}^{N_J} {}^J\epsilon_{j+1-i}^{i-1} p_i^J, \\ x_j^J &\rightarrow x_1^J + \sum_{i=2}^{N_J} {}^J\epsilon_{j+1-i}^{i-1} x_i^J, \end{aligned} \quad (2.21)$$

where

$${}^J\epsilon_{j+1-i}^{i-1} = \prod_{\ell=2}^i ({}^J\epsilon_{j-1}^1 - {}^J\epsilon_{\ell-2}^1), \quad a_j^J = 1 / [\prod_{k \neq j} ({}^J\epsilon_{j-1}^1 - {}^J\epsilon_{k-1}^1)], \quad k=1, \dots, N_J$$

and $N_1 + \dots + N_p = n+1$. (We require ${}^J\epsilon_0^i = 0$ and take the limit as the ${}^J\epsilon_h^1 \rightarrow 0$ for $h=1, \dots, N_J - 1$.) In particular

$$\mathbf{L}(u) = \sum_{j=1}^p \sum_{j=1}^{N_j} \frac{\mathbf{z}_j^J}{(u - e_j)^{N_j - J + 1}}, \quad (2.22)$$

where

$$\mathbf{z}_j^J = \left\langle \frac{1}{4} \sum_i (p_i^J p_{j+1-i}^J + x_i^J x_{j+1-i}^J), \frac{i}{4} \sum_i (p_i^J p_{j+1-i}^J - x_i^J x_{j+1-i}^J), \frac{i}{2} \sum_i p_i^J x_{j+1-i}^J \right\rangle. \quad (2.23)$$

The $(\mathbf{z}_j^J)_\ell$ satisfy the Poisson bracket relations

$$\{(\mathbf{z}_j^J)_\ell, (\mathbf{z}_k^J)_m\} = -\delta_{JJ'} (\mathbf{z}_{j+k-N_J}^J)_{\ell+m}, \quad (2.24)$$

and we also have

$$\mathbf{L} \cdot \mathbf{L}(u) = \frac{1}{4} \sum (x_{jp}^J p_k^L - x_k^L p_j^J) (x_{mp}^J p_n^L - x_n^L p_m^J) \frac{1}{(u - e_j)^p (u - e_L)^q}, \quad (2.25)$$

where $1 \leq p \leq N_J$, $1 \leq q \leq N_L$, $1 \leq j, m \leq N_J$, $1 \leq k, n \leq N_L$, subject to $j + m = N_J + 2 - p$, $k + n = N_L + 2 - q$ and the summation extends over the indices J, L, p, q, j, m, k, n subject to these restrictions.

From expressions (2.24) we can verify that relations (2.12) are satisfied, so that

$$\{\mathbf{L} \cdot \mathbf{L}(u), \mathbf{L} \cdot \mathbf{L}(v)\} = 0.$$

The constants of the motion can be read off from the partial fraction decomposition of Eq. (2.25). This clearly illustrates the compactness of the r matrix formulation for the operators describing the integrable systems examined so far.

In dealing with the case of Euclidean space $E_{n\mathbb{C}}$ the most transparent way to proceed is as follows. The generic ellipsoidal coordinates in n dimensional complex Euclidean space are given by^{1,11}

$$x_\alpha^2 = \frac{\prod_{j=1}^n (u_j - e_\alpha)}{\prod_{\beta \neq \alpha} (e_\beta - e_\alpha)}, \quad j, \alpha, \beta = 1, \dots, n, \quad (2.26)$$

with coordinate curves

$$\sum_{\alpha=1}^n \frac{x_\alpha^2}{(u - e_\alpha)} = 1, \quad u = u_j, \quad j = 1, \dots, n. \quad (2.27)$$

Proceeding in analogy with Eqs. (2.2)–(2.10) we can obtain the r matrix algebra. The corresponding $\mathbf{L}(u)$ operator is

$$\mathbf{L}(u) = \sum_{\alpha=1}^n \frac{\mathbf{s}_\alpha}{u - e_\alpha} + \frac{1}{4} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}. \quad (2.28)$$

[Indeed the equation $L^1(u) + iL^2(u) = 0$ is just Eq. (2.27). Moreover it is obvious, due to the fact that expressions (2.11) satisfy (2.12), that expressions (2.28) also satisfy (2.12).] The conjugate variables v_j are defined by $v_j = -iL^3(u_j)$ and they must satisfy the canonical relations (2.4). The integrals of motion H_α are determined from

$$\mathbf{L}^2(u) = \sum_{\alpha=1}^n \frac{H_{\alpha}}{u - e_{\alpha}}, \quad (2.29)$$

where

$$H_{\alpha} = 2 \sum_{\beta=1}^n \frac{(\mathbf{s}_{\alpha}, \mathbf{s}_{\beta})}{e_{\alpha} - e_{\beta}} - \frac{1}{2} (s_{\alpha}^1 + s_{\alpha}^2) = \frac{1}{4} \left(\sum_{\beta=1}^n \frac{M_{\alpha\beta}^2}{e_{\alpha} - e_{\beta}} - p_{\alpha}^2 \right),$$

with $\sum_{\alpha} H_{\alpha} = -\mathbf{p}^2/4$. The separation equations are of the form

$$v_j^2 + \mathbf{L}^2(u_j) = 0, \quad j = 1, \dots, n$$

as in Eq. (2.8).

III. CYCLIDIC COORDINATES

Associated with the separation of variables problem for the Hamilton Jacobi equation (1.1) with $E \neq 0$ is the corresponding $E=0$ problem. In this case the equation is

$$\sum_{\alpha, \beta=1}^n g^{\alpha\beta} p_{\alpha} p_{\beta} = 0, \quad p = \frac{\partial W}{\partial x_{\alpha}}, \quad \alpha = 1, \dots, n \quad (3.1)$$

and we consider only complex Euclidean space. While it is true that all the coordinate systems discussed for E_{nC} with $E \neq 0$ will provide a separation of variables of this equation, there are coordinates that provide an additive separation of variables *only* when $E=0$. This is related to the fact that the $E=0$ equation admits a conformal symmetry algebra.^{1,12,13}

The most convenient way to proceed is to introduce hyperspherical coordinates $\{x_1, \dots, x_{n+2}\}$

$$x_1 = t^2 \left(\sum_{j=1}^n z_j^2 - 1 \right), \quad x_2 = it^2 \left(\sum_{j=1}^n z_j^2 + 1 \right), \quad x_{k+2} = 2z_k t^2, \quad k = 1, \dots, n$$

related to the usual Cartesian coordinates $\{z_1, \dots, z_n\}$ according to

$$z_k = \frac{x_{k+2}}{(-x_1 - ix_2)}, \quad k = 1, \dots, n, \quad \sum_{j=1}^{n+2} x_j^2 = 0.$$

We consider the system of Sec. I in $n+2$ dimensions, where $\mathbf{J}=0$ and the $c_{\alpha}=0$. The general (separable) cyclidic coordinates are specified by

$$\Omega = \sum_{j=1}^{n+2} \frac{x_j^2}{\lambda - e_i} = 0, \quad \Phi = \sum_{j=1}^{n+2} x_j^2 = 0, \quad \lambda = u_1, \dots, u_n, \quad e_k \neq e_j.$$

Furthermore

$$\sigma x_{\alpha}^2 = \frac{\prod_{j=1}^n (u_j - e_{\alpha})}{\prod_{\beta \neq \alpha} (e_{\beta} - e_{\alpha})}, \quad \sigma = - \left[\sum_{j=1}^{n+2} e_j x_j^2 \right]^{-1}.$$

The Hamilton Jacobi equation is given by

$$\mathbf{J}^2 = -\sigma^2 \sum_{j=1}^n \left(\frac{\partial W}{\partial u_j} \right)^2 \frac{\prod_{\gamma=1}^{n+2} (u_j - e_{\gamma})}{\prod_{i \neq j} (u_j - u_i)} = 0. \quad (3.2)$$

The quadratic forms Ω and Φ have elementary divisors $[11 \cdots 1]$, see Refs. 2,12. It is known that the geometry of these fourth order coordinate curves is unchanged under birational transformations of the form

$$e_k \rightarrow \frac{\alpha e_k + \beta}{\gamma e_k + \delta}, \quad u_j \rightarrow \frac{\alpha u_j + \beta}{\gamma u_j + \delta}, \quad \lambda \rightarrow \frac{\alpha \lambda + \beta}{\gamma \lambda + \delta}$$

for $\alpha\delta - \beta\gamma \neq 0$ and $k=1, \dots, n+2$, $j=1, \dots, n$.

Now we can mimic the exposition given for the Gaudin magnet integrable systems using the hyperspherical coordinates and the Poisson bracket

$$\{F(x_i, p_j), G(x_i, p_j)\}_h = \sum_{k=1}^{n+2} \left(\frac{\partial F}{\partial p_k} \frac{\partial G}{\partial x_k} - \frac{\partial F}{\partial x_k} \frac{\partial G}{\partial p_k} \right) \quad (3.3)$$

the x_i , p_j now being regarded as independent. The analysis then proceeds much as in the construction (2.2)–(2.12), but with n replaced by $n+1$; indeed the coordinates are defined as zeros of the off diagonal element

$$B(u) = \sum_{\alpha=1}^{n+2} \frac{x_\alpha^2}{u - e_\alpha}$$

subject to the restriction $\Phi=0$. The crucial difference is that $\mathbf{J}=0$. The expressions for p_α and $M_{\alpha\beta}$ are altered by a factor σ

$$p_\alpha = 2\sigma x_\alpha \left[\sum_{j=1}^n \frac{v_j}{e_\alpha - u_j} \frac{\prod_{\gamma=1}^n (u_j - e_\gamma)}{\prod_{i \neq j} (u_j - u_i)} \right],$$

which together with x_α^2 in terms of u_j gives $C(u)$ in the new variables. Another useful formula is

$$M_{\alpha\beta} = 2\sigma x_\alpha x_\beta (e_\alpha - e_\beta) \sum_{j=1}^n v_j \frac{\hat{\Pi}_{\gamma=1}^{n+2} (u_j - e_\gamma)}{\prod_{i \neq j} (u_j - u_i)}.$$

In fact the Poisson bracket $\{, \}_h$ can be identified with the Poisson bracket $\{, \}$ for functions defined in the n dimensional space spanned by z_1, \dots, z_n . This can readily be seen by noting that $F(z_1, \dots, z_n, p_{z_1}, \dots, p_{z_n}) = F(-x_3/(x_1 + ix_2), \dots, -x_{n+2}/(x_1 + ix_2), -(x_1 + ix_2)p_{x_3} + x_3(p_{x_1} + ip_{x_2}), \dots, -(x_1 + ix_2)p_{x_{n+2}} + x_{n+2}(p_{x_1} + ip_{x_2}))$ from which the equality of the Poisson brackets follows identically. The infinitesimal distance for general cyclidic coordinates is

$$ds^2 = \frac{1}{4} (x_1 + ix_2)^{-2} \left(\sum_{i=1}^n \prod_{j \neq i} (u_i - u_j) \frac{(du_i)^2}{\prod_{j=1}^{n+2} (u_i - e_j)} \right). \quad (3.4)$$

We denote the coordinates defined by this graph as

$$[E_n \mathbf{C}, E=0 | e_1 | \cdots | e_{n+2} |.$$

Other coordinates of this type are those corresponding to the graphs

$$(S_{p\mathbf{C}} | e_1 | \cdots | e_{p+1} | \oplus (S_{q\mathbf{C}} | f_1 | \cdots | f_{q+1} |,$$

where $p+q=n+2$. These coordinates are given by

$$\sigma x_k^2 = \frac{\prod_{j=1}^p (u_j - e_\alpha)}{\prod_{\beta \neq \alpha} (e_\beta - e_\alpha)}, \quad \alpha, \beta = 1, \dots, p+1, k = 1, \dots, p,$$

$$\sigma x_k^2 = \frac{\prod_{j=1}^q (v_j - f_\alpha)}{\prod_{\beta \neq \alpha} (f_\beta - f_\alpha)}, \quad \alpha, \beta = 1, \dots, q+1, k = p+1, \dots, p+q+2, \quad (3.5)$$

$$\sigma = - \left[\sum_{i=1}^{p+1} e_i x_i^2 + \sum_{j=1}^{q+1} f_j x_j^2 \right]^{-1}.$$

We note here that e_i the f_j are pairwise distinct, for if they were not then for any of these quantities which occurred with multiplicity more than 1 a birational transformation could transform it to ∞ and hence to a graph corresponding to E_{nC} .

It can happen just as in the case of generic ellipsoidal integrable systems on the sphere that some of the e_i in Eq. (3.5) are equal to some of the f_j also. In this case the rules for obtaining the corresponding L matrix are summarized in the following theorem.

Theorem 2: Denote the generic ellipsoidal coordinates by the graph

$$[E_{nC} | e_1 | \cdots | e_n], \quad e_i \neq e_j$$

and generic cyclidic coordinates by the graph

$$\{CE_{nC} | e_1 | \cdots | e_{n+2}\}, \quad e_i \neq e_j.$$

Separable coordinates for the Hamilton Jacobi equation (3.1) corresponding to generic graphs with multiplicities

$$[E_{nC} | e_1^{n_1} | \cdots | e_p^{n_p}], \quad n_1 + \cdots + n_p = n,$$

$$\{CE_{nC} | e_1^{n_1} | \cdots | e_q^{n_q}\}, \quad n_1 + \cdots + n_q = n+2,$$

(Ref. 2) can be obtained via the transformations of Theorem 1 applied to the quadratic forms

$$\Omega = \sum_{i=1}^m \frac{a_i x_i^2}{\lambda - e_i} = 0, \quad \Phi = \sum_{i=1}^m a_i x_i^2 = 0,$$

where $m=n$ for generic ellipsoidal coordinates in E_{nC} and $m=n+2$ for the corresponding cyclidic coordinates.

For coordinates corresponding to the direct sum of two spheres, viz., Eq. (3.5), we can merely apply the result of the previous theorem for spherical coordinates to each of the pairs of quadratic forms

$$\Omega'_1 = \sum_{\alpha=1}^{p+1} \frac{a_\alpha x_\alpha^2}{u - e_\alpha}, \quad \Phi'_1 = \sum_{\alpha=1}^{p+1} a_\alpha x_\alpha^2,$$

$$\Omega'_2 = \sum_{\alpha=p+2}^{n+2} \frac{a_\alpha x_\alpha^2}{u - e_\alpha}, \quad \Phi'_2 = \sum_{\alpha=p+2}^{n+2} a_\alpha x_\alpha^2.$$

Indeed the freedom to subject the coordinates u_j and e_i to birational transformations in the expressions for generic cyclidic coordinates allows us to let $e_1 \rightarrow \infty$. (In this particular case the resulting coordinates can be identified with generic elliptical coordinates on the n sphere.) The process described in Theorem 1 enables one to pass from the elementary divisors $[11 \cdots 1]$ to

$[N_1, N_2, \dots, N_p], N_i \geq 1, i=1, \dots, p$, see Refs. 2, 12. It is then possible (via Theorem 2) to take $e_1 \rightarrow \infty$ in which case $[\tilde{N}_1, N_2, \dots, N_p]$ corresponds to the various generic coordinate systems in Euclidean space if $N_1 > 1$. To illustrate how this works for the Gaudin XXX magnet model consider the quadratic forms $\hat{\Omega}, \hat{\Phi}$ corresponding to elementary divisors $[21 \cdots 1]$, viz.,

$$\hat{\Omega} = \frac{x_1^2}{(\lambda - e_1)^2} + \frac{2x_1x_2}{\lambda - e_1} + \cdots + \frac{x_{n+2}^2}{\lambda - e_{n+2}} = 0, \quad \hat{\Phi} = 2x_1x_2 + x_3^2 + \cdots + x_{n+2}^2 = 0.$$

Putting $\lambda \rightarrow 1/\lambda$, $e_i \rightarrow 1/e_i$ we find (with the use of $\hat{\Phi} = 0$)

$$\hat{\Omega} = \lambda^2 \left[\frac{e_1^2 x_1^2}{(e_1 - \lambda)^2} + \sum_{k=3}^{n+2} \frac{(e_1 - e_k)}{(e_1 - \lambda)} \frac{x_k^2}{(e_k - \lambda)} \right]$$

or, in the limit as $e_1 \rightarrow \infty$

$$x_1^2 - \sum_{k=3}^{n+2} \frac{x_k^2}{\lambda - e_k} = 0.$$

Now if we perform the limiting procedure (2.13)–(2.15) on $L(u)$ (in $n+2$ dimensions), let $u \rightarrow 1/u$, $e_i \rightarrow 1/e_i$, let $e_1 \rightarrow \infty$, and then evaluate at $p_1 = 0$, $x_1 = 1$ we find (up to a common factor u)

$$L(u) = \sum_{\alpha=3}^{n+2} \frac{s_\alpha}{u - e_\alpha} + \frac{1}{4} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

Here, we have made use of the fact that $\mathbf{J} = 0$. This agrees with Eq. (2.28) and with the $L(u)$ operator given by Kuznetsov,⁵ and corresponds to elementary divisors $[\tilde{2}, 1, \dots, 1]$.

This process can be generalized for quadratic forms corresponding to elementary divisors $[\tilde{N}, 1, \dots, 1]$, $N > 1$. We have that

$$\hat{\Omega} = u^{N-2}x_1^2 + u^{N-3}2x_1x_2 + \sum_i x_i x_{N-i} - \sum_{k=N+1}^{n+2} \frac{x_k^2}{u - e_k}, \quad \hat{\Phi} = \sum_{i=1}^N x_i x_{N+1-i} + \sum_{k=N+1}^{n+2} x_k^2.$$

The corresponding $L(u)$ operator is

$$L(u) = \sum_{\alpha=N+1}^{n+2} \frac{s_\alpha}{u - e_\alpha} + s_1,$$

where

$$s_1^1 = \frac{1}{4} [\hat{\Omega}(0, p_2, \dots, p_{n+2}) - \hat{\Omega}(1, x_2, \dots, x_{n+2})],$$

$$s_1^2 = \frac{1}{4} [\hat{\Omega}(0, p_2, \dots, p_{n+2}) + \hat{\Omega}(1, x_2, \dots, x_{n+2})],$$

$$s_1^3 = \frac{1}{2} [S'(0, p_2, \dots, p_{n+2}; 1, x_2, \dots, x_{n+2})], \quad S' = \sum_{j=2}^N u^{N-j} \sum_{\ell+m=j} x_\ell p_m.$$

IV. BRANCHING RULES FOR THE CONSTRUCTION OF ORTHOGONAL NONGENERIC COMPLEX INTEGRABLE SYSTEMS ON S_{nC} AND E_{nC}

To deal with the nongeneric separable coordinate systems in E_{nC} and S_{nC} we must combine coordinate systems for both manifolds.² (See Refs. 13,14 for tabulations of all cases for small values of n .) The branching laws for graphs on these manifolds are summarized below

$$(S_{nC} | \cdots | e_i | \cdots | \downarrow_{S_{p_i C}} \quad (1)$$

$$(S_{nC} | \cdots | e_i^{\lambda_i} | \cdots | \downarrow_{E_{p_i C}} \quad (2)$$

$$(E_{nC} | \cdots | e_i | \cdots | \downarrow_{S_{p_i C}} \quad (3)$$

$$(E_{nC} | \cdots | e_i^{\lambda_i} | \cdots | \downarrow_{E_{p_i C}} \quad (4)$$

As an example consider the coordinate system given by the graph on S_{4C}

$$(S_{2C} | e_1^2 | e_5 | \downarrow [E_{2C} | f_3 | f_4 |$$

The coordinates for this graph can be obtained from those of the generic graph

$$(S_{4C} | e_1^2 | e_3 | e_4 | e_5 |$$

via the limiting transformations

$$e_j = e_1 + \epsilon + \epsilon^2 f_j, \quad j=3,4, \quad f_3 \neq f_4,$$

$$u_j = e_1 + \epsilon + \epsilon^2 \bar{u}_j, \quad j=3,4,$$

where $\epsilon \rightarrow 0$.

The corresponding coordinates are then given implicitly by

$$\begin{aligned} -x_1^2 &= \frac{(u_1 - e_1)(u_2 - e_1)}{e_5 - e_1}, \quad x_3^2 = \frac{(u_1 - e_5)(u_2 - e_5)}{(e_1 - e_5)^2}, \\ -2x_1 x_2 &= -\frac{(u_1 - e_1)}{e_5 - e_1} - \frac{(u_2 - e_1)}{e_5 - e_1} + \frac{(u_1 - e_1)(u_2 - e_1)}{(e_5 - e_1)^2} - \frac{(u_1 - e_1)(u_2 - e_1)}{e_5 - e_1} (f_3 + f_4 - \bar{u}_3 - \bar{u}_4), \\ -x_3^2 &= \frac{(u_1 - e_1)(u_2 - e_1)}{(e_5 - e_1)} \frac{(\bar{u}_3 - f_3)(\bar{u}_4 - f_3)}{(f_3 - f_4)}, \\ -x_4^2 &= \frac{(u_1 - e_1)(u_2 - e_1)}{(e_5 - e_1)} \frac{(\bar{u}_3 - f_4)(\bar{u}_4 - f_4)}{(f_4 - f_3)}. \end{aligned}$$

Using the nomenclature given previously we have that $L(u)$ has the two different forms

$$L_1(u) = \frac{z_1}{(u-e_1)^2} + \frac{z_2}{u-e_1} + \frac{z_3+z_4}{u-e_1} + \frac{z_5}{u-e_5}, \quad u=u_1, u_2,$$

$$L_2(u) = z_1 + \frac{z_3}{u-f_3} + \frac{z_4}{u-f_4}, \quad u=\bar{u}_3, \bar{u}_4.$$

As usual, the separation variables are the zeros of the equation $L_\lambda^1(u) + iL_\lambda^2(u) = 0$ with $u=u_1, u_2$ when $\lambda=1$ and $u=\bar{u}_3, \bar{u}_4$ when $\lambda=2$. Each of the L_λ separately satisfy Eq. (2.12) for $\lambda=1, 2$. In addition we have

$$\{L_1 \cdot L_1(u), L_2 \cdot L_2(v)\} = 0.$$

This example illustrates how to derive the substitutions that enable the various branching laws to be obtained from a generic form. For the sphere S_{nC} and generic coordinates

$$(S_{nC} | \cdots | e_0^{\lambda+q-s} | e_1 | \cdots | e_s | \cdots |, \quad q > s, \quad \lambda > 1,$$

if we make the substitutions

$$e_k = e_0 + \epsilon^{q-s+1} + e^{\lambda+q-s} f_k, \quad k=1 \cdots s,$$

$$u_j = e_0 + \epsilon^{q-s+1} + \epsilon^{\lambda+q-s} \bar{u}_j, \quad j=1 \cdots q,$$

where $\epsilon \rightarrow 0$, then the graph illustrated transforms into

$$\begin{array}{c} (S_{nC} | \cdots | e_0^\lambda | \cdots | \\ \downarrow \\ [E_{qC} | f_1 | \cdots | f_s |. \end{array}$$

The remaining branching rule is obtained from a graph of the type

$$(S_{nC} | \cdots | e_0 | e_1 | \cdots | e_s |.$$

By means of the substitutions

$$e_j = e_0 + \epsilon(f_0 - f_j), \quad j=1, \dots, p+1, \quad u_k = e_0 + \epsilon(\bar{u}_k - f_0), \quad k=1, \dots, p$$

we obtain the coordinate system coming from the graph

$$\begin{array}{c} (S_{nC} | \cdots | e_0 | \cdots | \\ \downarrow \\ (S_{pC} | f_0 | \cdots | f_p |. \end{array}$$

The corresponding substitutions for the analogous Euclidean space branching rules are essentially identical. To completely specify the coordinate systems on these manifolds we need a few more substitution rules. Firstly consider the graph

$$[E_{pC} | h_1 | \cdots | h_p] + [E_{qC} | f_1 | \cdots | f_q].$$

This graph can be obtained from the generic graph for $E_{(p+q)C}$ via the substitutions

$$u_i = \frac{K_1}{\epsilon} + \bar{u}_i, \quad e_i = \frac{K_1}{\epsilon} + h_i, \quad i = 1, \dots, p,$$

$$u_i = \frac{K_2}{\epsilon} + \bar{u}_i, \quad e_i = \frac{K_2}{\epsilon} + f_{i-p}, \quad i = p+1, \dots, p+q, \quad K_1 \neq K_2.$$

The graph $[E_{qC}|h_1|\dots|h_q|]$, $q < n$ can be obtained from $[E_{nC}|h_1|\dots|h_n|]$ via the substitutions

$$u_i = e_1 + \epsilon^{q-n-1} + \epsilon^{q-n}\bar{u}_i, \quad i = 1, \dots, n,$$

$$e_j = e_1 + \epsilon^{q-n-1} + \epsilon^{q-n}h_j, \quad j = 1, \dots, q, \quad e_k = h_k, \quad k = q+1, \dots, n.$$

Given these substitution rules all the corresponding graphs for E_{nC} and S_{nC} can be constructed together with their corresponding L matrices.

V. QUANTUM INTEGRABLE SYSTEMS ON COMPLEX CONSTANT CURVATURE SPACES AND THE QUANTUM GAUDIN MAGNET

To deal with the quantum version of this description of separation of variables we consider the Schrödinger or Helmholtz equation.⁶

$$\mathcal{H}\Psi = \frac{1}{\sqrt{g}} \frac{\partial}{\partial y_\alpha} \left[\sqrt{g} g^{\alpha\beta} \frac{\partial}{\partial y_\beta} \right] \Psi = E\Psi, \quad (5.1)$$

where $E \neq 0$ for the moment. Separation of variables means (roughly) the solution of this equation of the form

$$\Psi = \prod_{\alpha=1}^n \psi_\alpha(y_\alpha; h_1, \dots, h_n), \quad (5.2)$$

where the quantum numbers h_j are the eigenvalues of mutually commuting operators

$$\mathcal{A}_i = \sum_{\alpha, \beta=1}^n a_{\alpha\beta}^{(j)} \frac{\partial^2}{\partial y_\alpha \partial y_\beta} + \sum_{\beta=1}^n b_\beta \frac{\partial}{\partial y_\beta}, \quad j = 1, \dots, n,$$

$$[\mathcal{A}_i, \mathcal{A}_j] = 0, \quad i, j = 1, \dots, n, \quad \mathcal{A}_n = \mathcal{H}, \quad \mathcal{A}_j \Psi = h_j \Psi. \quad (5.3)$$

Furthermore these operators can be represented as symmetric quadratic elements in the enveloping algebra of the symmetry algebra of Eq. (5.1) in the case of S_{nC} and E_{nC} . The standardized representations of these symmetry algebras are

$$\text{SO}(n+1): \mathcal{M}_{\alpha\beta} = \hat{x}_\alpha \hat{p}_\beta - \hat{x}_\beta \hat{p}_\alpha, \quad \hat{p}_\alpha = \frac{\partial}{\partial x_\alpha}, \quad \alpha, \beta = 1, \dots, n+1,$$

$$[\mathcal{M}_{\alpha\beta}, \mathcal{M}_{\gamma\delta}] = \delta_{\alpha\gamma} \mathcal{M}_{\delta\beta} + \delta_{\alpha\delta} \mathcal{M}_{\beta\gamma} + \delta_{\beta\gamma} \mathcal{M}_{\alpha\delta} + \delta_{\beta\delta} \mathcal{M}_{\gamma\alpha}, \quad (5.4)$$

$$E(n): \mathcal{M}_{\alpha\beta}, \mathcal{P}_\gamma = \frac{\partial}{\partial x_\gamma},$$

$$[\mathcal{M}_{\alpha\beta}, \mathcal{P}_\gamma] = \delta_{\beta\gamma} \mathcal{P}_\alpha - \delta_{\alpha\gamma} \mathcal{P}_\beta, \quad [\mathcal{P}_\alpha, \mathcal{P}_\beta] = 0, \quad (5.5)$$

where $[\cdot, \cdot]$ is the commutator bracket. Much of the analysis goes through as it did in the classical case with, of course, some critical differences. For the quantum Gaudin magnet one considers

the sum of rank 1 Lie algebras $\mathcal{A} = \oplus_{\alpha=1}^m \mathfrak{so}_{\alpha}(3)$ where the generators of the algebra satisfy the commutation relations (1.6) and the inner product is defined as in Sec. I. The Casimir elements of \mathcal{A} have the form

$$(\mathbf{s}_{\alpha}, \mathbf{s}_{\alpha}) = k_{\alpha}(k_{\alpha} + 1), \quad (5.6)$$

where k_{α} is a constant when the generators of $\mathfrak{so}_{\alpha}(3)$ determine an irreducible representation.¹⁵ The quantum Gaudin magnet is the quantum integrable Hamiltonian system on \mathcal{A} given by m commuting integrals of motion

$$\mathcal{H}_{\alpha} = 2 \sum_{\beta=1}^m \frac{(\mathbf{s}_{\alpha}, \mathbf{s}_{\beta})}{e_{\alpha} - e_{\beta}}, \quad [\mathcal{H}_{\alpha}, \mathcal{H}_{\beta}] = 0.$$

This is the m -site $\mathrm{SO}(3, \mathbb{C})$ -XXX quantum Gaudin magnet.⁶ These integrals (operators) satisfy

$$\sum_{\alpha=1}^m \mathcal{H}_{\alpha} = 0, \quad \sum_{\alpha=1}^m e_{\alpha} \mathcal{H}_{\alpha} = \mathbf{J}^2 - \sum_{\alpha=1}^m k_{\alpha}(k_{\alpha} + 1), \quad (5.7)$$

where $\mathbf{J} = \sum_{\alpha=1}^m \mathbf{s}_{\alpha}$ is the total momentum operator. The complete set of commuting operators consists of \mathcal{H}_{α} , \mathbf{J}^2 , and \mathcal{J}^3 . The integrals are generated by the 2×2 operator $L(u)$ given by Eq. (1.11) understood in the operator sense. The quantum determinant is

$$q\text{-det } L(u) = -A(u)^2 - \frac{1}{2} \{B(u), C(u)\} = - \sum_{\alpha=1}^m \frac{\mathcal{H}_{\alpha}}{u - e_{\alpha}} - \sum_{\alpha=1}^m \frac{k_{\alpha}(k_{\alpha} + 1)}{(u - e_{\alpha})^2}, \quad (5.8)$$

with L operator satisfying the r matrix algebra

$$[L(u), L(v)] = \frac{i}{u-v} [P, L(v)], \quad i = \sqrt{-1}, \quad (5.9)$$

[compare with (1.13)]. The total momentum \mathbf{J} has components

$$\mathcal{J}^1 = \frac{1}{4} (\hat{\mathbf{p}} \cdot \hat{\mathbf{p}} + \hat{\mathbf{x}} \cdot \hat{\mathbf{x}}), \quad \mathcal{J}^2 = \frac{i}{4} (\hat{\mathbf{p}} \cdot \hat{\mathbf{p}} - \hat{\mathbf{x}} \cdot \hat{\mathbf{x}}), \quad \mathcal{J}^3 = \frac{i}{4} (\hat{\mathbf{p}} \cdot \hat{\mathbf{x}} + \hat{\mathbf{x}} \cdot \hat{\mathbf{p}}). \quad (5.10)$$

The Lie symmetries $\mathcal{M}_{\alpha\beta}$ and \mathbf{J} form the direct sum $\mathfrak{so}(m) \oplus \mathfrak{so}(3)$ with commutation relations

$$[\mathcal{M}_{\alpha\beta}, \mathcal{J}^i] = 0, \quad [\mathcal{J}^i, \mathcal{J}^j] = i\epsilon_{jkl} \mathcal{J}^l \quad (5.11)$$

subject to the constraints

$$\mathbf{J}^2 = \frac{1}{4} \sum_{\alpha < \beta} \mathcal{M}_{\alpha\beta}^2 + \frac{1}{16} m(m-4), \quad 2(\mathcal{J}^1 + i\mathcal{J}^2) = \hat{\mathbf{x}} \cdot \hat{\mathbf{x}}. \quad (5.12)$$

Considering again separation of variables using the coordinates of the irreducible block (2.1), the separation variables are defined, as before, as the zeros of the off diagonal elements $B(u)$ of the L matrix. The q -determinant is the generating function of the commuting integrals of motion

$$[q\text{-det } L(u), q\text{-det } L(v)] = 0. \quad (5.13)$$

On the n sphere the algebra \mathcal{A} is realized by taking the canonical operators

$$s_\alpha^1 = \frac{1}{4}(\hat{p}_\alpha^2 + \hat{x}_\alpha^2), \quad s_\alpha^2 = \frac{i}{4}(\hat{p}_\alpha^2 - \hat{x}_\alpha^2), \quad s_\alpha^3 = \frac{i}{4}\{\hat{p}_\alpha, \hat{x}_\alpha\}, \quad \alpha = 1, \dots, n+1, \quad (5.14)$$

where $(s_\alpha, s_\alpha) = -\frac{3}{16}$. Furthermore

$$(s_\alpha, s_\beta) = \frac{1}{8}(\mathcal{M}_{\alpha\beta}^2 + \frac{1}{2}) \quad (5.15)$$

and this establishes the relationship between the s_α of \mathcal{A} and the $\mathcal{M}_{\alpha\beta}$ of $\mathfrak{so}(n+1)$. The integrals transform into the family of integrals

$$\mathcal{H} = \sum_{\alpha=1}^{n+1} h_\alpha \mathcal{H}_\alpha = \frac{1}{4} \sum_{\alpha < \beta} \frac{h_\alpha - h_\beta}{e_\alpha - e_\beta} \left(\mathcal{M}_{\alpha\beta}^2 + \frac{1}{2} \right) \quad (5.16)$$

and the coordinates are given by

$$B(u_j) = 0: \sum_{\alpha=1}^{n+1} \frac{x_\alpha^2}{u - e_\alpha} = 0.$$

For each u_j variable there is defined the conjugate variable (operator)

$$v_j = -iA(u_j) = \frac{1}{4} \sum_{\alpha=1}^{n+1} \frac{1}{u_j - e_\alpha} \{\hat{p}_\alpha, \hat{x}_\alpha\}. \quad (5.17)$$

Separation of variables is then the process of changing to the new variables v_j , u_i , c , and \mathcal{J}^3 . In fact we have that

$$\frac{1}{4}\{\hat{p}_\alpha, \hat{x}_\alpha\} = \frac{\hat{x}_\alpha^2}{c} \left[-i\mathcal{J}^3 - \sum_{j=1}^n \frac{1}{e_\alpha - u_j} D_j v_j \right], \quad D_j = -\frac{\prod_{\alpha=1}^{n+1} (u_j - e_\alpha)}{\prod_{i \neq j} (u_j - u_i)}. \quad (5.18)$$

The separation equations can be obtained by substituting $u = u_j$ into the q -det $L(u)$, making the choice

$$v_j = i \left(\frac{\partial}{\partial u_j} + \frac{1}{4} \sum_{\alpha=1}^{n+1} \frac{1}{u_j - e_\alpha} \right) \quad (5.19)$$

and looking for the solutions of the spectral problem $\mathcal{H}_\alpha \Psi = h_\alpha \Psi$. The separation equations are then

$$\frac{d^2}{du_j^2} \psi_j + \frac{1}{2} \left(\sum_{\alpha=1}^n \frac{1}{u_j - e_\alpha} \right) \frac{d}{du_j} \psi_j = \sum_{\alpha=1}^{n+1} \frac{h'_\alpha}{u_j - e_\alpha} \psi_j, \quad (5.20)$$

where

$$\Psi = \prod_{j=1}^{n+1} \psi_j(u_j), \quad h'_\alpha = h_\alpha + \frac{1}{8} \sum_{\beta \neq \alpha} \frac{1}{e_\alpha - e_\beta}.$$

These are the separation equations for the Helmholtz equation in generic ellipsoidal coordinates. Note that for this choice of v_j we have taken $(s_\alpha, s_\alpha) = -3/16$. The L operator is given in direct analogy with the classical case

$$L(u) = \sum_{\alpha=1}^{n+1} \frac{s_\alpha}{u - e_\alpha}. \quad (5.21)$$

The treatment of generic ellipsoidal coordinates on E_{nC} follows similar lines, with the L operator given in this case by

$$L(u) = \sum_{\alpha=1}^n \frac{s_{\alpha}}{u-e_{\alpha}} + \frac{1}{4} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}. \quad (5.22)$$

The limiting procedures described in the classical case work also in the quantum case. All coordinate systems that were obtained in the classical case appear again. In the case of cyclidic coordinates we can adopt the same strategy as in Sec. III: we impose the conditions $\mathcal{J}=0$ and proceed as before. The natural setting in this case is again to use hyperspherical coordinates. The total momentum has components as in Eq. (5.10). If \hat{p}_{α} , \hat{x}_{α} are now vectors in hyperspherical coordinates then we can derive the standard quantum r matrix algebra as above with the same formulas valid. The constraints are now $2(\mathcal{J}^1 + i\mathcal{J}^2) = \hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = 0$, $\mathbf{J}^2 = \frac{1}{8}n(n-4)$. Coordinates u_j and their conjugate operators v_j can be chosen as before. In particular if we make the choice

$$v_j = \frac{\partial}{\partial u_j} + \frac{1}{4} \sum_{\alpha=1}^{n+2} \frac{1}{u_j - e_{\alpha}} \quad (5.23)$$

and look for solutions of $H\Psi=0$ of the form $\Psi = \sigma^{(n-2)/2} \prod_{i=1}^n \psi_i(u_i)$, then we obtain the separation equations which coincide with the equations given by Bocher¹², viz.,

$$\begin{aligned} \frac{d^2}{du_j^2} \psi_j + \frac{1}{2} \sum_{\alpha=1}^{n+2} \frac{1}{u_j - e_{\alpha}} \frac{d}{du_j} \psi_j \\ = \frac{(-1/16)(n^2-4)u_j^n - (1/16)(2n-n^2)(\sum_{\alpha=1}^{n+2} e_{\alpha})u_j^{n-1} + \sum_{\beta=0}^{n-2} \lambda_{\beta} u_j^{\beta}}{\prod_{\gamma=1}^{n+2} (u_j - e_{\gamma})} \psi_j \end{aligned}$$

for suitable γ_{β} . Note that the solutions Ψ are not strictly separable in this case but are what is termed R separable, i.e., separable to within a nonseparable factor $\sigma^{(n-2)/2}$.¹

We conclude this work by pointing out that one can obtain a complete set of constants of the motion associated with an orthogonal separable coordinate system $\{u_1, \dots, u_n\}$ for the Schrödinger equation (5.1) directly from Eq. (5.1) itself.^{16,17} It is well known that all orthogonal separable coordinate systems for the Schrödinger equation on a Riemannian space are obtainable via the Stäckel construction, e.g., Refs. 18, 19. Thus if $\{u\}$ is a separable orthogonal coordinate system for Eq. (5.1) there exists an $n \times n$ nonsingular matrix $S = (S_{\gamma\beta}(u_{\alpha}))$ such that $\partial_{u_{\gamma}} S_{\alpha\beta} = 0$ if $\gamma \neq \alpha$, and such that the nonzero components of the contravariant metric tensor in the coordinates $\{u\}$ are $g^{\alpha\alpha}(u) = T^{\alpha\alpha}(u)$, $\alpha = 1, \dots, n$, where T is the inverse matrix to S

$$\sum_{\beta=1}^n T^{\alpha\beta}(u) S_{\beta\gamma}(u_{\beta}) = \delta_{\gamma}^{\alpha}. \quad (5.24)$$

The constants of the motion are

$$\mathcal{A}_{\beta} = \sum_{\alpha=1}^n T^{\beta\alpha}(u) (\partial_{u_{\alpha}}^2 + f_{\alpha}(u_{\alpha}) \partial_{u_{\alpha}}), \quad \mathcal{A}_n = \mathcal{H}. \quad (5.25)$$

Here, $f_{\alpha}(u_{\alpha}) = \partial_{u_{\alpha}} \ln(\sqrt{g} g^{\alpha\alpha})$. (The fact that

$$\partial_{u_\alpha u_\beta} \ln(\sqrt{g} g^{\alpha\alpha}) = 0, \quad \text{for } \alpha \neq \beta$$

follows for any space of constant curvature by noting that it is equivalent to the statement that the off diagonal elements of the Ricci tensor must vanish for an orthogonal coordinate system: $R_{\alpha\beta} = 0$, $\alpha \neq \beta$.¹⁸⁾ One can show that $[\mathcal{A}_\alpha, \mathcal{A}_\beta] = 0$.¹⁸⁻²⁰ Furthermore, if $\Psi = \prod_{\alpha=1}^n \psi_\alpha(y_\alpha; h_1, \dots, h_n)$ satisfies the *separation equations*

$$\partial_{u_\alpha}^2 \Psi + f_\alpha(u_\alpha) \partial_{u_\alpha} \Psi = \sum_{\beta=1}^n S_{\alpha\beta}(u_\alpha) h_\beta \Psi, \quad \alpha = 1, \dots, n \quad (5.26)$$

then

$$\mathcal{A}_\beta \Psi = h_\beta \Psi, \quad \beta = 1, \dots, n. \quad (5.27)$$

Now fix γ , $1 \leq \gamma \leq n$, and denote by \mathbf{y} the coordinate choice

$$\mathbf{y} = (u_1, \dots, u_{\gamma-1}, \tau, u_{\gamma+1}, \dots, u_n),$$

where τ is a parameter. We see from Eq. (5.26) that if Ψ satisfies the separation equations then

$$\psi_\gamma^{-1} (\partial_{u_\gamma}^2 \psi_\gamma + f_\alpha(u_\gamma) \partial_{u_\gamma} \psi_\gamma) |_{u_\gamma=\tau} \Psi(\mathbf{u}) = \sum_{\beta=1}^n S_{\gamma\beta}(\tau) \mathcal{A}_\beta \Psi(\mathbf{u}). \quad (5.28)$$

On the other hand, from Eq. (5.1) we have

$$\psi_\gamma^{-1} (\partial_{u_\gamma}^2 \psi_\gamma + f_\alpha(u_\gamma) \partial_{u_\gamma} \psi_\gamma) |_{u_\gamma=\tau} \Psi(\mathbf{u}) = \frac{1}{g^{\gamma\gamma}(\mathbf{y})} (\mathcal{H}(\mathbf{u}) - \mathcal{H}^\gamma(\mathbf{y})) \Psi(\mathbf{u}),$$

where

$$\mathcal{H}^\gamma(\mathbf{y}) = \frac{1}{\sqrt{g(\mathbf{y})}} \sum_{\alpha \neq \gamma} \partial_{u_\alpha} (\sqrt{g(\mathbf{y})} g^{\alpha\alpha}(\mathbf{y}) \partial_{u_\alpha}).$$

This suggests the operator identity

$$\frac{1}{g^{\gamma\gamma}(\mathbf{y})} (\mathcal{H}(\mathbf{u}) - \mathcal{H}^\gamma(\mathbf{y})) = \sum_{\beta=1}^n S_{\gamma\beta}(\tau) \mathcal{A}_\beta, \quad (5.29)$$

i.e., that the left-hand side of Eq. (5.29) is a one-parameter family of constants of the motion and that as τ runs over a range of values and $\gamma = 1, \dots, n$ the full space of constants of motion associated with this separable system is spanned.^{16,17}

Now Eq. (5.29) is equivalent to the conditions

$$\frac{T^{n\alpha}(\mathbf{u})}{T^{n\gamma}(\mathbf{y})} - \frac{T^{n\alpha}(\mathbf{y})}{T^{n\gamma}(\mathbf{y})} (1 - \delta_\gamma^\alpha) = \sum_{\beta=1}^n S_{\gamma\beta}(\tau) T^{\beta\alpha}(\mathbf{u})$$

and these conditions are easily seen to follow from Eq. (5.24) and from

$$\sum_{\beta=1, \beta \neq \gamma}^n T^{n\beta}(\mathbf{y}) S_{\beta\xi}(u_\beta) + T^{n\gamma}(\mathbf{y}) S_{\gamma\xi}(\tau) = \delta_\xi^n.$$

Similarly, for the classical case an expression for the Hamiltonian analogous to Eq. (5.29) generates the constants of the motion.^{16,21}

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