FAMILIES OF ORTHOGONAL AND BIORTHOGONAL POLYNOMIALS ON THE N-SPHERE

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ABSTRACT. We study the Laplace-Beltrami eigenvalue equation $H\Phi = \lambda\Phi$ on the n-sphere, with an added vector potential term motivated by the differential equations for the polynomial Lauricella functions F_A. The operator H is self-adjoint with respect to the natural inner product induced on the sphere and, in certain special coordinates, it admits a spectral decomposition with eigenspaces composed entirely of polynomials. The eigenvalues are degenerate but the degeneracy can be broken through use of the possible separable coordinate systems on the n-sphere. Then a basis for each eigenspace can be selected in terms of the simultaneous eigenfunctions of a family of commuting second order differential operators that also commute with H. The results provide a multiplicity of n-variable orthogonal and biorthogonal families of polynomials that generalize classical results for one and two variable families of Jacobi polynomials on intervals, disks, and paraboloids.

1. Introduction. Orthogonal polynomials in one variable which also satisfy second order ordinary differential or difference equations have proven extraordinarily useful in the development of special function theory and in the practical approximation of functions, e.g. [R. Askey 1975]. Orthogonal and biorthogonal families of polynomials in several variables which satisfy second order partial differential or difference equations are similarly very useful but there is as yet no general theory and more examples are needed. In this paper we will study such families which are related to the Laplace-Beltrami eigenvalue equation on the n-sphere. Our procedure provides a uniform setting within which to classify several known examples related to the n-sphere and to generate many new examples. Our approach falls within the theory of Dunkl's differential-difference operators [C. Dunkl 1988, 1989]; the main contribution of our paper is to point out the power of separation of variable methods in this theory. (Note: There is also a considerable literature on discrete analogs of the Laplace-Beltrami eigenvalue equation on the sphere in which the symmetry groups are finite, e.g., [D. Stanton 1984].)

It was shown by [Lam and Tratnik 1985] that the Lauricella functions

$$\Phi = F_A \left[\begin{matrix} M+G-1; & -m_1, \cdots, -m_n \\ & \gamma_1, \cdots, \gamma_n \end{matrix} ; x_1, \cdots, x_n \right] \tag{1.1}$$

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and

$$(1.2)\qquad (1-x)^M F_A\left[\begin{matrix}-M-\gamma_{n+1}+1; & -m_1,\cdots,-m_n\\ & \gamma_1,\cdots,\gamma_n\end{matrix};-\frac{x_1}{1-x},\cdots,-\frac{x_n}{1-x}\right]$$

form a biorthogonal polynomial family where $m_i = 0,1,2,\cdots$, $M=\sum_{k=1}^n m_i$, $G=\sum_{\ell=1}^{n+1}\gamma_\ell$, $x=\sum_{k=1}^n x_i$ and the γ_ℓ are positive real numbers. (We will derive the inner product later.) Here, the Lauricella function F_A is defined by the series

$$F_A\left[\begin{matrix}a; & b_1,\cdots,b_n\\ & c_1,\cdots,c_n\end{matrix};z_1,\cdots,z_n\right]$$

$$(1.3)\qquad = \sum_{m_1,\cdots,m_n=0}^{\infty}\frac{(a)_{m_1+\cdots+m_n}(b_1)_{m_1}\cdots(b_n)_{m_n}z_1^{m_1}\cdots z_n^{m_n}}{(c_1)_{m_1}\cdots(c_n)_{m_n}m_1!\cdots m_n!},$$

where

$$(a)_m=\begin{cases}1 & \text{if } m=0\\ a(a+1)\dots(a+m-1) & \text{if } m\geq 1.\end{cases}$$

As is easily verified by adding the standard partial differential equations for the F_A, [Appell and Kampe de Feriet 1926], these polynomial functions Φ satisfy the eigenvalue equation

$$(1.4)\qquad H\Phi=-M(M+G-1)\Phi$$

where

$$(1.5)\qquad H=\sum_{i,j=1}^n(x_i\delta_{ij}-x_ix_j)\partial_{x_ix_j}+\sum_{i=1}^n(\gamma_i-Gx_i)\partial_{x_i}.$$

Here δ_{ij} is the Kronecker delta. Note that H maps polynomials of maximum order m_i in x_i to polynomials of the same type. It is easy to see that as the m_i range over all nonnegative integers the functions (1.1) form a basis for the space of all polynomials in variables $x_1,\cdots,x_n$, and that the spectrum of H acting on this space is exactly

$$\{-M(M+G-1): M=0,1,2,\cdots\}.$$

(For $n=2$ equation (1.4) appears in the classification by [Krall and Sheffer 1967] of all second order partial differential operators such that the Mth order orthogonal polynomials in two variables, with respect to some weight function, are eigenfunctions of the operator.) We will look for other bases of solutions to equation (1.4), both orthogonal and biorthogonal with respect to a natural inner product.

Equation (1.4) is closely related to the Laplace-Beltrami eigenvalue equation on the n-sphere, [Eisenhart 1949]. To see this consider the contravariant metric determined by the second derivative terms in H:

$$(1.6)\qquad g^{ij}=\delta_{ij}x_i-x_ix_j,\qquad 1\leq i,j\leq n.$$

Then $\det(g^{ij})=g^{-1}=x_1x_2\cdots x_n(1-x)$ and

$$(1.7)\qquad g_{ij}=\frac{1}{1-x}+\frac{\delta_{ij}}{x_i}.$$

Note that

$$\sum_{j=1}^{n} g^{ij} g_{jk} = \delta_k^i = \begin{cases} 1 \text{ if } & i = k \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$ds^2 = \sum_{i,j=1}^{n} g_{ij} dx_i dx_j$$

determines a metric on a Riemannian space with associated Laplace-Beltrami operator

$$\Delta_n = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \partial_{x_i} (g^{ij} \sqrt{g}\ \partial_{x_j}). \tag{1.8}$$

A straightforward computation yields

$$H = \Delta_n + \Lambda_n \tag{1.9}$$

where

$$\Lambda_n = \sum_{j=1}^{n} [\gamma_j - \frac{1}{2} + (\frac{n+1}{2} - G) x_j] \partial_{x_j}. \tag{1.10}$$

Thus if $\gamma_1 = \cdots = \gamma_{n+1} = 1/2$ then $H \equiv \Delta_n$, but in general H differs from Δ_n by the first order differential operator Λ_n.

To identify the Riemannian space we introduce Cartesian coordinates $z_0, z_1, \cdots, z_n$ in $n+1$ dimensional Euclidean space and restrict these coordinates by the conditions

$$\begin{aligned} z_0^2 &= 1 - \sum_{i=1}^{n} x_i = 1 - x \\ z_1^2 &= x_1 \\ z_2^2 &= x_2 \\ &\vdots \\ z_n^2 &= x_n. \end{aligned} \tag{1.11}$$

Note that $z_0^2 + z_1^2 + \cdots + z_n^2 = 1$. Defining a metric ds^2 by

$$ds^2 = \sum_{m=0}^{n} (dz_m)^2$$

we find

$$ds^2 = \frac{1}{4} \sum_{i,j=1}^{n} (\frac{1}{1-x} + \frac{\delta_{ij}}{x_i}) dx_i dx_j. \tag{1.12}$$

Thus the space corresponds to a portion of the n-sphere S^n. We can consider the coordinates $\{x_i\}$ for $0 \leq x_i$ and $x \leq 1$ as covering the portion of the n-sphere given by $0 \leq z_i$, $\sum_{k=1}^{n} z_k^2 = 1$.

One can transfer the Schrödinger equation (1.4) with vector potential Λ_n to one with a scalar potential V_n through the use of a multiplier transformation ρ. Setting $\Phi(\mathbf{x}) = \rho(\mathbf{x})\Psi(\mathbf{x})$ for a nonzero scalar function ρ we find

$$(\Delta_n + \Lambda_n)\Phi = -M(M+G-1)\Phi \iff (\Delta_n + V_n(\mathbf{x}))\Psi = -M(M+G-1)\Psi,$$

provided

$$\rho^{-1} = x_1^{\gamma_1/2-1/4}\cdots x_n^{\gamma_n/2-1/4}(1-x)^{\gamma_{n+1}/2-1/4}. \tag{1.13}$$

A straightforward but tedious computation gives for the scalar potential:

$$V_n = -\frac{1}{4}\sum_{i=1}^{n}\frac{(\gamma_i-\frac{1}{2})(\gamma_i-\frac{3}{2})}{x_i} - \frac{1}{4}\frac{(\gamma_{n+1}-\frac{1}{2})(\gamma_{n+1}-\frac{3}{2})}{1-x} + \frac{1}{4}\left[(1-G)^2 - 1 - \frac{(n-3)(n+1)}{4}\right] \tag{1.14}$$

or, in terms of Cartesian coordinates,

$$V_n = -\frac{1}{4}\sum_{i=1}^{n}\frac{(\gamma_i-\frac{1}{2})(\gamma_i-\frac{3}{2})}{z_i^2} - \frac{1}{4}\frac{(\gamma_{n+1}-\frac{1}{2})(\gamma_{n+1}-\frac{3}{2})}{z_0^2} + \frac{1}{4}\left[(1-G)^2 - 1 - \frac{(n-3)(n+1)}{4}\right]. \tag{1.15}$$

The equation $H'\Psi \equiv (\Delta_n + V_n)\Psi = \lambda\Psi$ has a natural Riemannian metric

$$d\omega = g^{1/2}dx_1\cdots dx_n = x_1^{-1/2}\cdots x_n^{-1/2}(1-x)^{-1/2}dx_1\cdots dx_n, \tag{1.16}$$

[Eisenhart 1949]. Furthermore, the operator $H' = \rho^{-1}H\rho = \Delta_n + V_n$ is formally self-adjoint with respect to the inner product

$$< \Psi_1, \Psi_2 > = \int\cdots\int_{x_i>0, x<1}\Psi_1(\mathbf{x})\overline{\Psi_2}(\mathbf{x})d\omega \tag{1.17}$$

where Ψ_1, Ψ_2 are twice continuously differentiable functions of the x_j which take complex values:

$$< H'\Psi_1, \Psi_2 > = < \Psi_1, H'\Psi_2 > .$$

This induces an inner product on the space of polynomial functions $\Phi(\mathbf{x}) = \rho\Psi$, with respect to which H is self-adjoint:

$$\begin{aligned}(\Phi_1, \Phi_2) &\equiv < \Psi_1, \Psi_2 > \\ &= \int\cdots\int_{x_i>0, x<1}\Phi_1(\mathbf{x})\overline{\Phi_2}(\mathbf{x})\rho^{-2}(\mathbf{x})\ d\omega \\ &= \int\cdots\int_{x_i>0, x<1}\Phi_1\overline{\Phi_2}\ d\tilde{\omega},\end{aligned} \tag{1.18}$$

$$d\tilde{\omega} = x_1^{\gamma_1 - 1} \dots x_n^{\gamma_n - 1}(1-x)^{\gamma_{n+1}-1} dx_1 \dots dx_n,$$

$$(H\Phi_1, \Phi_2) = (\Phi_1, H\Phi_2).$$

(Indeed, H is clearly formally self-adjoint and the boundary terms obviously vanish for the γ_i sufficiently large. The result can then be extended to all $\gamma_i > 0$ by analytic continuation.) Thus $(\cdot, \cdot)$ is the natural inner product associated with equation (1.4).

A *first order symmetry operator* for the equation $H\Phi = \lambda\Phi$ is a differential operator

$$K = \sum_{i=1}^{n} f_i(\mathbf{x})\partial_{x_i} + g(\mathbf{x})$$

such that

$$[H, K] \equiv HK - KH = 0,$$

[Miller 1977]. The first order symmetry operators form a real Lie algebra under addition of operators, multiplication of an operator by a real scalar, and the commutator bracket $[A, B] = AB - BA$. If $\gamma_1 = \gamma_2 = \cdots = \gamma_{n+1} = 1/2$ then $H = \Delta_n$ and it is well-known [Eisenhart 1949, 1961] that the Lie algebra of real symmetry operators of Δ_n is so$(n+1)$, with dimension $n(n+1)/2$ and a basis of the form $\{L_{\ell k}\}$ where $0 \leq \ell < k \leq n$, and $L_{\ell k} = -L_{k\ell}$. Explicitly,

$$L_{\ell k} = z_\ell \partial_{z_k} - z_k \partial_{z_\ell} \tag{1.19}$$

and

$$\begin{aligned} L_{ij} &= 2\sqrt{x_i x_j}(\partial_{x_j} - \partial_{x_i}), \quad 1 \leq i, j \leq n \\ L_{0i} &= 2\sqrt{x_i(1-x)}\partial_{x_i}, \quad 1 \leq i \leq n. \end{aligned} \tag{1.20}$$

Furthermore, all real second-order differential operators S that commute with Δ_n can be expressed as linear combinations over $\mathbb{R}$ of real constants, elements $L_{\ell k}$ and elements $L_{\ell k} L_{\ell' k'}$. For $\gamma_1, \dots, \gamma_{n+1}$ arbitrary, however, we have

Lemma 1. *If K is a first order operator such that $[K, H] = 0$ then $K = c$, multiplication by the real constant c. The second order operators*

$$\begin{aligned} S_{ij} &\equiv 4x_i x_j(\partial_{x_i} - \partial_{x_j})^2 + 4(\gamma_i x_j - \gamma_j x_i)(\partial_{x_i} - \partial_{x_j}) \\ &= L_{ij}^2 + 4[(\gamma_i - \frac{1}{2})x_j - (\gamma_j - \frac{1}{2})x_i](\partial_{x_i} - \partial_{x_j}) \\ &= S_{ji}, \qquad 1 \leq i < j \leq n, \end{aligned} \tag{1.21}$$

$$\begin{aligned} S_{0i} &\equiv 4x_i(1-x)\partial_{x_i}^2 + 4[\gamma_i(1-x) - \gamma_{n+1}x_i]\partial_{x_i} \\ &= L_{0i}^2 + 4[(\gamma_i - \frac{1}{2})(1-x) - (\gamma_{n+1} - \frac{1}{2})x_i]\partial_{x_i} \\ &= S_{i0}, \qquad 1 \leq i \leq n, \end{aligned} \tag{1.22}$$

do commute with H: $[S_{ij}, H] = [S_{0i}, H] = 0$. Also

$$8H \equiv \sum_{i,j=1}^{n} S_{ij} + 2\sum_{i=1}^{n} S_{0i}. \tag{1.23}$$

We conjecture, but have not proven, that linear combinations of the S_{ij} and S_{0i} are the only second order operators commuting with H.

If S is a second order symmetry operator for H then $S' = \rho^{-1}S\rho$ is a second order symmetry for $H' = \Delta_n + V_n$ and, necessarily, $S' = \Upsilon + f$ where Υ is a second order symmetry for Δ_n and f is a real-valued function. Thus S' is a formally self-adjoint operator with respect to the inner product $< \cdot, \cdot >$ and S is formally self-adjoint with respect to $(\cdot, \cdot)$.

2. Orthogonal bases of separable solutions. In the paper [Kalnins and Miller 1986] and in the book [Kalnins 1986] all separable coordinates for the equation $\Delta_n \Psi = \lambda \Psi$ are constructed, where Δ_n is the Laplace-Beltrami operator on S^n. It is shown that all separable coordinates are orthogonal and that for each separable coordinate system the corresponding separated solutions are characterized as simultaneous eigenfunctions of a set of n second order commuting symmetry operators for Δ_n. These operators are real linear combinations of the symmetries L_{ij}^2, $1 \le i < j \le n+1$, where L_{ij} is a rotational generator in so$(n+1)$. For $n = 2$ there are two separable systems (ellipsoidal and spherical coordinates), while for $n = 3$ there are 6 systems. The number of separable systems grows rapidly with n, but all systems can be constructed through a simple graphical procedure. (In general, the possible separable systems are the various polyspherical coordinates [Vilenkin 1968], the basic ellipsoidal coordinates, and combinations of polyspherical and ellipsoidal coordinates.) Moreover, the equation $(\Delta_n + V_n)\Psi = \lambda\Psi$ where the scalar potential takes the form

$$V_n = \sum_{i=1}^{n} \frac{\alpha_i}{z_i^2} + \frac{\alpha_0}{z_0^2}, \quad \alpha_0, \alpha_1, \ldots, \alpha_n \text{ const.}, \tag{2.1}$$

is separable in *all* the coordinate systems in which the Laplace-Beltrami eigenvalue equation is separable. (That is, V_n of this form is a *Stäckel multiplier* for all separable coordinate systems on S^n; see [Boyer, Kalnins and Miller 1986].) Indeed, the equation with potential (2.1) is separable in general ellipsoidal coordinates. Since all other coordinates are limiting cases of ellipsoidal coordinates, the conclusion follows. [NOTE: If each $\alpha_j = -\frac{1}{4}k_j(k_j + m_j - 1)$ where k_j and m_j are non-negative integers with $m_j \ge 1$, then the equation $(\Delta_n + V_n)\Psi = \lambda\Psi$ can be viewed as a resriction of the Laplace-Beltrami eigenvalue equation $(\Delta_M + V_M)\Psi' = \lambda\Psi'$ on the N-sphere where $N = \sum_{j=0}^{n} m_j + n$, in which the variable dependence on the subspheres S^{m_j} has already been factored out. Moreover, using the canonical equation technique found in [Kalnins, Manocha and Miller 1980] one can show that all solutions of the above equation for general γ_i are solutions of the flat-space wave equation in $2n + 2$ dimensions with signature $(n + 1, n + 1)$. Thus the conformal symmetry algebra of the wave equation can be expected to transform solutions of the eigenvalue equations among themselves. Lemma 2 and Corollary 1 below are examples of this action.]

The results of Kalnins and Miller, characterizing separable systems by symmetry operators, can easily be translated to the present case. In those references (for

$V_n = 0$) the symmetry operators are given explicitly as linear combinations of the symmetries L_{ij}^2. The results for the potential (1.14) are similar: one replaces L_{ij}^2 by $S'_{ij} = \rho^{-1} S_{ij} \rho$ and takes the same linear combinations. Moreover, since the defining symmetry operators for a separable system are real linear combinations of the L_{ij}^2 plus scalar functions, they are formally self-adjoint with respect to the inner product $< \cdot, \cdot >$.

These results can now easily be extended to results for solutions of

$$(\Delta_n + \Lambda_n)\Phi = \lambda\Phi \tag{2.2}$$

through the mappings

$$\begin{aligned} \Delta_n + \Lambda_n &= \rho(\Delta_n + V_n)\rho^{-1} \\ S_{ij} &= \rho S'_{ij} \rho^{-1} \\ \Phi &= \rho\Psi. \end{aligned} \tag{2.3}$$

Thus all separable solutions Ψ map to R-separable solutions Φ of (2.2), [Miller 1977]. The R-separable coordinates and solutions are determined by commuting symmetry operators S of $\Delta_n + \Lambda_n$ which are obtained from expressions in [Kalnins and Miller 1986] or [Kalnins 1986] where each occurrence of L_{ij}^2 is replaced by S_{ij}. The defining symmetry operators are all formally self-adjoint with respect to the inner product $(\cdot, \cdot)$. Finally, since each S_{ij} maps polynomials of maximum order m_k in x_k to polynomials of the same type, it follows that a basis of separated solutions can be expressed as *polynomials* in the x_i. Since the symmetry operators are self-adjoint, the basis of simultaneous eigenfunctions can be chosen to be *orthogonal*.

We conclude from this argument that every separable coordinate system for the Laplace-Beltrami eigenvalue equation on the n-sphere yields an orthogonal basis of polynomial solutions of equation (1.4), hence an orthogonal basis for all n-variable polynomials with inner product (1.18).

As an example we work out the separation equations for spherical coordinates $\{u_i\}$ on S^n:

$$\begin{aligned} z_0^2 &= 1 - x = 1 - u_n \\ z_1^2 &= x_1 = u_1 u_2 \dots u_n \\ z_2^2 &= x_2 = (1 - u_1) u_2 \dots u_n \\ &\vdots \\ z_{n-1}^2 &= x_{n-1} = (1 - u_{n-2}) u_{n-1} u_n \\ z_n^2 &= x_n = (1 - u_{n-1}) u_n. \end{aligned} \tag{2.4}$$

(Note that in terms of angles $\{\theta_i\}$ one usually sets $u_i = \sin^2\theta_i$.) It follows that

$$u_j = \begin{cases} w_j / w_{j+1}, & j = 1, \dots, n-1 \\ w_n, & j = n \end{cases} \tag{2.5}$$

where

$$w_\ell = \sum_{i=1}^{\ell} x_i.$$

In terms of the $\{u_i\}$, the operator (1.5) becomes

$$H = \sum_{i=1}^{n} \frac{1}{u_{i+1}\dots u_n} \left[u_i(1-u_i)\partial^2_{u_i} + \left(\sum_{j=1}^{i} \gamma_j - (\sum_{p=1}^{i+1} \gamma_p) u_i \right) \partial_{u_i} \right]. \tag{2.6}$$

Equation (1.4) is separable in these coordinates with separation equations

$$\begin{aligned} &u_1(1-u_1)\partial^2_{u_1}\Theta_1 + [\gamma_1 - (\gamma_1+\gamma_2)u_1]\,\partial_{u_1}\Theta_1 = c_1\Theta_1, \\ &\left[\frac{c_{k-1}}{u_k} + u_k(1-u_k)\partial^2_{u_k}\right]\Theta_k + \left[\sum_{j=1}^{k}\gamma_j - (\sum_{p=1}^{k+1}\gamma_p)u_k\right]\partial_{u_k}\Theta_k = c_k\Theta_k, \\ &\qquad k = 2,3,\dots,n. \end{aligned} \tag{2.7}$$

Here $\Theta = \prod_{k=1}^n \Theta_k(u_k)$ and the c_i are the separation constants, with $c_n = -M(M+G-1)$.

Noting that the hypergeometric equation

$$u(1-u)\frac{d^2 g}{du^2} + [c - (a+b+1)u]\,\frac{dg}{du} - abg = 0$$

admits the solution

$$g = {}_2F_1\left(\begin{matrix} a, & b \\ & c \end{matrix}; u\right) = \sum_{m=1}^{\infty} \frac{(a)_m (b)_m}{(c)_m m!} u^m,$$

a polynomial for $a = 0,-1,-2,\dots,$ and requiring that Θ be a polynomial in the $\{x_i\}$ we obtain the solutions

$$\begin{aligned} &\Theta_1(u_1) = {}_2F_1\left(\begin{matrix} -\ell_1, & \ell_1+\gamma_1+\gamma_2-1 \\ & \gamma_1 \end{matrix}; u_1\right) \\ &\quad c_1 = -\ell_1(\ell_1+\gamma_1+\gamma_2-1), \\ &\Theta_k(u_k) = u_k^{\ell_1+\ell_2+\cdots+\ell_{k-1}}\, {}_2F_1\left(\begin{matrix} -\ell_k, & 2(\ell_1+\cdots+\ell_{k-1})+\ell_k+\gamma_1+\cdots+\gamma_{k+1}-1 \\ & 2(\ell_1+\cdots+\ell_{k-1})+\gamma_1+\cdots+\gamma_k \end{matrix}; u_k\right), \\ &\quad c_k = -(\ell_1+\cdots+\ell_k)(\ell_1+\cdots+\ell_k+\gamma_1+\cdots+\gamma_{k+1}-1), \\ &\quad k = 2,3,\dots,n, \end{aligned} \tag{2.8}$$

where $\sum_{i=1}^n \ell_i = M$ and $\ell_i = 0,1,2\dots.$ This determines Θ to within a normalization factor.

In the special case $n = 2$ we have the result of [Proriol 1957] and of [Karlin and McGregor 1964]:

$$\begin{aligned} \Theta_{\ell_1,\ell_2}(x_1,x_2) =& {}_2F_1\left(\begin{matrix} -\ell_1, & \ell_1+\gamma_1+\gamma_2-1 \\ & \gamma_1 \end{matrix}; \frac{x_1}{x_1+x_2}\right)(x_1+x_2)^{\ell_1}\times \\ & {}_2F_1\left(\begin{matrix} -\ell_2, & 2\ell_1+\ell_2+\gamma_1+\gamma_2+\gamma_3-1 \\ & 2\ell_1+\gamma_1+\gamma_2 \end{matrix}; x_1+x_2\right) \\ \sim\ & P_{\ell_2}^{(\gamma_3-1,\gamma_1+\gamma_2+2\ell_1-1)}(2x_1+2x_2-1)\ (x_1+x_2)^{\ell_1}\times \\ & P_{\ell_1}^{(\gamma_2-1,\gamma_1-1)}\left(\frac{2x_1}{x_1+x_2}-1\right). \end{aligned} \tag{2.9}$$

where $P_k^{(\alpha,\beta)}(x)$ is a Jacobi polynomial.

Returning to the general case, we have the eigenvalue equations

$$S_\ell \Theta_\ell = c_\ell \Theta_\ell, \quad \ell = 1, \dots, n \tag{2.10}$$

where

$$\begin{aligned} S_1 &= u_1(1-u_1)\partial^2_{u_1} + [\gamma_1 - (\gamma_1+\gamma_2)u_1]\,\partial_{u_1}, \\ S_k &= \frac{1}{u_k} S_{k-1} + u_k(1-u_k)\partial^2_{u_k} + [\gamma_1 + \dots + \gamma_k - (\gamma_1 + \dots + \gamma_{k+1})u_k]\,\partial_{u_k}, \\ & k = 2, 3, \dots, n, \end{aligned} \tag{2.11}$$

and $S_n = H$. Furthermore, $[S_i, S_j] = 0$ and the S_i are self-adjoint with respect to the inner product $(\cdot,\cdot)$. It follows immediately that

$$(\Theta_\ell, \Theta_{\mathbf{m}}) = 0$$

unless $\ell_1 = m_1$, $\ell_2 = m_2$, $\dots$, $\ell_n = m_n$. The measure $d\tilde{\omega}$ becomes in these coordinates

$$d\tilde{\omega} = u_1^{\gamma_1-1} u_2^{\gamma_1+\gamma_2-1} \dots u_n^{\gamma_1+\dots+\gamma_n-1} (1-u_1)^{\gamma_2-1}(1-u_2)^{\gamma_3-1} \dots (1-u_n)^{\gamma_{n+1}-1} du_1 \dots du_n,$$

where $0 < u_i < 1$. In terms of the symmetries S_{ij}, S_{0i}, (1.21-22), we have:

$$\begin{aligned} S_k &= \frac{1}{8} \sum_{i,j=1}^{k+1} S_{ij}, \quad k = 1, \dots, n-1 \\ S_n &= H = \frac{1}{8}\Big(\sum_{h,p=0}^{n} S_{hp}\Big), \end{aligned} \tag{2.12}$$

where we set $S_{hh} = 0$.

3. Orthogonal bases for another space of polynomials. Now we make the change of coordinates $x_i = y_i^2$, $1 \le i \le n$, and look for solutions of (1.4) that are polynomials in the y_i. In general, H doesn't map polynomials in the y_i to polynomials, but in the special case $\gamma_1 = \gamma_2 = \dots = \gamma_n = 1/2$, $G = \gamma_{n+1} + n/2 = s/2 + (n+1)/2$, we have

$$H = \frac{1}{4} \sum_{i,j=1}^{n} (\delta_{ij} - y_i y_j)\partial_{y_i y_j} + \frac{1}{2}\Big(\frac{1}{2} - G\Big) \sum_{j=1}^{n} y_j \partial_{y_j}, \tag{3.1}$$

and H does map polynomials to polynomials of at most the same degree. Moreover, the differential operators

$$L_{ij} = -L_{ji} = y_i \partial_{y_j} - y_j \partial_{y_i}, \quad 1 \le i < j \le n, \tag{3.2}$$

commute with H and form a basis for the symmetry algebra so(n). The special second order symmetries take the form $S_{ij} = L_{ij}^2$, $1 \le i < j \le n$, and

$$S_{0i} = L_{0i}^2 - 2\Big(G - \frac{1}{2}\Big) y_i \partial_{y_i} = \Big(1 - \sum_{j=1}^{n} y_j^2\Big)\partial^2_{y_i} - 2G y_i \partial_{y_i},$$

and clearly map polynomials to polynomials of at most the same degree. The measure takes the form

$$d\tilde{\omega} = (1 - y_1^2 - \cdots - y_n^2)^{s/2-1/2} dy_1 \dots dy_n, \tag{3.3}$$

where $-1 \le y_i \le 1$ and

$$\rho^{-1} = (1 - y_1^2 - \cdots - y_n^2)^{s/4}. \tag{3.4}$$

Again, H and the $S_{\ell k}$ are formally self-adjoint with respect to the inner product

$$(\Phi_1, \Phi_2) = \int \cdots \int_{\sum y_i^2 < 1} \Phi_1(\mathbf{y})\overline{\Phi_2}(\mathbf{y})\ d\tilde{\omega}, \tag{3.5}$$

where Φ_1, Φ_2 are polynomials in the y_i.

Every separable coordinate system for the equation

$$H\Phi = -M(M + G - 1)\Phi, \quad 2M \text{ a nonnegative integer}, \tag{3.6}$$

where H is given by (3.1) yields an orthogonal basis of multivariable polynomials with respect to the inner product $(\cdot, \cdot)$. (For $n = 2$ this equation is also on the list of [Krall and Sheffer 1967].) Indeed, for spherical coordinates $u_i = \sin^2\theta_i$ we obtain the orthogonal basis of polynomials in $\mathbf{y}$:

$$\begin{aligned} &e^{\pm 2i\ell_1\theta_1} \prod_{k=2}^{n-1} [\sin\theta_k]^{2(\ell_1+\cdots+\ell_{k-1})} C_{2\ell_k}^{2(\ell_1+\cdots+\ell_{k-1})+(k-1)/2}(\cos\theta_k) \times \\ &u_n^{\ell_1+\cdots+\ell_{n-1}} {}_2F_1\left(\begin{matrix} -\ell_n, & 2(\ell_1+\cdots+\ell_{n-1}) + \ell_n + (n-1)/2 + s/2 \\ & 2(\ell_1+\cdots+\ell_{n-1}) + n/2 \end{matrix} ; u_n \right), \end{aligned} \tag{3.7}$$

where $2\ell_i = 0, 1, 2, \dots$ for $1 \le i \le n-1$, $\ell_n = 0, 1, 2, \dots$, and the $C_k^\lambda(x)$ are Gegenbauer polynomials

$$C_k^\lambda(x) = \frac{(2\lambda)_k}{k!} {}_2F_1\left(\begin{matrix} -k, & k+2\lambda \\ & \lambda + 1/2 \end{matrix} ; 1/2 - x/2 \right),$$

[Erdelyi et al. 1951]. (The eigenvalues are defined as before.)

Using the results of [Kalnins 1986] or [Kalnins and Miller 1986], many other orthogonal bases can be worked out. Moreover the symmetry group SO(n) permits the derivation of addition theorems for the basis elements, related to the addition theorem for Gegenbauer polynomials and Koornwinder's addition theorem, [Koornwinder 1972, 1975].

Next we relate the Cartesian coordinates z_ℓ and the y_a via

$$\begin{aligned} z_0^2 &= y_1^2 \\ z_1^2 &= y_2^2 \\ &\vdots \\ z_{n-1}^2 &= y_n^2 \\ z_n^2 &= 1 - y_1^2 - \cdots - y_n^2, \end{aligned} \tag{3.8}$$

a simple permutation of the relations (2.4), so that the (separable) spherical coordinates v_i are associated with the y_a through

$$v_1 = \frac{y_2^2}{y_2^2+y_3^2}, \quad v_2 = \frac{y_2^2+y_3^2}{y_2^2+y_3^2+y_4^2}, \quad \cdots, \quad v_{n-2} = \frac{y_2^2+\cdots+y_{n-1}^2}{y_2^2+\cdots+y_n^2},$$

$$v_{n-1} = \frac{y_2^2+\cdots+y_n^2}{1-y_1^2}, \quad v_n = 1-y_1^2. \tag{3.9}$$

From the point of view of separability for the Laplace-Beltrami eigenvalue equation, these v_i coordinates are equivalent to the u_i coordinates introduced earlier, since one system can be obtained from the other through the action of an element of the $\mathrm{SO}(n+1)$ symmetry group for this equation. However, the term Λ_n breaks this symmetry so from the viewpoint of the eigenvalue equation for H, with $\gamma_1 = \gamma_2 = \ldots \gamma_n = 1/2$, these are distinct coordinates. The separation equations for the v_i are identical to those for the u_i if we interchange $\gamma_n = 1/2$ and $\gamma_{n+1} = s/2+1/2$. For $n=2$ the orthogonal basis of polynomials is

$$C_{2\ell_1}^{s/2}(\sin\theta_1) C_{2\ell_2}^{2\ell_1+s/2+1/2}(\cos\theta_2)\sin^{2\ell_1}\theta_2 \tag{3.10}$$

where $\ell_1, \ell_2 = 0, 1/2, 1, 3/2, \ldots$, $\ell_1+\ell_2 = N$ and $v_j = \sin^2\theta_j$. This is in agreement with the basis of [Koschmieder 1951, 1957].

For $n>2$ we have an orthogonal basis of polynomials of the form $\Theta = \prod_{k=1}^n \Theta_k$ where

$$\begin{aligned}
&\Theta_1 = e^{\pm 2i\ell_1\theta_1}, \quad \ell_1 = 0, 1/2, 1, 3/2, \ldots, \\
&\Theta_k = [\sin\theta_k]^{2(\ell_1+\cdots+\ell_{k-1})} C_{2\ell_k}^{2(\ell_1+\cdots+\ell_{k-1})+(k-1)/2}(\cos\theta_k) \\
&\qquad \ell_k = 0, 1/2, 1, \ldots, \quad 1<k<n-1, \\
&\Theta_{n-1} = v_{n-1}^{\ell_1+\cdots+\ell_{n-2}}\, {}_2F_1\left(\begin{matrix} -\ell_{n-1}, & 2(\ell_1+\cdots+\ell_{n-2})+\ell_{n-1}+(n-2)/2+s/2 \\ & 2(\ell_1+\cdots+\ell_{n-2})+(n-1)/2 \end{matrix}; v_{n-1}\right) \\
&\qquad \ell_{n-1} = 0, 1, 2, \ldots, \\
&\Theta_n = [\sin\theta_n]^{2(\ell_1+\cdots+\ell_{n-1})} C_{2\ell_n}^{2(\ell_1+\cdots+\ell_{n-1})+(n-1)/2+s/2}(\cos\theta_n) \\
&\qquad \ell_n = 0, 1/2, 1, \ldots,
\end{aligned} \tag{3.11}$$

where $\ell_1+\cdots+\ell_n = M$ and $v_j = \sin^2\theta_j$.

4. The "mixed" case. Next we consider the more general mixed case with variables $x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_2}$, $n_1+n_2 = n$ where

$$\begin{aligned}
z_0^2 &= 1 - \sum_{i=1}^{n_1} x_i - \sum_{a=1}^{n_2} y_a^2, \\
z_1^2 &= x_1, \\
&\vdots \\
z_{n_1}^2 &= x_{n_1}, \\
z_{n_1+1}^2 &= y_1^2, \\
&\vdots \\
z_n^2 &= y_{n_2}^2,
\end{aligned} \tag{4.1}$$

and look for polynomial solutions in x_i, y_a of the equation

$$H\Phi(\mathbf{x},\mathbf{y}) = -M(M+G-1)\Phi(\mathbf{x},\mathbf{y}), \tag{4.2}$$

where $\gamma_{n_1+1} = \gamma_{n_1+2} = \cdots = \gamma_n = 1/2$ and

$$\begin{aligned} H &= \frac{1}{4}\sum_{a,b}(\delta_{ab} - y_a y_b)\partial_{y_a y_b} + \sum_{i,j}(\delta_{ij}x_i - x_i x_j)\partial_{x_i x_j} \\ &\quad - \sum_{a,i} y_a x_i \partial_{y_a x_i} + \sum_i (\gamma_i - G x_i)\partial_{x_i} + \frac{1}{2}\sum_a (\frac{1}{2} - G) y_a \partial_{y_a}, \\ G &= \frac{n_2+1}{2} + \sum_i \gamma_i + s/2, \quad 2M \text{ a nonnegative integer.} \end{aligned} \tag{4.3}$$

For reference,

$$\begin{aligned} \Delta_n &= \frac{1}{4}\sum_{a,b}(\delta_{ab} - y_a y_b)\partial_{y_a y_b} + \sum_{i,j}(\delta_{ij}x_i - x_i x_j)\partial_{x_i x_j} \\ &\quad - \sum_{a,i} y_a x_i \partial_{y_a x_i} + \frac{1}{2}\sum_i \left(1 - (n+1)x_i\right)\partial_{x_i} \\ &\quad - \frac{n}{4}\sum_a y_a \partial_{y_a}, \\ \Lambda_n &= \sum_i \left[\gamma_i - \frac{1}{2} + (\frac{n+1}{2} - G)x_i\right]\partial_{x_i} + \frac{1}{2}\sum_a (\frac{n+1}{2} - G) y_a \partial_{y_a}. \end{aligned} \tag{4.4}$$

Note that H maps polynomials in x_i, y_a to polynomials of at most the same order. The induced measure is

$$\begin{aligned} d\tilde{\omega} &= x_1^{\gamma_1 - 1}\dots x_{n_1}^{\gamma_{n_1}-1}(1 - \sum x_i - \sum y_a^2)^{s/2-1/2} dx_1 \dots dx_{n_1} dy_1 \dots dy_{n_2}, \\ & 0 < x_i,\ -1 < y_a < 1,\ \sum_i x_i + \sum_a y_a^2 < 1, \end{aligned} \tag{4.5}$$

and

$$\rho^{-1} = x_1^{\gamma_1/2-1/4}\dots x_{n_1}^{\gamma_{n_1}/2-1/4}(1 - \sum_i x_i - \sum_a y_a^2)^{s/4}.$$

Equation (4.2) admits the symmetry algebra so(n_2) with basis

$$L_{ab} = -L_{ba} = y_a \partial_{y_b} - y_b \partial_{y_a}, \quad 1 \le a < b \le n_2.$$

The operators H and S_{mk} are formally self-adjoint on the space of polynomials in x_i, y_a with respect to the inner product

$$(\Phi_1, \Phi_2) = \int \cdots \int_{0 < x_i, \sum_i x_i + \sum_a y_a^2 < 1} \Phi_1(\mathbf{x},\mathbf{y})\overline{\Phi_2}(\mathbf{x},\mathbf{y})\ d\tilde{\omega}.$$

However, in general the S_{mk} don't map a polynomial to one of the same or lower order in each variable, e.g.,

$$S_{ia} = 4x_i y_a^2 \partial_{x_i}^2 + x_i \partial_{y_a}^2 - 4x_i y_a \partial_{x_i y_a} + 4\gamma_i y_a^2 \partial_{x_i} - 2x_i \partial_{x_i} - 2\gamma_i y_a \partial_{y_a},$$

although they do map polynomials to polynomials. It is still true that each symmetry operator S maps a polynomial eigenspace of H into itself.

It follows that all separable coordinate systems for the n-sphere yield bases of orthogonal polynomials in the mixed case, (indeed multiple sets of such bases, depending on the ordering of the variables x_i, y_a). For example, if we choose spherical coordinates $u_\ell = \sin^2\theta_\ell$ in the form

$$u_\ell = \frac{w_\ell}{w_{\ell+1}}$$

where

$$w_\ell = \begin{cases} \sum_{i=1}^{\ell} x_i, & \ell = 1,\dots,n_1 \\ \sum_{i=1}^{n_1} x_i + \sum_{a=1}^{\ell-n_1} y_a^2, & \ell = n_1+1,\dots,n_1+n_2 \\ 1, & \ell = n_1+n_2+1 \end{cases} \tag{4.6}$$

we find the orthogonal basis of polynomials:

$$\Theta = \prod_{k=1}^{n} \Theta_k(u_k),$$

where

$$\Theta_k(u_k) = u_k^{\ell_1+\cdots+\ell_{k-1}} {}_2F_1\left(\begin{matrix} -\ell_k, & 2(\ell_1+\cdots+\ell_{k-1}) + \ell_k + \gamma_1 + \cdots + \gamma_{k+1} - 1 \\ & 2(\ell_1+\cdots+\ell_{k-1}) + \gamma_1 + \cdots + \gamma_k \end{matrix} ; u_k \right),$$

$$c_k = -(\ell_1+\cdots+\ell_k)(\ell_1+\cdots+\ell_k+\gamma_1+\cdots+\gamma_{k+1}-1), \quad k = 1,\dots,n_1, \tag{4.7}$$

$$\Theta_k(u_k) = [\sin\theta_k]^{2(\ell_1+\cdots+\ell_{k-1})} C_{2\ell_k}^{2(\ell_1+\cdots+\ell_{k-1})+\sum_{i=1}^{k}\gamma_i - 1/2}(\cos\theta_k),$$

$$c_k = -(\ell_1+\cdots+\ell_k)(\ell_1+\cdots+\ell_k+\gamma_1+\cdots+\gamma_{k+1}-1), \quad k = n_1+1,\dots,n_1+n_2-1,$$

$$\Theta_n(u_n) = u_n^{\ell_1+\cdots+\ell_{n-1}} {}_2F_1\left(\begin{matrix} -\ell_n, & 2(\ell_1+\cdots+\ell_{n-1}) + \ell_n + G - 1 \\ & 2(\ell_1+\cdots+\ell_{n-1}) + G - s/2 - 1/2 \end{matrix} ; u_n \right).$$

Here $\ell_1,\dots,\ell_{n_1}$, ℓ_n and $2\ell_{n_1+1},\dots,2\ell_{n_1+n_2-1}$ are nonnegative integers. (Recall that $\gamma_{n_1+1} = \gamma_{n_1+2} = \cdots = \gamma_{n_1+n_2} = 1/2$.)

5. Biorthogonal families of polynomials on S^n. We begin this section with a simplified proof of the biorthogonality of the polynomials (1.1) and (1.2) with respect to the inner product $(\cdot,\cdot)_\gamma$, see (1.18), (1.19). Let $\mathcal{S}_\gamma$ be the space of all polynomials in $x_1,\cdots,x_n$ with respect to this inner product and let $\mathcal{H}_{\gamma,M}$ be the subspace of $\mathcal{S}_\gamma$ consisting of solutions Φ to the eigenvalue equation

$$H\Phi = -M(M+G-1)\Phi, \tag{5.1}$$

where H is given by (1.5). Since the functions

$$D^\gamma_{\mathbf{m}}(\mathbf{x}) = F_A\left[\begin{matrix} M+G-1; & -m_1,\cdots,-m_n \\ & \gamma_1,\cdots,\gamma_n\end{matrix}; x_1,\cdots,x_n\right] \tag{5.2}$$

clearly satisfy (5.1) for $M=\sum_{i=1}^n m_i$ and since the highest order monomial in these solutions is $x_1^{m_1}\ldots x_n^{m_n}$ it follows that the $D^\gamma_{\mathbf{m}}$ for $m_1+\cdots+m_n\equiv m = M$ form a basis for $\mathcal{H}_{\gamma,M}$ and, as the m_i range over all nonnegative integers, a basis for $\mathcal{S}_\gamma$. [Note: $\dim\mathcal{H}_{\gamma,M} = \binom{M+n-1}{n-1}$.] Since H is self-adjoint we have $\mathcal{H}_{\gamma,M}\perp\mathcal{H}_{\gamma,M'}$ for $M'\neq M$. Thus

$$(D^\gamma_{\mathbf{m}'}, D^\gamma_{\mathbf{m}})_\gamma = 0 \quad \text{for } m'\neq m. \tag{5.3}$$

It is simple to verify the recurrence relation

$$\partial_{x_i} D^\gamma_{\mathbf{m}}(\mathbf{x}) = \frac{(M+G-1)(-m_i)}{\gamma_i} D^{\hat\gamma}_{\hat{\mathbf{m}}}(\mathbf{x}), \tag{5.4}$$

where

$$\hat\gamma_j = \begin{cases} \gamma_j & \text{for } j\neq i,\ 1\le j\le n \\ \gamma_i+1 & \text{for } j=i \\ \gamma_{n+1}+1 & \text{for } j=n+1\end{cases}$$

$$\hat m_j = \begin{cases} m_j & \text{for } j\neq i,\ 1\le j\le n \\ m_i-1 & \text{for } j=i\end{cases} \tag{5.5}$$

$$\hat M = M-1, \quad \hat G = G+2.$$

We can consider $P_i=\partial_{x_i}$ as an operator

$$P_i : \mathcal{S}_\gamma \to \mathcal{S}_{\hat\gamma}.$$

Indeed we have

Lemma 2. *P_i, $(1\le i\le n)$, maps $\mathcal{H}_{\gamma,M}$ onto $\mathcal{H}_{\hat\gamma,\hat M}$.*

Proof. Immediate from (5.4). For a basis free proof we can easily verify the operator identity

$$\hat H P_i = GP_i + P_i H \tag{5.6}$$

where $\hat H$ is the operator H with the γ_j replaced by $\hat\gamma_j$. Then if $H\Phi = -M(M+G-1)\Phi$ we have $\hat H(P_i\Phi) = -\hat M(\hat M+\hat G-1)(P_i\Phi)$. The null space of P_i acting on $\mathcal{H}_{\gamma,M}$ is of dimension $\binom{M+n-2}{n-2}$ for $n\ge 2$, hence the dimension of the range of P_i is

$$\binom{M+n-1}{n-1} - \binom{M+n-2}{n-2} = \binom{M+n-2}{n-1} = \dim\mathcal{H}_{\hat\gamma,\hat M}.$$

□

Corollary 1. *The operator $P_i - P_j$ maps $\mathcal{H}_{\gamma,M}$ into $\mathcal{H}_{\tilde{\gamma},\tilde{M}}$, where $1 \le i < j \le n$ and*

$$\tilde{\gamma}_k = \begin{cases} \gamma_k & \text{for } 1 \le k \le n+1,\ k \ne i,j \\ \gamma_i + 1 & \text{for } k = i \\ \gamma_j + 1 & \text{for } k = j \end{cases}$$

$$\tilde{M} = M - 1, \quad \tilde{G} = G + 2.$$

Proof.

$$\tilde{H}(P_i - P_j) = G(P_i - P_j) + (P_i - P_j)H$$

Thus if $H\Phi = -M(M+G-1)\Phi$ we have $\tilde{H}([P_i - P_j]\Phi) = -\tilde{M}(\tilde{M}+\tilde{G}-1)[P_i - P_j]\Phi$. □

The operator P_i induces an adjoint operator $P_i^* : \mathcal{S}_{\hat{\gamma}} \to \mathcal{S}_\gamma$, defined by

$$(P_i^*\Phi, \Phi')_\gamma = (\Phi, P_i\Phi')_{\hat{\gamma}}$$

for all $\Phi \in \mathcal{S}_{\hat{\gamma}}$, $\Phi' \in \mathcal{S}_\gamma$. A straightforward computation yields

$$P_i^* = -x_i(1-x)\partial_{x_i} - \gamma_i(1-x) + \gamma_{n+1}x_i. \tag{5.7}$$

Theorem 1. *P_i^* is a 1-1 map of $\mathcal{H}_{\hat{\gamma},\hat{M}}$ into $\mathcal{H}_{\gamma,M}$.*

Proof. Taking the adjoint of the relation (5.4) we obtain

$$P_i^*\hat{H} = GP_i^* + HP_i^*.$$

Furthermore, P_i^* is 1-1 since P_i is onto. □

Let

$$C_{\mathbf{m}}^\gamma(\mathbf{x}) = (P_1^*)^{m_1} \dots (P_n^*)^{m_n} 1 \in \mathcal{S}_\gamma$$

be the result of applying m_n operators $P_n^*, \dots, m_1$ operators P_1^*, one at a time, to the function $1 \in \mathcal{S}_{\gamma'}$ where

$$\gamma_i' = \gamma_i + m_i, \quad 1 \le i \le n,$$
$$\gamma_{n+1}' = \gamma_{n+1} + m.$$

(Each time an operator P_j^* is applied it lowers γ_j and γ_{n+1} by 1 and leaves the other γ_k's unchanged. The order in which these operators are applied makes no difference in the result.) It follows from the recurrence relation

$$\left(x_i\sum_{j=1}^n x_j\partial_{x_j} - x_i\partial_{x_i} + x_i(-M-\gamma_{n+1}+1) - \gamma_i + 1\right) \times$$

$$F_A\left[\begin{matrix} -M-\gamma_{n+1}+1; & -m_1, \cdots, -m_n \\ & \gamma_1, \cdots, \gamma_n \end{matrix}; x_1, \cdots, x_n\right]$$

(5.8)
$$= (1-\gamma_i)F_A\left[\begin{matrix}-(M+1)-(\gamma_{n+1}-1)+1; & -m_1,\cdots,-(m_i+1),\cdots,-m_n \\ & \gamma_1,\cdots,\gamma_i-1,\cdots,\gamma_n\end{matrix};x_1,\cdots,x_n\right]$$

and a simple induction argument that
(5.9)
$$C^{\gamma}_{\mathbf{m}}(\mathbf{x}) = c_{\gamma,m}(1-x)^M F_A\left[\begin{matrix}-M-\gamma_{n+1}+1; & -m_1,\cdots,-m_n \\ & \gamma_1,\cdots,\gamma_n\end{matrix};-\frac{x_1}{1-x},\cdots,-\frac{x_n}{1-x}\right]$$

where $c_{\gamma,m}$ is a nonzero constant. It follows from Theorem 1 that the $C^{\gamma}_{\mathbf{m}}$ belong to $\mathcal{H}_{\gamma,M}$ for $M = m_1+\cdots+m_n$. Since there are $\binom{M+n-1}{n-1}$ of these functions for fixed M and since they are clearly linearly independent, they form a basis for $\mathcal{H}_{\gamma,M}$.

Now consider the inner product

$$(C^{\gamma}_{\mathbf{m}}, D^{\gamma}_{\mathbf{m}'})_{\gamma}.$$

If $m = M \neq m' = M'$ the inner product vanishes, since $\mathcal{H}_{\gamma,M} \perp \mathcal{H}_{\gamma,M'}$. If $m = m'$ but $\mathbf{m} \not\equiv \mathbf{m}'$ then $m_i > m'_i$ for some i. Thus

$$(C^{\gamma}_{\mathbf{m}}, D^{\gamma}_{\mathbf{m}'})_{\gamma} = \kappa\ (1, P_1^{m_1}\ldots P_n^{m_n} D^{\gamma}_{\mathbf{m}'})_{\gamma'} = 0$$

since $P_i^{m_i} D^{\gamma}_{\mathbf{m}'} = 0$. (Here, κ is a nonzero constant.) We conclude that the set $\{C^{\gamma}_{\mathbf{m}}, D^{\gamma}_{\mathbf{m}'}\}$ is biorthogonal. (This family is a generalization of biorthogonal polynomals in two variables studied by [P. Appell and J. Kampé de Fériet 1926] and extended by [E.D. Fackerell and R.A. Littler 1974].)

Note that the norm of the weight function is

$$\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \cdots \int_0^{1-x_1-\cdots-x_{n-1}} dx_n\ [\prod_{k=1}^{n} x_k^{\gamma_k-1}](1-x)^{\gamma_{n+1}-1}$$

(5.10)
$$= (1,1)_{\gamma} = \frac{\left[\prod_{k=1}^{n+1}\Gamma(\gamma_k)\right]}{\Gamma(G)}.$$

The relation
$$(P_i^* C^{\hat\gamma}_{\hat{\mathbf{m}}}, D^{\gamma}_{\mathbf{m}'})_{\gamma} = (C^{\hat\gamma}_{\hat{\mathbf{m}}}, P_i D^{\gamma}_{\mathbf{m}'})_{\hat\gamma}$$

yields (for $\mathbf{m} = \mathbf{m}'$) the recurrence relation

$$(C^{\gamma}_{\mathbf{m}}, D^{\gamma}_{\mathbf{m}})_{\gamma} = -\frac{m_i(M+G-1)}{\gamma_i}(C^{\hat\gamma}_{\hat{\mathbf{m}}}, D^{\hat\gamma}_{\hat{\mathbf{m}}})_{\hat\gamma}.$$

The normalization of the biorthogonal basis can be obtained from this result and (5.10).

Now we extend the biorthogonality relations to the full n-sphere. We make the change of variables
$$x_k = y_k^2, \quad k = 1, 2, \cdots, n$$

in (5.10) and extend the domain of integration to negative values of y_k, since the integrand is even in all variables, to get

$$\int_{-1}^{1} dy_1 \int_{-\sqrt{1-y_1^2}}^{\sqrt{1-y_1^2}} dy_2 \cdots \int_{-\sqrt{1-y_1^2-\cdots-y_{n-1}^2}}^{\sqrt{1-y_1^2-\cdots-y_{n-1}^2}} dy_n \, [\prod_{k=1}^{n} (y_k^2)^{\gamma_k - \frac{1}{2}}](1-y_1^2-\cdots-y_n^2)^{s/2-1/2}$$

(5.11)

$$= (1,1)'_\gamma = \frac{[\prod_{k=1}^n \Gamma(\gamma_k)]\Gamma(\frac{s}{2}+\frac{1}{2})}{\Gamma(\gamma_1+\cdots+\gamma_n+\frac{s}{2}+\frac{1}{2})}.$$

Here we have set $\gamma_{n+1} = s/2+1/2$. (This is a generalization of the weight function for the biorthogonal family $\{V_m^{(s)}(\mathbf{x}), U_m^{(s)}(\mathbf{x})\}$ on the n-sphere of [Appell and Kampé de Fériet 1926], which is obtained by setting $\gamma_1 = \cdots = \gamma_n = 1/2$.) Under this change of variables the polynomials $\{C^\gamma_{\mathbf{m}}, D^\gamma_{\mathbf{m}'}\}$ become

$$U_{2m}^{(\gamma,s)}(\mathbf{y}) = (1-y_1^2-\cdots-y_n^2)^M \times$$
$$F_A\left(\begin{matrix}-M-s/2+1/2; & -m_1,\ldots,-m_n \\ & \gamma_1,\ldots,\gamma_n\end{matrix}; \frac{-y_1^2}{1-y_1^2-\cdots-y_n^2},\ldots,\frac{-y_n^2}{1-y_1^2-\cdots-y_n^2}\right),$$

(5.12)

$$V_{2m}^{(\gamma,s)}(\mathbf{y}) = F_A\left(\begin{matrix}M+\gamma_1+\cdots+\gamma_n+s/2-1/2; & -m_1,\ldots,-m_n \\ & \gamma_1,\ldots,\gamma_n\end{matrix}; y_1^2,\ldots,y_n^2\right).$$

In the special case $\gamma_1 = \cdots = \gamma_n = 1/2$ these are exactly the $U_m^{(s)}(\mathbf{y})$ and $V_m^{(s)}(\mathbf{y})$ of [Appell and Kampé de Fériet 1926, page 269]. (To see this transform $m_k \to m'_k/2$, reverse the order of the sums in F_A by transforming the summation indices as $j_k \to m'_k/2 - j_k$, and then use the reflection formula $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ to represent these polynomials in terms of F_B, as given on page 269.) The biorthogonality demonstration given above immediately implies

$$(5.13) \qquad (V_{2m}^{(\gamma,s)}, U_{2m'}^{(\gamma,s)})'_\gamma \sim \prod_{k=1}^{n} \delta_{m_k m'_k}$$

where

(5.14)

$$(V,U)'_\gamma = \int \cdots \int_{y_1^2+\cdots+y_n^2<1} [\prod_{k=1}^{n}(y_k^2)^{\gamma_k-1/2}](1-y_1^2-\cdots-y_n^2)^{s/2-1/2} V(\mathbf{y})\overline{U}(\mathbf{y}) dy_1 \ldots dy_n.$$

Also, since the operator H is self-adjoint with respect to this inner product and since $U_m^{(s)}$ and $V_m^{(s)}$ are eigenfunctions of H we have

$$(5.15) \qquad (V_{2m}^{(\gamma,s)}, V_{2m'}^{(\gamma,s)})'_\gamma = (U_{2m}^{(\gamma,s)}, U_{2m'}^{(\gamma,s)})'_\gamma = 0 \quad \text{if } M \neq M'.$$

Here, $U_{2m}^{(s)}(\mathbf{y})$ and $V_{2m}^{(s)}(\mathbf{y})$ are strictly even degree in all the variables y_k with total degree $2M$. We define odd degree polynomials as follows:

$$V_{2m+1}^{(\gamma,s)}(\mathbf{y}) \equiv [\prod_{k\in Q} y_k] V_{2m}^{(\gamma',s)}(\mathbf{y})$$

$$(5.16) \qquad U_{2m+1}^{(\gamma,s)}(\mathbf{y}) \equiv [\prod_{k\in Q} y_k] U_{2m}^{(\gamma',s)}(\mathbf{y}),$$

where Q is any subset of $(1, 2, \ldots, n)$, and

$$\gamma'_k = \gamma_k + 1 \quad \text{if } k \in Q$$
$$\gamma'_k = \gamma_k \quad \text{if } k \notin Q. \tag{5.17}$$

Since the weight function is even in all variables and the odd degree polynomials are odd in the variables y_k, $k \in Q$, we immediately deduce by parity

$$(V^{(\gamma,s)}_{2m+1}, V^{(\gamma,s)}_{2m'})'_\gamma = (U^{(\gamma,s)}_{2m+1}, U^{(\gamma,s)}_{2m'})'_\gamma = 0,$$
$$(V^{(\gamma,s)}_{2m+1}, U^{(\gamma,s)}_{2m'})'_\gamma = (V^{(\gamma,s)}_{2m}, U^{(\gamma,s)}_{2m'+1})'_\gamma = 0. \tag{5.18}$$

Also, $(V^{(\gamma,s)}_{2m+1}, U^{(\gamma,s)}_{2m'+1})'_\gamma$ vanishes by parity unless both polynomials are odd in exactly the same variables, in which case it is easy to verify that

$$(V^{(\gamma,s)}_{2m+1}, U^{(\gamma,s)}_{2m'+1})'_\gamma = (V^{(\gamma',s)}_{2m+1}, U^{(\gamma',s)}_{2m'+1})'_{\gamma'} \sim \prod_{k=1}^{n} \delta_{m_k m'_k}. \tag{5.19}$$

Similarly,

$$(V^{(\gamma,s)}_{2m+1}, V^{(\gamma,s)}_{2m'+1})'_\gamma = (U^{(\gamma,s)}_{2m}, U^{(\gamma,s)}_{2m'})'_\gamma = 0 \quad \text{if } M \neq M'. \tag{5.20}$$

Theorem 2. *Let*

$$V^{(\gamma,s)}_{m}(\mathbf{y}) \equiv \begin{cases} V^{(\gamma,s)}_{2q}(\mathbf{y}) \\ V^{(\gamma,s)}_{2q+1}(\mathbf{y}) \end{cases}$$
$$U^{(\gamma,s)}_{m}(\mathbf{y}) \equiv \begin{cases} U^{(\gamma,s)}_{2q}(\mathbf{y}) \\ U^{(\gamma,s)}_{2q+1}(\mathbf{y}). \end{cases}$$

Then

$$(V^{(\gamma,s)}_{m}, U^{(\gamma,s)}_{m'})'_\gamma \sim \prod_{k=1}^{n} \delta_{m_k m'_k},$$
$$(V^{(\gamma,s)}_{m}, V^{(\gamma,s)}_{m'})'_\gamma = (U^{(\gamma,s)}_{m}, U^{(\gamma,s)}_{m'})'_\gamma = 0 \quad \textit{if } M \neq M'.$$

In the case $n = 1$ the biorthogonal polynomials are orthogonal:

$$V^{(\gamma,s)}_{2m}(y) = U^{(\gamma,s)}_{2m}(y) = {}_2F_1\left(\begin{matrix} m+\gamma+s/2-1/2, -m \\ \gamma \end{matrix}; y^2\right)$$
$$V^{(\gamma,s)}_{2m+1}(y) = U^{(\gamma,s)}_{2m+1}(y) = y\ {}_2F_1\left(\begin{matrix} m+\gamma+s/2+1/2, -m \\ \gamma+1 \end{matrix}; y^2\right). \tag{5.21}$$

The measure on the interval $-1 \leq y \leq 1$ is

$$d\omega(y) = (y^2)^{\gamma-1/2}(1-y^2)^{s/2-1/2}\ dy.$$

For $\gamma = 1/2$ these are exactly the Gegenbauer polynomials. For general γ they are a generalization of these polynomials [Chihara 1978, page 156].

The same construction with $U_m = V_m$ can be carried out for *all* the orthogonal systems of polynomials in the variables x_k as found in §2 to obtain orthogonal polynomials in the variables y_k on the full n-sphere. In general, something is lost in this construction, however. The polynomials $U_m = V_m$ are (except for the even case) no longer eigenfunctions of H. Indeed, we have

Lemma 3. *Let $\Phi(\mathbf{y})$ be a polynomial eigenfunction of H:*

$$H\Phi = -M(M+G-1)\Phi$$

in the coordinates y_k, where $x_k = y_k^2$, $1 \le k \le n$, and let Q be a subset of $\{1,2,\ldots,n\}$ with $|Q|>0$ elements. Then $\Psi_Q \equiv [\prod_{i\in Q} y_i]\Phi(\mathbf{y})$ is an eigenfunction of the operator H' corresponding to parameters $\gamma_k', \gamma_{n+1}', G'$ if and only if

$$\gamma_k = \frac{3}{2} \quad \text{for } k \in Q,$$

and

$$\begin{aligned} \gamma_k' &= \frac{1}{2} \quad \text{for } k \in Q, \\ \gamma_k' &= \gamma_k \quad \text{for } k \notin Q \\ G' &= G - |Q|, \quad M' = M + \frac{|Q|}{2}. \end{aligned}$$

Then

$$H'\Psi_Q = -M'(M'+G'-1)\Psi_Q.$$

It follows from this result that in the case where $\gamma_1 = \cdots = \gamma_n = 1/2$, the construction leading to Theorem 2 yields the biorthogonal polynomials $U_m^{(s)}(\mathbf{y})$ and $V_m^{(s)}(\mathbf{y})$ of [Appell and Kampé de Fériet 1926]. These polynomials are all eigenfunctions of H. Similarly, for $\gamma_1 = \cdots = \gamma_n = 1/2$ the same construction applied to the families of orthogonal polynomials in x_k, found in §2, leads to the families of orthogonal polynomials in y_k, found in §3, all eigenfunctions of H.

As a referee has kindly pointed out, Lemma 3 can be generalized if one uses Dunkl's differential-difference operator [Dunkl 1988]. In the coordinates y_i and for general $\gamma_1, \cdots, \gamma_{n+1}$, Dunkl's operator $\tilde{H}$ is defined as

$$\begin{aligned} \tilde{H}p(\mathbf{y}) =& \frac{1}{4}\Big[\sum_{i,j=1}^{n}(\delta_{ij} - y_i y_j)\partial_{y_i y_j} p + (1-2G)\sum_{j=1}^{n} y_j \partial_{y_j} p \\ &+ \sum_{j=1}^{n}(\gamma_j - \frac{1}{2})\Big(\frac{2}{y_j}\partial_{y_j} p - \frac{p(\mathbf{y}) - p(\cdots, -y_j, \cdots)}{y_j^2}\Big)\Big]. \end{aligned}$$

(This differs from the operator (1.5) with $x_j = y_j^2$ only in the last term.) The eigenvalue equation is

$$\tilde{H}p(\mathbf{y}) = -M(M+G-1)p(\mathbf{y}).$$

Note that $\tilde{H}$ always maps polynomials in the y_i to polynomials and that $\tilde{H}p \equiv Hp$ for polynomials p which are even in each of the variables y_j and $\tilde{H} \equiv H$ if $\gamma_j = \frac{1}{2}$ for all j. Furthermore, since the operators I_j which map $p(\mathbf{y})$ to $p(y_1, \cdots, -y_j, \cdots, y_n)$ for $j = 1, \cdots, n$, commute with $\tilde{H}$, we can assume, without loss of generality, that each eigenfunction is either even or odd in every one of its variables y_j. We have the following generalization of Lemma 3.

Lemma 3′. *Let $\Phi(\mathbf{y})$ be a polynomial eigenfunction of $\tilde{H}$:*

$$\tilde{H}\Phi = -M(M+G-1)\Phi$$

in the coordinates y_k, where $x_k = y_k^2$, $1 \le k \le n$, and let Q be a subset of $\{1,2,\ldots,n\}$ with $|Q|>0$ elements. Then $\Psi_Q \equiv [\prod_{i\in Q} y_i]\Phi(\mathbf{y})$ is an eigenfunction of the operator $\tilde{H}'$ corresponding to parameters $\gamma'_k, \gamma'_{n+1}, G'$ if and only if

$$\gamma'_k = \gamma_k - 1 \quad \text{for } k \in Q,$$

and

$$\gamma'_k = \gamma_k \quad \text{for } k \notin Q$$
$$G' = G - |Q|, \quad M' = M + \frac{|Q|}{2}.$$

Then

$$\tilde{H}'\Psi_Q = -M'(M'+G'-1)\Psi_Q.$$

Similar comments apply to the "mixed" case in §6.

6. The "mixed" biorthogonal case. Using the techniques introduced in §5 it is now easy to determine a biorthogonal basis of polynomials in the mixed case with coordinates (4.1). We set

$$n_1+n_2=n, \quad x \equiv \sum_{k=1}^{n_1} x_k, \quad y^2 \equiv \sum_{k=1}^{n_2} y_k^2,$$
$$M \equiv \sum_{k=1}^{n_1} m_k, \quad \tilde{M} \equiv \sum_{k=1}^{n_2} \tilde{m}_k.$$

The basic building blocks are the polynomials

$$C^{(\gamma,s)}_{m,2\tilde{m}}(\mathbf{x},\mathbf{y}) = \tag{6.1}$$

$$(1-x-y^2)^{M+\tilde{M}} F_A\left(\begin{matrix}-M-\tilde{M}-s/2+1/2; -m_k, -\tilde{m}_k \\ \gamma_k, \quad s_k\end{matrix}; \frac{-x_k}{1-x-y^2}, \frac{-y_k^2}{1-x-y^2}\right)$$

and

$$D^{(\gamma,s)}_{m,2\tilde{m}}(\mathbf{x},\mathbf{y}) = \tag{6.2}$$

$$F_A\left(\begin{matrix}M+\tilde{M}+\gamma_1+\cdots+\gamma_{n_1}+s_1+\cdots+s_{n_2}+s/2-1/2; -m_k, -\tilde{m}_k \\ \gamma_k, \quad s_k\end{matrix}; x_k, y_k^2\right).$$

The weight function is

$$w(\mathbf{x},\mathbf{y}) = \left[\prod_{k=1}^{n_1} x_k^{\gamma_k-1}\right]\left[\prod_{k=1}^{n_2} (y_k^2)^{s_k-1/2}\right](1-x-y^2)^{s/2-1/2} \tag{6.3}$$

with $\gamma_k, s_k > 0$ and $s > -1$. The inner product is
(6.4)

$$< \Phi_1, \Phi_2 >_{\gamma,s} = \int \cdots \int_{0<x_i, x+y^2<1} \Phi_1(\mathbf{x},\mathbf{y})\overline{\Phi_2}(\mathbf{x},\mathbf{y})\, w(\mathbf{x},\mathbf{y}) dx_1 \dots dx_{n_1} dy_1 \dots dy_{n_2}.$$

Furthermore,

$$< 1, 1 >_{\gamma,s} = \frac{[\prod_{k=1}^{n_1} \Gamma(\gamma_k)][\prod_{k=1}^{n_2} \Gamma(s_k)]\Gamma(s/2+1/2)}{\Gamma(\gamma_1 + \cdots + \gamma_{n_1} + s_1 + \cdots + s_{n_2} + s/2 + 1/2)}. \tag{6.5}$$

It follows from the results immediately preceding (5.10) that the polynomial sets (6.1) and (6.2) are biorthogonal. However, since they are even functions of the y_k they don't form a basis for all polynomial functions in the variables x_k, y_k. To construct such a basis we define functions

$$\begin{aligned} C^{(\gamma,s)}_{m,2\tilde{m}+1}(\mathbf{x},\mathbf{y}) &= [\prod_{k\in Q} y_k] C^{(\gamma,s_k+1,s)}_{m,2\tilde{m}}(\mathbf{x},\mathbf{y}), \\ D^{(\gamma,s)}_{m,2\tilde{m}+1}(\mathbf{x},\mathbf{y}) &= [\prod_{k\in Q} y_k] D^{(\gamma,s_k+1,s)}_{m,2\tilde{m}}(\mathbf{x},\mathbf{y}), \end{aligned} \tag{6.6}$$

where Q is any nonempty subset of $(1, 2, \dots, n_2)$.

By parity we have

$$\begin{aligned} &< C^{(\gamma,s)}_{m,2\tilde{m}+1}, D^{(\gamma,s)}_{m',2\tilde{m}'} >_{\gamma,s} = 0, \quad < C^{(\gamma,s)}_{m,2\tilde{m}}, D^{(\gamma,s)}_{m',2\tilde{m}'+1} >_{\gamma,s} = 0, \\ &< C^{(\gamma,s)}_{m,2\tilde{m}+1}, D^{(\gamma,s)}_{m',2\tilde{m}'+1} >_{\gamma,s} = 0 \quad \text{if } Q \neq Q'. \end{aligned}$$

If $Q = Q'$ a simple computation yields

$$< C^{(\gamma,s)}_{m,2\tilde{m}+1}, D^{(\gamma,s)}_{m',2\tilde{m}'+1} >_{\gamma,s} = < C^{(\gamma,s_k+1,s)}_{m,2\tilde{m}}, D^{(\gamma,s_k+1,s)}_{m',2\tilde{m}'} >_{\gamma,s_k+1,s} \sim \prod_{k=1}^{n_1} \delta_{m_k m'_k} \prod_{k=1}^{n_2} \delta_{\tilde{m}_k,\tilde{m}'_k}.$$

Since $C^{(\gamma,s)}_{m,2\tilde{m}}$ and $D^{(\gamma,s)}_{m,2\tilde{m}}$ are eigenfunctions of H there are additional orthogonality relations obeyed by the C's alone and by the D's alone. Collecting all these results we have

Theorem 3. *Let*

$$C^{(\gamma,s)}_{m,\tilde{m}}(\mathbf{x},\mathbf{y}) \equiv \begin{cases} C^{(\gamma,s)}_{m,2\tilde{q}}(\mathbf{x},\mathbf{y}) \\ C^{(\gamma,s)}_{m,2\tilde{q}+1}(\mathbf{x},\mathbf{y}), \end{cases}$$

$$D^{(\gamma,s)}_{m,\tilde{m}}(\mathbf{x},\mathbf{y}) \equiv \begin{cases} D^{(\gamma,s)}_{m,2\tilde{q}}(\mathbf{x},\mathbf{y}) \\ D^{(\gamma,s)}_{m,2\tilde{q}+1}(\mathbf{x},\mathbf{y}). \end{cases}$$

Then

$$\begin{aligned} &< C^{(\gamma,s)}_{m,\tilde{m}}, D^{(\gamma,s)}_{m',\tilde{m}'} >_{\gamma,s} \sim \prod_{k=1}^{n_1} \delta_{m_k m'_k} \prod_{k=1}^{n_2} \delta_{\tilde{m}_k \tilde{m}'_k}, \\ &< C^{(\gamma,s)}_{m,\tilde{m}}, C^{(\gamma,s)}_{m',\tilde{m}'} >_{\gamma,s} = 0 \quad \textit{if } M + \tilde{M} \neq M' + \tilde{M}', \\ &< D^{(\gamma,s)}_{m,\tilde{m}}, D^{(\gamma,s)}_{m',\tilde{m}'} >_{\gamma,s} = 0 \quad \textit{if } M + \tilde{M} \neq M' + \tilde{M}'. \end{aligned}$$

In general, the biorthogonal polynomials listed in Theorem 3 are not eigenfunctions of H. However, in the case $s_1 = \cdots = s_{n_2} = 1/2$ it follows from Lemma 3 that each of the polynomials satisfies the eigenvalue equation

$$H\Phi = -(M + \tilde{M})(M + \tilde{M} + G - 1)\Phi$$

where $G = \sum_{k=1}^{n_1} \gamma_k + (n_2 + 1)/2 + s$.

Similarly, the above procedure when applied to any one of the orthogonal bases discussed in §2 leads to an orthogonal polynomial basis with respect to the inner product $< \cdot, \cdot >_{\gamma,s}$. Restriction to the case $s_1 = \cdots = s_{n_2} = 1/2$ yields eigenfunctions of H and coincides with the results of §4.

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