K_1-CONGRUENCES FOR THREE-DIMENSIONAL LIE GROUPS

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Abstract: *We completely describe $K_1(\mathbb{Z}_p[\![\mathcal{G}_\infty]\!])$ and its localisations by using an infinite family of p-adic congruences, where $\mathcal{G}_\infty$ is any solvable p-adic Lie group of dimension 3. This builds on earlier work of Kato when $\dim(\mathcal{G}_\infty) = 2$, and of the first named author and Lloyd Peters when $\mathcal{G}_\infty \cong \mathbb{Z}_p^\times \ltimes \mathbb{Z}_p^d$ with a scalar action of $\mathbb{Z}_p^\times$. The method exploits the classification of 3-dimensional p-adic Lie groups due to González-Sánchez and Klopsch, as well as the fundamental ideas of Kakde, Burns, etc. in non-commutative Iwasawa theory.*

Contents

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1 Introduction

Over the last twenty years, the study of non-commutative Iwasawa theory for motives has progressed rapidly, due to the work of many mathematicians [2, 3, 6, 17, 18, 19, 20, 23]. Fix an odd prime p, and an infinite algebraic extension F_∞/F of some number field F. We assume that $G_\infty = \mathrm{Gal}(F_\infty/F)$ is a p-adic Lie group with no element of order p; we further suppose that F_∞ contains the cyclotomic $\mathbb{Z}_p$-extension F^{cyc} of the base field F. Clearly if $H_\infty = \mathrm{Gal}\big(F_\infty/F^{\mathrm{cyc}}\big)$, then the quotient $\Gamma = G_\infty/H_\infty$ will be isomorphic to an open subgroup of $1 + p\mathbb{Z}_p$, under the p-th cyclotomic character 'κ_F'.

For a motive M with good ordinary reduction at p, the work of Coates et al [6] associates (under the $\mathfrak{M}_H(G)$-conjecture) a characteristic element $\xi_M \in K_1\big(\mathbb{Z}_p[\![G_\infty]\!]_{\mathcal{S}^*}\big)$, where $K_1(-)$ denotes the first algebraic K-group, and $\mathcal{S}^*$ is the p-saturation of the Ore set

$$\mathcal{S} \ := \ \left\{ f \in \mathbb{Z}_p[\![G_\infty]\!] \ \middle| \ \mathbb{Z}_p[\![G_\infty]\!]/\mathbb{Z}_p[\![G_\infty]\!]f \ \text{ is a finitely-generated } \mathbb{Z}_p[\![H_\infty]\!]\text{-module} \right\}.$$

The "Non-commutative Iwasawa Main Conjecture" predicts that there exists an element $\mathcal{L}_M^{\mathrm{an}} \in K_1\big(\mathbb{Z}_p[\![G_\infty]\!]_{\mathcal{S}^*}\big)$ of the exact form $\mathcal{L}_M^{\mathrm{an}} = \mathfrak{u}\cdot\xi_M$ with $\mathfrak{u}$ in the image of $K_1\big(\mathbb{Z}_p[\![G_\infty]\!]\big)$; for any Artin representation $\rho : G_\infty \to \mathrm{GL}(V)$, its evaluation at $\rho\otimes\kappa_F^k$ should then satisfy

$$\mathcal{L}_M^{\mathrm{an}}\big(\rho\kappa_F^k\big) \ = \ \text{the value of the } p\text{-adic } L\text{-function } \mathbf{L}_p(M,\rho,s) \text{ at } s=k,$$

as the variable k ranges over the p-adic integers. Note that the existence of $\mathbf{L}_p(M,\rho,s)$ is in most cases still conjectural, although its interpolation properties are easy to describe.

Remark: The strategy of Burns and Kato [2, 20] reduces this conjecture to the following: (1) prove the abelian Iwasawa Main Conjectures for M over all finite layers; (2) describe $K_1\big(\mathbb{Z}_p[\![G_\infty]\!]_{\mathcal{S}^*}\big)$ via a system of non-commutative congruences; and (3) show that each of the abelian fragments, $\mathbf{L}_p(M,\rho,-)$, in combination satisfy this system of congruences.

There seem to be two approaches to (2), either using congruences modulo trace ideals [1, 17, 20, 21, 23], or instead by deriving p-adic congruences [10, 11, 12, 16, 18, 19]. Naturally both approaches should be equivalent to one another.

To illustrate precisely what is meant by the terminology 'p-adic congruences' above, for the moment suppose that G_∞ is a two-dimensional p-adic Lie group of the form

$$G_\infty \ \cong \ \mathbb{Z}_p^\times \ltimes \mathbb{Z}_p \ \cong \ \big(\mathbb{F}_p^\times \times \Gamma\big) \ltimes \mathbb{Z}_p$$

where $\Gamma = 1+p\mathbb{Z}_p$, and the first factor $\mathbb{Z}_p^\times$ acts on the second $\mathbb{Z}_p$ via scalar multiplication. Let $\varphi : \mathbb{Z}_p[\![\Gamma]\!] \to \mathbb{Z}_p[\![\Gamma]\!]$, $\varphi : \gamma \mapsto \gamma^p$ denote the linear extension of the p-power map on Γ. At integers $m \geq m' \geq 0$, we also write $\mathcal{N}_{m',m} : \mathbb{Z}_p[\![\Gamma^{p^{m'}}]\!] \to \mathbb{Z}_p[\![\Gamma^{p^m}]\!]$ for the norm map.

Kato's Theorem. *([19, 8.12]) A sequence $\big(\mathbf{y}_m\big) \in \prod_{m\geq 0}\mathbb{Z}_p\big[\big[\Gamma^{p^m}\big]\big]^\times_{(p)}$ arises from an element in $K_1\big(\mathbb{Z}_p[\![G_\infty]\!]_{\mathcal{S}}\big)$ only if the system of p-adic congruences*

$$\prod_{m'=1}^{m} \mathcal{N}_{m',m}\left(\frac{\mathbf{y}_{m'}}{\varphi\big(\mathbf{y}_{m'-1}\big)}\cdot\frac{\varphi\big(\mathcal{N}_{0,m'-1}(\mathbf{y}_0)\big)}{\mathcal{N}_{0,m'}(\mathbf{y}_0)}\right)^{p^{m'}} \ \equiv \ 1 \mod p^{2m}\cdot\mathbb{Z}_p\big[\big[\Gamma^{p^m}\big]\big]_{(p)}$$

hold at every integer $m \geq 1$.

Kato has obtained similar congruences when G_∞ is replaced by any of the groups $\Gamma^{p^s} \ltimes \mathbb{Z}_p$. His work completely describes the two-dimensional situation, since any non-commutative torsion-free pro-p-group G with $\dim(G) = 2$ is isomorphic to $\Gamma^{p^s} \ltimes \mathbb{Z}_p$ for some $s \geq 0$.

Question. *Can the analogue of Kato's p-adic congruences be proven when* $\dim(G) > 2$*?*

Our goal here is to give a positive answer when $\dim(G) = 3$ and $G \neq \mathrm{SL}_2(\mathbb{Z}_p), \mathrm{SL}_1(\mathbb{D}_p)$. We exclude the two insolvable cases as the representation theory is unpleasant, although recent work of Kakde [18] provides hope that an answer for $\mathrm{GL}_2(\mathbb{Z}_p)$ is not too far away.

1.1 Preliminaries

Fix a number field F and a prime number $p \neq 2$. We shall assume that F_∞ denotes a p-adic Lie extension of F satisfying:

(i) $\mathrm{Gal}(F_\infty/F)$ is a pro-p-group without any p-torsion;

(ii) F_∞ contains the cyclotomic $\mathbb{Z}_p$-extension F^{cyc} of F.

The examples we have in mind here are solvable three-dimensional Galois groups arising from algebraic geometry, or alternatively the direct product of a two-dimensional Galois group with a group of diamond operators (in the context of Hida's deformation theory). We therefore suppose that either

(iiia) $\mathcal{G}_\infty = \mathrm{Gal}(F_\infty/F)$ where $\dim\big(\mathrm{Gal}(F_\infty/F)\big) = 3$ and $\mathcal{G}_\infty \ncong \mathrm{SL}_2(\mathbb{Z}_p), \mathrm{SL}_1(\mathbb{D}_p)$;

or (iiib) $\mathcal{G}_\infty = \mathrm{Gal}(F_\infty/F) \times \mathcal{W}_\infty$ where $\dim\big(\mathrm{Gal}(F_\infty/F)\big) = 2$ and $\mathcal{W}_\infty \cong \mathbb{Z}_p$.

In both (iiia) and (iiib), the p-adic Lie group $\mathcal{G}_\infty$ is three-dimensional and also solvable; in fact $\mathcal{G}_\infty$ is a semi-direct product of $\mathbb{Z}_p$ with an abelian subgroup $\mathcal{H}_\infty$ of $\mathbb{Z}_p$-rank two. The following result classifies such groups.

Classification Theorem. *(González-Sánchez and Klopsch [15]) If the pro-p-group* $\mathcal{G}_\infty$ *is solvable and torsion-free with* $\dim(\mathcal{G}_\infty) = 3$*, then* $\mathcal{G}_\infty$ *must be isomorphic to one of the following possibilities:*

(I) the abelian group $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$*;*

(II) an open subgroup of the p-adic Heisenberg group, i.e. a group given by the presentation $\big\langle \gamma, h_1, h_2 : [h_1, h_2] = 1, [h_1, \gamma] = 1, [h_2, \gamma] = h_1^{p^s} \big\rangle$ *for some* $s \in \mathbb{N}_0$*;*

(III) the group $\big\langle \gamma, h_1, h_2 : [h_1, h_2] = 1, [h_1, \gamma] = h_1^{p^s}, [h_2, \gamma] = h_2^{p^s} \big\rangle$ *for some* $s \in \mathbb{N}$*;*

(IV) $\big\langle \gamma, h_1, h_2 : [h_1, h_2] = 1, [h_1, \gamma] = h_1^{p^s} h_2^{p^{s+r} d}, [h_2, \gamma] = h_1^{p^{s+r}} h_2^{p^s} \big\rangle$ *for some* $s, r \in \mathbb{N}$ *with* $d \in \mathbb{Z}_p$*;*

(V) $\big\langle \gamma, h_1, h_2 : [h_1, h_2] = 1, [h_1, \gamma] = h_2^{p^s d}, [h_2, \gamma] = h_1^{p^s} h_2^{p^{s+r}} \big\rangle$ *where* $s, r \in \mathbb{N}_0$ *and* $d \in \mathbb{Z}_p$*, such that either* $s \geq 1$*, or instead* $r \geq 1$ *and* $d \in p\mathbb{Z}_p$*;*

(VI) either one of the groups: (a) $\big\langle \gamma, h_1, h_2 : [h_1, h_2] = 1, [h_1, \gamma] = h_2^{p^{s+r}}, [h_2, \gamma] = h_1^{p^s} \big\rangle$

or (b) $\big\langle \gamma, h_1, h_2 : [h_1, h_2] = 1, [h_1, \gamma] = h_2^{p^{s+r} t}, [h_2, \gamma] = h_1^{p^s} \big\rangle$

where $s, r \in \mathbb{N}_0$ such that $s + r \geq 1$, and $t \in \mathbb{Z}_p^\times$ is not a square modulo p.

Let $\Gamma = \{\gamma^z \mid z \in \mathbb{Z}_p\}$ where γ is as in the previous theorem (if $\mathcal{G}_\infty = \mathrm{Gal}(F_\infty/F)$ satisfies condition (iiia) above, we shall identify its quotient $\mathrm{Gal}(F^{\mathrm{cyc}}/F) \cong \mathbb{Z}_p$ with Γ). One defines a decreasing sequence of normal subgroups for $\mathcal{G}_\infty$ by

$$\mathcal{U}_m \;:=\; \Gamma^{p^m} \ltimes \mathcal{H}_\infty \quad \text{at each } m \geq 0.$$

Recall from [24, Prop 25], every irreducible $\mathcal{G}_\infty$-representation with finite image is of the form $\psi \otimes \mathrm{Ind}_{\mathcal{U}_m}^{\mathcal{G}_\infty}(\chi)$ for some $m \geq 0$, with characters $\chi : \mathcal{U}_m^{\mathrm{ab}} \to \mu_{p^\infty}$ and $\psi : \Gamma^{p^m} \to \overline{\mathbb{Q}}_p^\times$.

If G is a pro-p-group, then we write $\Lambda(G) = \varprojlim_P \mathbb{Z}_p[G/P]$ for its Iwasawa algebra where the inverse limit runs over open subgroups $P \lhd G$. If $\mathcal{O}$ contains $\mathbb{Z}_p$ as a subring then $\Lambda_{\mathcal{O}}(G) := \Lambda(G) \otimes_{\mathbb{Z}_p} \mathcal{O}$. Lastly for a canonical Ore set $\mathcal{S}$, we use $\Lambda(G)_{\mathcal{S}}$ and $\Lambda(G)_{\mathcal{S}^*}$ for the localisation of $\Lambda(G)$ at $\mathcal{S}$, and at its p-saturation $\mathcal{S}^* = \bigcup_{n\geq 0} p^n\mathcal{S}$, respectively.

Remark: Let us write $\mathcal{N}_{\mathcal{U}_m} : \Lambda(\mathcal{G}_\infty) \to \Lambda(\mathcal{U}_m)$ for the norm mapping on Iwasawa algebras. If $[\mathcal{U}_m, \mathcal{U}_m]$ denotes the commutator subgroup of $\mathcal{U}_m$, there is a commutative diagram

$$\begin{array}{ccccc}
K_1\big(\Lambda(\mathcal{G}_\infty)\big) & \overset{\prod \mathcal{N}_{\mathcal{U}_m}(-) \bmod [\mathcal{U}_m,\mathcal{U}_m]}{\longrightarrow} & \prod\limits_{m\geq 0} K_1\big(\Lambda(\mathcal{U}_m^{\mathrm{ab}})\big) & \overset{\prod \chi_*}{\longrightarrow} & \prod\limits_{m\geq 0}\prod\limits_{\rho_\chi} \Lambda_{\mathcal{O}_\chi}\big(\Gamma^{p^m}\big)^\times \\
\downarrow & & \downarrow & & \downarrow \\
K_1\big(\Lambda(\mathcal{G}_\infty)_{\mathcal{S}}\big) & \overset{\prod \mathcal{N}_{\mathcal{U}_m}(-) \bmod [\mathcal{U}_m,\mathcal{U}_m]}{\longrightarrow} & \prod\limits_{m\geq 0} K_1\big(\Lambda(\mathcal{U}_m^{\mathrm{ab}})_{\overline{\mathcal{S}}}\big) & \overset{\prod \chi_*}{\longrightarrow} & \prod\limits_{m\geq 0}\prod\limits_{\rho_\chi} \Lambda_{\mathcal{O}_\chi}\big(\Gamma^{p^m}\big)^\times_{(p)} \\
\downarrow & & \downarrow & & \downarrow \\
K_1\big(\Lambda(\mathcal{G}_\infty)_{\mathcal{S}^*}\big) & \overset{\prod \mathcal{N}_{\mathcal{U}_m}(-) \bmod [\mathcal{U}_m,\mathcal{U}_m]}{\longrightarrow} & \prod\limits_{m\geq 0} K_1\big(\Lambda(\mathcal{U}_m^{\mathrm{ab}})_{\overline{\mathcal{S}}^*}\big) & \overset{\prod \chi_*}{\longrightarrow} & \prod\limits_{m\geq 0}\prod\limits_{\rho_\chi} \mathrm{Quot}\big(\Lambda_{\mathcal{O}_\chi}(\Gamma^{p^m})\big)^\times
\end{array}$$

where the vertical arrows are induced from the inclusions $\Lambda(\mathcal{G}_\infty) \hookrightarrow \Lambda(\mathcal{G}_\infty)_{\mathcal{S}} \hookrightarrow \Lambda(\mathcal{G}_\infty)_{\mathcal{S}^*}$, and the right-most products range over *irreducible* non-isomorphic $\mathcal{G}_\infty$-representations. One can then define three separate theta-maps $\Theta_{\infty,\underline{\chi}}$, $\Theta_{\infty,\underline{\chi},\mathcal{S}}$ and $\Theta_{\infty,\underline{\chi},\mathcal{S}^*}$ by composing (respectively) the first, second and third rows in the above diagram, so that

$$\Theta_{\infty,\underline{\chi}} : K_1\big(\Lambda(\mathcal{G}_\infty)\big) \longrightarrow \prod_{\rho_\chi} \Lambda_{\mathcal{O}_\chi}\big(\Gamma^{p^{\mathbf{m}_\chi}}\big)^\times,$$

$$\Theta_{\infty,\underline{\chi},\mathcal{S}} : K_1\big(\Lambda(\mathcal{G}_\infty)_{\mathcal{S}}\big) \longrightarrow \prod_{\rho_\chi} \Lambda_{\mathcal{O}_\chi}\big(\Gamma^{p^{\mathbf{m}_\chi}}\big)^\times_{(p)}$$

$$\text{and} \qquad \Theta_{\infty,\underline{\chi},\mathcal{S}^*} : K_1\big(\Lambda(\mathcal{G}_\infty)_{\mathcal{S}^*}\big) \longrightarrow \prod_{\rho_\chi} \mathrm{Quot}\big(\Lambda_{\mathcal{O}_\chi}(\Gamma^{p^{\mathbf{m}_\chi}})\big)^\times.$$

The Main Goal. *To describe the images of $\Theta_{\infty,\underline{\chi}}$, $\Theta_{\infty,\underline{\chi},\mathcal{S}}$ and $\Theta_{\infty,\underline{\chi},\mathcal{S}^*}$ by using a family of p-adic congruences linking together the abelian fragments $\mathbf{y}_{\rho_\chi} \in \mathrm{Quot}\big(\Lambda_{\mathcal{O}_\chi}(\Gamma^{p^{\mathbf{m}_\chi}})\big)^\times$.*

Note that Case (I) is devoid of any content since $\mathcal{G}_\infty \cong \Gamma \times \mathcal{H}_\infty$ is abelian, in which case

$$K_1\big(\Lambda(\mathcal{G}_\infty)\big) \;=\; K_1\big(\Lambda(\Gamma \times \mathcal{H}_\infty)\big) \;\cong\; \Lambda(\Gamma \times \mathcal{H}_\infty)^\times$$

by Morita invariance. **Hence one may ignore Case (I) completely, since there are no non-abelian congruences to consider here.**

1.2 The main results

In order to describe the congruences in each of the non-empty Cases (II-VI), we first need some means to keep track of those Artin representations induced from characters on $\mathcal{H}_\infty$. If χ is a finite order character on $\mathcal{H}_\infty$ then χ extends naturally to $\mathrm{Stab}_\Gamma(\chi)\ltimes\mathcal{H}_\infty$, hence

$$\rho_\chi \ := \ \mathrm{Ind}_{\mathrm{Stab}_\Gamma(\chi)\ltimes\mathcal{H}_\infty}^{\mathcal{G}_\infty}(\chi)$$

is an irreducible $\mathcal{G}_\infty$-representation of dimension $p^{\mathbf{m}_\chi}$, where $\mathbf{m}_\chi=\mathrm{ord}_p\big([\Gamma:\mathrm{Stab}_\Gamma(\chi)]\big)$. In all cases $\star\in\{\text{II,III,IV,V,VI}\}$, one constructs characters $\chi_{1,n},\ \chi_{2,n}:\mathcal{H}_\infty\to\mu_{p^\infty}$ via

$$\chi_{1,n}\big(h_1^xh_2^y\big) \ = \ \exp\big(2\pi\sqrt{-1}\,x/p^n\big) \quad\text{and}\quad \chi_{2,n}\big(h_1^xh_2^y\big) \ = \ \exp\big(2\pi\sqrt{-1}\,y/p^n\big)$$

for each $x,y\in\mathbb{Z}_p$. In particular, $\chi_{1,n}$ and $\chi_{2,n}$ together generate a basis for $\mathrm{Hom}(\mathcal{H}_\infty,\mu_{p^n})$.

Case (II). For simplicity, let us initially assume we are in Case (II). Then for each character $\chi=\chi_{2,n}^a\cdot\chi_{1,s+m'}^b$ and group element $h=h_1^xh_2^y\in\mathcal{H}_\infty$, one defines $\mathbf{e}^*_{\chi,h}\in\mathbb{Z}[\mu_{p^n}]$ by the formula

$$\mathbf{e}^*_{\chi,h} \ := \ \begin{cases}\chi^{-1}(\overline{h})\cdot p^{\max\{0,m'-\mathrm{ord}_p(b)\}} & \text{if } p^{m'}\mid by\\ 0 & \text{if } p^{m'}\nmid by.\end{cases}$$

Theorem 1. *If we are in Case (II), then a sequence $\big(\mathbf{y}_{\rho_\chi}\big)\in\prod_{\rho_\chi}\Lambda_{\mathcal{O}_\chi}\big(\Gamma^{p^{\mathbf{m}_\chi}}\big)^\times_{(p)}$ belongs to the image of $\Theta_{\infty,\underline{\chi},\mathcal{S}}$ only if*

$$\prod_{m'=0}^{m}\ \prod_{a=1}^{p^{n-m'}}\ \prod_{\substack{b=1,\\ p\nmid b \text{ if } m'>0}}^{p^{s+m'}} \mathcal{N}_{\mathbf{m}_\chi,m}\left(\frac{\mathbf{y}_{\rho_\chi}}{\varphi\big(\mathbf{y}_{\rho_{\chi^p}}\big)}\cdot\frac{\varphi\big(\mathcal{N}_{0,\mathbf{m}_\chi-1}(\mathbf{y_1})\big)}{\mathcal{N}_{0,\mathbf{m}_\chi}(\mathbf{y_1})}\right)^{\mathbf{e}^*_{\chi,h}}\Bigg|_{\chi=\chi_{2,n}^a\cdot\chi_{1,s+m'}^b}$$

$$\equiv\ 1 \mod p^{s+m+n+\mathrm{ord}_p(y)}\cdot\mathbb{Z}_p[\![\Gamma^{p^m}]\!]_{(p)} \qquad (1)$$

for all integer pairs $m,n\geq 0$ with $m\leq n-s$, and at every choice of $h=h_1^xh_2^y\in\mathcal{H}_\infty$ with $x\in\{1,\dots,p^n\}$ and $y\in\{1,\dots,p^m\}$.

We should point out that, a priori, it is not clear whether the p-adic power $\mathcal{N}_{\mathbf{m}_\chi,m}(\dots)^{\mathbf{e}^*_{\chi,h}}$ above should even exist, as the exponent $\mathbf{e}^*_{\chi,h}\in\mathbb{Z}[\mu_{p^n}]$ is frequently not a rational integer!

Remarks: (i) For any function $f(X)\in 1+p\cdot\mathcal{O}_{\mathbb{C}_p}[\![X]\!]$, and provided that $s\in\mathbb{C}_p$ is chosen to lie inside the disk $|s|_p<p^{(p-2)/(p-1)}$, the p-adic power series defined as

$$f(X)^s \ := \ \exp_p\big(s\log_p\big(f(X)\big)\big)$$

converges to an element of $1+p\cdot\mathcal{O}_{\mathbb{C}_p}[\![X]\!]$. In particular, if $s\in\mathbb{Z}$ then $f(X)^s$ coincides with the standard definition of the s-th power.

(ii) Furthermore, this construction extends after localisation at the multiplicatively closed set $\mathcal{O}_{\mathbb{C}_p}[\![X]\!]-p\cdot\mathcal{O}_{\mathbb{C}_p}[\![X]\!]$, i.e. if $f(X)\in 1+p\cdot\mathcal{O}_{\mathbb{C}_p}[\![X]\!]_{(p)}$ then $f(X)^s\in 1+p\cdot\mathcal{O}_{\mathbb{C}_p}[\![X]\!]_{(p)}$.

(iii) Although not explicitly stated, it is nevertheless inbuilt into Theorem 1 that each of the fractions $\frac{\mathbf{y}_{\rho_\chi}}{\varphi(\mathbf{y}_{\rho_{\chi^p}})}\cdot\frac{\varphi(\mathcal{N}_{0,\mathbf{m}_\chi-1}(\mathbf{y_1}))}{\mathcal{N}_{0,\mathbf{m}_\chi}(\mathbf{y_1})}$ belongs to the multiplicative group $1+p\cdot\mathcal{O}_\chi[\![\Gamma^{p^m}]\!]_{(p)}$. In lieu of this discussion, one deduces that each term $\mathcal{N}_{\mathbf{m}_\chi,m}(\dots)^{\mathbf{e}^*_{\chi,h}}$ in the above theorem exists as a well-defined element of the multiplicative group $1+p\cdot\mathcal{O}_{\mathbb{C}_p}[\![\Gamma^{p^m}]\!]_{(p)}$.

Cases (III)-(VI). Let us now instead suppose we are in Case ($\star$) with $\star \in \{$III,IV,V,VI$\}$. We define a non-negative integer $\epsilon_{\star,p}$ by the rule

$$\epsilon_{\star,p} \;=\; \begin{cases} 0 & \text{if } \star = \text{(III) or (IV)} \\ \mathrm{ord}_p(d) & \text{if } \star = \text{(V)} \\ r + \mathrm{ord}_p(t) & \text{if } \star = \text{(VI)}. \end{cases}$$

It will be shown (in Proposition 7) that the abelianization of $\mathcal{U}_m$ yields the tricyclic group

$$\mathcal{U}_m^{\mathrm{ab}} \;:=\; \frac{\mathcal{U}_m}{[\mathcal{U}_m,\mathcal{U}_m]} \;\cong\; \Gamma^{p^m} \times C_{p^{s+m+\epsilon_{\star,p}}} \times C_{p^{s+m}}$$

where C_d denotes the cyclic group of order d.

Note that the commutator $[\mathcal{U}_m\ ,\ \mathcal{U}_m]$ is actually a subgroup of $\mathcal{H}_\infty$, while Γ acts on $\mathcal{U}_m^{\mathrm{ab}}$ through the finite quotient Γ/Γ^{p^m}; we can then partition

$$\overline{\mathcal{H}}_\infty^{(m)} \;:=\; \frac{\mathcal{H}_\infty}{[\mathcal{U}_m\ ,\ \mathcal{U}_m]} \;\cong\; C_{p^{s+m+\epsilon_{\star,p}}} \times C_{p^{s+m}}$$

into a finite disjoint union of its Γ-orbits. Similarly, the dual group $\mathrm{Hom}\big(\overline{\mathcal{H}}_\infty^{(m)}, \mathbb{C}^\times\big)$ also has an action of Γ/Γ^{p^m}; let '$\mathfrak{R}_m$' denote a set of representatives for its Γ-orbits.

For each orbit $\varpi_{\overline{h}} = \big\{\gamma^{-j}\overline{h}\gamma^j \mid j \in \mathbb{Z}/p^m\mathbb{Z}\big\}$, $\overline{h} \in \overline{\mathcal{H}}_\infty^{(m)}$ and character $\chi : \overline{\mathcal{H}}_\infty^{(m)} \to \mathbb{C}^\times$, we generalise the definition of $\mathbf{e}^*_{\chi,h}$ by computing the trace of $\overline{h}$ over the orbits of χ:

$$\mathbf{e}^*_{\chi,\varpi_{\overline{h}}} \;=\; \mathrm{Tr}(\mathrm{Ind}\chi^*)\big(\varpi_{\overline{h}}\big) \;:=\; \sum_{\chi' \in \{\chi^g \ \mid\ g \in \Gamma\}} (\chi')^{-1}(\overline{h}).$$

In fact, it is easy to check that $\mathbf{e}^*_{\chi,\varpi_{\overline{h}}}$ depends only on the image of χ within the set $\mathfrak{R}_m$ and on the orbit $\varpi_{\overline{h}}$ generated by $\overline{h}$, but not on the individual choices of χ and $\overline{h}$. Although these quantities might seem abstract, they are all computable (see Lemma 35).

Theorem 2. *If we are in Cases (III)–(VI), then a sequence $\big(\mathbf{y}_{\rho_\chi}\big) \in \prod_{\rho_\chi} \Lambda_{\mathcal{O}_\chi}\big(\Gamma^{p^{\mathbf{m}_\chi}}\big)^\times_{(p)}$ belongs to the image of $\Theta_{\infty,\underline{\chi},\mathcal{S}}$ only if*

$$\prod_{\chi \in \mathfrak{R}_m} \mathcal{N}_{\mathbf{m}_\chi,m}\left(\frac{\mathbf{y}_{\rho_\chi}}{\varphi\big(\mathbf{y}_{\rho_{\chi^p}}\big)} \cdot \frac{\varphi\big(\mathcal{N}_{0,\mathbf{m}_\chi-1}(\mathbf{y_1})\big)}{\mathcal{N}_{0,\mathbf{m}_\chi}(\mathbf{y_1})}\right)^{\mathbf{e}^*_{\chi,\varpi}}$$
$$\equiv\ 1 \mod p^{2s+3m+\epsilon_{\star,p}-\mathrm{ord}_p(\#\varpi)} \cdot \mathbb{Z}_p\big[\big[\Gamma^{p^m}\big]\big]_{(p)} \qquad (2)$$

for every $m \geq 0$, and over all Γ-orbits ϖ inside the group $\overline{\mathcal{H}}_\infty^{(m)} \cong C_{p^{s+m+\epsilon_{\star,p}}} \times C_{p^{s+m}}$.

Note in both of these theorems, if one additionally knows that $\big(\mathbf{y}_{\rho_\chi}\big) \in \prod_{\rho_\chi} \Lambda_{\mathcal{O}_\chi}\big(\Gamma^{p^{\mathbf{m}_\chi}}\big)^\times$, the modified statement should read: '$\big(\mathbf{y}_{\rho_\chi}\big) \in \mathrm{Im}\big(\Theta_{\infty,\underline{\chi}}\big)$ **if and only if** the same congruences in (1), (2) hold after replacing $p^\bullet \cdot \mathbb{Z}_p[\![\Gamma^{p^m}]\!]_{(p)}$ with its unlocalised version $p^\bullet \cdot \mathbb{Z}_p[\![\Gamma^{p^m}]\!]$'.

We also remark that Burns and Venjakob [3, Prop 3.4] have constructed a splitting

$$K_1\big(\Lambda(\mathcal{G}_\infty)_{\mathcal{S}^*}\big) \;\cong\; K_1\big(\Lambda(\mathcal{G}_\infty)_{\mathcal{S}}\big) \oplus K_0\big(\mathbb{F}_p[\![\mathcal{G}_\infty]\!]\big)$$

so one can reduce the existence of elements in $K_1\big(\Lambda(\mathcal{G}_\infty)_{\mathcal{S}^*}\big)$ to those in $K_1\big(\Lambda(\mathcal{G}_\infty)_{\mathcal{S}}\big)$, combined with a precise growth formula for the μ-invariant of the individual $\mathbf{y}_{\rho_\chi}$'s.

1.3 Some arithmetic examples

Before explaining the strategy to prove our two main theorems, we first discuss some applications to non-commutative Iwasawa theory that arise from these K_1-congruences.

Totally real extensions. Let us initially suppose that F is a totally real field, and further:

- $F_\infty = \bigcup_{n\geq 1} F_n$ is a union of totally real fields;
- only finitely many primes of F ramify inside F_∞/F;
- F_∞ contains the cyclotomic $\mathbb{Z}_p$-extension F^{cyc} of F;
- the cyclotomic μ-invariant of $F\big(e^{2\pi i/p}\big)$ vanishes.

We denote by Σ the primes ramifying inside F_∞/F. One also defines $F^{(m)}$ to be the unique extension of degree p^m contained in F^{cyc}, so that $\Gamma = \mathrm{Gal}\big(F^{\mathrm{cyc}}/F\big) \cong \varprojlim_m \mathrm{Gal}\big(F^{(m)}/F\big)$.

Let $\mathcal{G}_\infty = \mathrm{Gal}\big(F_\infty/F\big)$, and write $\kappa_F : \Gamma \to \mathbb{Z}_p^\times$ for the p-th cyclotomic character. By seminal work of Burns, Kakde and Ritter-Weiss [2, 17, 23], there exists an element $\zeta_{F_\infty/F} \in K_1\big(\Lambda(\mathcal{G}_\infty)_{\mathcal{S}^*}\big)$ such that, at any Artin representation $\rho : \mathcal{G}_\infty \to \mathrm{GL}(V)$, one has

$$\zeta_{F_\infty/F}\big(\rho\kappa_F^k\big) \;=\; L_\Sigma(\rho, 1-k)$$

for each $k \in \mathbb{N}$ satisfying $k \equiv 0 \pmod{[F(\mu_p) : F]}$. By deforming the k-variable p-adically, the above values interpolate to the Iwasawa function $L_{p,\Sigma}(\rho, -) : \mathbb{Z}_p \to \overline{\mathbb{Q}}_p$ constructed by Cassou-Noguès and Deligne-Ribet [4, 14].

Corollary 3. *Let F_∞/F be an infinite solvable Lie extension as above, with* $\dim(\mathcal{G}_\infty) = 3$. *If the representation $\rho_\chi = \mathrm{Ind}^{\mathcal{G}_\infty}_{\mathrm{Stab}_\Gamma(\chi)\ltimes\mathcal{H}_\infty}(\chi)$ has dimension equal to $p^{\mathbf{m}_\chi}$ say, then write $\mathbf{L}^{\text{D-R}}_{p,\Sigma}\big(\rho_\chi\big) \in \mathrm{Quot}\big(\Lambda_{\mathcal{O}_\chi}(\Gamma^{p^{\mathbf{m}_\chi}})\big)^\times$ for the unique element satisfying*

$$\kappa_F^k \circ \mathbf{L}^{\text{D-R}}_{p,\Sigma}\big(\rho_\chi\big) \;=\; L_{p,\Sigma}\big(\rho_\chi, 1-k\big) \quad \textit{for all } k \in \mathbb{Z}_p.$$

(a) If we are in Case (II), then the system of congruences (1) holds for $\mathbf{y}_{\rho_\chi} = \mathbf{L}^{\text{D-R}}_{p,\Sigma}\big(\rho_\chi\big)$.

(b) In Case ($\star$) with $\star \in \{III, IV, V, VI\}$, the congruences (2) hold for $\mathbf{y}_{\rho_\chi} = \mathbf{L}^{\text{D-R}}_{p,\Sigma}\big(\rho_\chi\big)$.

Proof. Note that the infinite sequence $\big(\mathbf{L}^{\text{D-R}}_{p,\Sigma}\big(\rho_\chi\big)\big) \in \prod_{\rho_\chi} \mathrm{Quot}\big(\Lambda_{\mathcal{O}_\chi}(\Gamma^{p^{\mathbf{m}_\chi}})\big)^\times$ coincides with $\Theta_{\infty,\underline{\chi},\mathcal{S}^*}\big(\zeta_{F_\infty/F}\big)$, as they both interpolate the same L-values. Therefore the necessity of the congruences (1) and (2) follows directly from Theorems 1 and 2, respectively. □

Let us now digress momentarily, and assume we are given a congruence of the form

$$\frac{F(X)}{G(X)} \;\equiv\; 1 \mod p^v \cdot \mathbb{Z}_p[\![X]\!]_{(p)} \quad \text{with } F, G \in \mathcal{O}_{\mathbb{C}_p}[\![X]\!] \text{ and } v \geq 1.$$

Then $\frac{F(X)}{G(X)} = 1 + p^v \cdot \frac{R(X)}{T(X)}$ for some $R, T \in \mathbb{Z}_p[\![X]\!]$ where the μ-invariant of T equals zero. It follows that $F \cdot T = G \cdot (T + p^v \cdot R)$, and one works out that

$$\mu(F) \;=\; \mu(F \cdot T) \;=\; \mu(G) + \mu(T + p^v \cdot R) \;=\; \mu(G) + 0,$$

i.e. $\mu(F) = \mu(G)$. Also $F = G + \frac{p^v \cdot RG}{T} \in \mathcal{O}_{\mathbb{C}_p}[\![X]\!]$ so that $T \big| RG$, whence $F \equiv G \pmod{p^v}$. Certainly if $\mu(F) = \mu(G) = 0$, then the leading terms of F and G are congruent mod p^v.

However even if $\mu(F)=\mu(G)>0$, their leading terms must still be congruent modulo p^v, as one can repeat the above argument with $\tilde{F}=p^{-\mu(F)}\cdot F$ and $\tilde{G}=p^{-\mu(F)}\cdot G$ instead.

Conclusion: If $\frac{F(X)}{G(X)}\equiv 1 \bmod p^v\cdot \mathbb{Z}_p[\![X]\!]_{(p)}$, the leading terms of F,G agree modulo p^v.

We are going to apply this to the congruences (1) and (2) at the trivial orbit $\varpi=\{\mathrm{id}\}$: specifically, F will denote the numerator of (1) and (2) while G will be the denominator, so that $\frac{F(X)}{G(X)}\equiv 1 \bmod p^v\cdot\mathbb{Z}_p[\![X]\!]_{(p)}$ with $X=\gamma^{p^m}-1$, and $v=s+2m+n$ when $\star=$II whilst $v=2s+3m+\epsilon_{\star,p}$ when $\star\neq$II.

To individually describe the leading terms, if $r(\rho,x_0)=\mathrm{order}_{x=x_0}\big(L_{p,\Sigma}(\rho,x)\big)$ then

$$L_\Sigma^{(p)}(\rho,1-k) \ := \ \begin{cases} L_\Sigma(\rho,1-k) & \text{if } r(\rho,1-k)=0 \\ \lim_{x\to 1-k}\big(x^{-r(\rho,1-k)}\cdot L_{p,\Sigma}(\rho,x)\big) & \text{if } r(\rho,1-k)>0\end{cases}$$

yields the p-adic residue of $L_{p,\Sigma}(\rho,x)$ at the non-positive critical value $x=1-k$.

Notations: (i) At integers $m\geq m'\geq 0$, let us define $\mathbf{r}_{m',m}=\mathrm{Ind}_{F^{(m)}}^{F^{(m')}}(\mathbf{1})$ to be the regular representation for $\mathrm{Gal}\big(F^{(m)}/F^{(m')}\big)$.

(ii) Furthermore, we shall write $\mathbf{r}_{0,m}^{(m')}$ as an abbreviation for $\mathrm{Ind}_{F^{(m'-1)}}^{F}\Big(\psi_p\circ \mathbf{r}_{m',m}\big|_{F^{(m')}}\Big)$, where ψ_p is the p-th Adams operator (strictly speaking ψ_p only acts on the trace of a virtual representation, but the abuse of notation makes sense in the context of ζ-functions).

(iii) Lastly set $\rho_\chi^{(m)}:=\mathrm{Ind}_{F^{(m)}}^{F}\big(\chi\big|_{F^{(m)}}\big)$ and $\rho_{\chi^p}^{(m)}:=\mathrm{Ind}_{F^{(\mathbf{m}_\chi-1)}}^{F}\Big(\psi_p\circ\mathrm{Ind}_{F^{(m)}}^{F^{(\mathbf{m}_\chi)}}\big(\chi\big|_{F^{(m)}}\big)\Big)$.

Theorem 4. *Let F_∞/F be as above, with $\dim(\mathcal{G}_\infty)=3$ and also $\zeta_{F_\infty/F}\in K_1\big(\Lambda(\mathcal{G}_\infty)_{\mathcal{S}}\big)$.*

(a) If we are in Case (II), then for every $m,n,k\in\mathbb{N}$:

$$\prod_{m'=0}^{m}\prod_{a=1}^{p^{n-m'}}\prod_{\substack{b=1,\\ p\nmid b \text{ if } m'>0}}^{p^{s+m'}}\Big(L_\Sigma^{(p)}\big(\rho_\chi^{(m)},1-k\big)\cdot L_\Sigma^{(p)}\big(\mathbf{r}_{0,m}^{(\mathbf{m}_\chi)},1-k\big)\Big)^{p^{\mathbf{m}_\chi}}\Bigg|_{\chi=\chi_{2,n}^a\cdot\chi_{1,s+m'}^b}$$
$$\equiv\ \prod_{m'=0}^{m}\prod_{a=1}^{p^{n-m'}}\prod_{\substack{b=1,\\ p\nmid b \text{ if } m'>0}}^{p^{s+m'}}\Big(L_\Sigma^{(p)}\big(\rho_{\chi^p}^{(m)},1-k\big)\cdot L_\Sigma^{(p)}\big(\mathbf{r}_{0,m},1-k\big)\Big)^{p^{\mathbf{m}_\chi}}\Bigg|_{\chi=\chi_{2,n}^a\cdot\chi_{1,s+m'}^b}$$

modulo p^{s+2m+n}.

(b) In Case ($\star$) with $\star\in\{$III,IV,V,VI$\}$, for every $m,k\in\mathbb{N}$:

$$\prod_{\chi\in\mathfrak{R}_m}\Big(L_\Sigma^{(p)}\big(\rho_\chi^{(m)},1-k\big)\cdot L_\Sigma^{(p)}\big(\mathbf{r}_{0,m}^{(\mathbf{m}_\chi)},1-k\big)\Big)^{p^{\mathbf{m}_\chi}}$$
$$\equiv\ \prod_{\chi\in\mathfrak{R}_m}\Big(L_\Sigma^{(p)}\big(\rho_{\chi^p}^{(m)},1-k\big)\cdot L_\Sigma^{(p)}\big(\mathbf{r}_{0,m},1-k\big)\Big)^{p^{\mathbf{m}_\chi}}\quad \bmod p^{2s+3m+\epsilon_{\star,p}}.$$

Because p-adic zeta-functions of totally real fields do not vanish at odd negative integers, a nice consequence is that whenever $k\equiv 0 \pmod{[F(\mu_p):F]}$, these congruences actually involve bona fide *complex zeta-values*, not simply their p-adic residues.

Heisenberg extensions. Let us now suppose we are in Case (II) with the parameter $s \geq 0$, in which case $\mathcal{G}_\infty$ is an open subgroup of the Heisenberg group, i.e.

$$\mathcal{G}_\infty \ \lhd \ H_3(\mathbb{Z}_p) := \left(\begin{array}{ccc} 1 & \mathbb{Z}_p & \mathbb{Z}_p \\ 0 & 1 & \mathbb{Z}_p \\ 0 & 0 & 1 \end{array} \right) \quad \text{where} \ \ \left[H_3(\mathbb{Z}_p) : \mathcal{G}_\infty\right] = p^s.$$

In an unpublished preprint [20], Kato derives different but equivalent congruences to (1), as ideal congruences in the group algebras associated to finite sub-quotients of $H_3(\mathbb{Z}_p)$. Thus Theorem 4(a) gives a concrete description for the most basic of these ideal relations, as a congruence modulo p^{s+2m+n} connecting the special values of Artin L-functions.

False-Tate extensions. Fix $s \geq 1$. We set $F = \mathbb{Q}(\mu_{p^s})$ and $F_\infty = \mathbb{Q}\big(\mu_{p^\infty}, q_1^{1/p^\infty}, q_2^{1/p^\infty}\big)$ where $q_1, q_2 > 1$ are distinct p-power free integers satisfying $\gcd(p, q_1 q_2) = \gcd(q_1, q_2) = 1$. Then $\mathcal{G}_\infty = \mathrm{Gal}\big(F_\infty/F\big)$ is a three-dimensional pro-p-group, corresponding to Case (III) in the Classification Theorem (note that F_∞ is **not** a union of totally real fields so there is no element $\zeta_{F_\infty/F} \in K_1\big(\Lambda(\mathcal{G}_\infty)_{\mathcal{S}^*}\big)$ available, and therefore no Iwasawa Main Conjecture can be formulated for Tate motives here).

Now if $s = 1$, the congruences (2) specialise down to yield the congruences labelled $(1.1)_{m,\underline{\mathfrak{h}}}$ and $(1.2)_m$ in [10, p3]. If $E_{/\mathbb{Q}}$ denotes a semistable elliptic curve with good ordinary reduction at p, then p-adic L-functions $\mathbf{L}_p(E, \rho_\chi) \in \Lambda\big(\Gamma^{p^{\mathbf{m}_\chi}}\big)\big[1/p\big]$ interpolating the algebraic part of $L_{\{pq_1q_2\}}(E, \rho_\chi, 1)$ have been constructed in Theorem 1.5 of *op. cit.* Furthermore, there are three 'first layer congruences' to check for each tuple (E, p, q_1, q_2). These were verified numerically for the elliptic curves 11a3, 77c1, 19a3 and 56a1 using MAGMA at the primes $p = 3, 5$ and at small values of q_1 and q_2, in §6 of *op. cit.*

On the algebraic side, let us further assume that q_1 and q_2 are both chosen to be primes of non-split multiplicative reduction for E, such that

$$(-1)^{(p-1)/2} \times \prod_{l|\mathrm{cond}(E),\ l \neq q_1, q_2} \left(\frac{l}{p}\right) \ = \ -1$$

where $\left(\frac{-}{p}\right)$ denotes the Legendre symbol at p. Then if the cyclotomic λ-invariant of $\mathrm{Sel}_{p^\infty}\big(E/\mathbb{Q}(\mu_{p^\infty})\big)$ equals one and if $\mathrm{Sel}_{p^\infty}(E/F_\infty)^\wedge$ belongs to the category $\mathfrak{M}_{\mathcal{H}_\infty}(\mathcal{G}_\infty)$, it is shown in [9, Corollary 2.6] that

$$\mathrm{rank}_{\mathbb{Z}}\big(E(F_n)\big) \ = \ p^{2n-1} \ \text{or} \ \ p^{2n},$$

provided the p-Sylow subgroup of $\text{Ш}(E/F_n)$ is finite at each layer $F_n = \mathbb{Q}\big(\mu_{p^n}, q_1^{1/p^n}, q_2^{1/p^n}\big)$. Alternatively, by studying the λ-invariants of each χ-part $\mathrm{Sel}_{p^\infty}(E/F_n(\mu_{p^\infty}))^\wedge \otimes_{\mathbb{Z}_p, \chi} \mathcal{O}_\chi$ using the congruences in Theorem 2, one can produce the same estimate for the rank (current work of the first named author [13]).

Heegner-type extensions. Consider an imaginary quadratic field $k = \mathbb{Q}\big(\sqrt{-D}\big)$ and let us suppose k_∞ denotes its $\mathbb{Z}_p^2$-extension, so that $\mathrm{Gal}(k_\infty/k) \cong \Gamma \times \mathcal{H}_{1,\infty}$ where $\mathcal{H}_{1,\infty}$ is the Galois group of the anticyclotomic $\mathbb{Z}_p$-extension of k. For any choice of odd prime $q \neq p$ with $q \nmid D$, one may set $F = \mathbb{Q}\big(\sqrt{-D}, \mu_p\big)$ and $F_\infty = k_\infty\big(\mu_p, q^{1/p^\infty}\big)$, in which case

$$\mathcal{G}_\infty := \mathrm{Gal}(F_\infty/F) \ \cong \ \Gamma \ltimes \big(\mathcal{H}_{1,\infty} \times \mathcal{H}_{2,\infty}\big) \ \cong \ \big(\Gamma \times \mathcal{H}_{1,\infty}\big) \ltimes \mathcal{H}_{2,\infty}.$$

Here h_1 acts trivially on $\mathcal{H}_{2,\infty} = \overline{\langle h_2 \rangle} = \mathrm{Gal}\big(F_\infty/k_\infty(\mu_p)\big)$, while γ acts on h_2 through multiplication by $1 + p$ (we must therefore be in Case (V) with $s = d = 0$ and $r = 1$).

Let $E_{/\mathbb{Q}}$ be a semistable elliptic curve with ordinary reduction at p, split multiplicative reduction at q, and with non-split multiplicative reduction at all other primes dividing the conductor of E. We also suppose that q generates $\left(\mathbb{Z}/p^2\mathbb{Z}\right)^\times$ so that q is inert in $\mathbb{Q}(\mu_{p^\infty})$, and that the various Heegner conditions **(DT1)**–**(DT7)** described in [9, Sect 2.4] hold. Then it is shown in Proposition 2.14 of *op. cit.* that for $n \gg 0$,

$$p^{2n} \cdot \left(1 - \frac{2p^2+2p+1}{(p+1)^3}\right) \ \leq \ \mathrm{rank}_{\mathbb{Z}}\big(E(F_n)\big) \ \leq \ p^{2n}+4$$

with no hypotheses whatsoever on the finiteness of $\text{Ш}(E/F_n)[p^\infty]$.

The upper bound essentially comes from a growth formula for the λ-invariant of $\mathrm{Sel}_{p^\infty}(E/F_n(\mu_{p^\infty}))^\wedge$ as n becomes large. In fact if one exploits the congruences (2), this yields another way to obtain the upper bound on $\mathrm{rank}_{\mathbb{Z}}\big(E(F_n)\big)$, and establishes finer bounds on the χ-part of $E(F_n)$. However the lower bound relies heavily on the properties of Heegner points, following the same approach as Darmon and Tian [8] in dimension 2.

p^n-division fields of CM curves. Let $E_{/\mathbb{Q}}$ be an elliptic curve with complex multiplication by $k = \mathbb{Q}\big(\sqrt{-D}\big)$, and select a good ordinary prime $p \neq 2$ for E which splits inside $\mathbb{Z}\big(\sqrt{-D}\big)$. If one takes $F = \mathbb{Q}\big(\sqrt{-D}, \mu_p\big)$, $F_n = \mathbb{Q}\big(E[p^n], q^{1/p^n}\big)$ and $F_\infty = \bigcup_{n\geq 1} F_n$ for an auxiliary prime q not dividing $\mathrm{cond}(E)$, then $\mathcal{G}_\infty := \mathrm{Gal}(F_\infty/F)$ corresponds to Case (V) with $s = d = 0$ and $r = 1$ again. By using the congruences (2) to study the λ-invariants of $\mathrm{Sel}_{p^\infty}(E/F_n)^\wedge$, one can bound the rank of $E(F_n)$ from above by p^{2n} if the cyclotomic λ-invariant is one. Whilst Heegner points are no longer useful here, a lower bound on the $\mathbb{Z}$-rank of $E(F_n)$ of the form $c_p \times p^{2n}$ (with $c_p \neq 0$ and $c_p \sim 1$ if $p \gg 0$) should still be feasible, if one exploits the non-triviality of the Euler system of elliptic units in place of the Heegner points.

Here is a brief plan of the article. In Section 2 we begin by choosing an appropriate system of subgroups with which to define our theta-map. The choice we make differs from that made in [17] – ours is a coarser system than Kakde's choice, yet better suited to the specific representation theory of $\mathcal{G}_\infty$. We then write down bases for each piece of the image of the theta-map, and also introduce auxiliary homomorphisms Ver and π which allow us to pass between adjacent subgroups in this directed system.

The additive component of the proof is contained in Section 3, where we describe the image of the additive theta-map through its special values at Artin representations ρ_χ. We next formulate four conditions (C1)–(C4), which are just strong enough to determine whether or not an element lies in the image of this homomorphism.

In Section 4, we pass from the additive to the multiplicative world by means of the Taylor-Oliver logarithm; for those familiar with the details of [7, p79-123], under this logarithm the conditions '(M1)–(M4)' transform into our additive conditions (C1)–(C4). Because our subgroup system is coarser than in *op. cit.*, the proof of the converse statement "(C1)–(C4) $\Longrightarrow$ (M1)–(M4)" is far from immediate and occupies much of this article. Finally in Section 5, we develop an algorithm to compute the quantities $\mathfrak{R}_m$, $\mathbf{e}^*_{\chi,\varpi}$, $\#\varpi$ in Theorem 2 explicitly, using Case (II) as a worked example to trial the algorithm.

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2 The general set-up in dimension three

We shall begin by reviewing the representation theory behind these semi-direct products. Let us first observe that the subgroup $\mathcal{H}_\infty = \mathcal{H}_{1,\infty} \times \mathcal{H}_{2,\infty} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ is generated by $h_1 = (1,0)^T$ and $h_2 = (0,1)^T$ topologically. The action of each $g = \gamma^z \in \Gamma$ on an arbitrary element $(x,y)^T = h_1^x h_2^y \in \mathcal{H}_\infty$ can be described through a 2×2-matrix of the form $I_2 + M$:

$$\gamma^z\big((x,y)^T\big) \;=\; \gamma^{-z}\big(h_1^x h_2^y\big)\gamma^z \;=\; \big(I_2+M\big)^z \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{for all } g=\gamma^z \in \Gamma$$

where $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity, and $M \in \mathrm{Mat}_{2\times 2}\big(\mathbb{Z}_p\big)$ is topologically nilpotent. Applying the Classification Theorem for $\mathcal{G}_\infty$, the matrix M equals

$$\begin{pmatrix} 0 & p^s \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} p^s & 0 \\ 0 & p^s \end{pmatrix}, \begin{pmatrix} p^s & p^{s+r} \\ p^{s+r}d & p^s \end{pmatrix}, \begin{pmatrix} 0 & p^s \\ p^s d & p^{s+r} \end{pmatrix} \text{ and } \begin{pmatrix} 0 & p^s \\ p^{s+r}t & 0 \end{pmatrix} \tag{3}$$

in Cases (II), (III), (IV), (V) and (VI) respectively (note in Case (VIa) we have set $t=1$).

2.1 Determining the stabilizer of a character on $\mathcal{H}_\infty$

Note that each element $g \in \Gamma$ acts naturally on $\chi \in \mathrm{Hom}(\mathcal{H}_\infty, \mu_{p^\infty})$ by sending $\chi \mapsto g * \chi$, where $g*\chi(h) := \chi(g^{-1}hg)$ for all $h \in \mathcal{H}_\infty$. The Γ-stabilizer of χ is given by the subgroup

$$\mathrm{Stab}_\Gamma(\chi) \;:=\; \Big\{ g \in \Gamma \;\Big|\; \chi\big(g^{-1}(h_1^x h_2^y)g\big) = \chi\big(h_1^x h_2^y\big) \text{ for all } h = h_1^x h_2^y \in \mathcal{H}_\infty \Big\}.$$

Proposition 5. *If $\chi = \chi_{1,n}^{e_1} \times \chi_{2,n}^{e_2} : \mathcal{H}_\infty \twoheadrightarrow \mu_{p^n}$ is a surjective character, then*

$$\big[\Gamma : \mathrm{Stab}_\Gamma(\chi)\big] = p^{\mathbf{m}_\chi} \quad \textit{where } \mathbf{m}_\chi := \max\{0, \tilde{\mathbf{m}}_\chi\}$$

and, using the case-by-case description in the Classification Theorem, one has respectively:

(II) $\tilde{\mathbf{m}}_\chi = n - s - \mathrm{ord}_p(\boldsymbol{e}_1)$; *(III)* $\tilde{\mathbf{m}}_\chi = n-s$; *(IV)* $\tilde{\mathbf{m}}_\chi = n-s$;

(V) $\tilde{\mathbf{m}}_\chi = n - s - \min\big\{\mathrm{ord}_p(\boldsymbol{e}_2) + \mathrm{ord}_p(d), \mathrm{ord}_p(\boldsymbol{e}_1 + p^r\boldsymbol{e}_2)\big\}$; *and*

(VI) $\tilde{\mathbf{m}}_\chi = n - s - \min\big\{r + \mathrm{ord}_p(\boldsymbol{e}_2), \mathrm{ord}_p(\boldsymbol{e}_1)\big\}$.

Proof. Firstly, let us denote by ζ_{p^n} the primitive p^n-th root of unity $\exp(2\pi\sqrt{-1}/p^n)$.

Case (II). Here $I_2 + M = \begin{pmatrix} 1 & p^s \\ 0 & 1 \end{pmatrix}$, so that $\gamma^{-p^i}(h_1^x h_2^y)\gamma^{p^i} = h_1^{x+p^{s+i}y} h_2^y$. Consequently

$$\chi\left(\gamma^{-p^i}(h_1^x h_2^y)\gamma^{p^i}\right) \;=\; \chi_{1,n}\big(h_1^{x+p^{s+i}y}h_2^y\big)^{\mathbf{e}_1} \times \chi_{2,n}\big(h_1^{x+p^{s+i}y}h_2^y\big)^{\mathbf{e}_2} \;=\; \zeta_{p^n}^{\mathbf{e}_1 x + (\mathbf{e}_2 + \mathbf{e}_1 \times p^{s+i})y}$$

equals $\chi\big(h_1^x h_2^y\big) = \zeta_{p^n}^{\mathbf{e}_1 x + \mathbf{e}_2 y}$ for all $x, y \in \mathbb{Z}$, if and only if $\mathbf{e}_1 \times p^{s+i} \equiv 0 \pmod{p^n}$.

Case (III). Here $I_2 + M = \begin{pmatrix} 1+p^s & 0 \\ 0 & 1+p^s \end{pmatrix}$ with repeated eigenvalue $\lambda_{III,\pm} = 1 + p^s$, and it follows that

$$\chi\left(\gamma^{-p^i}(h_1^x h_2^y)\gamma^{p^i}\right) \;=\; \chi(h_1^x h_2^y)^{(1+p^s)^{p^i}} \;=\; \zeta_{p^n}^{(\mathbf{e}_1 x + \mathbf{e}_2 y)\times(1+p^s)^{p^i}}.$$

However $(1+p^s)^{p^i} \equiv 1 \pmod{p^{s+i}}$ but $(1+p^s)^{p^i} \not\equiv 1 \pmod{p^{s+i+1}}$, in which case $\chi\left(\gamma^{-p^i}(h_1^x h_2^y)\gamma^{p^i}\right)$ equals $\chi(h_1^x h_2^y) = \zeta_{p^n}^{\mathbf{e}_1 x + \mathbf{e}_2 y}$ for all $x, y \in \mathbb{Z}$, if and only if

$$\mathrm{ord}_p\big((1+p^s)^{p^i} - 1\big) = s + i \ \geq \ n, \qquad \text{i.e. if and only if } \ i \ \geq \ n - s.$$

Case (IV). Here $I_2 + M = \begin{pmatrix} 1+p^s & p^{s+r} \\ p^{s+r}d & 1+p^s \end{pmatrix}$; let $\lambda_{IV,\pm} := 1 + p^s \pm p^{s+r}\sqrt{d}$ be the two distinct eigenvalues of $I_2 + M$, so that

$$I_2 + M = P_{IV} D_{IV} P_{IV}^{-1} \quad \text{with } D_{IV} = \begin{pmatrix} \lambda_{IV,+} & 0 \\ 0 & \lambda_{IV,-} \end{pmatrix} \text{and } P_{IV} = \begin{pmatrix} 1 & 1 \\ \sqrt{d} & -\sqrt{d} \end{pmatrix}.$$

Since $(I_2+M)^{p^i} = P_{IV} D_{IV}^{p^i} P_{IV}^{-1}$, one readily computes

$$\gamma^{-p^i}(h_1^x h_2^y)\gamma^{p^i} \ = \ h_1^{\left(\frac{\lambda_{IV,+}^{p^i}+\lambda_{IV,-}^{p^i}}{2}\right)x + \left(\frac{\lambda_{IV,+}^{p^i}-\lambda_{IV,-}^{p^i}}{2\sqrt{d}}\right)y} \times h_2^{\left(\frac{\lambda_{IV,+}^{p^i}-\lambda_{IV,-}^{p^i}}{2}\right)\sqrt{d}\,x + \left(\frac{\lambda_{IV,+}^{p^i}+\lambda_{IV,-}^{p^i}}{2}\right)y}. \tag{4}$$

To study both $\frac{\lambda_{IV,+}^{p^i}+\lambda_{IV,-}^{p^i}}{2}$ and $\frac{\lambda_{IV,+}^{p^i}-\lambda_{IV,-}^{p^i}}{2}$, note that

$$\lambda_{IV,\pm}^p \ = \ \big(1 + p^s(1 \pm p^r\sqrt{d})\big)^p \ = \ 1 + \binom{p}{1} p^s(1 \pm p^r\sqrt{d}) + \sum_{j=2}^{p} \binom{p}{j} p^{sj}(1 \pm p^r\sqrt{d})^j$$

and $(1 \pm p^r\sqrt{d})^j = 1 \pm jp^r\sqrt{d} + O\big(p^{2r+\delta_p}\big)$ where $\delta_p = \mathrm{ord}_p(d)$, hence

$$\lambda_{IV,\pm}^p = 1 + p^{s+1} \pm p^{s+r+1}\sqrt{d} + \sum_{j=2}^{p} \binom{p}{j} p^{sj} \pm p^r\sqrt{d} \sum_{j=2}^{p} \binom{p}{j} jp^{sj} + O\big(p^{2s+2r+1+\delta_p}\big)$$

$$= \ 1 + p^{s+1} \pm p^{s+r+1}\sqrt{d} + \Big((1+p^s)^p - 1 - p^{s+1}\Big) \pm O\big(p^{2s+r+1+\delta_p/2}\big) + O\big(p^{2s+2r+1+\delta_p}\big).$$

It follows that $\frac{\lambda_{IV,+}^p+\lambda_{IV,-}^p}{2}$ will equal $1 + p^{s+1} + \Big((1+p^s)^p - 1 - p^{s+1}\Big) + O\big(p^{2s+r+1+\delta_p/2}\big)$, or less accurately $\frac{\lambda_{IV,+}^p+\lambda_{IV,-}^p}{2} = 1 + p^{s+1} + O\big(p^{2s+1}\big)$; applying an induction argument:

$$\frac{\lambda_{IV,+}^{p^i} + \lambda_{IV,-}^{p^i}}{2} \ = \ 1 + p^{s+i} + O\big(p^{2s+i}\big). \tag{5}$$

On the other hand, the difference term $\frac{\lambda_{IV,+}^p-\lambda_{IV,-}^p}{2}$ equals $p^{s+r+1}\sqrt{d} + O\big(p^{2s+r+1+\delta_p/2}\big)$, and therefore $\frac{\lambda_{IV,+}^p-\lambda_{IV,-}^p}{2\sqrt{d}} = p^{s+r+1} + O\big(p^{2s+r+1}\big)$; applying induction again:

$$\frac{\lambda_{IV,+}^{p^i} - \lambda_{IV,-}^{p^i}}{2\sqrt{d}} \ = \ p^{s+r+i} + O\big(p^{2s+r+i}\big). \tag{6}$$

Recalling the chosen character $\chi = \chi_{1,n}^{\mathbf{e}_1} \times \chi_{2,n}^{\mathbf{e}_2}$, from Equation (4) one obtains

$$\chi\left(\gamma^{-p^i}(h_1^x h_2^y)\gamma^{p^i}\right) \ = \ \chi_{1,n}\left(\gamma^{-p^i}(h_1^x h_2^y)\gamma^{p^i}\right)^{\mathbf{e}_1} \times \chi_{2,n}\left(\gamma^{-p^i}(h_1^x h_2^y)\gamma^{p^i}\right)^{\mathbf{e}_2}$$

$$= \ \zeta_{p^n}^{\left(\mathbf{e}_1\left(\frac{\lambda_{IV,+}^{p^i}+\lambda_{IV,-}^{p^i}}{2}\right) + \mathbf{e}_2\left(\frac{\lambda_{IV,+}^{p^i}-\lambda_{IV,-}^{p^i}}{2}\right)\sqrt{d}\right)x + \left(\mathbf{e}_1\left(\frac{\lambda_{IV,+}^{p^i}-\lambda_{IV,-}^{p^i}}{2\sqrt{d}}\right) + \mathbf{e}_2\left(\frac{\lambda_{IV,+}^{p^i}+\lambda_{IV,-}^{p^i}}{2}\right)\right)y}.$$

As a corollary of our estimates in (5) and (6), $\gamma^{p^i}\star\chi(h_1^x h_2^y)=\chi\left(\gamma^{-p^i}(h_1^x h_2^y)\gamma^{p^i}\right)$ equals $\chi(h_1^x h_2^y)=\zeta_{p^n}^{\mathbf{e}_1x+\mathbf{e}_2y}$ for all $x,y\in\mathbb{Z}$, if and only if

$$\mathbf{e}_1p^{s+i}+\mathbf{e}_2p^{s+r+i}d\equiv 0\ (\bmod\ p^n)\quad\text{and}\quad \mathbf{e}_1p^{s+r+i}+\mathbf{e}_2p^{s+i}\equiv 0\ (\bmod\ p^n),$$

i.e. if and only if $i\ \ge\ n-s-\min\big\{\mathrm{ord}_p(\mathbf{e}_1+p^r d\mathbf{e}_2)\,,\mathrm{ord}_p(p^r\mathbf{e}_1+\mathbf{e}_2)\big\}=n-s$.

Case (V). Here $I_2+M=\begin{pmatrix}1 & p^s\\ p^s d & 1+p^{s+r}\end{pmatrix}$; let $\lambda_{V,\pm}:=1+\frac{p^{s+r}}{2}\pm p^s\sqrt{\Delta_V}$ with $\Delta_V=d+p^{2r}/4$ denote the eigenvalues of I_2+M. Indeed for all $i\ge 0$, one may write

$$(I_2+M)^{p^i}\ =\ P_V\begin{pmatrix}\lambda_{V,+}^{p^i} & 0\\ 0 & \lambda_{V,-}^{p^i}\end{pmatrix}P_V^{-1}$$

where $P_V=\begin{pmatrix}1 & 1\\ \frac{p^r}{2}+\sqrt{\Delta_V} & \frac{p^r}{2}-\sqrt{\Delta_V}\end{pmatrix}$, and its inverse $P_V^{-1}=\frac{1}{2}\begin{pmatrix}1-\frac{p^r}{2\sqrt{\Delta_V}} & \frac{1}{\sqrt{\Delta_V}}\\ 1+\frac{p^r}{2\sqrt{\Delta_V}} & -\frac{1}{\sqrt{\Delta_V}}\end{pmatrix}$. Using this decomposition, we next deduce

$$\begin{aligned}\gamma^{-p^i}(h_1^x h_2^y)\gamma^{p^i}\ &=\ h_1^{\left(\frac{\lambda_{V,+}^{p^i}+\lambda_{V,-}^{p^i}}{2}-\frac{\lambda_{V,+}^{p^i}-\lambda_{V,-}^{p^i}}{2\sqrt{\Delta_V}}\times\frac{p^r}{2}\right)x+\left(\frac{\lambda_{V,+}^{p^i}-\lambda_{V,-}^{p^i}}{2\sqrt{\Delta_V}}\right)y}\\ &\quad\times h_2^{\left(\frac{\lambda_{V,+}^{p^i}-\lambda_{V,-}^{p^i}}{2\sqrt{\Delta_V}}\right)dx+\left(\frac{\lambda_{V,+}^{p^i}+\lambda_{V,-}^{p^i}}{2}+\frac{\lambda_{V,+}^{p^i}-\lambda_{V,-}^{p^i}}{2\sqrt{\Delta_V}}\times\frac{p^r}{2}\right)y}.\end{aligned}\tag{7}$$

Now from the binomial theorem,

$$\lambda_{V,\pm}^p\ =\ 1+\frac{p^{s+r+1}}{2}\pm p^{s+1}\sqrt{\Delta_V}\ +\ \sum_{j=2}^{p}\begin{pmatrix}p\\ j\end{pmatrix}p^{sj}\Big(\frac{p^r}{2}\pm\sqrt{\Delta_V}\Big)^j.$$

- If $\mathrm{ord}_p(\sqrt{\Delta_V})\ge r$ then $\left(\frac{p^r}{2}\pm\sqrt{\Delta_V}\right)^j=\left(\frac{p^r}{2}\right)^j\pm j\left(\frac{p^r}{2}\right)^{j-1}\sqrt{\Delta_V}+O\big(p^{r(j-2)+\delta'_p}\big)$ where $\delta'_p=\mathrm{ord}_p(\Delta_V)$, hence

$$\sum_{j=2}^{p}\begin{pmatrix}p\\ j\end{pmatrix}p^{sj}\Big(\frac{p^r}{2}\pm\sqrt{\Delta_V}\Big)^j=\sum_{j=2}^{p}\begin{pmatrix}p\\ j\end{pmatrix}p^{sj}\left(\left(\frac{p^r}{2}\right)^j\pm j\left(\frac{p^r}{2}\right)^{j-1}\sqrt{\Delta_V}\right)+O\big(p^{2s+1+\delta'_p}\big)$$

$$=\left(1+\frac{p^{r+s}}{2}\right)^p-\left(1+\frac{p^{r+s+1}}{2}\right)\pm p^{s+1}\sqrt{\Delta_V}\times\left(\left(1+\frac{p^{r+s}}{2}\right)^{p-1}-1\right)\ +\ O\big(p^{2s+1+\delta'_p}\big).$$

It follows that $\frac{\lambda_{V,+}^p+\lambda_{V,-}^p}{2}=1+\frac{p^{s+r+1}}{2}+O\big(p^{2s+2r+1}\big)$ and $\frac{\lambda_{V,+}^p-\lambda_{V,-}^p}{2\sqrt{\Delta_V}}=p^{s+1}+O\big(p^{2s+r+1}\big)$ upon using the condition $\delta'_p\ge 2r$, so by induction:

$$\frac{\lambda_{V,+}^{p^i}+\lambda_{V,-}^{p^i}}{2}=1+\frac{p^{s+r+i}}{2}+O\big(p^{2s+2r+i}\big)\ \text{ and }\ \frac{\lambda_{V,+}^{p^i}-\lambda_{V,-}^{p^i}}{2\sqrt{\Delta_V}}=p^{s+i}+O\big(p^{2s+r+i}\big).\tag{8}$$

- Alternatively, if $r\ge\mathrm{ord}_p(\sqrt{\Delta_V})$ then

$$\Big(\frac{p^r}{2}\pm\sqrt{\Delta_V}\Big)^j\ =\ \Big(\pm\sqrt{\Delta_V}\Big)^j+\frac{jp^r}{2}\Big(\pm\sqrt{\Delta_V}\Big)^{j-1}+O\big(p^{\delta'_p(j-2)/2+2r}\big)$$

and arguing in an identical fashion to before, one deduces that

$$\frac{\lambda_{V,+}^{p^i}+\lambda_{V,-}^{p^i}}{2} = 1+\frac{p^{s+r+i}}{2}+O\big(p^{2s+\delta'_p+i}\big) \quad\text{and}\quad \frac{\lambda_{V,+}^{p^i}-\lambda_{V,-}^{p^i}}{2\sqrt{\Delta_V}} = p^{s+i}+O\big(p^{2s+\delta'_p/2+i}\big). \tag{9}$$

Again as $\chi=\chi_{1,n}^{\mathbf{e}_1}\times\chi_{2,n}^{\mathbf{e}_2}$, this time Equation (7) implies

$$\chi\left(\gamma^{-p^i}(h_1^x h_2^y)\gamma^{p^i}\right) = \zeta_{p^n}^{\left(\mathbf{e}_1\left(\frac{\lambda_{V,+}^{p^i}+\lambda_{V,-}^{p^i}}{2}-\frac{\lambda_{V,+}^{p^i}-\lambda_{V,-}^{p^i}}{2\sqrt{\Delta_V}}\times\frac{p^r}{2}\right)+\mathbf{e}_2 d\left(\frac{\lambda_{V,+}^{p^i}-\lambda_{V,-}^{p^i}}{2\sqrt{\Delta_V}}\right)\right)x}$$
$$\times\ \zeta_{p^n}^{\left(\mathbf{e}_1\left(\frac{\lambda_{V,+}^{p^i}-\lambda_{V,-}^{p^i}}{2\sqrt{\Delta_V}}\right)+\mathbf{e}_2\left(\frac{\lambda_{V,+}^{p^i}+\lambda_{V,-}^{p^i}}{2}+\frac{\lambda_{V,+}^{p^i}-\lambda_{V,-}^{p^i}}{2\sqrt{\Delta_V}}\times\frac{p^r}{2}\right)\right)y}.$$

Exploiting our eigenvalue estimates in Equations (8) and (9) appropriately, it follows that $\chi\left(\gamma^{-p^i}(h_1^x h_2^y)\gamma^{p^i}\right)$ equals $\chi(h_1^x h_2^y)=\zeta_{p^n}^{\mathbf{e}_1x+\mathbf{e}_2y}$ for all $x,y\in\mathbb{Z}$, if and only if

$$\mathbf{e}_2 d\times p^{s+i}\equiv 0\ (\bmod\ p^n)\quad\text{and}\quad \mathbf{e}_1\times p^{s+i}+\mathbf{e}_2\times p^{s+i+r}\equiv 0\ (\bmod\ p^n);$$

the latter holds precisely when $s+i\ \geq\ n-\mathrm{ord}_p(\mathbf{e}_2 d)$ and $s+i\ \geq n-\mathrm{ord}_p(\mathbf{e}_1+\mathbf{e}_2p^r)$.

Case (VI). Here $I_2+M=\begin{pmatrix}1 & p^s\\ p^{s+r}t & 1\end{pmatrix}$; let $\lambda_{VI,\pm}:=1\pm p^s\sqrt{p^rt}$ be its eigenvalues (note that $t=1$ in (a) of the Classification Theorem, and $t\in\mathbb{Z}_p^\times$ is not a square in (b)). Then

$$(I_2+M)^{p^i}=P_{VI}D_{VI}^{p^i}P_{VI}^{-1}\quad\text{with}\quad D_{VI}=\begin{pmatrix}\lambda_{VI,+} & 0\\ 0 & \lambda_{VI,-}\end{pmatrix}\text{ and } P_{VI}=\begin{pmatrix}1 & 1\\ \sqrt{p^rt} & -\sqrt{p^rt}\end{pmatrix}.$$

A straightforward calculation shows

$$\gamma^{-p^i}(h_1^xh_2^y)\gamma^{p^i} = h_1^{\left(\frac{\lambda_{VI,+}^{p^i}+\lambda_{VI,-}^{p^i}}{2}\right)x+\left(\frac{\lambda_{VI,+}^{p^i}-\lambda_{VI,-}^{p^i}}{2\sqrt{p^rt}}\right)y}\times h_2^{\sqrt{p^rt}\left(\frac{\lambda_{VI,+}^{p^i}-\lambda_{VI,-}^{p^i}}{2}\right)x+\left(\frac{\lambda_{VI,+}^{p^i}+\lambda_{VI,-}^{p^i}}{2}\right)y} \tag{10}$$

and clearly $\lambda_{V,\pm}^p = 1\pm p^{s+1}\sqrt{p^rt}+p^{2s+1}\left(\frac{p-1}{2}\right)p^rt+\ldots = 1\pm p^{s+1}\sqrt{p^rt}+O\big(p^{2s+r+1}\big)$. Using a now familiar mathematical induction,

$$\frac{\lambda_{VI,+}^{p^i}+\lambda_{VI,-}^{p^i}}{2} = 1+O\big(p^{2s+r+i}\big)\quad\text{and}\quad \frac{\lambda_{VI,+}^{p^i}-\lambda_{VI,-}^{p^i}}{2\sqrt{p^rt}} = p^{s+i}+O\big(p^{2s+r/2+i}\big). \tag{11}$$

If the character $\chi=\chi_{1,n}^{\mathbf{e}_1}\times\chi_{2,n}^{\mathbf{e}_2}$, by Equation (10) the value $\chi\left(\gamma^{-p^i}(h_1^xh_2^y)\gamma^{p^i}\right)$ equals

$$\zeta_{p^n}^{\left(\mathbf{e}_1\left(\frac{\lambda_{VI,+}^{p^i}+\lambda_{VI,-}^{p^i}}{2}\right)+\mathbf{e}_2\sqrt{p^rt}\left(\frac{\lambda_{VI,+}^{p^i}-\lambda_{VI,-}^{p^i}}{2}\right)\right)x+\left(\mathbf{e}_1\left(\frac{\lambda_{VI,+}^{p^i}-\lambda_{VI,-}^{p^i}}{2\sqrt{p^rt}}\right)+\mathbf{e}_2\left(\frac{\lambda_{VI,+}^{p^i}+\lambda_{VI,-}^{p^i}}{2}\right)\right)y}.$$

Plugging Equation (11) into the above, one can then deduce $\chi\left(\gamma^{-p^i}(h_1^xh_2^y)\gamma^{p^i}\right)=\chi\left(h_1^xh_2^y\right)$ for all $x,y\in\mathbb{Z}$, if and only if both

$$\mathbf{e}_2\times p^{s+i}\times\left(\sqrt{p^rt}\right)^2\equiv 0\ (\bmod\ p^n)\quad\text{and}\quad \mathbf{e}_1\times p^{s+i}\equiv 0\ (\bmod\ p^n),$$

which is itself equivalent to ensuring that

$$s+i\ \geq\ n-\mathrm{ord}_p(\mathbf{e}_2p^rt)=n-r-\mathrm{ord}_p(\mathbf{e}_2)\quad\text{and}\quad s+i\ \geq\ n-\mathrm{ord}_p(\mathbf{e}_1).$$

□

2.2 A "coarse but clean" system of subgroups

The theory in [7, 16, 17, 23] operates best in the setting of one-dimensional Lie groups. Throughout we choose an integer n, and work with the p-adic group $\mathcal{G}_{\infty,n} := \Gamma \ltimes \left(\frac{\mathcal{H}_\infty}{\mathcal{H}_\infty^{p^n}}\right)$. In later sections we will allow n to vary, but for the time being n is fixed.

Lemma 6. *If $\mathcal{Z}(G)$ denotes the centre of a group G, then*

$$\mathcal{Z}(\mathcal{G}_{\infty,n}) \;=\; \begin{cases} \Gamma^{p^{n-s}} \times \frac{\mathcal{H}_{1,\infty} \times \mathcal{H}_{2,\infty}^{p^{n-s}}}{\mathcal{H}_\infty^{p^n}} & \text{in Case (II)} \\ \Gamma^{p^{n-s}} \times \frac{\mathcal{H}_\infty^{p^{n-s}}}{\mathcal{H}_\infty^{p^n}} & \text{in Cases (III) and (IV)} \\ \Gamma^{p^{n-s}} \times \frac{\mathcal{H}_{1,\infty}^{p^{n-s-\mathrm{ord}_p(d)}} \times \mathcal{H}_{2,\infty}^{p^{n-s}}}{\mathcal{H}_\infty^{p^n}} & \text{in Case (V)} \\ \Gamma^{p^{n-s}} \times \frac{\mathcal{H}_{1,\infty}^{p^{n-s-r-\mathrm{ord}_p(t)}} \times \mathcal{H}_{2,\infty}^{p^{n-s}}}{\mathcal{H}_\infty^{p^n}} & \text{in Case (VI).} \end{cases}$$

In particular, $\mathcal{Z}(\mathcal{G}_\infty) \cong \varprojlim_n \mathcal{Z}(\mathcal{G}_{\infty,n}) = \begin{cases} \mathcal{H}_{1,\infty} & \text{in Case (II)} \\ \{1\} & \text{otherwise.} \end{cases}$

Proof. We first note from the semi-direct product structure on $\mathcal{G}_{\infty,n}$ that

$$\mathcal{Z}(\mathcal{G}_{\infty,n}) \;=\; \mathrm{Stab}_\Gamma\left(\frac{\mathcal{H}_\infty}{\mathcal{H}_\infty^{p^n}}\right) \times \frac{\left\{ h_1^x h_2^y \;\middle|\; (I_2+M)\begin{pmatrix} x \\ y \end{pmatrix} \equiv \begin{pmatrix} x \\ y \end{pmatrix} \bmod p^n\mathbb{Z}_p^2 \right\}}{\mathcal{H}_\infty^{p^n}}.$$

One then computes the right-hand side on a case-by-case basis, using the form of the matrix M listed in Equation (3) (see [5] for the full details of each calculation). $\square$

Bearing in mind Kakde's subgroups should always contain the centre of $\mathcal{G}_{\infty,n}$, we define

$$\mathcal{U}_{m,n} \;:=\; \Gamma^{p^m} \ltimes \left(\frac{\mathcal{H}_\infty}{\mathcal{H}_\infty^{p^n}}\right) \quad \text{where the integer } m \in \{0,\dots,n-s\},$$

so: (i) $\mathcal{Z}(\mathcal{G}_{\infty,n}) \subset \mathcal{U}_{m,n}$, and (ii) $\Gamma^{p^{n-s}} \subset \mathrm{Stab}_\Gamma(\chi)$ for any $\chi : \mathcal{H}_\infty \twoheadrightarrow \mu_{p^m}$ by Proposition 5. It follows that such χ extend to $\mathcal{U}_{m,n}$ if $m \in \{\mathbf{m}_\chi,\dots,n-s\}$, and will thus factor through

$$\mathcal{U}_{m,n}^{\mathrm{ab}} \;=\; \frac{\mathcal{U}_{m,n}}{[\mathcal{U}_{m,n},\mathcal{U}_{m,n}]} \;=\; \frac{\Gamma^{p^m} \ltimes \mathcal{H}_\infty/\mathcal{H}_\infty^{p^n}}{\left\langle \left[h_1^x h_2^y \bmod \mathcal{H}_\infty^{p^n},\, \gamma^{p^m}\right] \;\middle|\; x,y \in \mathbb{Z} \right\rangle}.$$

Therefore, by determining the nature of $\mathcal{U}_{m,n}^{\mathrm{ab}}$ in each case, we may calculate the number of irreducible representations $\psi \otimes \mathrm{Ind}_{\mathcal{U}_{m,n}}^{\mathcal{G}_{\infty,n}}(\chi)$ with $\psi : \Gamma \to \mathbb{C}^\times$ of finite order. (Remember that every irreducible Artin representation ρ on $\mathcal{G}_\infty$ is of this form for suitable m,n,χ,ψ.)

Proposition 7. *For each pair $m,n \in \mathbb{Z}$ with $0 \le m \le n-s$,*

$$\mathcal{U}_{m,n}^{\mathrm{ab}} \;\cong\; \begin{cases} \Gamma^{p^m} \times \frac{\mathcal{H}_{1,\infty}}{\mathcal{H}_{1,\infty}^{p^{s+m}}} \times \frac{\mathcal{H}_{2,\infty}}{\mathcal{H}_{2,\infty}^{p^n}} & \text{in Case (II)} \\ \mathcal{U}_{m,s+m} & \text{in Cases (III) and (IV)} \\ \Gamma^{p^m} \times \frac{\mathbb{Z}}{p^{\min\{n,s+m+\mathrm{ord}_p(d)\}}\mathbb{Z}} \times \frac{\mathbb{Z}}{p^{s+m}\mathbb{Z}} & \text{in Case (V)} \\ \Gamma^{p^m} \times \frac{\mathbb{Z}}{p^{\min\{n,s+m+r+\mathrm{ord}_p(t)\}}\mathbb{Z}} \times \frac{\mathbb{Z}}{p^{s+m}\mathbb{Z}} & \text{in Case (VI);} \end{cases}$$

in fact, the first two lines are actual equalities, not just isomorphisms.

Proof. We proceed by working through the different cases (II)–(VI) in numerical order.

Case (II). Here one simply exploits the commutator relation $\left[h_1^x h_2^y, \gamma^{p^m}\right] = (h_1^y)^{p^{s+m}}$.

Case (III). Here we use $\left[h_1^x h_2^y, \gamma^{p^m}\right] = (h_1^x h_2^y)^{(1+p^s)^{p^m}-1}$ and $\mathrm{ord}_p\big((1+p^s)^{p^m}-1\big) = s+m$.

Case (IV). Recall from Equation 4 that

$$\begin{aligned}\gamma^{-p^m}(h_1^x h_2^y)\gamma^{p^m} \;&=\; h_1^{\left(\frac{\lambda_{IV,+}^{p^m}+\lambda_{IV,-}^{p^m}}{2}\right)x+\left(\frac{\lambda_{IV,+}^{p^m}-\lambda_{IV,-}^{p^m}}{2\sqrt{d}}\right)y} \times h_2^{\left(\frac{\lambda_{IV,+}^{p^m}-\lambda_{IV,-}^{p^m}}{2}\right)\sqrt{d}\,x+\left(\frac{\lambda_{IV,+}^{p^m}+\lambda_{IV,-}^{p^m}}{2}\right)y}\\ &=\; \left(h_1^{p^{s+m}+\dots} \times h_2^{p^{s+r+m}d+\dots}\right)^x \times \left(h_1^{p^{s+r+m}+\dots} \times h_2^{p^{s+m}+\dots}\right)^y \times h_1^x h_2^y\end{aligned}$$

upon using the estimates in (5) and (6); consequently

$$\frac{\mathcal{H}_\infty}{\left\langle [h_1, \gamma^{p^m}], [h_2, \gamma^{p^m}]\right\rangle} \cong \frac{\mathbb{Z}_p \,\oplus\, \mathbb{Z}_p}{\mathbb{Z}_p \cdot \left\{(p^{s+m} + \dots\, , p^{s+r+m}d + \dots), (p^{s+r+m} + \dots\, , p^{s+m} + \dots)\right\}}$$

which means $\mathcal{U}_{m,n}^{\mathrm{ab}} = \frac{\mathcal{U}_{m,n}}{\left\langle [h_1,\gamma^{p^m}],[h_2,\gamma^{p^m}]\right\rangle} \cong \Gamma^{p^m} \times \frac{\mathbb{Z}_p}{p^{s+m}\mathbb{Z}_p} \times \frac{\mathbb{Z}_p}{p^{s+m}\mathbb{Z}_p}$.

Case (V). This time Equation (7) combined with the estimates (8) and (9) yields

$$\begin{aligned}\gamma^{-p^m}(h_1^x h_2^y)\gamma^{p^m} \;&=\; h_1^{\left(\frac{\lambda_{V,+}^{p^m}+\lambda_{V,-}^{p^m}}{2}-\frac{\lambda_{V,+}^{p^m}-\lambda_{V,-}^{p^m}}{2\sqrt{\Delta_V}}\times\frac{p^r}{2}\right)x+\left(\frac{\lambda_{V,+}^{p^m}-\lambda_{V,-}^{p^m}}{2\sqrt{\Delta_V}}\right)y}\\ &\qquad \times h_2^{\left(\frac{\lambda_{V,+}^{p^m}-\lambda_{V,-}^{p^m}}{2\sqrt{\Delta_V}}\right)dx+\left(\frac{\lambda_{V,+}^{p^m}+\lambda_{V,-}^{p^m}}{2}+\frac{\lambda_{V,+}^{p^m}-\lambda_{V,-}^{p^m}}{2\sqrt{\Delta_V}}\times\frac{p^r}{2}\right)y}\\ &= \left(h_1^{\frac{p^{s+r+m}}{2}-\frac{p^{s+r+m}}{2}+\dots} \times h_2^{p^{s+m}d+\dots}\right)^x \times \left(h_1^{p^{s+m}+\dots} \times h_2^{\frac{p^{s+r+m}}{2}+\frac{p^{s+r+m}}{2}+\dots}\right)^y \times h_1^x h_2^y\end{aligned}$$

so that $\mathcal{U}_{m,n}^{\mathrm{ab}} = \frac{\mathcal{U}_{m,n}}{\left\langle [h_1,\gamma^{p^m}],[h_2,\gamma^{p^m}]\right\rangle} \cong \Gamma^{p^m} \times \frac{\mathbb{Z}_p}{p^n\mathbb{Z}_p \cup\, p^{s+m}d\mathbb{Z}_p} \times \frac{\mathbb{Z}_p}{p^{s+m}\mathbb{Z}_p}$.

Case (VI). Lastly, Equation (10) in tandem with the estimates in (11) implies

$$\begin{aligned}\gamma^{-p^m}(h_1^x h_2^y)\gamma^{p^m} &= h_1^{\left(\frac{\lambda_{VI,+}^{p^m}+\lambda_{VI,-}^{p^m}}{2}\right)x+\left(\frac{\lambda_{VI,+}^{p^m}-\lambda_{VI,-}^{p^m}}{2\sqrt{p^r t}}\right)y} \times h_2^{\sqrt{p^r t}\left(\frac{\lambda_{VI,+}^{p^m}-\lambda_{VI,-}^{p^m}}{2}\right)x+\left(\frac{\lambda_{VI,+}^{p^m}+\lambda_{VI,-}^{p^m}}{2}\right)y}\\ &= \left(h_1^{0+\dots} \times h_2^{p^{s+m+r}t+\dots}\right)^x \times \left(h_1^{p^{s+m}+\dots} \times h_2^{0+\dots}\right)^y \times h_1^x h_2^y\,,\end{aligned}$$

hence $\mathcal{U}_{m,n}^{\mathrm{ab}} = \frac{\mathcal{U}_{m,n}}{\left\langle [h_1,\gamma^{p^m}],[h_2,\gamma^{p^m}]\right\rangle} \cong \Gamma^{p^m} \times \frac{\mathbb{Z}_p}{p^n\mathbb{Z}_p \cup\, p^{s+m+r}t\mathbb{Z}_p} \times \frac{\mathbb{Z}_p}{p^{s+m}\mathbb{Z}_p}$. $\square$

We remark in Cases (II-VI), each $\mathcal{U}_{m,n}^{\mathrm{ab}}$ has the form $\Gamma^{p^m} \times \overline{\mathcal{H}}_\infty^{(m,n)}$ where $\overline{\mathcal{H}}_\infty^{(m,n)}$ is obtained from quotienting $\mathcal{H}_\infty/\mathcal{H}_\infty^{p^n} = \langle \overline{h}_1, \overline{h}_2\rangle$ with the subgroup generated by $\left\{[\overline{h}_1, \gamma^{p^m}], [\overline{h}_2, \gamma^{p^m}]\right\}$.

Definition 8. *Let "*$\mathrm{orb}_\Gamma\big(\overline{\mathcal{H}}_\infty^{(m,n)}\big)$*" denote the orbits under the action of* Γ/Γ^{p^m} *in* $\overline{\mathcal{H}}_\infty^{(m,n)}$. *In particular, if* $\overline{h} \in \overline{\mathcal{H}}_\infty^{(m,n)}$ *then* $\varpi_{\overline{h}} \in \mathrm{orb}_\Gamma\big(\overline{\mathcal{H}}_\infty^{(m,n)}\big)$ *consists of the set* $\left\{\gamma^{-i}\overline{h}\gamma^i \;\middle|\; i \in \mathbb{Z}\right\}$; *we shall sometimes abuse notation, and write* $\overline{h}$ *in place of* $\varpi_{\overline{h}}$.

2.3 Maps between the abelianizations of $\mathcal{U}_{m,n}$

We now outline the various mappings that appear in the description of Ψ and Φ in [7, 17]. Rather than give their full definitions, we specialise them to the specific three-dimensional situation we are considering.

The conditions (A1)-(A3) and (M1)-(M4) in the exposition [7, p79-123] degenerate into some fairly simple rules, which can be expressed in terms of an explicit basis for the image of Kakde's map "$\sigma_U^{N(U)}$". In subsequent sections we will then study how these expressions transform, once the completed group algebras $\Lambda(\mathcal{U}_{m,n}^{\mathrm{ab}})$ are evaluated at a system of characters χ on $\mathcal{H}_\infty$.

The mapping σ_m: Note that the normaliser of each subgroup $U = \mathcal{U}_{m,n} \subset \mathcal{G}_{\infty,n}$ is the whole of $\mathcal{G}_{\infty,n}$, so the $\mathbb{Z}_p$-linear map labelled $\sigma_U^{N(U)}$ in [7, p85] becomes

$$\sigma_{\mathcal{U}_{m,n}}^{\mathcal{G}_{\infty,n}} : \Lambda(\mathcal{U}_{m,n}^{\mathrm{ab}}) \longrightarrow \Lambda(\mathcal{U}_{m,n}^{\mathrm{ab}}) \quad \text{where } f \mapsto \sum_{i=0}^{p^m-1} \gamma^{-i} f \gamma^i.$$

If we use the shorthand σ_m for this linear mapping, clearly $\sigma_m(f) \in H^0\left(\Gamma, \Lambda(\mathcal{U}_{m,n}^{\mathrm{ab}})\right)$ corresponds to the sum over the orbits of f under the action of the finite group Γ/Γ^{p^m}.

Definition 9. *For any $\overline{h} = h_1^x h_2^y \mod \left[\mathcal{U}_{m,n}, \mathcal{U}_{m,n}\right]$, one defines $\mathcal{A}_{\overline{h}}^{(m,n)} \in \mathbb{Z}_p\left[\mathcal{U}_{m,n}^{\mathrm{ab}}\right]$ by*

$$\mathcal{A}_{\overline{h}}^{(m,n)} := \sum_{i=0}^{p^m-1} \overline{h}_1^{x_i} \overline{h}_2^{y_i} \quad \textit{where} \begin{pmatrix} x_i \\ y_i \end{pmatrix} \equiv (I_2 + M)^i \begin{pmatrix} x \\ y \end{pmatrix} \mod p^n.$$

In fact, we could alternatively have defined $\mathcal{A}_{\overline{h}}^{(m,n)}$ to be the summation $\sum_{i=0}^{p^m-1} \gamma^{-i}\overline{h}\gamma^i$ which coincides, of course, with $\sigma_m(\overline{h})$; we will see that these form a basis for $\mathrm{Im}(\sigma_m)$.

Proposition 10. *(i) Each element $\mathcal{A}_{\overline{h}}^{(m,n)}$ depends only on the Γ-orbit of $\overline{h}$ inside $\overline{\mathcal{H}}_\infty^{(m,n)}$;*

(ii) The image of σ_m is freely generated over $\mathbb{Z}_p[[\Gamma^{p^m}]]$ by the $\mathcal{A}_{\overline{h}}^{(m,n)}$'s, in other words

$$\mathrm{Im}(\sigma_m) \;\cong\; \mathbb{Z}_p[[\Gamma^{p^m}]] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p\left\{\mathcal{A}_{\overline{h}}^{(m,n)} \;\middle|\; \overline{h} = h_1^x h_2^y \mod \left[\mathcal{U}_{m,n}, \mathcal{U}_{m,n}\right]\right\};$$

(iii) If $r_{\sigma_m}^{(n)} := \mathrm{rank}_{\mathbb{Z}_p[\![\Gamma^{p^m}]\!]}\big(\mathrm{Im}(\sigma_m)\big)$, then

$$r_{\sigma_m}^{(n)} = \begin{cases} p^{n+s-1} \times (mp + p - m) & \textit{in Case (II)} \\ p^{2s-1} \times (p^{m+1} + p^m - 1) & \textit{in Cases (III) and (IV)} \\ p^{\min\{n-m, s+\mathrm{ord}_p(d)\}+s-1} \times (p^{m+1} + p^m - 1) & \textit{in Case (V)} \\ p^{\min\{n-m, s+r+\mathrm{ord}_p(t)\}+s-1} \times (p^{m+1} + p^m - 1) & \textit{in Case (VI).} \end{cases}$$

Proof. Statement (i) is self-evident. To establish (ii), first note that $\mathcal{U}_{m,n}^{\mathrm{ab}} = \Gamma^{p^m} \times \overline{\mathcal{H}}_\infty^{(m,n)}$ where $\overline{\mathcal{H}}_\infty^{(m,n)}$ is the previous quotient of $\mathcal{H}_\infty$ equipped with the action of the group Γ/Γ^{p^m}; part (ii) now follows because $\overline{\mathcal{H}}_\infty^{(m,n)}$ is generated by $h_1^x h_2^y \mod \left[\mathcal{U}_{m,n}, \mathcal{U}_{m,n}\right]$ for $x, y \in \mathbb{Z}$.

To prove (iii) we just need to count the number of distinct $\mathcal{A}_{\overline{h}}^{(m,n)}$'s, which coincides with the total number of (Γ/Γ^{p^m})-orbits inside $\overline{\mathcal{H}}_\infty^{(m,n)}$. In fact by Burnside's lemma,

$$\#\{\Gamma\text{-orbits in } \overline{\mathcal{H}}_\infty^{(m,n)}\} \;=\; \#\big(\Gamma/\Gamma^{p^m}\big)^{-1} \times \sum_{j=1}^{p^m} \#\left\{\overline{h} \in \overline{\mathcal{H}}_\infty^{(m,n)} \;\Big|\; \gamma^{-j}\overline{h}\gamma^j = \overline{h}\right\}.$$

From Proposition 7, in each case $\star \in \{\mathrm{II,III,IV,V,VI}\}$ one knows $\overline{\mathcal{H}}_\infty^{(m,n)} \cong \frac{\mathbb{Z}}{p^{N_{\star,1}^{(m)}}\mathbb{Z}} \times \frac{\mathbb{Z}}{p^{N_{\star,2}^{(m)}}\mathbb{Z}}$ where $N_{\star,1}^{(m)}, N_{\star,2}^{(m)} \in \mathbb{N}$ satisfy $m+s \le N_{\star,1}^{(m)} \le n$ and $m+s \le N_{\star,2}^{(m)} \le n$ in all five scenarios.

- Assuming that $\star \ne \mathrm{II}$, one discovers

$$\begin{aligned}\#\{\Gamma\text{-orbits in } \overline{\mathcal{H}}_\infty^{(m,n)}\} \;&=\; p^{-m} \times \sum_{j=1}^{p^m} p^{N_{\star,1}^{(m)}+\mathrm{ord}_p(j)-m} \times p^{N_{\star,2}^{(m)}+\mathrm{ord}_p(j)-m} \\ &=\; p^{\left(N_{\star,1}^{(m)}-m\right)+\left(N_{\star,2}^{(m)}-m\right)-1} \times \left(p^{m+1}+p^m-1\right).\end{aligned}$$

- Alternatively, if $\star = \mathrm{II}$ then γ acts trivially on the first direct factor in $\overline{\mathcal{H}}_\infty^{(m,n)}$, whence

$$\begin{aligned}\#\{\Gamma\text{-orbits in } \overline{\mathcal{H}}_\infty^{(m,n)}\} \;&=\; p^{-m} \times \sum_{j=1}^{p^m} p^{N_{II,1}^{(m)}} \times p^{N_{II,2}^{(m)}+\mathrm{ord}_p(j)-m} \\ &=\; p^{\left(N_{II,1}^{(m)}-m\right)+\left(N_{II,2}^{(m)}-m\right)-1} \times \left((m+1)p^{m+1}-mp^m\right).\end{aligned}$$

The result follows upon plugging in values of $N_{\star,1}^{(m)}$ and $N_{\star,2}^{(m)}$ listed in Proposition 7. $\square$

Corollary 11. *The number of irreducible representations of the form* $\mathrm{Ind}_{\mathrm{Stab}_\Gamma(\chi)\ltimes\mathcal{H}_\infty/p^n}^{\mathcal{G}_{\infty,n}}(\chi)$ *where* χ *factors through* $\overline{\mathcal{H}}_\infty^{(m,n)}$ *but not through* $\overline{\mathcal{H}}_\infty^{(m-1,n)}$ *is given by* $r_{\sigma_m}^{(n)} - r_{\sigma_{m-1}}^{(n)}$.

Proof. Note that any two characters χ, χ' as above induce the same $\mathcal{G}_{\infty,n}$-representation, if and only if χ' belongs to the Γ-orbit of χ inside $\mathrm{Hom}\big(\overline{\mathcal{H}}_\infty^{(m,n)}, \mathbb{C}^\times\big)$; since the latter group is (non-canonically) isomorphic to $\overline{\mathcal{H}}_\infty^{(m,n)}$, its Γ-orbits are in one-to-one correspondence with the finite set $\mathrm{orb}_\Gamma\big(\overline{\mathcal{H}}_\infty^{(m,n)}\big)$. It follows immediately that

$$\text{"the no. of } \mathrm{Ind}(\chi)\text{'s primitive on } \overline{\mathcal{H}}_\infty^{(m,n)}\text{"} \;=\; \#\mathrm{orb}_\Gamma\big(\overline{\mathcal{H}}_\infty^{(m,n)}\big) - \#\mathrm{orb}_\Gamma\big(\overline{\mathcal{H}}_\infty^{(m-1,n)}\big),$$

which equals $r_{\sigma_m}^{(n)} - r_{\sigma_{m-1}}^{(n)}$ because $\mathrm{Im}(\sigma_m) = \mathbb{Z}_p[\![\Gamma^{p^m}]\!]\cdot\{\mathcal{A}_{\overline{h}}^{(m,n)} \mid \varpi_{\overline{h}} \in \mathrm{orb}_\Gamma\big(\overline{\mathcal{H}}_\infty^{(m,n)}\big)\}$. $\square$

The transfer map $\mathrm{Ver}_{m,m'}$. Consider the subgroups $\mathcal{U}_{m,n} \subset \mathcal{U}_{m',n}$ of $\mathcal{G}_{\infty,n}$ with $m > m'$. The transfer homomorphism (Verlagerung) $\mathrm{Ver}_{\mathcal{U}_{m,n}}^{\mathcal{U}_{m',n}}$ relative to these subgroups maps $\mathcal{U}_{m',n}^{\mathrm{ab}} \longrightarrow \mathcal{U}_{m,n}^{\mathrm{ab}}$ by sending

$$g\left[\mathcal{U}_{m',n}, \mathcal{U}_{m',n}\right] \;\mapsto\; \prod_{\tau\in\mathcal{R}} c_{g,\tau}\left[\mathcal{U}_{m,n}, \mathcal{U}_{m,n}\right]$$

where $\mathcal{R}$ is a fixed set of left coset representatives for $\mathcal{U}_{m',n}/\mathcal{U}_{m,n}$, and $g\tau = r_g c_{g,\tau}$ with $c_{g,\tau} \in \mathcal{U}_{m,n}$ and $r_g \in \mathcal{R}$.

Henceforth one writes $\mathrm{Ver}_{m',m} : \Lambda\big(\mathcal{U}_{m',n}^{\mathrm{ab}}\big) \to \Lambda\big(\mathcal{U}_{m,n}^{\mathrm{ab}}\big)$ for the $\mathbb{Z}_p$-linear and continuous extension of the transfer map to the completed group algebras.

Lemma 12. *Suppose $g \in \mathcal{U}_{m',n}^{\mathrm{ab}}$, and let $\hat{g} = (\gamma^{p^{m'}})^j \cdot (h_1^x h_2^y) \in \Gamma^{p^{m'}} \ltimes \mathcal{H}_\infty$ be any lift. Then*

$$\mathrm{Ver}_{m',m}(g) \ \equiv \ (\gamma^{p^m})^j \cdot h_1^{x'} h_2^{y'} \mod \big[\mathcal{U}_{m,n}, \mathcal{U}_{m,n}\big]$$

where $(x', y') = \Big(p^{m-m'}x,\ p^{m-m'}y\Big)$ in Case (II), and in the same notation as the proof of Proposition 5:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} \ = \ P_\star \begin{pmatrix} \frac{\lambda_{\star,+}^{p^m}-1}{\lambda_{\star,+}^{p^{m'}}-1} & 0 \\ 0 & \frac{\lambda_{\star,-}^{p^m}-1}{\lambda_{\star,-}^{p^{m'}}-1} \end{pmatrix} P_\star^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \quad \textit{in Case } (\star), \quad \textit{with } \star \in \{III, IV, V, VI\}.$$

Proof. Since $\mathcal{U}_{m',n}/\mathcal{U}_{m,n} \cong \Gamma^{p^{m'}}/\Gamma^{p^m}$, its coset representatives are $\big\{r_0, r_1, \ldots, r_{p^{m-m'}-1}\big\}$ where $r_i = \gamma^{p^{m'}i}$. One can represent $\hat{g}$ in the form $\gamma^{p^{m'}j} \cdot (h_1^x h_2^y)$ for some choice of $j \in \mathbb{Z}_p$, in which case

$$\hat{g}\, r_i \ = \ \gamma^{p^{m'}j}(h_1^x h_2^y)\gamma^{p^{m'}i} \ = \ \gamma^{p^{m'}(j+i)}\big(\gamma^{-p^{m'}i}(h_1^x h_2^y)\gamma^{p^{m'}i}\big) \ = \ \gamma^{p^{m'}(j+i)} \cdot \Big(h_1^{x_{p^{m'}i}} h_2^{y_{p^{m'}i}}\Big)$$

where $\begin{pmatrix} x_{p^{m'}i} \\ y_{p^{m'}i} \end{pmatrix} = \big(I_2 + M\big)^{p^{m'}i} \begin{pmatrix} x \\ y \end{pmatrix}$. In fact, if $\iota : \mathbb{Z}_p \to \{0, 1, \ldots, p^{m-m'} - 1\}$ so that $\iota(z) \equiv z \bmod p^{m-m'}$, then $\gamma^{p^{m'}(j+i)} = r_{\iota(j+i)} \cdot \gamma^{p^{m'}(j+i-\iota(j+i))}$; consequently

$$\hat{g}\, r_i \ = \ r_{\iota(j+i)} \ \Big(\gamma^{p^{m'}(j+i-\iota(j+i))} \cdot \Big(h_1^{x_{p^{m'}i}} h_2^{y_{p^{m'}i}}\Big)\Big).$$

By definition, the transfer is congruent to

$$\mathrm{Ver}_{m',m}(g) \ \equiv \ \prod_{i=0}^{p^{m-m'}-1} \gamma^{p^{m'}(j+i-\iota(j+i))} \cdot h_1^{x_{p^{m'}i}} h_2^{y_{p^{m'}i}} \mod \big[\mathcal{U}_{m,n}, \mathcal{U}_{m,n}\big]$$

and as $j + i \equiv \iota(j+i) \bmod p^{m-m'}$ clearly $\gamma^{p^{m'}(j+i-\iota(j+i))} \in \Gamma^{p^m}$, hence $\gamma^{p^{m'}(j+i-\iota(j+i))}$ and $h_1^{x_i} h_2^{y_i}$ commute modulo $\big[\mathcal{U}_{m,n}, \mathcal{U}_{m,n}\big]$. It follows that

$$\mathrm{Ver}_{m',m}(g) \ \equiv \ \gamma^{p^{m'}c} \cdot h_1^{x'} h_2^{y'} \mod \big[\mathcal{U}_{m,n}, \mathcal{U}_{m,n}\big]$$

where $c = \sum_{i=0}^{p^{m-m'}-1} j + i - \iota(j+i)$, and the vector

$$\begin{pmatrix} x' \\ y' \end{pmatrix} \ = \ \begin{pmatrix} \sum x_{p^{m'}i} \\ \sum y_{p^{m'}i} \end{pmatrix} \ = \ \sum_{i=0}^{p^{m-m'}-1} \big(I_2 + M\big)^{p^{m'}i} \begin{pmatrix} x \\ y \end{pmatrix}.$$

To calculate the term c, without loss of generality assume $j \in \mathbb{Z}$, which implies

$$c \ = \ \sum_{i=0}^{p^{m-m'}-1} j + i - \iota(j+i) \ = \ p^{m-m'} \times \sum_{i=0}^{p^{m-m'}-1} \Big\lfloor \frac{j+i}{p^{m-m'}} \Big\rfloor.$$

The right-hand sum then yields

$$\sum_{i=0}^{p^{m-m'}-1} \left\lfloor \frac{j+i}{p^{m-m'}} \right\rfloor \;=\; p^{m-m'} \left\lfloor \frac{j}{p^{m-m'}} \right\rfloor + \sum_{i=0}^{p^{m-m'}-1} \left\lfloor \frac{\iota(j)+i}{p^{m-m'}} \right\rfloor$$

$$=\; p^{m-m'} \left\lfloor \frac{j}{p^{m-m'}} \right\rfloor + \sum_{i=0}^{p^{m-m'}-\iota(j)-1} 0 \;+ \sum_{i=p^{m-m'}-\iota(j)}^{p^{m-m'}-1} 1 \;=\; p^{m-m'} \left\lfloor \frac{j}{p^{m-m'}} \right\rfloor + \iota(j) \;=\; j$$

and as an immediate consequence, $c = p^{m-m'} \times j$ so that $\gamma^{p^{m'}c} = \gamma^{p^m j}$ as required.

To compute x' and y', in Case (II) we find that

$$\sum_{i=0}^{p^{m-m'}-1} \big(I_2 + M\big)^{p^{m'} i} = \sum_{i=0}^{p^{m-m'}-1} \begin{pmatrix} 1 & p^s \times i p^{m'} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p^{m-m'} & p^{s+m} \times \frac{p^{m-m'}-1}{2} \\ 0 & p^{m-m'} \end{pmatrix}.$$

In all other cases $\star \in \{\text{III,IV,V,VI}\}$ one has $\big(I_2+M\big)^{p^{m'} i} = P_\star \begin{pmatrix} \lambda_{\star,+}^{p^{m'} i} & 0 \\ 0 & \lambda_{\star,-}^{p^{m'} i} \end{pmatrix} P_\star^{-1}$, which means

$$\sum_{i=0}^{p^{m-m'}-1} \big(I_2+M\big)^{p^{m'} i} \begin{pmatrix} x \\ y \end{pmatrix} \;=\; P_\star \begin{pmatrix} \sum_{i=0}^{p^{m-m'}-1} \lambda_{\star,+}^{p^{m'} i} & 0 \\ 0 & \sum_{i=0}^{p^{m-m'}-1} \lambda_{\star,-}^{p^{m'} i} \end{pmatrix} P_\star^{-1}.$$

Note that $P_{III} = I_2$ because $I_2 + M$ is already diagonalised. The result follows upon summing up the relevant geometric progression, i.e. $\sum_{i=0}^{p^{m-m'}-1} \lambda_{\star,\pm}^{p^{m'} i}$ equals $\frac{\lambda_{\star,\pm}^{p^m}-1}{\lambda_{\star,\pm}^{p^{m'}}-1}$. $\square$

The shift $\pi_{m,m'}$. For integers $m > m'$, we now look for a reverse mapping to $\mathrm{Ver}_{m',m}$. The commutator $[h_1^x h_2^y, \gamma^{p^m}]$ corresponds to $\big((I_2+M)^{p^m} - I_2\big)\begin{pmatrix} x \\ y \end{pmatrix}$ as a vector in $\mathbb{Z}_p^2$; however $X^{p^m} - 1 = (X^{p^{m'}} - 1) \times \prod_{d=m'+1}^{m} \phi_{p^d}(X)$ where ϕ_{p^d} denotes the p^d-th cyclotomic polynomial, therefore

$$[h_1^x h_2^y, \gamma^{p^m}] \;=\; [h_1^{x''} h_2^{y''}, \gamma^{p^{m'}}] \quad \text{with} \quad \begin{pmatrix} x'' \\ y'' \end{pmatrix} = \prod_{d=m'+1}^{m} \phi_{p^d}\big(I_2+M\big) \begin{pmatrix} x \\ y \end{pmatrix}.$$

As a consequence, we have the containments $\big[\mathcal{U}_{m,n}, \mathcal{U}_{m,n}\big] \subset \big[\mathcal{U}_{m',n}, \mathcal{U}_{m',n}\big] \subset \mathcal{H}_\infty / \mathcal{H}_\infty^{p^n}$. The natural inclusion $\mathcal{U}_{m,n} \hookrightarrow \mathcal{U}_{m',n}$ then yields the composition

$$\pi_{m,m'} \;:\; \mathcal{U}_{m,n}^{\mathrm{ab}} \;=\; \frac{\mathcal{U}_{m,n}}{\big[\mathcal{U}_{m,n}, \mathcal{U}_{m,n}\big]} \;\hookrightarrow\; \frac{\mathcal{U}_{m',n}}{\big[\mathcal{U}_{m,n}, \mathcal{U}_{m,n}\big]} \;\overset{\text{proj}}{\twoheadrightarrow}\; \frac{\mathcal{U}_{m',n}}{\big[\mathcal{U}_{m',n}, \mathcal{U}_{m',n}\big]} \;=\; \mathcal{U}_{m',n}^{\mathrm{ab}}.$$

Moreover this shift homomorphism induces a map $(\pi_{m,m'})_* : \mathrm{orb}_\Gamma\big(\overline{\mathcal{H}}_\infty^{(m,n)}\big) \to \mathrm{orb}_\Gamma\big(\overline{\mathcal{H}}_\infty^{(m',n)}\big)$, sending each orbit $\varpi_{\overline{h}} = \big\{\gamma^{-i}\overline{h}\gamma^i \;\big|\; i \in \mathbb{Z}\big\}$ to the direct image $\varpi_{\pi_{m,m'}(\overline{h})}$.

Recall from Proposition 10(ii) that a typical element of $\mathrm{Im}(\sigma_m)$ has the form

$$\sum_{\varpi \in \mathrm{orb}_\Gamma(\overline{\mathcal{H}}_\infty^{(m,n)})} f_\varpi(\gamma^{p^m} - 1) \cdot \mathcal{A}_\varpi^{(m,n)} \;=\; \sum_{\varpi} f_\varpi \cdot \mathcal{A}_\varpi^{(m,n)} \;\text{ say,}$$

where each $f_\varpi(X) \in \mathbb{Z}_p[\![X]\!]$ and $\mathcal{A}_\varpi^{(m,n)} := \sum_{i=0}^{p^m-1} \gamma^{-i}\overline{h}\gamma^i$ for any $\overline{h} \in \varpi$.

Lemma 13. *If $m > m'$, then $\pi_{m,m'}\left(\sum_{\varpi} f_{\varpi}\cdot\mathcal{A}_{\varpi}^{(m,n)}\right) = p^{m-m'}\times\sum_{\varpi} f_{\varpi}\cdot\mathcal{A}^{(m',n)}_{(\pi_{m,m'})_*(\varpi)}$.*

Proof. If $\overline{h}\in\varpi$ with $\varpi\in\mathrm{orb}_{\Gamma}\big(\overline{\mathcal{H}}_{\infty}^{(m,n)}\big)$, then within the algebra $\Lambda\big(\mathcal{U}^{\mathrm{ab}}_{m',n}\big)$ one has

$$\begin{aligned}\pi_{m,m'}\left(f_{\varpi}\cdot\sum_{i=0}^{p^m-1}\gamma^{-i}\overline{h}\gamma^{i}\right) &= f_{\varpi}(\gamma^{p^m}-1)\cdot\pi_{m,m'}\left(\sum_{i_1=0}^{p^{m-m'}-1}\sum_{i_2=0}^{p^{m'}-1}\gamma^{-p^{m'}i_1-i_2}\overline{h}\gamma^{p^{m'}i_1+i_2}\right)\\ &= f_{\varpi}(\gamma^{p^m}-1)\cdot\sum_{i_1=0}^{p^{m-m'}-1}\sum_{i_2=0}^{p^{m'}-1}\gamma^{-i_2}\pi_{m,m'}(\overline{h})\gamma^{i_2}\end{aligned}$$

since $\gamma^{-p^{m'}}\pi_{m,m'}(\overline{h})\gamma^{p^{m'}}=\pi_{m,m'}(\overline{h})$ inside $\mathcal{U}^{\mathrm{ab}}_{m',n}$, which gives the result. $\square$

The norm and trace homomorphisms. We now introduce two final maps that occur in the definition of both of Kakde's groups Ψ and Φ. Firstly, if G is a group and $\mathrm{Conj}(G)$ denotes it set of conjugacy classes, then $\Lambda\big(\mathrm{Conj}(G)\big)\cong\Lambda(G)/\overline{[\Lambda(G),\Lambda(G)]}$ as an isomorphism of $\mathbb{Z}_p$-modules [7, §2]. For an integer pair m, m' with $m\geq m'$:

- the norm mapping $K_1\Big(\Lambda\big(\mathcal{U}^{\mathrm{ab}}_{m',n}\big)\Big)\longrightarrow K_1\Big(\Lambda\big(\mathcal{U}_{m,n}/[\mathcal{U}_{m',n},\mathcal{U}_{m',n}]\big)\Big)$ relative to the subgroup $\frac{\mathcal{U}_{m,n}}{[\mathcal{U}_{m',n},\mathcal{U}_{m',n}]}\subset\frac{\mathcal{U}_{m',n}}{[\mathcal{U}_{m',n},\mathcal{U}_{m',n}]}=\mathcal{U}^{\mathrm{ab}}_{m',n}$ is abbreviated by $\mathcal{N}_{m',m}$; and
- similarly, the additive trace map $\Lambda\Big(\mathrm{Conj}\big(\mathcal{U}^{\mathrm{ab}}_{m',n}\big)\Big)\longrightarrow\Lambda\Big(\mathrm{Conj}\big(\mathcal{U}_{m,n}/[\mathcal{U}_{m',n},\mathcal{U}_{m',n}]\big)\Big)$ relative to $\frac{\mathcal{U}_{m,n}}{[\mathcal{U}_{m',n},\mathcal{U}_{m',n}]}\subset\frac{\mathcal{U}_{m',n}}{[\mathcal{U}_{m',n},\mathcal{U}_{m',n}]}=\mathcal{U}^{\mathrm{ab}}_{m',n}$ is denoted by $\mathrm{Tr}_{m',m}$.

The following lemma describes the effect of the second of these maps on the image of $\sigma_{m'}$. Let $\mathrm{char}_{\Gamma^{p^m}}:\Lambda(\Gamma)\to\Lambda(\Gamma^{p^m})$ denote the $\mathbb{Z}_p$-linear and continuous extension of the map which sends $\gamma^i\mapsto\gamma^i$ if p^m divides i, and sends $\gamma^i\mapsto 0$ if p^m does not divide i.

Lemma 14. *For a typical element $\mathbf{a}_{m'}=\sum_{\varpi'}f_{\varpi'}(\gamma^{p^{m'}}-1)\cdot\mathcal{A}_{\varpi'}^{(m',n)}\in\mathrm{Im}\big(\sigma_{m'}\big)\big[1/p\big]$,*

$$\mathrm{Tr}_{m',m}\big(\mathbf{a}_{m'}\big) = p^{m-m'}\times\sum_{\varpi'}\mathrm{char}_{\Gamma^{p^m}}\left(f_{\varpi'}\big(\gamma^{p^{m'}}-1\big)\right)\cdot\mathcal{A}_{\varpi'}^{(m',n)} \in \Lambda\left(\frac{\mathcal{U}_{m,n}}{[\mathcal{U}_{m',n},\mathcal{U}_{m',n}]}\right)$$

where the sum is taken over all $\varpi'\in\mathrm{orb}_{\Gamma}\big(\overline{\mathcal{H}}_{\infty}^{(m',n)}\big)$.

Proof. From [7, Rk iii], one knows $\mathrm{Tr}_{m',m}\big(\gamma^{p^{m'}j}\overline{h}\big)=\begin{cases}p^{m-m'}\times\gamma^{p^{m'}j}\overline{h} & \text{if } \gamma^{p^{m'}j}\in\Gamma^{p^m}\\ 0 & \text{if } \gamma^{p^{m'}j}\notin\Gamma^{p^m},\end{cases}$

so that for any $\overline{h}\in\varpi'$:

$$\mathrm{Tr}_{m',m}\left(\gamma^{p^{m'}j}\cdot\mathcal{A}_{\varpi'}^{(m',n)}\right) = \begin{cases}p^{m-m'}\times\sum_{i=0}^{p^{m'}-1}\gamma^{p^{m'}j}\cdot\big(\gamma^{-i}\overline{h}\gamma^{i}\big) & \text{if } \gamma^{p^{m'}j}\in\Gamma^{p^m}\\ 0 & \text{otherwise.}\end{cases}$$

The stated formula then follows by linearity and continuity. $\square$

3 The additive calculations

We begin by recalling Kakde's definition of the subset $\Psi \subset \prod_m \mathbb{Q}_p\big[\big[\mathcal{U}^{\mathrm{ab}}_{m,n}\big]\big]$ given in [17]. For a fixed $n \geq s$, the $\mathbb{Z}_p$-module Ψ consists of sequences $\big(\mathbf{a}_m\big)$ satisfying the conditions:

(A1) $\mathrm{Tr}_{m',m}\big(\mathbf{a}_{m'}\big) = \pi_{m,m'}\big(\mathbf{a}_m\big)$ for any $m > m'$;

(A2) $\mathbf{a}_m = g\mathbf{a}_m g^{-1}$ at every $g \in \mathcal{G}_{\infty,n}$;

(A3) $\mathbf{a}_m \in \mathrm{Im}(\sigma_m)$ for each $m \in \{0, \ldots, n-s\}$.

In fact, the general definition of Ψ involves more than just this system of sub-quotients. However the "coarse but clean" choice of subgroups we made is sufficient for our purposes, as every irreducible representation of $\mathcal{G}_{\infty,n}$ is a finite twist of a representation obtained from inducing down a character χ on $\mathcal{U}_{m,n}$, for an appropriate choice of m and χ.

3.1 The image of Ψ under the characters on $\overline{\mathcal{H}}_\infty^{(m,n)}$

The main task is to see how Ψ transforms if we evaluate its constituent elements at a system of characters $\underline{\chi} = \{\chi\}$ on $\mathcal{H}_\infty/\mathcal{H}_\infty^{p^n}$. In particular, we want to translate the conditions (A1)–(A3) involving the $\mathbf{a}_m$'s into equivalent conditions involving $\mathbf{a}_\chi := \chi(\mathbf{a}_{\mathbf{m}_\chi})$ instead, and thereby complete the middle square in the diagram

$$\begin{tikzcd}
K_1'\big(\mathbb{Z}_p[\![\mathcal{G}_{\infty,n}]\!]\big) \arrow[r, "\Theta_{\infty,n}"] \arrow[dr, "\mathrm{Ev}_{\underline{\chi}}"'] & \Omega \arrow[r, "\text{``twisted log''}"] \arrow[d, "\underline{\chi}"] & \Psi \arrow[r, hook] \arrow[d, "\underline{\chi}"] & \mathbb{Q} \otimes \left(\prod_{0 \leq m \leq n-s} \mathbb{Z}_p\big[\big[\mathcal{U}^{\mathrm{ab}}_{m,n}\big]\big] \right) \arrow[d, "\underline{\chi}"] \\
 & \underline{\chi}(\Omega) \arrow[r, dashed, "??"] & \underline{\chi}(\Psi) \arrow[r, hook] & \mathbb{Q} \otimes \left(\prod_{m,\chi} \mathcal{O}_\chi\big[\big[\mathrm{Stab}_\Gamma(\chi)\big]\big] \right).
\end{tikzcd}$$

The following key result describes $\underline{\chi}(\Psi) \subset \prod_\chi \mathbb{C}_p\big[\big[\mathrm{Stab}_\Gamma(\chi)\big]\big]$ using p-adic congruences.

Theorem 15. *A collection of elements $\mathbf{a}_\chi \in \mathcal{O}_{\mathbb{C}_p}\big[\big[\mathrm{Stab}_\Gamma(\chi)\big]\big]$ arises from a sequence $(\mathbf{a}_m) \in \Psi \cap \prod_{0 \leq m \leq n-s} \mathbb{Z}_p\big[\big[\mathcal{U}^{\mathrm{ab}}_{m,n}\big]\big]$, if and only if for each $m \geq 0$ and $\varpi \in \mathrm{orb}_\Gamma\big(\overline{\mathcal{H}}_\infty^{(m,n)}\big)$:*

(C1) *the compatibility $\chi(\mathbf{a}_m) = \mathrm{Tr}_{\mathrm{Stab}_\Gamma(\chi)/\Gamma^{p^m}}(\mathbf{a}_\chi)$ holds if $m \in \{\mathbf{m}_\chi, \ldots, n-s\}$,*

(C2) *the equality $\mathbf{a}_{\chi'} = \mathbf{a}_\chi$ holds at each character $\chi' \in \Gamma * \chi$,*

(C3) $\displaystyle\sum_{\chi \in \mathfrak{R}_{m,n}} \mathrm{Tr}_{\mathrm{Stab}_\Gamma(\chi)/\Gamma^{p^m}}(\mathbf{a}_\chi) \cdot \mathrm{Tr}\big(\mathrm{Ind}\chi^*\big)(\varpi) \ \in \ \mathbb{Z}_p\big[\big[\Gamma^{p^m}\big]\big],$ *and*

(C4) $\displaystyle\sum_{\chi \in \mathfrak{R}_{m,n}} \mathrm{Tr}_{\mathrm{Stab}_\Gamma(\chi)/\Gamma^{p^m}}(\mathbf{a}_\chi) \cdot \mathrm{Tr}\big(\mathrm{Ind}\chi^*\big)(\varpi) \equiv 0 \mod p^{\mathrm{ord}_p(\#\overline{\mathcal{H}}_\infty^{(m,n)}) + m - \mathrm{ord}_p(\#\varpi)}$

where $\mathfrak{R}_{m,n}$ denotes a set of representatives for the Γ-orbits inside $\mathrm{Hom}\big(\overline{\mathcal{H}}_\infty^{(m,n)}, \mathbb{C}^\times\big)$.

To calculate $\#\overline{\mathcal{H}}_\infty^{(m,n)}$ in property (C4) above, one just applies Proposition 7. On the other hand, to calculate $\#\varpi$ we use the orbit-stabilizer theorem, so that for any $\overline{h} \in \varpi$ one obtains

$$\#\varpi \ = \ \big[\Gamma/\Gamma^{p^m} : \mathrm{Stab}_{\Gamma/\Gamma^{p^m}}(\overline{h})\big] \ = \ \big[\Gamma : \mathrm{Stab}_\Gamma(\overline{h})\big].$$

Also by property (C2), an element $\mathbf{a}_\chi$ depends only on the representative for χ in $\mathfrak{R}_{m,n}$, hence the last two summations in the above theorem are independent of any choices.

Proof. We begin with the 'only if' part of the argument. Suppose we are given an arbitrary element $\mathbf{a}_m \in \mathbb{Z}_p\left[\left[\mathcal{U}^{\mathrm{ab}}_{m,n}\right]\right]$, and let us put $\mathbf{a}^{(m)}_\chi := \chi(\mathbf{a}_m)$ for any character $\chi : \mathcal{H}_\infty \to \mu_{p^n}$ (note that if $\mathrm{Stab}_\Gamma(\chi) = \Gamma^{p^m}$, then we will drop the superscript ${}^{(m)}$ above completely). Assuming that $(\mathbf{a}_m) \in \Psi \cap \prod_m \mathbb{Z}_p\left[\left[\mathcal{U}^{\mathrm{ab}}_{m,n}\right]\right]$, we claim the following statements hold:

(a) there are equalities $\mathbf{a}^{(m)}_\chi = \mathbf{a}^{(m)}_{\chi'}$ for any $\chi' \in \Gamma * \chi$, where $\Gamma * \chi := \{g * \chi \mid g \in \Gamma\}$;

(b) we can express $\mathbf{a}_m = \sum_{\varpi \in \mathrm{orb}_\Gamma(\overline{\mathcal{H}}^{(m,n)}_\infty)} C^{(m)}_\varpi \cdot \mathcal{A}^{(m,n)}_\varpi$, where for any $\overline{h} \in \varpi$ one has

$$C^{(m)}_\varpi = \frac{\#\varpi}{p^m \cdot \#\overline{\mathcal{H}}^{(m,n)}_\infty} \times \sum_{\chi \in \mathfrak{R}_{m,n}} \mathbf{a}^{(m)}_\chi \cdot \left(\frac{\#(\Gamma * \chi)}{p^m} \cdot \sum_{i=0}^{p^m-1} \chi^{-1}\left(\gamma^{-i}\overline{h}\gamma^i\right) \right) \in \Lambda\left(\Gamma^{p^m}\right);$$

(c) $-\mathrm{ord}_p\left(\frac{\#\varpi}{p^m \cdot \#\overline{\mathcal{H}}^{(m,n)}_\infty}\right) = \mathrm{ord}_p\left(\#\overline{\mathcal{H}}^{(m,n)}_\infty\right) + m - \mathrm{ord}_p(\#\varpi) \geq 0$;

(d) $\mathrm{Tr}\left(\mathrm{Ind}\chi^*\right)(\varpi) = \frac{\#(\Gamma * \chi)}{p^m} \cdot \sum_{i=0}^{p^m-1} \chi^{-1}\left(\gamma^{-i}\overline{h}\gamma^i\right)$;

(e) one has $\mathbf{a}^{(m)}_\chi = \mathrm{Tr}_{\mathrm{Stab}_\Gamma(\chi)/\Gamma^{p^m}}(\mathbf{a}_\chi)$ for each $m \geq \mathbf{m}_\chi$, i.e. (C1) is true.

Deferring their proof temporarily, let us first understand why they yield the three assertions in our theorem. Clearly statement (C2) is implied by (a) with $m = \mathrm{ord}_p[\Gamma : \mathrm{Stab}_\Gamma(\chi)]$. Moreover both (C3) and (C4) will now follow upon combining (b), (c), (d) and (e) together, and then observing that the p-integrality of the $C^{(m)}_\varpi$'s is equivalent to each sum

$$\sum_{\chi \in \mathfrak{R}_{m,n}} \mathbf{a}^{(m)}_\chi \cdot \left(\frac{\#(\Gamma * \chi)}{p^m} \cdot \sum_{i=0}^{p^m-1} \chi^{-1}\left(\gamma^{-i}\overline{h}\gamma^i\right) \right) = \sum_{\chi \in \mathfrak{R}_{m,n}} \mathrm{Tr}_{\mathrm{Stab}_\Gamma(\chi)/\Gamma^{p^m}}(\mathbf{a}_\chi) \cdot \mathrm{Tr}\left(\mathrm{Ind}\chi^*\right)(\varpi)$$

belonging to the lattice $\frac{p^m \cdot \#\overline{\mathcal{H}}^{(m,n)}_\infty}{\#\varpi} \cdot \mathbb{Z}_p\left[\left[\Gamma^{p^m}\right]\right] = p^{\mathrm{ord}_p(\#\overline{\mathcal{H}}^{(m,n)}_\infty) + m - \mathrm{ord}_p(\#\varpi)} \cdot \mathbb{Z}_p\left[\left[\Gamma^{p^m}\right]\right]$.

We are left to prove these five assertions. Part (a) is a consequence of property (A2). To prove statement (b), let us write $\mathbf{a}_m = \sum_{\overline{h} \in \overline{\mathcal{H}}^{(m,n)}_\infty} c^{(m)}_{\overline{h}} \cdot \overline{h}$ where each $c^{(m)}_{\overline{h}} \in \Lambda\left(\Gamma^{p^m}\right)$. Since the characteristic function of $\overline{h}$ can be decomposed into a sum over the characters of the abelian group $\overline{\mathcal{H}}^{(m,n)}_\infty$, one can express each coefficient above as

$$c^{(m)}_{\overline{h}} = \frac{1}{\#\overline{\mathcal{H}}^{(m,n)}_\infty} \times \sum_{\chi : \overline{\mathcal{H}}^{(m,n)}_\infty \to \mu_{p^n}} \chi^{-1}(\overline{h}) \cdot \mathbf{a}^{(m)}_\chi.$$

Using property (A3) and Proposition 10, we know that $\mathbf{a}_m$ is a $\Lambda(\Gamma^{p^m})$-linear combination of $\mathcal{A}^{(m,n)}_\varpi$'s, which indicates $c^{(m)}_{\overline{h}}$ is constant-valued for all $\overline{h}$ inside a prescribed orbit ϖ. If we denote this common value as '$c^{(m)}_\varpi$', then

$$\mathbf{a}_m = \sum_{\varpi \in \mathrm{orb}_\Gamma(\overline{\mathcal{H}}^{(m,n)}_\infty)} \sum_{\overline{h} \in \varpi} c^{(m)}_\varpi \cdot \overline{h} = \sum_\varpi c^{(m)}_\varpi \cdot \sum_{\overline{h} \in \varpi} \overline{h} = \sum_\varpi c^{(m)}_\varpi \cdot \frac{\#\varpi}{p^m} \cdot \mathcal{A}^{(m,n)}_\varpi.$$

N.B. In this situation, the term $c^{(m)}_\varpi \cdot \frac{\#\varpi}{p^m}$ corresponds to the coefficient $C^{(m)}_\varpi$ of $\mathcal{A}^{(m,n)}_\varpi$.

Now we can always break $\sum_{\chi:\overline{\mathcal{H}}_\infty^{(m,n)}\to\mu_{p^n}}$ into a double summation $\sum_{\chi\in\mathfrak{R}_{m,n}}\sum_{\chi'\in\Gamma*\chi}$. Furthermore, $\mathbf{a}_{\chi'}^{(m)}=\mathbf{a}_\chi^{(m)}$ whenever $\chi'\in\Gamma*\chi$ from (a), hence for any $\overline{h}\in\varpi$:

$$c_\varpi^{(m)} \;=\; \frac{1}{\#\overline{\mathcal{H}}_\infty^{(m,n)}}\cdot\sum_{\chi:\overline{\mathcal{H}}_\infty^{(m,n)}\to\mu_{p^n}}\chi^{-1}(\overline{h})\cdot\mathbf{a}_\chi^{(m)} \;=\; \frac{1}{\#\overline{\mathcal{H}}_\infty^{(m,n)}}\cdot\sum_{\chi\in\mathfrak{R}_{m,n}}\mathbf{a}_\chi^{(m)}\sum_{\chi'\in\Gamma*\chi}\left(\chi'\right)^{-1}(\overline{h}).$$

Splicing together these last two equations, we therefore conclude

$$\mathbf{a}_m \;=\; \sum_{\varpi\in\mathrm{orb}_\Gamma(\overline{\mathcal{H}}_\infty^{(m,n)})}\left(\frac{\#\varpi}{p^m\cdot\#\overline{\mathcal{H}}_\infty^{(m,n)}}\times\sum_{\chi\in\mathfrak{R}_{m,n}}\mathbf{a}_\chi^{(m)}\cdot\sum_{\chi'\in\Gamma*\chi}\left(\chi'\right)^{-1}(\overline{h})\right)\cdot\mathcal{A}_\varpi^{(m,n)}.$$

Lastly $\sum_{\chi'\in\Gamma*\chi}\left(\chi'\right)^{-1}(\overline{h})$ coincides with the scaled value $\frac{\#(\Gamma*\chi)}{p^m}\cdot\sum_{i=0}^{p^m-1}\chi^{-1}\left(\gamma^{-i}\overline{h}\gamma^i\right)$, which means (b) is also established.

To show part (c) is easy since the size of each orbit $\varpi\in\mathrm{orb}_\Gamma\big(\overline{\mathcal{H}}_\infty^{(m,n)}\big)$ divides into p^m. In order to establish (d) we define $\rho_m:=\mathrm{Ind}_{\Gamma^{p^m}\ltimes\mathcal{H}_\infty/p^n}^{G_{\infty,n}}(\chi)$, so that $\rho_m\cong\bigoplus_\psi\mathrm{Ind}(\chi)\otimes\psi$ where the sum is over all characters $\psi:\mathrm{Stab}_\Gamma(\chi)/\Gamma^{p^m}\to\mathbb{C}^\times$. Thus for $\overline{h}\in\varpi\subset\overline{\mathcal{H}}_\infty^{(m,n)}$,

$$\big[\mathrm{Stab}_\Gamma(\chi):\Gamma^{p^m}\big]\cdot\mathrm{Tr}\big(\mathrm{Ind}\chi^*\big)(\overline{h}) \;=\; \mathrm{Tr}\big(\rho_m^*\big)(\overline{h}) \;=\; \sum_{i=0}^{p^m-1}\chi^{-1}\left(\gamma^{-i}\overline{h}\gamma^i\right)$$

and the orbit-stabilizer theorem for Γ/Γ^{p^m} acting on $\mathrm{Hom}\big(\overline{\mathcal{H}}_\infty^{(m,n)},\mu_{p^n}\big)$ then implies

$$\big[\mathrm{Stab}_\Gamma(\chi):\Gamma^{p^m}\big] \;=\; \frac{\big[\Gamma:\Gamma^{p^m}\big]}{\big[\Gamma:\mathrm{Stab}_\Gamma(\chi)\big]} \;=\; \frac{\big[\Gamma:\Gamma^{p^m}\big]}{\big[\Gamma/\Gamma^{p^m}:\mathrm{Stab}_{\Gamma/\Gamma^{p^m}}(\chi)\big]} \;=\; \frac{p^m}{\#\big(\Gamma*\chi\big)}.$$

The assertion (e) follows from property (A1): if we set $m'=\mathbf{m}_\chi$ then

$$\mathrm{Tr}_{\mathrm{Stab}_\Gamma(\chi)/\Gamma^{p^m}}(\mathbf{a}_\chi) \;=\; \chi\left(\mathrm{Tr}_{m',m}\big(\mathbf{a}_{m'}\big)\right) \overset{\text{by (A1)}}{=} \chi\left(\pi_{m,m'}\big(\mathbf{a}_m\big)\right) \;=\; \mathbf{a}_\chi^{(m)}.$$

Finally, we must of course demonstrate the 'if' portion of the 'if and only if' statement. This amounts to showing the implication

"(A1) and (A2) and (A3) $\Longrightarrow$ (C1) and (C2) and (C3) and (C4)"

is in fact **reversible**, which is a tedious but relatively straightforward exercise involving Lemmas 13 and 14 – we refer the reader to [5] for further details. □

3.2 A transfer-compatible basis for the set $\mathfrak{R}_{m,n}$

Assume again that $\star\in\{\text{II,III,IV,V,VI}\}$. We can express $\overline{\mathcal{H}}_\infty^{(m,n)}$ as the double quotient

$$\overline{\mathcal{H}}_\infty^{(m,n)} \;\cong\; \frac{\mathcal{H}_\infty/\mathcal{H}_\infty^{p^n}}{\big\langle[\overline{h}_1,\gamma^{p^m}]\;,\;[\overline{h}_2,\gamma^{p^m}]\big\rangle}$$

where $\overline{h}_1$ and $\overline{h}_2$ denote the image inside $\mathcal{H}_\infty/\mathcal{H}_\infty^{p^n}$ of the subgroup generators $h_1,h_2\in\mathcal{H}_\infty$, as outlined in the Classification Theorem.

Clearly any character χ defined on $\overline{\mathcal{H}}_{\infty}^{(m,n)}$ must satisfy $\chi\big([\overline{h}_1,\gamma^{p^m}]\big) = \chi\big([\overline{h}_2,\gamma^{p^m}]\big) = 1$. Also $\overline{\mathcal{H}}_{\infty}^{(m,n)} \cong \frac{\mathbb{Z}}{p^{N_{\star,1}^{(m)}}\mathbb{Z}} \times \frac{\mathbb{Z}}{p^{N_{\star,2}^{(m)}}\mathbb{Z}}$ where $N_{\star,1}^{(m)}, N_{\star,2}^{(m)} \in \mathbb{N}$ can be read off from Proposition 7; one may then write

$$\big[h_1,\gamma^{p^m}\big] \;=\; \big(h_1^{\tilde{x}_1}h_2^{\tilde{y}_1}\big)^{p^{N_{\star,1}^{(m)}}} \quad \text{and} \quad \big[h_2,\gamma^{p^m}\big] \;=\; \big(h_1^{\tilde{x}_2}h_2^{\tilde{y}_2}\big)^{p^{N_{\star,2}^{(m)}}}$$

for integer pairs $(\tilde{x}_1,\tilde{y}_1)$ and $(\tilde{x}_2,\tilde{y}_2)$, neither of which is p-divisible in $\frac{\mathbb{Z}}{p^{N_{\star,1}^{(m)}}\mathbb{Z}} \times \frac{\mathbb{Z}}{p^{N_{\star,2}^{(m)}}\mathbb{Z}}$. To precisely determine them, we note that the commutator $\big[h_1^x h_2^y,\gamma^{p^m}\big]$ corresponds to the vector $\big((I_2+M)^{p^m}-I_2\big)\begin{pmatrix} x \\ y\end{pmatrix}$ inside $\mathbb{Z}_p\oplus\mathbb{Z}_p$, whence

$$\begin{pmatrix} \tilde{x}_1 & \tilde{x}_2 \\ \tilde{y}_1 & \tilde{y}_2 \end{pmatrix} \;=\; \Big(\big(I_2+M\big)^{p^m}-I_2\Big)\begin{pmatrix} p^{-N_{\star,1}^{(m)}} & 0 \\ 0 & p^{-N_{\star,2}^{(m)}}\end{pmatrix}. \tag{12}$$

To construct a basis for $\mathrm{Hom}\big(\overline{\mathcal{H}}_{\infty}^{(m,n)},\mathbb{C}^{\times}\big)$**, we therefore need a pair of characters** $\tilde{\chi}_1$ **and** $\tilde{\chi}_2$**, sending** $h_1^{\tilde{x}_j}h_2^{\tilde{y}_j}$ **to a primitive** $p^{N_{\star,j}^{(m)}}$**-th root of unity for each** $j\in\{1,2\}$**.**

Recall the definition of the generating characters $\chi_{1,n},\chi_{2,n}:\mathcal{H}_\infty\to\mu_{p^n}$ from §1.2, namely

$$\chi_{1,n}\big(h_1^x h_2^y\big) \;=\; \exp\big(2\pi\sqrt{-1}\;x/p^n\big) \quad \text{and} \quad \chi_{2,n}\big(h_1^x h_2^y\big) \;=\; \exp\big(2\pi\sqrt{-1}\;y/p^n\big).$$

As an illustration, in Case (II) we know $\overline{\mathcal{H}}_{\infty}^{(m,n)} \cong \frac{\mathcal{H}_{1,\infty}}{\mathcal{H}_{1,\infty}^{p^{s+m}}} \times \frac{\mathcal{H}_{2,\infty}}{\mathcal{H}_{2,\infty}^{p^n}}$ from Proposition 7, thus one may set

$$\tilde{\chi}_{1,N_{II,1}^{(m)}}\big(h_1^x h_2^y\big) := \chi_{2,n}\big(h_1^x h_2^y\big) = \zeta_{p^n}^{y} \quad \text{and} \quad \tilde{\chi}_{2,N_{II,2}^{(m)}}\big(h_1^x h_2^y\big) := \chi_{1,s+m}\big(h_1^x h_2^y\big) = \zeta_{p^{s+m}}^{x}. \tag{13}$$

We will now abuse our notation, and employ $\chi\begin{pmatrix} x \\ y\end{pmatrix}$ as an abbreviation for $\chi(h_1^x h_2^y)$.

Definition 16. *For $j\in\{1,2\}$, we define characters $\tilde{\chi}_{j,N_{\star,j}^{(m)}}:\overline{\mathcal{H}}_{\infty}^{(m,n)} \twoheadrightarrow \mu_{p^{N_{\star,j}^{(m)}}}$ through:*

- *if $\star\in\{III,IV,V,VI\}$, then*

$$\tilde{\chi}_{1,N_{\star,1}^{(m)}}\begin{pmatrix} x \\ y\end{pmatrix} \;:=\; \chi_{1,N_{\star,1}^{(m)}}\left(\begin{pmatrix} p^{N_{\star,1}^{(m)}} & 0 \\ 0 & 0\end{pmatrix}\Big(\big(I_2+M\big)^{p^m}-I_2\Big)^{-1}\begin{pmatrix} x \\ y\end{pmatrix}\right)$$

and

$$\tilde{\chi}_{2,N_{\star,2}^{(m)}}\begin{pmatrix} x \\ y\end{pmatrix} \;:=\; \chi_{2,N_{\star,2}^{(m)}}\left(\begin{pmatrix} 0 & 0 \\ 0 & p^{N_{\star,2}^{(m)}}\end{pmatrix}\Big(\big(I_2+M\big)^{p^m}-I_2\Big)^{-1}\begin{pmatrix} x \\ y\end{pmatrix}\right);$$

- *if $\star = II$, one uses Equation (13) instead to define $\tilde{\chi}_{1,N_{II,1}^{(m)}}$ and $\tilde{\chi}_{2,N_{II,2}^{(m)}}$.*

In particular, from Equation (12) we see that $\tilde{\chi}_{1,N_{\star,1}^{(m)}}\big(h_1^{\tilde{x}_1}h_2^{\tilde{y}_1}\big) = \chi_{1,N_{\star,1}^{(m)}}\big(h_1^1 h_2^0\big) = \zeta_{p^{N_{\star,1}^{(m)}}}$ and $\tilde{\chi}_{2,N_{\star,2}^{(m)}}\big(h_1^{\tilde{x}_2}h_2^{\tilde{y}_2}\big) = \chi_{2,N_{\star,2}^{(m)}}\big(h_1^0 h_2^1\big) = \zeta_{p^{N_{\star,2}^{(m)}}}$, which satisfies our stated requirement. The main reason why we prefer using the character set $\big\{\tilde{\chi}_{1,N_{\star,1}^{(m)}},\tilde{\chi}_{2,N_{\star,2}^{(m)}}\big\}$ over the more naive choice $\big\{\chi_{1,N_{\star,1}^{(m)}},\chi_{2,N_{\star,2}^{(m)}}\big\}$ is motivated by the following compatibility result.

Proposition 17. *(a) The elements of* $\mathrm{Hom}\big(\overline{\mathcal{H}}_\infty^{(m,n)},\mathbb{C}^\times\big)$ *are explicitly given by the set*

$$\left\{\tilde{\chi}^{e_1}_{1,N^{(m)}_{\star,1}}\cdot\tilde{\chi}^{e_2}_{2,N^{(m)}_{\star,2}}\quad\text{where } e_1\in\mathbb{Z}/p^{N^{(m)}_{\star,1}}\mathbb{Z}\text{ and } e_2\in\mathbb{Z}/p^{N^{(m)}_{\star,2}}\mathbb{Z}\right\}.$$

(b) If $\star=II$ *and* $m>m'$*, then*

$$\tilde{\chi}_{1,N^{(m)}_{\star,1}}\circ\mathrm{Ver}_{m',m}=\left(\tilde{\chi}_{1,N^{(m')}_{\star,1}}\right)^{p^{m-m'}}\quad\text{and}\quad\tilde{\chi}_{2,N^{(m)}_{\star,2}}\circ\mathrm{Ver}_{m',m}=\tilde{\chi}_{2,N^{(m')}_{\star,2}}.$$

(c) If $\star\in\{III,IV,V,VI\}$ *and* $m>m'$*, then* $\tilde{\chi}_{j,N^{(m)}_{\star,j}}\circ\mathrm{Ver}_{m',m}=\tilde{\chi}_{j,N^{(m')}_{\star,j}}$ *at each* $j\in\{1,2\}$*.*

Proof. Let us first suppose $\star=$ II. Here one has $[\overline{h}_1,\gamma^{p^m}]=1$ and $[\overline{h}_2,\gamma^{p^m}]=\overline{h}_1^{p^{s+m}}$ with $N^{(m)}_{II,1}=n$ and $N^{(m)}_{II,2}=s+m$, whilst $\tilde{\chi}_{1,N^{(m)}_{II,1}}(\overline{h}_1^x\overline{h}_2^y)=\zeta^y_{p^n}$ and $\tilde{\chi}_{2,N^{(m)}_{II,1}}(\overline{h}_1^x\overline{h}_2^y)=\zeta^x_{p^{s+m}}$. Part (a) then follows as $\tilde{\chi}_{1,N^{(m)}_{II,1}}$ and $\tilde{\chi}_{2,N^{(m)}_{II,1}}$ are independent, while $\#\overline{\mathcal{H}}_\infty^{(m,n)}=p^n\cdot p^{s+m}$. To show (b) one notes for $j=1,2$ that $\tilde{\chi}_{j,N^{(m)}_{II,j}}\circ\mathrm{Ver}_{m',m}\Big|_{\mathcal{H}_\infty^{(m',n)}}=\tilde{\chi}^{p^{m-m'}}_{j,N^{(m)}_{II,j}}$ by Lemma 12, in which case

$$\tilde{\chi}_{1,N^{(m)}_{II,1}}\big((\overline{h}_1^x\overline{h}_2^y)^{p^{m-m'}}\big)=\big(\zeta^y_{p^n}\big)^{p^{m-m'}}\quad\text{and}\quad\tilde{\chi}_{2,N^{(m)}_{II,2}}\big((\overline{h}_1^x\overline{h}_2^y)^{p^{m-m'}}\big)=\big(\zeta^x_{p^{s+m}}\big)^{p^{m-m'}}=\zeta^x_{p^{s+m'}}.$$

Let us instead suppose $\star\in\{\text{III,IV,V,VI}\}$. Since $\big(I_2+M\big)^{p^m}=P_\star\begin{pmatrix}\lambda^{p^m}_{\star,+}&0\\0&\lambda^{p^m}_{\star,-}\end{pmatrix}P_\star^{-1}$, we deduce that

$$\begin{pmatrix}p^{N^{(m)}_{\star,1}}&0\\0&0\end{pmatrix}\Big(\big(I_2+M\big)^{p^m}-I_2\Big)^{-1}=\begin{pmatrix}1&0\\0&0\end{pmatrix}P_\star\begin{pmatrix}\frac{p^{N^{(m)}_{\star,1}}}{\lambda^{p^m}_{\star,+}-1}&0\\0&\frac{p^{N^{(m)}_{\star,1}}}{\lambda^{p^m}_{\star,-}-1}\end{pmatrix}P_\star^{-1}.$$

On the other hand, again from Lemma 12 the matrix corresponding to $\mathrm{Ver}_{m',m}\Big|_{\mathcal{H}_\infty^{(m',n)}}$ is given by $P_\star\begin{pmatrix}\frac{\lambda^{p^m}_{\star,+}-1}{\lambda^{p^{m'}}_{\star,+}-1}&0\\0&\frac{\lambda^{p^m}_{\star,-}-1}{\lambda^{p^{m'}}_{\star,-}-1}\end{pmatrix}P_\star^{-1}$. An elementary calculation reveals the identities

$$\begin{aligned}&\begin{pmatrix}p^{N^{(m)}_{\star,1}}&0\\0&0\end{pmatrix}\Big(\big(I_2+M\big)^{p^m}-I_2\Big)^{-1}\cdot P_\star\begin{pmatrix}\frac{\lambda^{p^m}_{\star,+}-1}{\lambda^{p^{m'}}_{\star,+}-1}&0\\0&\frac{\lambda^{p^m}_{\star,-}-1}{\lambda^{p^{m'}}_{\star,-}-1}\end{pmatrix}P_\star^{-1}\begin{pmatrix}x\\y\end{pmatrix}\\
&=\begin{pmatrix}1&0\\0&0\end{pmatrix}P_\star\begin{pmatrix}\frac{p^{N^{(m)}_{\star,1}}}{\lambda^{p^m}_{\star,+}-1}&0\\0&\frac{p^{N^{(m)}_{\star,1}}}{\lambda^{p^m}_{\star,-}-1}\end{pmatrix}\begin{pmatrix}\frac{\lambda^{p^m}_{\star,+}-1}{\lambda^{p^{m'}}_{\star,+}-1}&0\\0&\frac{\lambda^{p^m}_{\star,-}-1}{\lambda^{p^{m'}}_{\star,-}-1}\end{pmatrix}P_\star^{-1}\begin{pmatrix}x\\y\end{pmatrix}\\
&=p^{N^{(m)}_{\star,1}-N^{(m')}_{\star,1}}\begin{pmatrix}p^{N^{(m')}_{\star,1}}&0\\0&0\end{pmatrix}\Big(\big(I_2+M\big)^{p^{m'}}-I_2\Big)^{-1}\begin{pmatrix}x\\y\end{pmatrix}.\end{aligned}$$

These matrix identities directly imply that $\tilde{\chi}_{1,N^{(m)}_{\star,1}} \circ \mathrm{Ver}_{m',m}\begin{pmatrix} x \\ y \end{pmatrix}$ equals

$$\left(\chi_{1,N^{(m)}_{\star,1}}\right)^{p^{N^{(m)}_{\star,1}-N^{(m')}_{\star,1}}}\left(\begin{pmatrix} p^{N^{(m')}_{\star,1}} & 0 \\ 0 & 0 \end{pmatrix}\left((I_2+M)^{p^{m'}}-I_2\right)^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right).$$

Since $\left(\chi_{1,N^{(m)}_{\star,1}}\right)^{p^{N^{(m)}_{\star,1}-N^{(m')}_{\star,1}}} = \chi_{1,N^{(m')}_{\star,1}}$ the above quantity is none other than $\tilde{\chi}_{1,N^{(m')}_{\star,1}}\begin{pmatrix} x \\ y \end{pmatrix}$, which establishes that $\tilde{\chi}_{1,N^{(m)}_{\star,1}} \circ \mathrm{Ver}_{m',m} = \tilde{\chi}_{1,N^{(m')}_{\star,1}}$.

The argument for the second composition $\tilde{\chi}_{2,N^{(m)}_{\star,2}} \circ \mathrm{Ver}_{m',m}$ follows identical lines. $\square$

Lemma 18. *(i) If $\overline{h}_1^x\overline{h}_2^y \in \overline{\mathcal{H}}_\infty^{(m',n)}$ and $f(X) \in \mathbb{Z}_p[\![X]\!]$, then*

$$\mathrm{Ver}_{m',m}\Big(f\big(\gamma^{p^{m'}}-1\big)\cdot\mathcal{A}^{(m',n)}_{\overline{h}_1^x\overline{h}_2^y}\Big) \;=\; p^{-(m-m')}\times f\big(\gamma^{p^m}-1\big)\cdot\mathcal{A}^{(m,n)}_{\overline{h}_1^{x'}\overline{h}_2^{y'}}$$

where x', y' are as in Lemma 12.

(ii) Using exactly the same notation,

$$\tilde{\chi}^{e_1}_{1,N^{(m')}_{\star,1}}\cdot\tilde{\chi}^{e_2}_{2,N^{(m')}_{\star,2}}\Big(\mathcal{A}^{(m',n)}_{\overline{h}_1^x\overline{h}_2^y}\Big) \;=\; p^{-(m-m')}\;\times\;\tilde{\chi}^{e_1}_{1,N^{(m)}_{\star,1}}\cdot\tilde{\chi}^{e_2}_{2,N^{(m)}_{\star,2}}\Big(\mathcal{A}^{(m,n)}_{\overline{h}_1^{x'}\overline{h}_2^{y'}}\Big)$$

unless $\star =$II, in which case one replaces $\tilde{\chi}^{e_1}_{1,N^{(m')}_{\star,1}}\cdot\tilde{\chi}^{e_2}_{2,N^{(m')}_{\star,2}}$ instead with $\tilde{\chi}^{e_1p^{m-m'}}_{1,N^{(m')}_{\star,1}}\cdot\tilde{\chi}^{e_2}_{2,N^{(m')}_{\star,2}}$ on the left-hand side of this formula.

Proof. Let us start by establishing (i). If $\begin{pmatrix} x_i \\ y_i \end{pmatrix} = \big(I_2+M\big)^i\begin{pmatrix} x \\ y \end{pmatrix}$ for all $i \geq 0$, then

$$\mathrm{Ver}_{m',m}\left(\gamma^{p^{m'}j}\cdot\mathcal{A}^{(m',n)}_{\overline{h}_1^x\overline{h}_2^y}\right) \;=\; \sum_{i=0}^{p^{m'}-1}\mathrm{Ver}_{m',m}\left(\gamma^{p^{m'}j}\cdot\overline{h}_1^{x_i}\overline{h}_2^{y_i}\right) \;=\; \gamma^{p^m j}\cdot\sum_{i=0}^{p^{m'}-1}\overline{h}_1^{x_i'}\overline{h}_2^{y_i'}$$

upon applying Lemma 12. Here in Case ($\star$) with $\star\in\{$III,IV,V,VI$\}$, the vector

$$\begin{aligned}\begin{pmatrix} x_i' \\ y_i' \end{pmatrix} \;&=\; P_\star\begin{pmatrix} \frac{\lambda_{\star,+}^{p^m}-1}{\lambda_{\star,+}^{p^{m'}}-1} & 0 \\ 0 & \frac{\lambda_{\star,-}^{p^m}-1}{\lambda_{\star,-}^{p^{m'}}-1} \end{pmatrix}P_\star^{-1}\begin{pmatrix} x_i \\ y_i \end{pmatrix} \\ &=\; P_\star\begin{pmatrix} \lambda_{\star,+}^i\cdot\frac{\lambda_{\star,+}^{p^m}-1}{\lambda_{\star,+}^{p^{m'}}-1} & 0 \\ 0 & \lambda_{\star,-}^i\cdot\frac{\lambda_{\star,-}^{p^m}-1}{\lambda_{\star,-}^{p^{m'}}-1} \end{pmatrix}P_\star^{-1}\begin{pmatrix} x \\ y \end{pmatrix} \;=\; \big(I_2+M\big)^i\begin{pmatrix} x' \\ y' \end{pmatrix}\end{aligned}$$

so that $\mathrm{Ver}_{m',m}\Big(\gamma^{p^{m'}j}\cdot\mathcal{A}^{(m',n)}_{\overline{h}_1^x\overline{h}_2^y}\Big)$ equals $\gamma^{p^m j}\cdot\sum_{i=0}^{p^{m'}-1}\gamma^{-i}\overline{h}_1^{x'}\overline{h}_2^{y'}\gamma^i = \gamma^{p^m j}\cdot p^{m'-m}\mathcal{A}^{(m,n)}_{\overline{h}_1^{x'}\overline{h}_2^{y'}}$ (the same identity for the Verlagerung holds in Case (II) also). The result extends to the completed group algebra by linearity and continuity.

Secondly to show part (ii) is true, we first set $f(X)=1$ and then evaluate the identity from (i) at the character $\tilde{\chi}^{e_1}_{1,N^{(m)}_{\star,1}}\cdot\tilde{\chi}^{e_2}_{2,N^{(m)}_{\star,2}}$. We next use Proposition 17(b)-(c) to rewrite the transformed left-hand side in terms of the powers of $\tilde{\chi}_{1,N^{(m')}_{\star,1}}$ and $\tilde{\chi}_{2,N^{(m')}_{\star,2}}$. $\square$

4 The multiplicative calculations

To complete the proof of the main theorem, our strategy is to establish the existence, commutativity and row-exactness of the diagram

$$\begin{array}{ccccccccc}
1 \to & \mathbb{F}_p^\times \times \mathcal{G}^{\mathrm{ab}}_{\infty,n} & \to & K_1'\big(\mathbb{Z}_p[\![\mathcal{G}_{\infty,n}]\!]\big) & \stackrel{\mathrm{LOG}}{\longrightarrow} & \mathbb{Z}_p\big[\big[\mathrm{Conj}(\mathcal{G}_{\infty,n})\big]\big] & \to & \mathcal{G}^{\mathrm{ab}}_{\infty,n} & \to 1 \\
& \| & & \downarrow \Theta_{\infty,n} & & \downarrow \Theta^+_{\infty,n} & & \| & \\
1 \to & \mathbb{F}_p^\times \times \mathcal{G}^{\mathrm{ab}}_{\infty,n} & \to & \Phi & \stackrel{\mathcal{L}}{\longrightarrow} & \Psi & \to & \mathcal{G}^{\mathrm{ab}}_{\infty,n} & \to 1 \\
& \| & & \downarrow \underline{\chi} & & \downarrow \underline{\chi} & & & \\
1 \to & \mathbb{F}_p^\times \times \mathcal{G}^{\mathrm{ab}}_{\infty,n} & \to & \underline{\chi}(\Phi) & \stackrel{\mathcal{L}_{\underline{\chi}}}{\longrightarrow} & \underline{\chi}(\Psi) & & & \\
& & & \hookdownarrow & & \hookdownarrow & & & \\
& & & \prod_{m,\chi} \mathcal{O}_{\mathbb{C}_p}\big[\big[\mathrm{Stab}_\Gamma(\chi)\big]\big]^\times & & \Big(\prod_{m,\chi} \mathcal{O}_{\mathbb{C}_p}\big[\big[\mathrm{Stab}_\Gamma(\chi)\big]\big]\Big)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p. & & &
\end{array} \tag{14}$$

The top two lines of this diagram are precisely those occurring in [7, p80]. The vertical arrows labelled as "$\underline{\chi}$" denote evaluation at a system of representatives $\mathfrak{R}_{m,n}$, and as $\mathcal{G}^{\mathrm{ab}}_{\infty,n} \cong \Gamma$, the whole ensemble $\underline{\chi}$ therefore restricts to being the identity map on $\mathbb{F}_p^\times \times \mathcal{G}^{\mathrm{ab}}_{\infty,n}$. At this preliminary stage, we make no attempt to explain the maps LOG, $\mathcal{L}$ and $\mathcal{L}_{\underline{\chi}}$.

From Section 3, the module $\Psi \subset \prod_m \mathbb{Z}_p\big[\big[\mathcal{U}^{\mathrm{ab}}_{m,n}\big]\big]$ will consist of elements satisfying Kakde's additive conditions (A1)-(A3). Analogously, $\Phi \subset \prod_m \mathbb{Z}_p\big[\big[\mathcal{U}^{\mathrm{ab}}_{m,n}\big]\big]^\times$ consists of those elements $\big(\mathbf{y}_m\big)$ satisfying the multiplicative conditions (M1)-(M4) below, which we have specialised from [7, p107] to our particular situation:

(M1) $\mathcal{N}_{m-1,m}\big(\mathbf{y}_{m-1}\big) = \pi_{m,m-1}\big(\mathbf{y}_m\big)$ for all $m \geq 1$;

(M2) $\mathbf{y}_m = g\mathbf{y}_m g^{-1}$ at every $g \in \mathcal{G}_{\infty,n}$;

(M3) $\mathbf{y}_m \equiv \mathrm{Ver}_{m-1,m}(\mathbf{y}_{m-1}) \mod \mathrm{Im}\big(\widetilde{\sigma_m}\big)$ for each $m \geq 1$;

(M4) $\dfrac{\big(\mathbf{y}_m^{(\nu)}\big)^p}{\mathcal{N}_{m,m+1}\big(\mathbf{y}_m^{(\nu)}\big)} - \varphi\left(\dfrac{\big(\mathbf{y}_{m-1}^{(\nu)}\big)^p}{\mathcal{N}_{m-1,m}\big(\mathbf{y}_{m-1}^{(\nu)}\big)}\right) \in p \cdot \mathrm{Im}\big(\sigma_m^{(\nu)}\big)$ for every $m \geq 0$.

Here in condition (M3), the homomorphism $\widetilde{\sigma_m} : \mathbb{Z}_p\big[\big[\mathcal{U}^{\mathrm{ab}}_{m,n}\big]\big] \to \mathbb{Z}_p\big[\big[\mathcal{U}^{\mathrm{ab}}_{m,n}\big]\big]$ denotes the additive map sending $f \mapsto \sum_{i=0}^{p-1} \gamma^{-p^{m-1}i} f \gamma^{p^{m-1}i}$.

Warning: If a sequence $\big(\mathbf{y}_m\big)$ satisfies conditions (M1)-(M4), then its image under $\mathcal{L}$ automatically satisfies (A1)-(A3) by [7, p107, Lemma 4.5]. Unfortunately, because the family of abelianizations $\big\{\mathcal{U}^{\mathrm{ab}}_{m,n}\big\}_{0 \leq m \leq n-s}$ we use is *coarser* than that considered in [7, 17], we cannot directly apply the results in *op. cit.* to obtain a converse statement such as

$$\mathcal{L}\big((\mathbf{y}_m)\big) \in \Big(\prod_m \mathbb{Z}_p\big[\big[\mathcal{U}^{\mathrm{ab}}_{m,n}\big]\big]\Big)_{\text{(A1)-(A3)}} \stackrel{?}{\Longrightarrow} (\mathbf{y}_m) \in \Big(\prod_m \mathbb{Z}_p\big[\big[\mathcal{U}^{\mathrm{ab}}_{m,n}\big]\big]^\times\Big)_{\text{(M1)-(M4)}}.$$

The salvage is to show that $K_1\big(\mathbb{Z}_p[\![\mathcal{G}_{\infty,n}]\!]\big)$ splits into a direct product of $K_1\big(\mathbb{Z}_p[\![\Gamma]\!]\big)$ with with a complementary factor $\mathcal{W}_\dagger$; we shall then construct a section $\mathcal{S} : p \cdot \Psi \to \Theta_{\infty,n}\big(\mathcal{W}_\dagger\big)$ for which $\mathcal{L} \circ \mathcal{S}$ and $\mathcal{S} \circ \mathcal{L}\big|_{\Theta_{\infty,n}(\mathcal{W}_\dagger)}$ are both identity maps. One concludes that $\big(\mathbf{y}_m\big)$ arises from $K_1'\big(\mathbb{Z}_p[\![\mathcal{G}_{\infty,n}]\!]\big)$ if and only if $\mathcal{L}\big((\mathbf{y}_m)\big) \in p \cdot \Psi$, which is itself equivalent to the sequence $\underline{\chi} \circ \mathcal{L}\big((\mathbf{y}_m)\big)$ satisfying constraints (C1)–(C4) from Theorem 15.

4.1 Convergence of the logarithm on $\mathrm{Im}(\sigma_m)$

We will shortly introduce the Taylor-Oliver logarithm, which is usually defined in terms of group algebras arising from finite groups. Since the profinite groups $\mathcal{G}_{\infty,n}$ and $\mathcal{U}_{m,n}$ are both infinite, one should instead consider their finite counterparts

$$\mathcal{G}_{\infty,n}^{(\nu)} := \Gamma/\Gamma^{p^\nu} \ltimes \mathcal{H}_\infty/\mathcal{H}_\infty^{p^n} \quad \text{and more generally } \mathcal{U}_{m,n}^{(\nu)} := \Gamma^{p^m}/\Gamma^{p^\nu} \ltimes \mathcal{H}_\infty/\mathcal{H}_\infty^{p^n},$$

at each integer triple $m,n,\nu \in \mathbb{Z}$ with $0 \le m \le n-s \le \nu$. For example, $\mathcal{U}_{0,n}^{(\nu)}$ equals $\mathcal{G}_{\infty,n}^{(\nu)}$.

Remark: Using Proposition 7, one has $\mathcal{U}_{n-s,n}^{\mathrm{ab}} \cong \mathcal{U}_{n-s,n}$; in other words $\mathcal{U}_{n-s,n}$ is abelian. It follows that Γ^{p^ν} acts trivially on $\mathcal{H}_\infty/\mathcal{H}_\infty^{p^n}$ for all $\nu \ge n-s$, so the semi-direct products above make good sense. Whenever we write the superscript $^{(\nu)}$ above an object or a map, we mean the analogue of that object/map for the corresponding finite group (providing the object/map descends to its finite version, of course).

Now recall from Proposition 10(ii) that $\mathrm{Im}(\sigma_m)$ is freely generated over $\mathbb{Z}_p[\![\Gamma^{p^m}]\!]$ by the elements $\mathcal{A}_\varpi^{(m,n)}$ with $\varpi \in \mathrm{orb}_\Gamma\big(\overline{\mathcal{H}}_\infty^{(m,n)}\big)$. It is therefore trivially true that $\mathrm{Im}\big(\sigma_m^{(\nu)}\big)$ must be generated over $\mathbb{Z}_p\big[\Gamma^{p^m}/\Gamma^{p^\nu}\big]$ by the same $\mathcal{A}_\varpi^{(m,n)}$'s. If $\varpi_1, \varpi_2 \in \mathrm{orb}_\Gamma\big(\overline{\mathcal{H}}_\infty^{(m,n)}\big)$ contain $\overline{h}_1$ and $\overline{h}_2$ respectively, then

$$\mathcal{A}_{\varpi_1}^{(m,n)} \cdot \mathcal{A}_{\varpi_2}^{(m,n)} = \sum_{i=0}^{p^m-1} \gamma^{-i}\overline{h}_1\gamma^i \cdot \sum_{j=0}^{p^m-1} \gamma^{-j}\overline{h}_2\gamma^j = \sum_{i=0}^{p^m-1}\sum_{j=0}^{p^m-1} \gamma^{-i}\big(\overline{h}_1\overline{h}_2^{\gamma^{j-i}}\big)\gamma^i = \sum_{t=0}^{p^m-1} \mathcal{A}_{\overline{h}_1\overline{h}_2^{\gamma^t}}^{(m,n)}$$

which belongs to the image of $\sigma_m^{(\nu)}$. It follows that $\mathrm{Im}\big(\sigma_m^{(\nu)}\big)$ is an ideal of $\mathbb{Z}_p\big[\mathcal{U}_{m,n}^{(\nu),\mathrm{ab}}\big]$. Iterating the above calculation N-times, one deduces that

$$\mathcal{A}_{\varpi_1}^{(m,n)} \cdot \mathcal{A}_{\varpi_2}^{(m,n)} \cdots \mathcal{A}_{\varpi_{N+1}}^{(m,n)} \;=\; \sum_{t_1=0}^{p^m-1}\sum_{t_2=0}^{p^m-1}\cdots\sum_{t_N=0}^{p^m-1} \mathcal{A}_{\overline{h}_1\overline{h}_2^{\gamma^{t_1}}\cdots\overline{h}_{N+1}^{\gamma^{t_N}}}^{(m,n)}$$

which means for each $\varpi \in \mathrm{orb}_\Gamma\big(\overline{\mathcal{H}}_\infty^{(m,n)}\big)$ and element $\overline{h} \in \varpi$,

$$\Big(\mathcal{A}_\varpi^{(m,n)}\Big)^{N+1} = \sum_{t_1=0}^{p^m-1}\cdots\sum_{t_N=0}^{p^m-1} \mathcal{A}_{\overline{h}\,\overline{h}^{\gamma^{t_1}}\cdots\overline{h}^{\gamma^{t_N}}}^{(m,n)} \;=\; \prod_{j=2}^{N+1}\frac{p^m}{\#\varpi}\cdot\sum_{w_2\in\varpi}\cdots\sum_{w_{N+1}\in\varpi}\mathcal{A}_{\overline{h}w_2\cdots w_{N+1}}^{(m,n)}.$$

- Clearly if $\#\varpi < p^m$, then $\Big(\mathcal{A}_\varpi^{(m,n)}\Big)^{N+1} \in p^N \cdot \mathrm{Im}\big(\sigma_m^{(\nu)}\big) \subset p\cdot\mathrm{Im}\big(\sigma_m^{(\nu)}\big)$.
- Alternatively, if $\#\varpi = p^m$ so that $\mathrm{Stab}_{\Gamma/\Gamma^{p^m}}(\overline{h}) = \big\{\gamma^{p^m}\big\}$, then

$$\Big(\mathcal{A}_\varpi^{(m,n)}\Big)^{N+1} = \sum_{w_2\in\varpi}\cdots\sum_{w_{N+1}\in\varpi}\mathcal{A}_{\overline{h}w_2\cdots w_{N+1}}^{(m,n)} \;=\; \sum_{(t_1,\ldots,t_N)\in(\mathbb{Z}/p^m\mathbb{Z})^{\oplus N}} \mathcal{A}_{\overline{h}\,\overline{h}^{\gamma^{t_1}}\cdots\overline{h}^{\gamma^{t_N}}}^{(m,n)}.$$

There are at most p^{mN} distinct elements of the form $\overline{h}\;\overline{h}^{\gamma^{t_1}}\cdots\overline{h}^{\gamma^{t_N}}$, whilst the total number of elements in $\overline{\mathcal{H}}_\infty^{(m,n)}$ is $p^{2s+2m+\epsilon_{\star,p}}$ if $(\star)\neq$(II), where by Proposition 7 the term

$$\epsilon_{\star,p} \;:=\; N_{\star,1}^{(m)} + N_{\star,2}^{(m)} - 2s - 2m \;=\; \begin{cases} 0 & \text{in Cases (III),(IV)} \\ \mathrm{ord}_p(d) & \text{in Case (V)} \\ r + \mathrm{ord}_p(t) & \text{in Case (VI)} \end{cases}$$

is independent of m and n.

Consequently for $mN \geq 2s + 2m + \epsilon_{\star,p}$ these elements $\overline{h}\,\overline{h}^{\gamma^{t_1}} \cdots \overline{h}^{\gamma^{t_N}}$ will start repeating, in which case $\left(\mathcal{A}_{\varpi}^{(m,n)}\right)^{N+1} \in p \cdot \mathrm{Im}\big(\sigma_m^{(\nu)}\big)$. Note that the latter inequality is equivalent to $N+1 \geq 3 + \frac{2s+\epsilon_{\star,p}}{m}$, so we arrive at the following estimate:

$$\frac{\left(\mathcal{A}_{\varpi}^{(m,n)}\right)^j}{j} \quad \in \quad p^{\left\lfloor \frac{j}{3+\frac{2s+\epsilon_{\star,p}}{m}} \right\rfloor - \frac{\log(j)}{\log(p)}} \cdot \mathrm{Im}\big(\sigma_m^{(\nu)}\big). \tag{15}$$

If one sets $\epsilon_{\star,p} = -s$ and $n = m$, a similar argument implies (15) also holds for $(\star) = $(II).

Proposition 19. *(a) The two formal power series* $\log(1+y) = \sum_{j=1}^{\infty}(-1)^{j+1}\frac{y^j}{j}$ *and* $(1+y)^{-1} = \sum_{j=0}^{\infty}(-1)^j y^j$ *converge for all* $y \in \mathrm{Im}\big(\sigma_m^{(\nu)}\big)$.

(b) If $\delta_m := \left\lceil \frac{3+\frac{2s+\epsilon_{\star,p}}{m}}{p} \right\rceil$ *then for every* $N \geq 1$*, the logarithm induces a natural isomorphism*

$$\overline{\log} : \frac{1 + \mathrm{Im}\big(\sigma_m^{(\nu)}\big)^{\delta_m \cdot N}}{1 + \mathrm{Im}\big(\sigma_m^{(\nu)}\big)^{\delta_m \cdot N+1}} \xrightarrow{\sim} \frac{\mathrm{Im}\big(\sigma_m^{(\nu)}\big)^{\delta_m \cdot N}}{\mathrm{Im}\big(\sigma_m^{(\nu)}\big)^{\delta_m \cdot N+1}} ;$$

in particular, if $p \geq 5$ *and one chooses* $m \geq 2s + \epsilon_{\star,p}$*, then* $\delta_m = 1$ *above.*

(c) There are isomorphisms $1+p\cdot\mathrm{Im}\big(\sigma_m^{(\nu)}\big) \overset{\log}{\to} p\cdot\mathrm{Im}\big(\sigma_m^{(\nu)}\big)$ *and* $p\cdot\mathrm{Im}\big(\sigma_m^{(\nu)}\big) \overset{\exp}{\to} 1+p\cdot\mathrm{Im}\big(\sigma_m^{(\nu)}\big)$ *which are mutually inverse maps to one another.*

Proof. To show (a) one uses the estimate (15) together with the fact that the exponent $\left\lfloor \frac{j}{3+\frac{2s+\epsilon_{\star,p}}{m}} \right\rfloor - \frac{\log(j)}{\log(p)} \to \infty$ as $j \to \infty$, which implies both $\lim_{j\to\infty}(-1)^{j+1}\frac{y^j}{j} = 0$ and $\lim_{j\to\infty}(-1)^j y^j = 0$. In fact, since $\mathrm{Im}\big(\sigma_m^{(\nu)}\big)^j \subset p \cdot \mathrm{Im}\big(\sigma_m^{(\nu)}\big)$ for $j \gg 0$, the topology induced by the neighborhoods $\big\{\mathrm{Im}\big(\sigma_m^{(\nu)}\big)^j\big\}_{j\in\mathbb{N}}$ coincides with the p-adic topology.

The assertion in (c) can be proved by following an identical argument to [7, p106], which leaves us to tackle (b).

For simplicity we suppose that $p \geq 5$ and $m \geq 2s+\epsilon_{\star,p}$, so that $\frac{(\mathcal{A}_{\varpi}^{(m,n)})^p}{p} \in \mathrm{Im}\big(\sigma_m^{(\nu)}\big)$ by the estimate (15), whence $\frac{y^p}{p} \in \mathrm{Im}\big(\sigma_m^{(\nu)}\big)$ for all $y \in \mathrm{Im}\big(\sigma_m^{(\nu)}\big)$. Consider the homomorphism

$$\log^{\dagger} : 1 + \mathrm{Im}\big(\sigma_m^{(\nu)}\big)^N \;\to\; \frac{\mathrm{Im}\big(\sigma_m^{(\nu)}\big)^N}{\mathrm{Im}\big(\sigma_m^{(\nu)}\big)^{N+1}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

given by $\log^{\dagger}(1+y) := \log(1+y) \mod \mathrm{Im}\big(\sigma_m^{(\nu)}\big)^{N+1}$. Assuming that $j > 1$, let us examine the p-integrality of $(-1)^{j+1}\frac{y^j}{j}$ for each $y = a_1 \cdots a_N \in \mathrm{Im}\big(\sigma_m^{(\nu)}\big)^N$:

- If $p \nmid j$ then $(-1)^{j+1}\frac{y^j}{j} = \pm\frac{a_1^j \cdots a_N^j}{j} \in \mathrm{Im}\big(\sigma_m^{(\nu)}\big)^{Nj} \subset \mathrm{Im}\big(\sigma_m^{(\nu)}\big)^{N+1}$;
- If $j = p$ then $(-1)^{p+1}\frac{y^p}{p} = \frac{a_1^p}{p} \cdot a_2^p \cdots a_N^p \in \mathrm{Im}\big(\sigma_m^{(\nu)}\big)^{1+p(N-1)} \subset \mathrm{Im}\big(\sigma_m^{(\nu)}\big)^{N+1}$;
- If $j = p^k$ with $k > 1$, then

$$(-1)^{p^k+1}\frac{y^{p^k}}{p^k} = \left(\frac{a_1^p}{p}\right)^k \cdot a_1^{p^k-pk} \cdot a_2^{p^k} \cdots a_N^{p^k} \in \mathrm{Im}\big(\sigma_m^{(\nu)}\big)^{k+p^kN-pk} \subset \mathrm{Im}\big(\sigma_m^{(\nu)}\big)^{N+1}.$$

Lastly, the general case where $j = p^k c$ with $p \nmid c$ and $j > 1$ reduces to the previous cases, upon replacing y with y^c throughout.

We therefore conclude $(-1)^{j+1}\frac{y^j}{j} \in \mathrm{Im}\big(\sigma_m^{(\nu)}\big)^{N+1}$ for every $y \in \mathrm{Im}\big(\sigma_m^{(\nu)}\big)^N$ and $j > 1$. Because $\log^\dagger(1+y) \equiv y \mod \mathrm{Im}(\sigma_m^{(\nu)})^{N+1}$, clearly $\log^\dagger : 1 + \mathrm{Im}\big(\sigma_m^{(\nu)}\big)^N \to \frac{\mathrm{Im}(\sigma_m^{(\nu)})^N}{\mathrm{Im}(\sigma_m^{(\nu)})^{N+1}}$ must be a surjective map; further, one easily checks that $1 + \mathrm{Im}\big(\sigma_m^{(\nu)}\big)^{N+1} \subset \mathrm{Ker}\big(\log^\dagger\big)$. Assertion (b) now follows immediately for $p \geq 5$ and $m \geq 2s + \epsilon_{\star,p}$.

Finally, to treat assertion (b) when $p = 3$ or $m < 2s + \epsilon_{\star,p}$, one simply observes that if $\delta_m \geq \frac{3 + \frac{2s+\epsilon_{\star,p}}{m}}{p}$ then $\frac{(y^{\delta_m})^p}{p} \in \mathrm{Im}\big(\sigma_m^{(\nu)}\big)$ for all $y \in \mathrm{Im}\big(\sigma_m^{(\nu)}\big)$, using the estimate (15) again. One then repeats the previous arguments, with y replaced by y^{δ_m} everywhere. $\square$

4.2 Interaction of the theta-maps with both φ and log

We now derive some technical results describing how the Frobenius mapping φ and the logarithm commute with the theta-homomorphisms. Let us recall that in our situation, the trace and norm maps from $\mathcal{G}_{\infty,n}^{(\nu)}$ down to $\mathcal{U}_{m,n}^{(\nu)}$ have the simple description

$$\mathrm{Tr}_{\mathcal{G}_{\infty,n}^{(\nu)}/\mathcal{U}_{m,n}^{(\nu)}}(\alpha) \;=\; \sum_{k=0}^{p^m-1} \gamma^{-k}\alpha\gamma^k \qquad \text{and} \qquad \mathrm{Norm}_{\mathcal{G}_{\infty,n}^{(\nu)}/\mathcal{U}_{m,n}^{(\nu)}}(x) \;=\; \prod_{k=0}^{p^m-1} \gamma^{-k}x\gamma^k.$$

Definition 20. *(a) The additive theta-map* $\theta_{m,n}^{(\nu),+} : \mathbb{Z}_p\big[\mathrm{Conj}(\mathcal{G}_{\infty,n}^{(\nu)})\big] \to \mathbb{Z}_p\big[\mathcal{U}_{m,n}^{(\nu),\mathrm{ab}}\big]$ *is given by the composition*

$$\theta_{m,n}^{(\nu),+}(-) \;:=\; \mathrm{Tr}_{\mathcal{G}_{\infty,n}^{(\nu)}/\mathcal{U}_{m,n}^{(\nu)}}(-) \mod \big[\mathcal{U}_{m,n}^{(\nu)}, \mathcal{U}_{m,n}^{(\nu)}\big].$$

(b) The multiplicative theta-map $\theta_{m,n}^{(\nu)} : K_1\big(\mathbb{Z}_p\big[\mathcal{G}_{\infty,n}^{(\nu)}\big]\big) \to \mathbb{Z}_p\big[\mathcal{U}_{m,n}^{(\nu),\mathrm{ab}}\big]^\times$ *is defined by*

$$\theta_{m,n}^{(\nu)}(-) \;:=\; \mathrm{Norm}_{\mathcal{G}_{\infty,n}^{(\nu)}/\mathcal{U}_{m,n}^{(\nu)}}(-) \mod \big[\mathcal{U}_{m,n}^{(\nu)}, \mathcal{U}_{m,n}^{(\nu)}\big].$$

Let $\iota : \mathbb{Z}_p\big[\Gamma/\Gamma^{p^\nu}\big] \hookrightarrow \mathbb{Z}_p\big[\mathcal{G}_{\infty,n}^{(\nu)}\big]$ be the map on group algebras induced from the sequence $\Gamma/\Gamma^{p^\nu} \overset{\sim}{\to} \Gamma/\Gamma^{p^\nu} \ltimes \{1\} \hookrightarrow \mathcal{G}_{\infty,n}^{(\nu)}$ that identifies Γ/Γ^{p^ν} with a non-normal subgroup of $\mathcal{G}_{\infty,n}^{(\nu)}$.

Lemma 21. *There exists a splitting of abelian groups*

$$K_1\big(\mathbb{Z}_p\big[\mathcal{G}_{\infty,n}^{(\nu)}\big]\big) \;\overset{\sim}{\longrightarrow}\; \mathbb{Z}_p\big[\Gamma/\Gamma^{p^\nu}\big]^\times \times \mathcal{W}_\dagger^{(\nu)} \quad \textit{sending } x \mapsto \big(x^{\mathrm{cy}}, x^\dagger\big),$$

where $x^{\mathrm{cy}} = \iota_* \circ \theta_{0,n}^{(\nu)}(x)$, $x^\dagger = \frac{x}{x^{\mathrm{cy}}}$, *and the complement* $\mathcal{W}_\dagger^{(\nu)} := \big\{x^\dagger \bigm| x \in K_1\big(\mathbb{Z}_p\big[\mathcal{G}_{\infty,n}^{(\nu)}\big]\big)\big\}$.

Proof. Firstly $\theta_{0,n}^{(\nu)}$ coincides with the quotient mapping modulo $\big[\mathcal{U}_{0,n}^{(\nu)}, \mathcal{U}_{0,n}^{(\nu)}\big] = \mathcal{H}_\infty/\mathcal{H}_\infty^{p^n}$. The composition $\Gamma/\Gamma^{p^\nu} \overset{\iota}{\hookrightarrow} \mathcal{G}_{\infty,n}^{(\nu)} \overset{\mathrm{mod}\ \mathcal{H}_\infty/p^n}{\twoheadrightarrow} \Gamma/\Gamma^{p^\nu}$ equals the identity, and this induces

$$K_1\big(\mathbb{Z}_p\big[\Gamma/\Gamma^{p^\nu}\big]\big) \;\overset{\iota_*}{\longrightarrow}\; K_1\big(\mathbb{Z}_p\big[\mathcal{G}_{\infty,n}^{(\nu)}\big]\big) \;\overset{\theta_{0,n}^{(\nu)}}{\longrightarrow}\; K_1\big(\mathbb{Z}_p\big[\Gamma/\Gamma^{p^\nu}\big]\big)$$

which must then be the identity map on $K_1\big(\mathbb{Z}_p\big[\Gamma/\Gamma^{p^\nu}\big]\big) \cong \mathbb{Z}_p\big[\Gamma/\Gamma^{p^\nu}\big]^\times$. The latter group is therefore isomorphic to a direct factor of $K_1\big(\mathbb{Z}_p\big[\mathcal{G}_{\infty,n}^{(\nu)}\big]\big)$, and the rest follows easily. $\square$

For a group G, the ring homomorphism $\varphi_G : \mathbb{Z}_p[\mathrm{Conj}(G)] \to \mathbb{Z}_p[\mathrm{Conj}(G)]$ denotes the linear extension of the map $[g] \mapsto [g^p]$ on $\mathrm{Conj}(G)$ (note if G is abelian, then $\mathrm{Conj}(G) = G$).

Lemma 22. *For all $\alpha \in \mathbb{Q}_p\big[\mathrm{Conj}\big(\mathcal{G}^{(\nu)}_{\infty,n}\big)\big]$,*

$$\theta^{(\nu),+}_{m,n} \circ \varphi_{\mathcal{G}^{(\nu)}_{\infty,n}}(\alpha) \;=\; \begin{cases} p \cdot \varphi_{\mathcal{U}^{(\nu)}_{m-1,n}} \circ \mathrm{Tr}_{\mathcal{G}^{(\nu)}_{\infty,n}/\mathcal{U}^{(\nu)}_{m-1,n}}(\alpha) \mod \big[\mathcal{U}^{(\nu)}_{m,n}, \mathcal{U}^{(\nu)}_{m,n}\big] & \text{if } m \geq 1 \\ \varphi_{\mathcal{G}^{(\nu)}_{\infty,n}}(\alpha) \mod \big[\mathcal{U}^{(\nu)}_{0,n}, \mathcal{U}^{(\nu)}_{0,n}\big] & \text{if } m = 0. \end{cases}$$

Proof. If $m = 0$, the formula is straightforward to establish.

We therefore suppose that $m \geq 1$. It is enough to consider conjugacy classes of the form $\alpha = [\gamma^j \cdot \overline{h}]$ with $j \in \mathbb{Z}/p^\nu\mathbb{Z}$ and $\overline{h} \in \frac{\mathcal{H}_\infty}{\mathcal{H}_\infty^{p^n}}$, since these will generate $\mathbb{Q}_p\big[\mathrm{Conj}\big(\mathcal{G}^{(\nu)}_{\infty,n}\big)\big]$.

Key Claims: (I) For all $j \in \mathbb{Z}/p^\nu\mathbb{Z}$, one has $\big(\gamma^j \cdot \overline{h}\big)^p = \gamma^{pj} \cdot \prod_{i=0}^{p-1} \overline{h}^{\gamma^{ji}}$ inside $\Gamma/\Gamma^{p^\nu} \ltimes \frac{\mathcal{H}_\infty}{\mathcal{H}_\infty^{p^n}}$.

(II) If $k, k' \in \mathbb{Z}$ satisfy $k \equiv k' \pmod{p^{m-1}}$, then

$$\varphi_{\mathcal{U}^{(\nu)}_{m-1,n}}\Big(\big[\gamma^j \cdot \overline{h}^{\gamma^k}\big]\Big) \;\equiv\; \varphi_{\mathcal{U}^{(\nu)}_{m-1,n}}\Big(\big[\gamma^j \cdot \overline{h}^{\gamma^{k'}}\big]\Big) \mod \big[\mathcal{U}^{(\nu)}_{m,n}\,,\, \mathcal{U}^{(\nu)}_{m,n}\big]. \tag{16}$$

Postponing their proof for the moment, one calculates that

$$\begin{aligned}
\theta^{(\nu),+}_{m,n} \circ \varphi_{\mathcal{G}^{(\nu)}_{\infty,n}}\big([\gamma^j \cdot \overline{h}]\big) &\overset{\text{by (I)}}{=} \theta^{(\nu),+}_{m,n}\left(\left[\gamma^{pj} \cdot \prod_{i=0}^{p-1} \overline{h}^{\gamma^{ji}}\right]\right) \\
&= \begin{cases} \gamma^{pj} \cdot \sum_{k=0}^{p^m-1} \gamma^{-k}\Big(\prod_{i=0}^{p-1} \overline{h}^{\gamma^{ji}}\Big)\gamma^k \mod \big[\mathcal{U}^{(\nu)}_{m,n}, \mathcal{U}^{(\nu)}_{m,n}\big] & \text{if } \gamma^{pj} \in \Gamma^{p^m} \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \gamma^{pj} \cdot \sum_{k=0}^{p^m-1} \prod_{i=0}^{p-1} \gamma^{-k}\overline{h}^{\gamma^{ji}}\gamma^k \mod \big[\mathcal{U}^{(\nu)}_{m,n}, \mathcal{U}^{(\nu)}_{m,n}\big] & \text{if } \gamma^{j} \in \Gamma^{p^{m-1}} \\ 0 & \text{otherwise} \end{cases} \\
&\overset{\text{by (I)}}{=} \begin{cases} \varphi_{\mathcal{U}^{(\nu)}_{m-1,n}}\Big(\gamma^j \cdot \sum_{k=0}^{p^m-1} \overline{h}^{\gamma^k}\Big) \mod \big[\mathcal{U}^{(\nu)}_{m,n}, \mathcal{U}^{(\nu)}_{m,n}\big] & \text{if } \gamma^{j} \in \Gamma^{p^{m-1}} \\ 0 & \text{otherwise} \end{cases} \\
&\overset{\text{by (II)}}{=} \begin{cases} \varphi_{\mathcal{U}^{(\nu)}_{m-1,n}}\Big(\gamma^j \cdot p \cdot \sum_{k'=0}^{p^{m-1}-1} \overline{h}^{\gamma^{k'}}\Big) \mod \big[\mathcal{U}^{(\nu)}_{m,n}, \mathcal{U}^{(\nu)}_{m,n}\big] & \text{if } \gamma^{j} \in \Gamma^{p^{m-1}} \\ 0 & \text{otherwise} \end{cases} \\
&= p \cdot \varphi_{\mathcal{U}^{(\nu)}_{m-1,n}} \circ \mathrm{Tr}_{\mathcal{G}^{(\nu)}_{\infty,n}/\mathcal{U}^{(\nu)}_{m-1,n}}\Big([\gamma^j \cdot \overline{h}]\Big) \mod \big[\mathcal{U}^{(\nu)}_{m,n}, \mathcal{U}^{(\nu)}_{m,n}\big].
\end{aligned}$$

The full lemma now follows for each $m \geq 1$, as $\mathbb{Q}_p\big[\mathrm{Conj}\big(\mathcal{G}^{(\nu)}_{\infty,n}\big)\big]$ is generated by $[\gamma^j \cdot \overline{h}]$'s.

It remains to establish Claims (I) and (II). To prove (I) we know that $\overline{h} \cdot \gamma^j = \gamma^j \cdot \overline{h}^{\gamma^j}$, in which case

$$\begin{aligned}
\big(\gamma^j \cdot \overline{h}\big)^p &= \gamma^j \cdot \big(\overline{h} \cdot \gamma^j\big) \cdot \overline{h} \cdot \gamma^j \cdot \overline{h} \cdots \gamma^j \cdot \overline{h} \;=\; \gamma^{2j} \cdot \overline{h}^{\gamma^j} \cdot \big(\overline{h} \cdot \gamma^j\big) \cdot \overline{h} \cdots \gamma^j \cdot \overline{h} \\
&= \gamma^{2j} \cdot \big(\overline{h}^{\gamma^j} \cdot \gamma^j\big) \cdot \overline{h}^{\gamma^j} \cdot \overline{h} \cdots \gamma^j \cdot \overline{h} \;=\; \gamma^{3j} \cdot \overline{h}^{\gamma^{2j}} \cdot \overline{h}^{\gamma^j} \cdot \overline{h} \cdots \gamma^j \cdot \overline{h} \\
&= \ldots \;=\; \gamma^{(p-1)j} \cdot \overline{h}^{\gamma^{(p-2)j}} \cdot \overline{h}^{\gamma^{(p-3)j}} \cdots \big(\overline{h} \cdot \gamma^j\big) \cdot \overline{h} \;=\; \ldots \;=\; \gamma^{pj} \cdot \prod_{i=0}^{p-1} \overline{h}^{\gamma^{ji}}.
\end{aligned}$$

To show (II) note that the L.H.S. of (16) $\overset{\text{by (I)}}{=}$ $\gamma^{pj}\cdot\prod_{i=0}^{p-1}(\overline{h}^{\gamma^k})^{\gamma^{ji}} \;=\; \gamma^{pj}\cdot\prod_{i=0}^{p-1}\overline{h}^{\gamma^{ji+k}}$, while the R.H.S. of (16) $=\gamma^{pj}\cdot\prod_{i=0}^{p-1}\overline{h}^{\gamma^{ji+k'}}$ by an identical argument; one deduces that

$$\begin{aligned}\frac{\text{L.H.S. of (16)}}{\text{R.H.S. of (16)}} &= \gamma^{pj}\cdot\left(\prod_{i=0}^{p-1}\overline{h}^{\gamma^{ji+k}}(\overline{h}^{-1})^{\gamma^{ji+k'}}\right)\cdot\gamma^{-pj}\\ &= \gamma^{pj}\cdot\left(\prod_{i=0}^{p-1}\gamma^{-(ji+k')}\cdot\left(\gamma^{k'-k}\cdot\overline{h}\cdot\gamma^{-(k'-k)}\cdot\overline{h}^{-1}\right)\cdot\gamma^{ji+k'}\right)\cdot\gamma^{-pj}.\end{aligned}$$

However $\overline{h}_{k,k'} := \gamma^{k'-k}\cdot\overline{h}\cdot\gamma^{-(k'-k)}\cdot\overline{h}^{-1} \in \left[\mathcal{U}^{(\nu)}_{m-1,n},\mathcal{U}^{(\nu)}_{m-1,n}\right]$ because $\gamma^{k-k'}\in\Gamma^{p^{m-1}}$ whenever $k\equiv k' \pmod{p^{m-1}}$, which in turn implies $\frac{\text{L.H.S. of (16)}}{\text{R.H.S. of (16)}}=\left(\prod_{i=0}^{p-1}\overline{h}_{k,k'}^{\gamma^{ji+k'}}\right)^{\gamma^{-pj}}$. This latter product is divisible by p, in fact

$$\frac{\text{L.H.S. of (16)}}{\text{R.H.S. of (16)}} \in \left[\mathcal{U}^{(\nu)}_{m-1,n}\,,\,\mathcal{U}^{(\nu)}_{m-1,n}\right]^p \subset \left[\mathcal{U}^{(\nu)}_{m,n}\,,\,\mathcal{U}^{(\nu)}_{m,n}\right].$$

Therefore L.H.S. $\equiv$ R.H.S. mod $\left[\mathcal{U}^{(\nu)}_{m,n}\,,\,\mathcal{U}^{(\nu)}_{m,n}\right]$, which establishes Claim (II) as well. $\square$

We now examine how the Frobenius map φ commutes with $\theta^{(\nu)}_{m-1,n}$. Consider the sequence

$$\frac{\Gamma^{p^{m-1}}}{\Gamma^{p^\nu}}\times\frac{\mathcal{H}_\infty}{\left[\mathcal{U}^{(\nu)}_{m-1,n},\mathcal{U}^{(\nu)}_{m-1,n}\right]} \overset{(-)^p}{\longrightarrow} \frac{\Gamma^{p^m}}{\Gamma^{p^\nu}}\times\frac{(\mathcal{H}_\infty)^p}{\left[\mathcal{U}^{(\nu)}_{m-1,n},\mathcal{U}^{(\nu)}_{m-1,n}\right]^p} \twoheadrightarrow \frac{\Gamma^{p^m}}{\Gamma^{p^\nu}}\times\frac{(\mathcal{H}_\infty)^p}{\left[\mathcal{U}^{(\nu)}_{m,n},\mathcal{U}^{(\nu)}_{m,n}\right]}$$

induced from the p-power map, and the containment $\left[\mathcal{U}^{(\nu)}_{m-1,n},\mathcal{U}^{(\nu)}_{m-1,n}\right]^p\hookrightarrow\left[\mathcal{U}^{(\nu)}_{m,n},\mathcal{U}^{(\nu)}_{m,n}\right]$. If we label the composition as $\widetilde{\varphi}:\mathcal{U}^{(\nu),\mathrm{ab}}_{m-1,n}\to\mathcal{U}^{(\nu),\mathrm{ab}}_{m,n}$, by linearly extending $\widetilde{\varphi}$ one obtains

$$\widetilde{\varphi}_{\mathcal{U}^{(\nu),\mathrm{ab}}_{m-1,n}}:\mathbb{Q}_p\left[\mathcal{U}^{(\nu),\mathrm{ab}}_{m-1,n}\right]\to\mathbb{Q}_p\left[\mathcal{U}^{(\nu),\mathrm{ab}}_{m,n}\right],\qquad \sum_{g\in\mathcal{U}^{(\nu),\mathrm{ab}}_{m-1,n}}c_g\cdot[g]\;\mapsto\;\sum_{g\in\mathcal{U}^{(\nu),\mathrm{ab}}_{m-1,n}}c_g\cdot\widetilde{\varphi}[g]$$

as a homomorphism of commutative algebras.

Lemma 23. *(i) For each integer $m\geq 1$ and every $x\in K_1\left(\mathbb{Z}_p\left[\mathcal{G}^{(\nu)}_{\infty,n}\right]\right)$,*

$$\begin{aligned}&\widetilde{\varphi}_{\mathcal{U}^{(\nu),\mathrm{ab}}_{m-1,n}}\circ\log_{\mathbb{Z}_p[\mathcal{U}^{(\nu),\mathrm{ab}}_{m-1,n}]}\circ\,\theta^{(\nu)}_{m-1,n}(x)\\ &\quad=\;\varphi_{\mathcal{U}^{(\nu),\mathrm{ab}}_{m-1,n}}\left(\log_{\mathbb{Z}_p[\mathcal{U}^{(\nu),\mathrm{ab}}_{m-1,n}]}\circ\,\mathrm{Norm}_{\mathcal{G}^{(\nu)}_{\infty,n}/\mathcal{U}^{(\nu)}_{m-1,n}}(x)\right)\mod\left[\mathcal{U}^{(\nu)}_{m,n}\,,\,\mathcal{U}^{(\nu)}_{m,n}\right].\end{aligned}$$

(ii) For each integer $m\geq 0$ and every $x\in K_1\left(\mathbb{Z}_p\left[\mathcal{G}^{(\nu)}_{\infty,n}\right]\right)$,

$$\theta^{(\nu)}_{m,n}\left(x^\dagger\right)=\frac{\theta^{(\nu)}_{m,n}(x)}{\tau^{(m,\nu)}_*\circ\mathcal{N}_{0,m}\left(\theta^{(\nu)}_{0,n}(x)\right)}\qquad\textit{and}\qquad\theta^{(\nu)}_{m,n}\left(x^{\mathrm{cy}}\right)=\tau^{(m,\nu)}_*\circ\mathcal{N}_{0,m}(\theta^{(\nu)}_{0,n}(x))$$

where $\tau^{(m,\nu)}$ denotes the natural inclusion $\mathbb{Q}_p\left[\Gamma^{p^m}/\Gamma^{p^\nu}\right]\hookrightarrow\mathbb{Q}_p\left[\mathcal{U}^{(\nu),\mathrm{ab}}_{m,n}\right]$.

At first glance these statements are rather technical in nature, and their demonstrations could easily be skipped on an initial reading. However they will become important tools for us in the next section, when we calculate the Taylor-Oliver logarithm composed with the family of theta-maps $\left\{\theta^{(\nu),+}_{m,n}\right\}_{0\leq m\leq n-s}$.

Proof. Starting with assertion (i), since $\big[\mathcal{U}^{(\nu)}_{m-1,n}\ ,\ \mathcal{U}^{(\nu)}_{m-1,n}\big]^p \subset \big[\mathcal{U}^{(\nu)}_{m,n}\ ,\ \mathcal{U}^{(\nu)}_{m,n}\big]$ one deduces

$$\begin{aligned}
&\varphi_{\mathcal{U}^{(\nu),\mathrm{ab}}_{m-1,n}} \circ \mathrm{Tr}_{\mathcal{G}^{(\nu)}_{\infty,n}/\mathcal{U}^{(\nu)}_{m-1,n}}(\alpha) \mod \big[\mathcal{U}^{(\nu)}_{m,n}\ ,\ \mathcal{U}^{(\nu)}_{m,n}\big] \\
&= \ \widetilde{\varphi}_{\mathcal{U}^{(\nu),\mathrm{ab}}_{m-1,n}} \Big(\mathrm{Tr}_{\mathcal{G}^{(\nu)}_{\infty,n}/\mathcal{U}^{(\nu)}_{m-1,n}}(\alpha) \mod \big[\mathcal{U}^{(\nu)}_{m-1,n}\ ,\ \mathcal{U}^{(\nu)}_{m-1,n}\big]\Big) \ = \ \widetilde{\varphi}_{\mathcal{U}^{(\nu),\mathrm{ab}}_{m-1,n}} \circ \theta^{(\nu),+}_{m-1,n}(\alpha)
\end{aligned} \tag{17}$$

for every $\alpha \in \mathbb{Q}_p\big[\mathrm{Conj}\big(\mathcal{G}^{(\nu)}_{\infty,n}\big)\big]$. Evaluating both sides at $\alpha = \log(x)$, it is easily verified

$$\begin{aligned}
\varphi_{\mathcal{U}^{(\nu),\mathrm{ab}}_{m-1,n}} \circ \log \circ \mathrm{Norm}_{\mathcal{G}^{(\nu)}_{\infty,n}/\mathcal{U}^{(\nu)}_{m-1,n}}(x) \ &\equiv \ \varphi_{\mathcal{U}^{(\nu),\mathrm{ab}}_{m-1,n}} \circ \ \mathrm{Tr}_{\mathcal{G}^{(\nu)}_{\infty,n}/\mathcal{U}^{(\nu)}_{m-1,n}}\big(\log(x)\big) \\
&\overset{\text{by (17)}}{=} \ \widetilde{\varphi}_{\mathcal{U}^{(\nu),\mathrm{ab}}_{m-1,n}} \circ \theta^{(\nu),+}_{m-1,n}\big(\log(x)\big) \ = \ \widetilde{\varphi}_{\mathcal{U}^{(\nu),\mathrm{ab}}_{m-1,n}} \circ \log \circ \theta^{(\nu)}_{m-1,n}(x).
\end{aligned}$$

To prove (ii), one simply observes that

$$\begin{aligned}
\tau_*^{(m,\nu)} \circ \mathcal{N}_{0,m}\big(\theta^{(\nu)}_{0,n}(x)\big) \ &= \ \tau_*^{(m,\nu)} \circ \mathrm{Norm}_{\Gamma/\Gamma^{p^m}}\big(x \mod \mathcal{H}_\infty/\mathcal{H}_\infty^{p^n}\big) \\
&= \ \mathrm{Norm}_{\mathcal{G}^{(\nu)}_{\infty,n}/\mathcal{U}^{(\nu)}_{m,n}}\Big(\tau_*^{(0,\nu)}\big(x \mod \mathcal{H}_\infty/\mathcal{H}_\infty^{p^n}\big)\Big) \mod \big[\mathcal{U}^{(\nu)}_{m,n},\mathcal{U}^{(\nu)}_{m,n}\big] \\
&= \ \theta^{(\nu)}_{m,n} \circ \iota_*\big(x \mod \mathcal{H}_\infty/\mathcal{H}_\infty^{p^n}\big) \ = \ \theta^{(\nu)}_{m,n}\big(x^{\mathrm{cy}}\big).
\end{aligned}$$

Consequently $\theta^{(\nu)}_{m,n}\big(x^\dagger\big) = \frac{\theta^{(\nu)}_{m,n}(x)}{\theta^{(\nu)}_{m,n}(x^{\mathrm{cy}})} = \frac{\theta^{(\nu)}_{m,n}(x)}{\tau_*^{(m,\nu)}\circ\mathcal{N}_{0,m}(\theta^{(\nu)}_{0,n}(x))}$, and the two identities follow. ☐

4.3 The image of the Taylor-Oliver logarithm

For a finite group G, the Taylor-Oliver logarithm $\mathrm{LOG}_G : K_1\big(\mathbb{Z}_p[G]\big) \to \mathbb{Z}_p\big[\mathrm{Conj}(G)\big]$ is defined by

$$\mathrm{LOG}_G(x) \ := \ \log_{\mathbb{Z}_p[G]}(x) - \frac{1}{p}\varphi_G\big(\log_{\mathbb{Z}_p[G]}(x)\big)$$

where $\log_{\mathbb{Z}_p[G]}$ is the unique extension of $\log_{\mathrm{Jac}(\mathbb{Z}_p[G])}$ (see [22] for more details). Note that G need not necessarily be a p-group, even though it happens to be so in this paper.

If $G = \mathcal{G}^{(\nu)}_{\infty,n}$ then $\mathrm{LOG}_{\mathcal{G}^{(\nu)}_{\infty,n}}$ denotes the ν-th layer of the map 'LOG' occurring in (14). Our task is to calculate the mappings $\mathcal{L}$ and $\mathcal{L}_{\underline{\chi}}$ which make that diagram commutative. The former of these maps may be determined from the following formulae.

Proposition 24. *(a) If $m \in \{1,\dots,n-s\}$ and $x \in K_1\big(\mathbb{Z}_p[\![\mathcal{G}^{(\nu)}_{\infty,n}]\!]\big)$, then*

$$\theta^{(\nu),+}_{m,n} \circ \mathrm{LOG}_{\mathcal{G}^{(\nu)}_{\infty,n}}(x) \ = \ \log_{\mathbb{Z}_p[\mathcal{U}^{(\nu),\mathrm{ab}}_{m,n}]}\left(\frac{\theta^{(\nu)}_{m,n}(x)}{\widetilde{\varphi}_{\mathcal{U}^{(\nu),\mathrm{ab}}_{m-1,n}} \circ \theta^{(\nu)}_{m-1,n}(x)}\right).$$

(b) Furthermore, if $x^\dagger = \frac{x}{x^{\mathrm{cy}}} \in \mathcal{W}^{(\nu)}_\dagger$ then

$$\begin{aligned}
&\theta^{(\nu),+}_{m,n} \circ \mathrm{LOG}_{\mathcal{G}^{(\nu)}_{\infty,n}}\big(x^\dagger\big) \\
&= \ \log_{\mathbb{Z}_p[\mathcal{U}^{(\nu),\mathrm{ab}}_{m,n}]}\left(\frac{\theta^{(\nu)}_{m,n}(x)}{\tau_*^{(m,\nu)} \circ \mathcal{N}_{0,m}\big(\theta^{(\nu)}_{0,n}(x)\big)} \cdot \widetilde{\varphi}_{\mathcal{U}^{(\nu),\mathrm{ab}}_{m-1,n}}\left(\frac{\tau_*^{(m-1,\nu)} \circ \mathcal{N}_{0,m-1}\big(\theta^{(\nu)}_{0,n}(x)\big)}{\theta^{(\nu)}_{m-1,n}(x)}\right)\right).
\end{aligned}$$

Proof. Using the definition of the Taylor-Oliver logarithm and our previous results,

$$\begin{aligned}
\theta^{(\nu),+}_{m,n}\circ \mathrm{LOG}_{\mathcal{G}^{(\nu)}_{\infty,n}}(x) \;&=\; \theta^{(\nu),+}_{m,n}\circ \log_{\mathbb{Z}_p[\mathcal{G}^{(\nu)}_{\infty,n}]}(x) - \frac{1}{p}\cdot \theta^{(\nu),+}_{m,n}\circ \varphi_{\mathcal{G}^{(\nu)}_{\infty,n}}\big(\log_{\mathbb{Z}_p[\mathcal{G}^{(\nu)}_{\infty,n}]}(x)\big)\\
&\overset{\text{by } 22}{=}\; \theta^{(\nu),+}_{m,n}\big(\log(x)\big) - \frac{1}{p}\cdot p\cdot \varphi_{\mathcal{U}^{(\nu)}_{m-1,n}}\circ \mathrm{Tr}_{\mathcal{G}^{(\nu)}_{\infty,n}/\mathcal{U}^{(\nu)}_{m-1,n}}\big(\log(x)\big) \mod \big[\mathcal{U}^{(\nu)}_{m,n},\mathcal{U}^{(\nu)}_{m,n}\big]\\
&=\; \theta^{(\nu),+}_{m,n}\big(\log(x)\big) - \varphi_{\mathcal{U}^{(\nu)}_{m-1,n}}\circ \log\Big(\mathrm{Norm}_{\mathcal{G}^{(\nu)}_{\infty,n}/\mathcal{U}^{(\nu)}_{m-1,n}}(x)\Big) \mod \big[\mathcal{U}^{(\nu)}_{m,n},\mathcal{U}^{(\nu)}_{m,n}\big]\\
&\overset{\text{by } 23(\mathrm{i})}{=}\; \theta^{(\nu),+}_{m,n}\big(\log_{\mathbb{Z}_p[\mathcal{G}^{(\nu)}_{\infty,n}]}(x)\big) - \widetilde{\varphi}_{\mathcal{U}^{(\nu),\mathrm{ab}}_{m-1,n}}\circ \log_{\mathbb{Z}_p[\mathcal{U}^{(\nu),\mathrm{ab}}_{m-1,n}]}\circ\, \theta^{(\nu)}_{m-1,n}(x)\\
&=\; \log_{\mathbb{Z}_p[\mathcal{U}^{(\nu),\mathrm{ab}}_{m,n}]}\big(\theta^{(\nu)}_{m,n}(x)\big) - \log_{\mathbb{Z}_p[\mathcal{U}^{(\nu),\mathrm{ab}}_{m,n}]}\Big(\widetilde{\varphi}_{\mathcal{U}^{(\nu),\mathrm{ab}}_{m-1,n}}\circ\, \theta^{(\nu)}_{m-1,n}(x)\Big)
\end{aligned}$$

which establishes assertion (a).

To prove (b), one simply combines part (a) with the formula from Lemma 23(ii). □

Remark: As a direct consequence, in order to make the left-hand square in the diagram

$$\begin{array}{ccccc}
K_1\big(\mathbb{Z}_p[\mathcal{G}^{(\nu)}_{\infty,n}]\big) & \overset{\prod \theta^{(\nu)}_{m,n}}{\longrightarrow} & \Phi^{(\nu)} & \overset{\prod \underline{\chi}}{\longrightarrow} & \underline{\chi}\big(\Phi^{(\nu)}\big)\\
\downarrow \mathrm{LOG}_{\mathcal{G}^{(\nu)}_{\infty,n}} & & \downarrow \mathcal{L}^{(\nu)} & & \downarrow \mathcal{L}^{(\nu)}_{\underline{\chi}}\\
\mathbb{Z}_p\big[\mathrm{Conj}(\mathcal{G}^{(\nu)}_{\infty,n})\big] & \overset{\prod \theta^{(\nu),+}_{m,n}}{\longrightarrow} & \Psi^{(\nu)} & \overset{\prod \underline{\chi}}{\longrightarrow} & \underline{\chi}\big(\Psi^{(\nu)}\big)
\end{array}$$

commutative, it follows from Proposition 24(a) that one should define

$$\mathcal{L}^{(\nu)}\Big(\big(\mathbf{y}^{(\nu)}_m\big)\Big)_m := \log_{\mathbb{Z}_p[\mathcal{U}^{(\nu),\mathrm{ab}}_{m,n}]}\left(\frac{\mathbf{y}^{(\nu)}_m}{\widetilde{\varphi}_{\mathcal{U}^{(\nu),\mathrm{ab}}_{m-1,n}}\big(\mathbf{y}^{(\nu)}_{m-1}\big)}\right) \quad \text{for all } \big(\mathbf{y}^{(\nu)}_m\big)\in \prod_{0\le m\le n-s}\mathbb{Z}_p\big[\mathcal{U}^{(\nu),\mathrm{ab}}_{m,n}\big]^{\times}. \tag{18}$$

To make the right-hand square commutative, we need to work out the map $\mathcal{L}^{(\nu)}_{\underline{\chi}}$ explicitly. Fix a finite order character $\chi:\mathcal{H}_\infty\to\mu_{p^\infty}$ factoring through the quotient group $\overline{\mathcal{H}}^{(m,n)}_\infty$, which one may interpret as a homomorphism

$$\chi:\mathcal{U}^{(\nu),\mathrm{ab}}_{m,n}\cong \Gamma^{p^m}/\Gamma^{p^\nu}\times \overline{\mathcal{H}}^{(m,n)}_\infty \longrightarrow \Gamma^{p^m}/\Gamma^{p^\nu}\times \mathrm{Im}(\chi)$$

sending an element $\gamma^j\cdot\overline{h}$ to $\gamma^j\cdot\chi(\overline{h})$. It follows that its extension to $\mathbb{Z}_p\big[\mathcal{U}^{(\nu),\mathrm{ab}}_{m,n}\big]$ satisfies

$$\chi\Big(\theta^{(\nu),+}_{m,n}\circ \mathrm{LOG}_{\mathcal{G}^{(\nu)}_{\infty,n}}(x)\Big) \;=\; \log_{\mathcal{O}_\chi\left[\frac{\Gamma^{p^m}}{\Gamma^{p^\nu}}\right]}\left(\frac{\chi\circ\theta^{(\nu)}_{m,n}(x)}{\varphi_{\frac{\Gamma^{p^{m-1}}}{\Gamma^{p^\nu}}}\big(\chi^p\circ\theta^{(\nu)}_{m-1,n}(x)\big)}\right).$$

Moreover by Proposition 24(b), for any $x^\dagger = x/x^{\mathrm{cy}}\in\mathcal{W}^{(\nu)}_\dagger$ one has

$$\chi\Big(\theta^{(\nu),+}_{m,n}\circ \mathrm{LOG}_{\mathcal{G}^{(\nu)}_{\infty,n}}(x^\dagger)\Big) = \log_{\mathcal{O}_\chi\left[\frac{\Gamma^{p^m}}{\Gamma^{p^\nu}}\right]}\left(\frac{\chi\circ\theta^{(\nu)}_{m,n}(x)}{\mathcal{N}_{0,m}\big(\theta^{(\nu)}_{0,n}(x)\big)}\cdot \varphi_{\frac{\Gamma^{p^{m-1}}}{\Gamma^{p^\nu}}}\left(\frac{\mathcal{N}_{0,m-1}\big(\theta^{(\nu)}_{0,n}(x)\big)}{\chi^p\circ\theta^{(\nu)}_{m-1,n}(x)}\right)\right)$$

as χ acts trivially on $\mathbb{Z}_p[\Gamma^{p^m}/\Gamma^{p^\nu}]$, and thus also on $\mathcal{N}_{0,m-1}\big(\theta^{(\nu)}_{0,n}(x)\big)$ and $\mathcal{N}_{0,m}\big(\theta^{(\nu)}_{0,n}(x)\big)$.

Since $\mathbf{y}_{m,\chi}^{(\nu)}$ corresponds to $\chi\circ\theta_{m,n}^{(\nu)}(x)$, the preceding formulae imply one should define

$$\mathcal{L}_{\underline{\chi}}^{(\nu)}\big((\mathbf{y}_{m,\chi}^{(\nu)})\big)_{m,\chi} := \log_{\mathcal{O}_\chi\left[\frac{\Gamma^{p^m}}{\Gamma^{p^\nu}}\right]}\left(\frac{\mathbf{y}_{m,\chi}^{(\nu)}}{\varphi_{\frac{\Gamma^{p^{m-1}}}{\Gamma^{p^\nu}}}\big(\mathbf{y}_{m-1,\chi^p}^{(\nu)}\big)}\right) \text{ where } \big(\mathbf{y}_{m,\chi}^{(\nu)}\big)\in\prod_{m,\chi}\mathcal{O}_\chi\left[\frac{\Gamma^{p^m}}{\Gamma^{p^\nu}}\right]^\times.$$

Indeed if $\big(\mathbf{y}_{m,\chi}^{(\nu)}\big)\in\prod_{m,\chi}\chi\circ\theta_{m,n}^{(\nu)}\big(\mathcal{W}_\dagger^{(\nu)}\big)$, then one can further say

$$\mathcal{L}_{\underline{\chi}}^{(\nu)}\big((\mathbf{y}_{m,\chi}^{(\nu)})\big)_{m,\chi} = \log_{\mathcal{O}_\chi\left[\frac{\Gamma^{p^m}}{\Gamma^{p^\nu}}\right]}\left(\frac{\mathbf{y}_{m,\chi}^{(\nu)}}{\mathcal{N}_{0,m}\big(\mathbf{y}_{0,\mathbf{1}}^{(\nu)}\big)}\cdot\varphi_{\frac{\Gamma^{p^{m-1}}}{\Gamma^{p^\nu}}}\left(\frac{\mathcal{N}_{0,m-1}\big(\mathbf{y}_{0,\mathbf{1}}^{(\nu)}\big)}{\mathbf{y}_{m-1,\chi^p}^{(\nu)}}\right)\right). \tag{19}$$

In fact $\frac{\mathbf{y}_{m,\chi}^{(\nu)}}{\mathcal{N}_{0,m}\big(\mathbf{y}_{0,\mathbf{1}}^{(\nu)}\big)}\in 1+p\cdot\mathcal{O}_\chi\big[\frac{\Gamma^{p^m}}{\Gamma^{p^\nu}}\big]$ for all m, so the full expression occurring inside the logarithm in Equation (19) must automatically be congruent to 1 modulo $p\cdot\mathcal{O}_{\mathbb{C}_p}\big[\frac{\Gamma^{p^m}}{\Gamma^{p^\nu}}\big]$.

Corollary 25. *If* $\big(\mathbf{y}_m^{(\nu)}\big)\in\Theta_{\infty,n}^{(\nu)}\big(\mathcal{W}_\dagger^{(\nu)}\big)$ *and one sets* $\big(\mathbf{y}_{m,\chi}^{(\nu)}\big)=\underline{\chi}\big((\mathbf{y}_m^{(\nu)})\big)$*, then both*

$\mathcal{L}^{(\nu)}\big((\mathbf{y}_m^{(\nu)})\big)\in\Psi^{(\nu)}\cap p\cdot\prod_m\mathrm{Im}\big(\sigma_m^{(\nu)}\big)$ *and* $\mathcal{L}_{\underline{\chi}}^{(\nu)}\big((\mathbf{y}_{m,\chi}^{(\nu)})\big)\in\underline{\chi}\big(\Psi^{(\nu)}\big)\cap p\cdot\prod_{m,\chi}\mathcal{O}_{\mathbb{C}_p}\big[\Gamma^{p^m}/\Gamma^{p^\nu}\big]$.

Proof. To address the first assertion, Proposition 24(b) implies that

$$\mathcal{L}^{(\nu)}\big((\mathbf{y}_m^{(\nu)})\big)_m = \log_{\mathbb{Z}_p[\mathcal{U}_{m,n}^{(\nu),\mathrm{ab}}]}\left(\frac{\mathbf{y}_m^{(\nu)}}{\mathcal{N}_{0,m}\big(\mathbf{y}_0^{(\nu)}\big)}\cdot\widetilde{\varphi}_{\mathcal{U}_{m-1,n}^{(\nu),\mathrm{ab}}}\left(\frac{\mathcal{N}_{0,m-1}\big(\mathbf{y}_0^{(\nu)}\big)}{\mathbf{y}_{m-1}^{(\nu)}}\right)\right)$$

and as each of the two fractions inside the logarithm belongs to the group $1+p\cdot\mathrm{Im}(\sigma_m^{(\nu)})$, the containment follows directly from Proposition 19(c).

To establish the second assertion, one combines the discussion after Equation (19) together with the isomorphism $\log: 1+p\cdot\mathcal{O}_{\mathbb{C}_p}\big[\Gamma^{p^m}/\Gamma^{p^\nu}\big]\xrightarrow{\sim}p\cdot\mathcal{O}_{\mathbb{C}_p}\big[\Gamma^{p^m}/\Gamma^{p^\nu}\big]$. □

4.4 A proof of Theorems 1 and 2

Recall from earlier that if a sequence $\big(\mathbf{y}_m^{(\nu)}\big)$ satisfies conditions (M1)-(M4), then its image under $\mathcal{L}^{(\nu)}$ always satisfies (A1)-(A3). We shall now establish a converse statement

$$\mathcal{L}^{(\nu)}\big((\mathbf{y}_m^{(\nu)})\big)\in p\cdot\Psi^{(\nu)} \implies \big(\mathbf{y}_m^{(\nu)}\big)\in\Phi^{(\nu)}.$$

If we are successful, the question as to whether or not $\big(\mathbf{y}_m^{(\nu)}\big)$ arises from $K_1\big(\mathbb{Z}_p[\mathcal{G}_{\infty,n}^{(\nu)}]\big)$ under the mapping $\Theta_{\infty,n}^{(\nu)}$ reduces to determining whether or not $\mathcal{L}_{\underline{\chi}}^{(\nu)}\big((\mathbf{y}_{m,\chi}^{(\nu)})\big)\in\underline{\chi}\big(\Psi^{(\nu)}\big)$. To achieve this goal, we will explicitly construct a section

$$\mathcal{S}^{(\nu)}:\left(\prod_{0\le m\le n-s}p\cdot\mathbb{Z}_p\big[\mathcal{U}_{m,n}^{(\nu),\mathrm{ab}}\big]\right)_{\text{(A1)-(A3)}}\longrightarrow\left(\prod_{0\le m\le n-s}1+p\cdot\mathbb{Z}_p\big[\mathcal{U}_{m,n}^{(\nu),\mathrm{ab}}\big]\right)_{\text{(M1)-(M4)}}$$

for which $\mathcal{L}^{(\nu)}\circ\mathcal{S}^{(\nu)}\Big|_{p\cdot\Psi^{(\nu)}}$ and $\mathcal{S}^{(\nu)}\circ\mathcal{L}^{(\nu)}\Big|_{\Theta_{\infty,n}^{(\nu)}(\mathcal{W}_\dagger^{(\nu)})}$ are the respective identity mappings.

To produce this map $\mathcal{S}^{(\nu)}$, let us first fix a sequence $\big(\mathbf{a}_m^{(\nu)}\big) \in \prod_{0\leq m\leq n-s} p\cdot\mathbb{Z}_p\big[\mathcal{U}_{m,n}^{(\nu),\mathrm{ab}}\big]$. Recall that $\exp : p\cdot\mathbb{Z}_p\big[\mathcal{U}_{m,n}^{(\nu),\mathrm{ab}}\big] \xrightarrow{\sim} 1+p\cdot\mathbb{Z}_p\big[\mathcal{U}_{m,n}^{(\nu),\mathrm{ab}}\big]$ is an isomorphism of abelian groups.

Definition 26. *Given the sequence $\big(\mathbf{a}_m^{(\nu)}\big)$ above, one recursively defines $\mathbf{y}_0^{(\nu)} := 1$ and*

$$\mathbf{y}_m^{(\nu)} \;:=\; \widetilde{\varphi}_{\mathcal{U}_{m-1,n}^{(\nu),\mathrm{ab}}}\big(\mathbf{y}_{m-1}^{(\nu)}\big) \times \exp_{\mathbb{Z}_p[\mathcal{U}_{m,n}^{(\nu),\mathrm{ab}}]}\big(\mathbf{a}_m^{(\nu)}\big) \text{ for each } m\geq 1,$$

so that $\big(\mathbf{y}_m\big) \in \prod_m 1+p\cdot\mathbb{Z}_p\big[\mathcal{U}_{m,n}^{(\nu),\mathrm{ab}}\big]$. We label this association $\big(\mathbf{a}_m^{(\nu)}\big) \mapsto \big(\mathbf{y}_m^{(\nu)}\big)$ by $\mathcal{S}^{(\nu)}$.

Lemma 27. *(i) The composition $\mathcal{L}^{(\nu)}\circ\mathcal{S}^{(\nu)}$ is the identity map on $\prod_m p\cdot\mathbb{Z}_p\big[\mathcal{U}_{m,n}^{(\nu),\mathrm{ab}}\big]$.*

(ii) The composition $\mathcal{S}^{(\nu)}\circ\mathcal{L}^{(\nu)}$ yields the identity map on $\prod_m 1+p\cdot\mathbb{Z}_p\big[\mathcal{U}_{m,n}^{(\nu),\mathrm{ab}}\big]$.

Proof. To establish the first assertion, one simply calculates that

$$\mathcal{L}^{(\nu)}\circ\mathcal{S}^{(\nu)}\big((\mathbf{a}_m^{(\nu)})\big)_m \;=\; \mathcal{L}^{(\nu)}\big((\mathbf{y}_m^{(\nu)})\big) \overset{\text{by (18)}}{=} \log_{\mathbb{Z}_p[\mathcal{U}_{m,n}^{(\nu),\mathrm{ab}}]}\left(\frac{\mathbf{y}_m^{(\nu)}}{\widetilde{\varphi}_{\mathcal{U}_{m-1,n}^{(\nu),\mathrm{ab}}}\big(\mathbf{y}_{m-1}^{(\nu)}\big)}\right)$$

$$\overset{\text{by 26}}{=} \log_{\mathbb{Z}_p[\mathcal{U}_{m,n}^{(\nu),\mathrm{ab}}]}\Big(\exp_{\mathbb{Z}_p[\mathcal{U}_{m,n}^{(\nu),\mathrm{ab}}]}\big(\mathbf{a}_m^{(\nu)}\big)\Big) \;=\; \mathbf{a}_m^{(\nu)}.$$

The proof of the second assertion follows along identical lines. □

For the rest of this section, we assume that $\big(\mathbf{a}_m^{(\nu)}\big) \in \prod_m p\cdot\mathbb{Z}_p\big[\mathcal{U}_{m,n}^{(\nu),\mathrm{ab}}\big]$ satisfies (A1)–(A3). The goal now is to prove that properties (M1)–(M4) all hold for $\big(\mathbf{y}_m^{(\nu)}\big) = \mathcal{S}^{(\nu)}\big((\mathbf{a}_m^{(\nu)})\big)$. Three of them are straightforward to deduce, but property (M3) requires more effort.

Establishing that $\mathcal{S}^{(\nu)}\big((\mathbf{a}_m^{(\nu)})\big)$ satisfies (M1),(M2),(M4). Let us begin by obtaining (M1). Since (A1) holds for the sequence $\big(\mathbf{a}_m^{(\nu)}\big)$, clearly

$$\mathcal{N}_{m-1,m}\circ\exp_{\mathbb{Z}_p[\mathcal{U}_{m-1,n}^{(\nu),\mathrm{ab}}]}\big(\mathbf{a}_{m-1}^{(\nu)}\big) \;=\; \exp_{\mathbb{Z}_p[\mathcal{U}_{m,n}^{(\nu),\mathrm{ab}}]}\circ\mathrm{Tr}_{m-1,m}\big(\mathbf{a}_{m-1}^{(\nu)}\big)$$

$$\overset{\text{by (A1)}}{=} \exp_{\mathbb{Z}_p[\mathcal{U}_{m,n}^{(\nu),\mathrm{ab}}]}\circ\pi_{m,m-1}\big(\mathbf{a}_m^{(\nu)}\big) \;=\; \pi_{m,m-1}\circ\exp_{\mathbb{Z}_p[\mathcal{U}_{m,n}^{(\nu),\mathrm{ab}}]}\big(\mathbf{a}_m^{(\nu)}\big)$$

i.e. $\dfrac{\mathcal{N}_{m-1,m}\big(\mathbf{y}_{m-1}^{(\nu)}\big)}{\mathcal{N}_{m-1,m}\big(\widetilde{\varphi}(\mathbf{y}_{m-2}^{(\nu)})\big)} = \dfrac{\pi_{m,m-1}\big(\mathbf{y}_m^{(\nu)}\big)}{\pi_{m,m-1}\big(\widetilde{\varphi}(\mathbf{y}_{m-1}^{(\nu)})\big)}$ for each $m\geq 1$. The latter is equivalent to

$$\mathcal{N}_{m-1,m}\big(\mathbf{y}_{m-1}^{(\nu)}\big) \;=\; \pi_{m,m-1}\big(\mathbf{y}_m^{(\nu)}\big) \;\times\; \widetilde{\varphi}_{\mathcal{U}_{m-1,n}^{(\nu),\mathrm{ab}}}\left(\frac{\mathcal{N}_{m-2,m-1}\big(\mathbf{y}_{m-2}^{(\nu)}\big)}{\pi_{m-1,m-2}\big(\mathbf{y}_{m-1}^{(\nu)}\big)}\right).$$

The equality between $\mathcal{N}_{m-1,m}\big(\mathbf{y}_{m-1}^{(\nu)}\big)$ and $\pi_{m,m-1}\big(\mathbf{y}_m^{(\nu)}\big)$ now follows by induction on m, thereby yielding (M1) as a consequence.

Focussing instead on (M2), the semi-direct product structure on $\mathcal{G}_{\infty,n}^{(\nu)} = \Gamma/\Gamma^{p^\nu} \ltimes \frac{\mathcal{H}_\infty}{\mathcal{H}_\infty^{p^n}}$ implies the subset of $\mathcal{G}_{\infty,n}^{(\nu)}$-invariant elements in $\mathbb{Z}_p[\mathcal{U}_{m,n}^{(\nu),\mathrm{ab}}]$ consists of

$$H^0\big(\mathcal{G}_{\infty,n}^{(\nu)}, \mathbb{Z}_p[\mathcal{U}_{m,n}^{(\nu),\mathrm{ab}}]\big) \;=\; H^0\big(\Gamma, \mathbb{Z}_p[\mathcal{U}_{m,n}^{(\nu),\mathrm{ab}}]\big) \;=\; \Big(\mathrm{Im}\big(\sigma_m^{(\nu)}\big)\big[1/p\big]\Big)\cap\mathbb{Z}_p\big[\mathcal{U}_{m,n}^{(\nu),\mathrm{ab}}\big].$$

Now (A2) states that $\mathbf{a}_m^{(\nu)}$ belongs to this subset, hence $\mathbf{y}_m^{(\nu)} \in \mathrm{Im}\big(\sigma_m^{(\nu)}\big)\big[1/p\big]\cap\mathbb{Z}_p[\mathcal{U}_{m,n}^{(\nu),\mathrm{ab}}]^\times$ upon combining the recurrence in Definition 26 with induction on m, and (M2) follows.

To show that (M4) holds true, consider the trace mapping $\mathrm{Tr}_{m,m+1}$ acting on $\mathbb{Z}_p\big[\mathcal{U}_{m,n}^{(\nu),\mathrm{ab}}\big]$. For each integer $m \geq 0$, one may decompose

$$\mathbb{Z}_p\big[\mathcal{U}_{m,n}^{(\nu),\mathrm{ab}}\big] \;\cong\; \mathbb{Z}_p\Big[\Gamma^{p^{m+1}}/\Gamma^{p^\nu} \times \overline{\mathcal{H}}_\infty^{(m,n)}\Big] \oplus \mathrm{Ker}\big(\mathrm{Tr}_{m,m+1}\big)$$

where by Lemma 14, the trace acts through multiplication by p on the first factor and kills off the second factor.

Note that $\mathbf{a}_m^{(\nu)} \in p \cdot \mathbb{Z}_p\big[\mathcal{U}_{m,n}^{(\nu),\mathrm{ab}}\big]$ so $\frac{1}{p}\mathrm{Tr}_{m,m+1}\big(\mathbf{a}_m^{(\nu)}\big) \equiv \mathbf{a}_m^{(\nu)} \bmod p \cdot \mathrm{Ker}\big(\mathrm{Tr}_{m,m+1}\big)$. Moreover the sequence $\big(\mathbf{a}_m^{(\nu)}\big)$ satisfies (A3), thus $p \cdot \mathbf{a}_m^{(\nu)} - \mathrm{Tr}_{m,m+1}\big(\mathbf{a}_m^{(\nu)}\big) \in p \cdot \mathrm{Im}\big(\sigma_m^{(\nu)}\big)$ and applying Proposition 19:

$$\exp_{\mathbb{Z}_p[\mathcal{U}_{m,n}^{(\nu),\mathrm{ab}}]}\Big(p \cdot \mathbf{a}_m^{(\nu)} - \mathrm{Tr}_{m,m+1}\big(\mathbf{a}_m^{(\nu)}\big)\Big) \in \; 1 + p \cdot \mathrm{Im}\big(\sigma_m^{(\nu)}\big).$$

It is easy to see $\exp\big(p \cdot \mathbf{a}_m^{(\nu)} - \mathrm{Tr}_{m,m+1}(\mathbf{a}_m^{(\nu)})\big) = \frac{\exp(\mathbf{a}_m^{(\nu)})^p}{\mathcal{N}_{m,m+1} \circ \exp(\mathbf{a}_m^{(\nu)})}$. Also, recalling from earlier that $\exp\big(\mathbf{a}_m^{(\nu)}\big) = \frac{\mathbf{y}_m^{(\nu)}}{\widetilde{\varphi}(\mathbf{y}_{m-1}^{(\nu)})}$, we therefore conclude

$$\frac{\big(\mathbf{y}_m^{(\nu)}\big)^p}{\widetilde{\varphi}_{\mathcal{U}_{m-1,n}^{(\nu),\mathrm{ab}}}\big(\mathbf{y}_{m-1}^{(\nu)}\big)^p} \times \left(\frac{\mathcal{N}_{m,m+1}\big(\mathbf{y}_m^{(\nu)}\big)}{\mathcal{N}_{m,m+1} \circ \widetilde{\varphi}_{\mathcal{U}_{m,n}^{(\nu),\mathrm{ab}}}\big(\mathbf{y}_{m-1}^{(\nu)}\big)}\right)^{-1} \in \; 1 + p \cdot \mathrm{Im}\big(\sigma_m^{(\nu)}\big).$$

Equivalently $\frac{\big(\mathbf{y}_m^{(\nu)}\big)^p}{\mathcal{N}_{m,m+1}\big(\mathbf{y}_m^{(\nu)}\big)} \times \widetilde{\varphi}_{\mathcal{U}_{m-1,n}^{(\nu),\mathrm{ab}}}\left(\frac{\big(\mathbf{y}_{m-1}^{(\nu)}\big)^p}{\mathcal{N}_{m-1,m}\big(\mathbf{y}_{m-1}^{(\nu)}\big)}\right)^{-1} \in 1 + p \cdot \mathrm{Im}\big(\sigma_m^{(\nu)}\big)$, so (M4) holds.

Establishing that $\mathcal{S}^{(\nu)}\big((\mathbf{a}_m^{(\nu)})\big)$ satisfies (M3). We begin with a technical result describing the image of the map $\widetilde{\sigma_m}^{(\nu)} \colon \mathbb{Z}_p\big[\mathcal{U}_{m,n}^{(\nu),\mathrm{ab}}\big] \to \mathbb{Z}_p\big[\mathcal{U}_{m,n}^{(\nu),\mathrm{ab}}\big]$ sending $f \mapsto \sum_{i=0}^{p-1} \gamma^{-p^{m-1}i} f \gamma^{p^{m-1}i}$.

Lemma 28. *For each $m \in \{0, \ldots, n-s\}$, the Γ-invariant submodule $H^0\big(\Gamma, \mathrm{Im}\big(\widetilde{\sigma_m}^{(\nu)}\big)\big)$ is finitely generated over $\mathbb{Z}_p\big[\Gamma/\Gamma^{p^\nu}\big]$ by the combined set*

$$\Big\{\mathcal{A}_\varpi^{(m,n)} \;\Big|\; \varpi \in \mathrm{orb}_\Gamma\big(\overline{\mathcal{H}}_\infty^{(m,n)}\big), \#\varpi = p^m\Big\} \cup \left\{\frac{\#\varpi}{p^{m-1}} \cdot \mathcal{A}_\varpi^{(m,n)} \;\Big|\; \varpi \in \mathrm{orb}_\Gamma\big(\overline{\mathcal{H}}_\infty^{(m,n)}\big), \#\varpi < p^m\right\}$$

and in particular, $\mathrm{Im}\big(\sigma_m^{(\nu)}\big) \subset H^0\big(\Gamma, \mathrm{Im}\big(\widetilde{\sigma_m}^{(\nu)}\big)\big) \subset \mathrm{Im}\big(\widetilde{\sigma_m}^{(\nu)}\big)$.

Proof. Because a generator $\gamma \in \Gamma$ acts trivially on $\Gamma^{p^m}/\Gamma^{p^\nu}$ and through $I_2 + M$ on $\overline{\mathcal{H}}_\infty^{(m,n)}$,

$$\begin{aligned} H^0\big(\Gamma, \mathbb{Z}_p\big[\mathcal{U}_{m,n}^{(\nu),\mathrm{ab}}\big]\big) \;&=\; \mathbb{Z}_p\big[\Gamma^{p^m}/\Gamma^{p^\nu}\big] \otimes_{\mathbb{Z}_p} H^0\Big(\big\langle I_2 + M\big\rangle, \mathbb{Z}_p\big[\overline{\mathcal{H}}_\infty^{(m,n)}\big]\Big) \\ &=\; \mathbb{Z}_p\big[\Gamma^{p^m}/\Gamma^{p^\nu}\big] \cdot \left\langle \sum_{\overline{h}' \in \varpi} \overline{h}' \;\middle|\; \varpi \in \mathrm{orb}_\Gamma\big(\overline{\mathcal{H}}_\infty^{(m,n)}\big) \right\rangle \\ &=\; \mathbb{Z}_p\big[\Gamma^{p^m}/\Gamma^{p^\nu}\big] \cdot \left\langle \frac{\#\varpi}{p^m} \cdot \mathcal{A}_\varpi^{(m,n)} \;\middle|\; \varpi \in \mathrm{orb}_\Gamma\big(\overline{\mathcal{H}}_\infty^{(m,n)}\big) \right\rangle \end{aligned}$$

where we have employed the basic identity $\mathcal{A}_{\varpi_{\overline{h}}}^{(m,n)} = \sum_{i=0}^{p^m-1} \gamma^{-i}\overline{h}\gamma^i = \frac{p^m}{\#\varpi_{\overline{h}}} \cdot \sum_{\overline{h}' \in \varpi_{\overline{h}}} \overline{h}'$.

Now pick an element $\frac{\#\varpi_{\overline{h}}}{p^m} \cdot \mathcal{A}^{(m,n)}_{\varpi_{\overline{h}}} = \sum_{\overline{h}' \in \varpi_{\overline{h}}} \overline{h}'$ belonging to $H^0\big(\langle I_2 + M\rangle, \mathbb{Z}_p\big[\overline{\mathcal{H}}^{(m,n)}_\infty\big]\big)$. Then one easily sees that

$$\frac{\#\varpi_{\overline{h}}}{p^m} \cdot \mathcal{A}^{(m,n)}_{\varpi_{\overline{h}}} \;=\; \frac{\#\varpi_{\overline{h}}}{p^m} \cdot \sum_{j=0}^{p^m-1} \gamma^{-j}\overline{h}\gamma^j \;=\; \sum_{i=0}^{p-1}\sum_{j=0}^{p^{m-1}-1} \frac{\#\varpi_{\overline{h}}}{p^m} \cdot \gamma^{-p^{m-1}i}\big(\gamma^{-j}\overline{h}\gamma^j\big)\gamma^{p^{m-1}i}$$

which coincides exactly with $\widetilde{\sigma_m}^{(\nu)}\big(f_{\overline{h}}\big)$, where $f_{\overline{h}} := \frac{\#\varpi_{\overline{h}}}{p^m} \cdot \sum_{j=0}^{p^{m-1}-1} \gamma^{-j}\overline{h}\gamma^j \in \mathbb{Q}_p\big[\overline{\mathcal{H}}^{(m,n)}_\infty\big]$. It follows that $p^z \cdot \Big(\frac{\#\varpi_{\overline{h}}}{p^m} \cdot \mathcal{A}^{(m,n)}_{\varpi_{\overline{h}}}\Big) \in \mathrm{Im}\big(\widetilde{\sigma_m}^{(\nu)}\big)$ if and only if $p^z \cdot f_{\overline{h}} \in \mathbb{Z}_p\big[\overline{\mathcal{H}}^{(m,n)}_\infty\big]$, and as

$$p^z \cdot f_{\overline{h}} \;=\; \begin{cases} p^z \cdot \sum_{j=0}^{p^{m-1}-1} \gamma^{-j}\overline{h}\gamma^j & \text{if } \#\varpi_{\overline{h}} = p^m \\ p^{z-1} \cdot \sum_{\overline{h}' \in \varpi_{\overline{h}}} \overline{h}' & \text{if } \#\varpi_{\overline{h}} < p^m, \end{cases}$$

the latter condition occurs when $z \geq 0$ if $\#\varpi = p^m$, or alternatively $z \geq 1$ if $\#\varpi < p^m$. Therefore the union of the sets $\big\{f_{\overline{h}} \;\big|\; \#\varpi_{\overline{h}} = p^m\big\}$ and $\big\{p \cdot f_{\overline{h}} \;\big|\; \#\varpi_{\overline{h}} < p^m\big\}$ will generate the Γ-invariant part of $\mathrm{Im}\big(\widetilde{\sigma_m}^{(\nu)}\big)$ over $\mathbb{Z}_p\big[\Gamma/\Gamma^{p^\nu}\big]$, as asserted.

Finally, the inclusion $\mathrm{Im}\big(\sigma_m^{(\nu)}\big) \hookrightarrow H^0\big(\Gamma, \mathrm{Im}\big(\widetilde{\sigma_m}^{(\nu)}\big)\big)$ occurs as the generators $\mathcal{A}^{(m,n)}_{\varpi}$ of the left-hand module are p-integral multiples of generators for the right-hand module. □

Proposition 29. *For each $m \geq 1$, the transfer sends* $p \cdot \mathrm{Im}(\sigma_{m-1}) \overset{\mathrm{Ver}_{m-1,m}}{\longrightarrow} \mathrm{Im}\big(\widetilde{\sigma_m}^{(\nu)}\big)$.

Proof. If we choose any $\overline{h} = \overline{h}_1^x \overline{h}_2^y \in \overline{\mathcal{H}}^{(m-1,n)}_\infty$ and $f(X) \in \mathbb{Z}_p[\![X]\!]$, then from Lemma 18:

$$\mathrm{Ver}_{m-1,m}\Big(f\big(\gamma^{p^{m-1}} - 1\big) \cdot \mathcal{A}^{(m-1,n)}_{\overline{h}_1^x\overline{h}_2^y}\Big) \;=\; p^{-1} \times f\big(\gamma^{p^m} - 1\big) \cdot \mathcal{A}^{(m,n)}_{\overline{h}_1^{x'}\overline{h}_2^{y'}}$$

where $\begin{pmatrix} x' \\ y' \end{pmatrix} \in \mathbb{Z}_p^2$ is given in Lemma 12. Setting $f(X) = p$, it follows immediately that

$$\mathrm{Ver}_{m-1,m}\Big(p \cdot \mathcal{A}^{(m-1,n)}_{\overline{h}_1^x\overline{h}_2^y}\Big) \;=\; \mathcal{A}^{(m,n)}_{\overline{h}_1^{x'}\overline{h}_2^{y'}} \;\in\; \mathrm{Im}\big(\sigma_m^{(\nu)}\big) \overset{\text{by } 28}{\hookrightarrow} \mathrm{Im}\big(\widetilde{\sigma_m}^{(\nu)}\big).$$

Lastly applying Proposition 10(ii), we know $p \cdot \mathrm{Im}\big(\sigma_{m-1}^{(\nu)}\big)$ is freely generated over the algebra $\mathbb{Z}_p\big[\Gamma^{p^{m-1}}/\Gamma^{p^\nu}\big]$ by the set of $p \cdot \mathcal{A}^{(m-1,n)}_{\overline{h}_1^x\overline{h}_2^y}$'s, hence the result is proven. □

Let us now establish that (M3) holds for $\big(\mathbf{y}_m^{(\nu)}\big) = \mathcal{S}^{(\nu)}\big(\big(\mathbf{a}_m^{(\nu)}\big)\big)$. For each integer $m \geq 2$,

$$\frac{\mathbf{y}_m^{(\nu)}}{\mathrm{Ver}_{m-1,m}\big(\mathbf{y}_{m-1}^{(\nu)}\big)} \overset{\text{by } 26}{=} \frac{\widetilde{\varphi}_{\mathcal{U}^{(\nu),\mathrm{ab}}_{m-1,n}}\big(\mathbf{y}_{m-1}^{(\nu)}\big) \times \exp_{\mathbb{Z}_p[\mathcal{U}^{(\nu),\mathrm{ab}}_{m,n}]}\big(\mathbf{a}_m^{(\nu)}\big)}{\mathrm{Ver}_{m-1,m}\Big(\widetilde{\varphi}_{\mathcal{U}^{(\nu),\mathrm{ab}}_{m-2,n}}\big(\mathbf{y}_{m-2}^{(\nu)}\big) \times \exp_{\mathbb{Z}_p[\mathcal{U}^{(\nu),\mathrm{ab}}_{m-1,n}]}\big(\mathbf{a}_{m-1}^{(\nu)}\big)\Big)}$$

$$= \widetilde{\varphi}_{\mathcal{U}^{(\nu),\mathrm{ab}}_{m-1,n}}\left(\frac{\mathbf{y}_{m-1}^{(\nu)}}{\mathrm{Ver}_{m-2,m-1}\big(\mathbf{y}_{m-2}^{(\nu)}\big)}\right) \times \exp_{\mathbb{Z}_p[\mathcal{U}^{(\nu),\mathrm{ab}}_{m,n}]}\Big(\mathbf{a}_m^{(\nu)} - \mathrm{Ver}_{m-1,m}\big(\mathbf{a}_{m-1}^{(\nu)}\big)\Big)$$

and the term $\mathbf{a}_m^{(\nu)} - \mathrm{Ver}_{m-1,m}\big(\mathbf{a}_{m-1}^{(\nu)}\big) \in \mathrm{Im}\big(\widetilde{\sigma_m}^{(\nu)}\big)$, using Lemma 28 and Proposition 29.

An identical argument to Proposition 19(b) shows that

$$\exp_{\mathbb{Z}_p[\mathcal{U}^{(\nu),\mathrm{ab}}_{m,n}]}:\ \frac{\mathrm{Im}(\widetilde{\sigma_m}^{(\nu)})^N}{\mathrm{Im}(\widetilde{\sigma_m}^{(\nu)})^{N+1}}\ \overset{\sim}{\longrightarrow}\ \frac{1+\mathrm{Im}(\widetilde{\sigma_m}^{(\nu)})^N}{1+\mathrm{Im}(\widetilde{\sigma_m}^{(\nu)})^{N+1}}$$

is an isomorphism for every $N\geq 1$, in which case

$$\frac{\mathbf{y}_m^{(\nu)}}{\mathrm{Ver}_{m-1,m}\big(\mathbf{y}_{m-1}^{(\nu)}\big)}\ =\ \widetilde{\varphi}_{\mathcal{U}^{(\nu),\mathrm{ab}}_{m-1,n}}\left(\frac{\mathbf{y}_{m-1}^{(\nu)}}{\mathrm{Ver}_{m-2,m-1}\big(\mathbf{y}_{m-2}^{(\nu)}\big)}\right)\times\big(1+\mathbf{d}_m\big)$$

for some $\mathbf{d}_m\in\mathrm{Im}\big(\widetilde{\sigma_m}^{(\nu)}\big)$.

Furthermore, one easily checks the containment $\widetilde{\varphi}_{\mathcal{U}^{(\nu),\mathrm{ab}}_{m-1,n}}\Big(\mathrm{Im}\big(\widetilde{\sigma_{m-1}}^{(\nu)}\big)\Big)\subset\mathrm{Im}\big(\widetilde{\sigma_m}^{(\nu)}\big)$. Therefore, if we inductively assume $\frac{\mathbf{y}_{m-1}^{(\nu)}}{\mathrm{Ver}_{m-2,m-1}\left(\mathbf{y}_{m-2}^{(\nu)}\right)}\in 1+\mathrm{Im}\big(\widetilde{\sigma_{m-1}}^{(\nu)}\big)$, one may conclude $\frac{\mathbf{y}_{m}^{(\nu)}}{\mathrm{Ver}_{m-1,m}\left(\mathbf{y}_{m-1}^{(\nu)}\right)}\in 1+\mathrm{Im}\big(\widetilde{\sigma_m}^{(\nu)}\big)$. Property (M3) then follows for all $m\geq 2$ by induction. (If $m=1$ the same argument works fine, except one omits the denominator terms above.)

Proof of Theorem 2. As mentioned earlier, now that we have constructed the section $\mathcal{S}^{(\nu)}$ mapping $p\cdot\Psi^{(\nu)}$ into $\Phi^{(\nu)}$, to check whether $\big(\mathbf{y}_m^{(\nu)}\big)$ arises from an element of $K_1\big(\mathbb{Z}_p[\mathcal{G}^{(\nu)}_{\infty,n}]\big)$ it is the same as verifying if $\mathcal{L}^{(\nu)}_{\underline{\chi}}\big((\mathbf{y}^{(\nu)}_{m,\chi})\big)\in\underline{\chi}\big(\Psi^{(\nu)}\big)$. However, the latter is equivalent to checking whether $\mathcal{L}^{(\nu)}_{\underline{\chi}}\big((\mathbf{y}^{(\nu)}_{m,\chi})\big)$ satisfies the conditions (C1)–(C4) listed in Theorem 15.

Theorem 30. *If $\star\in\{III,IV,V,VI\}$, then $\mathcal{L}^{(\nu)}_{\underline{\chi}}\big((\mathbf{y}^{(\nu)}_{m,\chi})\big)$ satisfies conditions (C1)–(C4) in Theorem 15 if and only if:*

(i) $\mathcal{N}_{\mathrm{Stab}_\Gamma(\chi)/\Gamma^{p^m}}\big(\mathbf{y}^{(\nu)}_{\mathbf{m}_\chi,\chi}\big)=\mathbf{y}^{(\nu)}_{m,\chi}$ *at each* $m\in\{\mathbf{m}_\chi,\dots,n-s\}$,

(ii) $\mathbf{y}^{(\nu)}_{m,\chi'}=\mathbf{y}^{(\nu)}_{m,\chi}$ *whenever* $\chi'\in\Gamma*\chi$, *and*

(iii)
$$\prod_{\chi\in\mathfrak{R}_{m,\infty}}\mathcal{N}_{\mathrm{Stab}_\Gamma(\chi)/\Gamma^{p^m}}\left(\frac{\mathbf{y}_\chi^{(\nu)}}{\varphi\big(\mathbf{y}_{\chi^p}^{(\nu)}\big)}\cdot\frac{\varphi\big(\mathcal{N}_{0,\mathbf{m}_\chi-1}\big(\mathbf{y}_{\mathbf{1}}^{(\nu)}\big)\big)}{\mathcal{N}_{0,\mathbf{m}_\chi}\big(\mathbf{y}_{\mathbf{1}}^{(\nu)}\big)}\right)^{\mathrm{Tr}(\mathrm{Ind}\chi^*)(\varpi)}$$
$$\equiv\ 1\mod p^{N^{(m)}_{\star,1}+N^{(m)}_{\star,2}+m-\mathrm{ord}_p(\#\varpi)}\cdot\mathbb{Z}_p\big[\Gamma^{p^m}/\Gamma^{p^\nu}\big]$$

for every integer $m\in\{0,\dots,\nu\}$, and every orbit $\varpi\in\mathrm{orb}_\Gamma\big(\overline{\mathcal{H}}^{(m,\infty)}_\infty\big)$.

Proof. If one chooses the sequence $\big(\mathbf{a}_\chi^{(m,\nu)}\big):=\mathcal{L}^{(\nu)}_{\underline{\chi}}\big((\mathbf{y}^{(\nu)}_{m,\chi})\big)$, then (C1) is readily seen to be equivalent to (i), while condition (C2) is equivalent to (ii). Focussing therefore on conditions (C3) and (C4), if one puts $\mathbf{e}^*_{\chi,\varpi}=\mathrm{Tr}\big(\mathrm{Ind}\chi^*\big)(\varpi)$ then

$$\sum_{\chi\in\mathfrak{R}_{m,n}}\mathrm{Tr}_{\mathrm{Stab}_\Gamma(\chi)/\Gamma^{p^m}}\big(\mathbf{a}_\chi^{(\nu)}\big)\cdot\mathrm{Tr}\big(\mathrm{Ind}\chi^*\big)(\varpi)\ =\ \sum_{\chi\in\mathfrak{R}_{m,n}}\mathbf{e}^*_{\chi,\varpi}\times\mathrm{Tr}_{\mathrm{Stab}_\Gamma(\chi)/\Gamma^{p^m}}\big(\mathbf{a}_\chi^{(\nu)}\big)$$
$$\overset{\text{by (19)}}{=}\sum_{\chi\in\mathfrak{R}_{m,n}}\mathbf{e}^*_{\chi,\varpi}\times\mathrm{Tr}_{\mathrm{Stab}_\Gamma(\chi)/\Gamma^{p^m}}\circ\log\left(\frac{\mathbf{y}_\chi^{(\nu)}}{\mathcal{N}_{0,\mathbf{m}_\chi}\big(\mathbf{y}_{\mathbf{1}}^{(\nu)}\big)}\cdot\varphi_{\frac{\Gamma^{p^{\mathbf{m}_\chi-1}}}{\Gamma^{p^\nu}}}\left(\frac{\mathcal{N}_{0,\mathbf{m}_\chi-1}\big(\mathbf{y}_{\mathbf{1}}^{(\nu)}\big)}{\mathbf{y}_{\chi^p}^{(\nu)}}\right)\right)$$
$$=\ \log_{\mathbb{Z}_p\left[\frac{\Gamma^{p^m}}{\Gamma^{p^\nu}}\right]}\left(\prod_{\chi\in\mathfrak{R}_{m,n}}\mathcal{N}_{\mathrm{Stab}_\Gamma(\chi)/\Gamma^{p^m}}\left(\frac{\mathbf{y}_\chi^{(\nu)}}{\mathcal{N}_{0,\mathbf{m}_\chi}\big(\mathbf{y}_{\mathbf{1}}^{(\nu)}\big)}\cdot\varphi_{\frac{\Gamma^{p^{\mathbf{m}_\chi-1}}}{\Gamma^{p^\nu}}}\left(\frac{\mathcal{N}_{0,\mathbf{m}_\chi-1}\big(\mathbf{y}_{\mathbf{1}}^{(\nu)}\big)}{\mathbf{y}_{\chi^p}^{(\nu)}}\right)\right)^{\mathbf{e}^*_{\chi,\varpi}}\right).$$

Recall that (C3) and (C4) together imply $\sum_{\chi\in\mathfrak{R}_{m,n}}\mathrm{Tr}_{\mathrm{Stab}_\Gamma(\chi)/\Gamma^{p^m}}\big(\mathbf{a}_\chi^{(\nu)}\big)\cdot\mathrm{Tr}\big(\mathrm{Ind}\chi^*\big)(\varpi)$ is congruent to zero modulo $p^{\mathrm{ord}_p(\#\overline{\mathcal{H}}_\infty^{(m,n)})+m-\mathrm{ord}_p(\#\varpi)}\cdot\mathbb{Z}_p[\Gamma^{p^m}/\Gamma^{p^\nu}]$, for $m\in\{0,\ldots,n-s\}$ and at each orbit $\varpi\in\mathrm{orb}_\Gamma\big(\overline{\mathcal{H}}_\infty^{(m,n)}\big)$. Now for all integers $i\geq 1$, the mappings $\log: 1+p^i\cdot\mathbb{Z}_p[\Gamma^{p^m}/\Gamma^{p^\nu}]\xrightarrow{\sim}p^i\cdot\mathbb{Z}_p[\Gamma^{p^m}/\Gamma^{p^\nu}]$ and $\exp: p^i\cdot\mathbb{Z}_p[\Gamma^{p^m}/\Gamma^{p^\nu}]\xrightarrow{\sim}1+p^i\cdot\mathbb{Z}_p[\Gamma^{p^m}/\Gamma^{p^\nu}]$ are inverse isomorphisms to each other. As an immediate consequence,

$$\sum_{\chi\in\mathfrak{R}_{m,n}}\mathrm{Tr}_{\mathrm{Stab}_\Gamma(\chi)/\Gamma^{p^m}}\big(\mathbf{a}_\chi^{(\nu)}\big)\cdot\mathrm{Tr}\big(\mathrm{Ind}\chi^*\big)(\varpi)\equiv 0\mod p^{\mathrm{ord}_p(\#\overline{\mathcal{H}}_\infty^{(m,n)})+m-\mathrm{ord}_p(\#\varpi)}\cdot\mathbb{Z}_p\left[\frac{\Gamma^{p^m}}{\Gamma^{p^\nu}}\right]$$

if and only if $\prod_{\chi\in\mathfrak{R}_{m,n}}\mathcal{N}_{\mathrm{Stab}_\Gamma(\chi)/\Gamma^{p^m}}\left(\frac{\mathbf{y}_\chi^{(\nu)}}{\varphi\big(\mathbf{y}_{\chi^p}^{(\nu)}\big)}\cdot\frac{\varphi\big(\mathcal{N}_{0,\mathbf{m}_\chi-1}\big(\mathbf{y}_\mathbf{1}^{(\nu)}\big)\big)}{\mathcal{N}_{0,\mathbf{m}_\chi}\big(\mathbf{y}_\mathbf{1}^{(\nu)}\big)}\right)^{\mathrm{Tr}(\mathrm{Ind}\chi^*)(\varpi)}$ belongs to $1+p^{\mathrm{ord}_p(\#\overline{\mathcal{H}}_\infty^{(m,n)})+m-\mathrm{ord}_p(\#\varpi)}\cdot\mathbb{Z}_p[\Gamma^{p^m}/\Gamma^{p^\nu}]$.

Finally, both $\overline{\mathcal{H}}_\infty^{(m,n)}\cong\overline{\mathcal{H}}_\infty^{(m,\infty)}$ and $\mathfrak{R}_{m,n}=\mathfrak{R}_{m,\infty}$ provided that $\star\in\{\mathrm{III,IV,V,VI}\}$; moreover $\mathrm{ord}_p\big(\#\overline{\mathcal{H}}_\infty^{(m,n)}\big)=N_{\star,1}^{(m)}+N_{\star,2}^{(m)}$, therefore the equivalence is fully established. □

The reader will notice that these congruences are independent of the choice of $n\geq m+s$. They also behave well if we take the projective limit as $\nu\to\infty$, hence one can obtain analogous congruences for the completed group algebras $\mathbb{Z}_p\big[\big[\Gamma^{p^m}\big]\big]=\varprojlim_\nu\mathbb{Z}_p[\Gamma^{p^m}/\Gamma^{p^\nu}]$, i.e. those congruences labelled Equation (2) in §1.2.

The proof of the 'non-$\mathcal{S}$-localised version' of Theorem 2 has therefore been completed, i.e. a sequence $\big(\mathbf{y}_{m,\chi}\big)\in\prod_{m,\chi}\Lambda_{\mathcal{O}_{\mathbb{C}_p}}\big(\Gamma^{p^m}\big)^\times$ belongs to $\Theta_{\infty,\underline{\chi}}\big(K_1'(\Lambda(\mathcal{G}_\infty))\big)$ if and only if $\mathcal{N}_{\mathrm{Stab}_\Gamma(\chi)/\Gamma^{p^m}}\big(\mathbf{y}_{\mathbf{m}_\chi,\chi}^{(\nu)}\big)=\mathbf{y}_{m,\chi}^{(\nu)}$ if $m\geq\mathbf{m}_\chi$, secondly $\mathbf{y}_{m,\chi'}^{(\nu)}=\mathbf{y}_{m,\chi}^{(\nu)}$ for $\chi'\in\Gamma*\chi$, and lastly

$$\prod_{\chi\in\mathfrak{R}_{m,\infty}}\mathcal{N}_{\mathrm{Stab}_\Gamma(\chi)/\Gamma^{p^m}}\left(\frac{\mathbf{y}_\chi}{\varphi\big(\mathbf{y}_{\chi^p}\big)}\cdot\frac{\varphi\big(\mathcal{N}_{0,\mathbf{m}_\chi-1}\big(\mathbf{y}_\mathbf{1}\big)\big)}{\mathcal{N}_{0,\mathbf{m}_\chi}\big(\mathbf{y}_\mathbf{1}\big)}\right)^{\mathrm{Tr}(\mathrm{Ind}\chi^*)(\varpi)}$$
$$\equiv 1\mod p^{N_{\star,1}^{(m)}+N_{\star,2}^{(m)}+m-\mathrm{ord}_p(\#\varpi)}\cdot\mathbb{Z}_p\big[\big[\Gamma^{p^m}\big]\big]$$

for every positive integer m, and at every orbit $\varpi\in\mathrm{orb}_\Gamma\big(\overline{\mathcal{H}}_\infty^{(m,\infty)}\big)$.

Remarks: (a) If $\star=$II, the proof of Theorem 1 runs along identical lines – the only point of departure is that $N_{II,1}^{(m)}=n$ and $N_{II,2}^{(m)}=s+m$, so $\mathfrak{R}_{m,n}$ is no longer independent of n. Nevertheless in Case (II), the multiplicative conditions equivalent to (C3) and (C4) are

$$\prod_{\chi\in\mathfrak{R}_{m,n}}\mathcal{N}_{\mathrm{Stab}_\Gamma(\chi)/\Gamma^{p^m}}\left(\frac{\mathbf{y}_\chi}{\varphi\big(\mathbf{y}_{\chi^p}\big)}\cdot\frac{\varphi\big(\mathcal{N}_{0,\mathbf{m}_\chi-1}\big(\mathbf{y}_\mathbf{1}\big)\big)}{\mathcal{N}_{0,\mathbf{m}_\chi}\big(\mathbf{y}_\mathbf{1}\big)}\right)^{\mathrm{Tr}(\mathrm{Ind}\chi^*)(\varpi)}$$
$$\equiv 1\mod p^{s+2m+n-\mathrm{ord}_p(\#\varpi)}\cdot\mathbb{Z}_p\big[\big[\Gamma^{p^m}\big]\big]\qquad(20)$$

for every positive integer $m\leq n-s$, and at every orbit $\varpi\in\mathrm{orb}_\Gamma\big(\overline{\mathcal{H}}_\infty^{(m,n)}\big)$.

(b) To transform these into the congruences labelled Equation (1), one must calculate each of $\mathfrak{R}_{m,n}$, $\#\varpi$ and $\mathrm{Tr}(\mathrm{Ind}\chi^*)(\varpi)$ precisely – we refer the reader to the worked example given later in §5.1, for the full details.

(c) Of course, this still only gives us a non-$\mathcal{S}$-localised version of Theorem 1, describing $\Theta_{\infty,\underline{\chi}}\big(K_1'(\Lambda(\mathcal{G}_\infty))\big)$ rather than $\Theta_{\infty,\mathcal{S},\underline{\chi}}\big(K_1'\big(\Lambda(\mathcal{G}_\infty)_\mathcal{S}\big)\big)$, which is an issue we address below.

Extending these congruences to the localisations. Finally, we explain how to extend these results from $K_1'\big(\Lambda(\mathcal{G}_\infty)\big)$, to both of the Ore localisations $K_1'\big(\Lambda(\mathcal{G}_\infty)_{\mathcal{S}}\big)$ and $K_1'\big(\Lambda(\mathcal{G}_\infty)_{\mathcal{S}^*}\big)$. Let us focus first on $K_1\big(\Lambda(\mathcal{G}_\infty)_{\mathcal{S}}\big)$, and write

$$\Theta_{\infty,\mathcal{S}} : K_1\big(\Lambda(\mathcal{G}_\infty)_{\mathcal{S}}\big) \to \prod_{m\geq 0} K_1\big(\Lambda(\mathcal{U}_m^{\mathrm{ab}})_{\overline{\mathcal{S}}}\big)$$

for the corresponding collection of morphisms $\prod \theta_{m,\mathcal{S}}$, with $\theta_{m,\mathcal{S}} := \mathcal{N}_{\mathcal{U}_m}(-) \bmod [\mathcal{U}_m,\mathcal{U}_m]$.

In order to extend the arguments in §4.1-§4.3 so as to produce non-abelian congruence conditions '$\Phi_{\mathcal{S}}$' describing $\mathrm{Im}\big(\Theta_{\infty,\mathcal{S}}\big)$, one must first extend the Taylor-Oliver logarithm to a homomorphism

$$\mathrm{LOG}_{\mathcal{G}_{\infty,n},\mathcal{S}} : K_1\Big(\Lambda(\widehat{\mathcal{G}_{\infty,n}})_{\mathcal{S}}\Big) \longrightarrow \frac{\Lambda(\widehat{\mathcal{G}_{\infty,n}})_{\mathcal{S}}}{\big[\Lambda(\widehat{\mathcal{G}_{\infty,n}})_{\mathcal{S}}, \Lambda(\widehat{\mathcal{G}_{\infty,n}})_{\mathcal{S}}\big]} \quad \text{for every } n \geq 1,$$

where $\Lambda(\widehat{\mathcal{G}_{\infty,n}})_{\mathcal{S}}$ denotes the $\mathrm{Jac}\big(\mathbb{Z}_p[\mathcal{H}_{\infty,n}]\big)$-adic completion of the localisation $\Lambda(\mathcal{G}_{\infty,n})_{\mathcal{S}}$. This task has already been partially accomplished (see for example [7, Section 5] or [17]), but not enough is known about the kernel and cokernel of these maps on the completion. Indeed by [7, Lemma 5.2], the extension of the logarithm sits inside a commutative square

$$\begin{array}{ccc} K_1\big(\Lambda(\mathcal{G}_{\infty,n})\big) & \longrightarrow & K_1\big(\Lambda(\widehat{\mathcal{G}_{\infty,n}})_{\mathcal{S}}\big) \\ \big\downarrow \scriptstyle{\mathrm{LOG}_{\mathcal{G}_{\infty,n}}} & & \big\downarrow \scriptstyle{\mathrm{LOG}_{\mathcal{G}_{\infty,n},\mathcal{S}}} \\ \mathbb{Z}_p\big[\big[\mathrm{Conj}(\mathcal{G}_{\infty,n})\big]\big] & \longrightarrow & \dfrac{\Lambda(\widehat{\mathcal{G}_{\infty,n}})_{\mathcal{S}}}{\big[\Lambda(\widehat{\mathcal{G}_{\infty,n}})_{\mathcal{S}}, \Lambda(\widehat{\mathcal{G}_{\infty,n}})_{\mathcal{S}}\big]} \end{array}$$

where the horizontal arrows are induced from the natural inclusion $\Lambda(\mathcal{G}_{\infty,n}) \hookrightarrow \Lambda(\widehat{\mathcal{G}_{\infty,n}})_{\mathcal{S}}$.

We simply observe that the properties of the Taylor-Oliver logarithm we derived in §4.3 extend to the $\mathrm{Jac}\big(\mathbb{Z}_p[\mathcal{H}_{\infty,n}]\big)$-adic completion if one ignores their kernels/cokernels, and omit the details (which are anyway identical to Section 5 of *op. cit.*). The remainder of the proof of Theorems 1 and 2 in the $\mathcal{S}$-localised situation then follows readily, albeit the congruences in Equations (1) and (2) are now taken modulo $p^\bullet \cdot \mathbb{Z}_p\big[\big[\Gamma^{p^m}\big]\big]_{(p)}$ rather than just modulo $p^\bullet \cdot \mathbb{Z}_p\big[\big[\Gamma^{p^m}\big]\big]$, and we unfortunately lose their sufficiency in the process.

We now turn our attention to the $\mathcal{S}^*$-localisation, $\Lambda(\mathcal{G}_\infty)_{\mathcal{S}^*}$, which is less problematic. Recall that $\mathcal{G}_\infty$ has no element of order p, in which case Burns and Venjakob [3, Prop 3.4] have constructed a splitting

$$K_1\big(\Lambda(\mathcal{G}_\infty)_{\mathcal{S}^*}\big) \;\cong\; K_1\big(\Lambda(\mathcal{G}_\infty)_{\mathcal{S}}\big) \oplus K_0\big(\mathbb{F}_p[\![\mathcal{G}_\infty]\!]\big).$$

Furthermore, there exists another commutative diagram

$$\begin{array}{ccc} K_1\big(\Lambda(\mathcal{G}_\infty)_{\mathcal{S}^*}\big) & \stackrel{\sim}{\longrightarrow} & K_1\big(\Lambda(\mathcal{G}_\infty)_{\mathcal{S}}\big) \oplus K_0\big(\mathbb{F}_p[\![\mathcal{G}_\infty]\!]\big) \\ \big\downarrow \scriptstyle{\Theta_{\infty,\mathcal{S}^*}} & & \big\downarrow \scriptstyle{(\Theta_{\infty,\mathcal{S}},\Theta_0)} \\ \prod\limits_{m\geq 0} K_1\big(\Lambda(\mathcal{U}_m^{\mathrm{ab}})_{\overline{\mathcal{S}}^*}\big) & \hookleftarrow & \prod\limits_{m\geq 0} K_1\big(\Lambda(\mathcal{U}_m^{\mathrm{ab}})_{\overline{\mathcal{S}}}\big) \oplus K_0\big(\mathbb{F}_p[\![\mathcal{U}_m^{\mathrm{ab}}]\!]\big) \end{array}$$

where the map $\Theta_0 : K_0\big(\mathbb{F}_p[\![\mathcal{G}_\infty]\!]\big) \to \prod_{m\geq 0} K_0\big(\mathbb{F}_p[\![\mathcal{U}_m^{\mathrm{ab}}]\!]\big)$ encodes how the non-commutative μ-invariant information in $K_0\big(\mathbb{F}_p[\![\mathcal{G}_\infty]\!]\big)$ gets distributed amongst its abelian fragments.

Thus a sequence $(\mathbf{y}_{\overline{\mathcal{S}}^*,m})$ lies in the image of $\Theta_{\infty,\mathcal{S}^*}$, if and only if each term factorises into $\mathbf{y}_{\overline{\mathcal{S}}^*,m} = \big(\mathbf{y}_{\overline{\mathcal{S}},m}, \mu_m\big)$ where the components $(\mathbf{y}_{\overline{\mathcal{S}},m}) \in \mathrm{Im}\big(\Theta_{\infty,\mathcal{S}}\big)$ and $(\mu_m) \in \mathrm{Im}(\Theta_0)$. Note that $\mathcal{G}_\infty$ is a pro-p-group so that $K_0\big(\mathbb{F}_p[\![\mathcal{G}_\infty]\!]\big) \cong \mathbb{Z}$, and similarly $K_0\big(\mathbb{F}_p[\![\mathcal{U}_m^{\mathrm{ab}}]\!]\big) \cong \mathbb{Z}$. Consequently a tuple $(\mu_m) \in \prod_m K_0\big(\mathbb{F}_p[\![\mathcal{U}_m^{\mathrm{ab}}]\!]\big)$ arises from the image of Θ_0 if and only if for every integer $m \geq 0$, one has $\mu_m = [\mathcal{G}_\infty : \mathcal{U}_m] \times \mu$ for some fixed $\mu \in \mathbb{Z}$.

Because the bottom arrow in the above diagram may possibly not be surjective, the most one can say is that any $(\mathbf{y}_{\overline{\mathcal{S}}^*,m}) \in \mathrm{Im}\big(\Theta_{\infty,\mathcal{S}^*}\big)$ must of necessity satisfy (M1)–(M4). If we denote this subset of $\prod_{m\geq 0} K_1\big(\Lambda(\mathcal{U}_m^{\mathrm{ab}})_{\overline{\mathcal{S}}^*}\big)$ satisfying (M1)–(M4) by '$\Phi_{\mathcal{S}^*}$', then this potential lack of surjectivity yields another obstruction to $\Theta_{\infty,\mathcal{S}^*} : K_1'\big(\Lambda(\mathcal{G}_\infty)_{\mathcal{S}^*}\big) \to \Phi_{\mathcal{S}^*}$ being an isomorphism. In terms of $\Theta_{\infty,\underline{\chi},\mathcal{S}^*} = \underline{\chi} \circ \Theta_{\infty,\mathcal{S}^*}$ from the Introduction, this translates into the necessity of the congruences written down in Theorems 1 and 2 holding for $\underline{\chi}(\mathbf{y}_{\overline{\mathcal{S}}^*,m}) \in \prod_{m,\chi} \mathrm{Quot}\big(\Lambda_{\mathcal{O}_\chi}(\Gamma^{p^m})\big)^\times$, but *not* their sufficiency regrettably.

5 Computing the terms in Theorems 1 and 2

The various quantities $\mathfrak{R}_{m,n}$, ϖ and $\mathbf{e}^*_{\chi,\varpi}$ occurring in the congruences (1) and (2) are easy to define in theory, but it is not quite so evident how to work them out in practice. We shall now give a step-by-step guide to calculating these terms algorithmically.

Step 1: We first explain how to express $\tilde{\chi}_{1,N^{(m)}_{\star,1}}$ and $\tilde{\chi}_{2,N^{(m)}_{\star,2}}$ in terms of $\chi_{1,n}$ and $\chi_{2,n}$.

Step 2: We next explicitly list representatives for $\mathfrak{R}_{m,n}$ in the form $\tilde{\chi}^a_{1,N^{(m)}_{\star,1}} \cdot \tilde{\chi}^b_{2,N^{(m)}_{\star,2}}$.

Step 3: We end by giving formulae to compute both $\#\varpi$ and $\mathbf{e}^*_{\chi,\varpi} = \mathrm{Tr}\big(\mathrm{Ind}\chi^*\big)(\varpi)$.

The technical results corresponding to Steps 1, 2, 3 in the text below are respectively Proposition 32, Lemma 34 and Lemma 35.

Definition 31. *(a) We set the non-negative integer pair* $\left(\mathbf{e}^{[1,m]}_{\star,1}, \mathbf{e}^{[1,m]}_{\star,2}\right)$ *equal to*

- $\big(0,1\big)$
- $\left(\dfrac{p^{s+m}}{\lambda^{p^m}_{III,\pm} - 1}\ ,\ 0\right)$
- $\left(\dfrac{p^{s+m}}{2}\left(\dfrac{1}{\lambda^{p^m}_{IV,+} - 1} + \dfrac{1}{\lambda^{p^m}_{IV,-} - 1}\right),\ \dfrac{p^{s+m}}{2\sqrt{d}}\left(\dfrac{1}{\lambda^{p^m}_{IV,+} - 1} - \dfrac{1}{\lambda^{p^m}_{IV,-} - 1}\right)\right)$
- $\left(\dfrac{p^{s+m+\mathrm{ord}_p(d)}}{2}\left(\dfrac{1 - \frac{p^r}{2\sqrt{\Delta_V}}}{\lambda^{p^m}_{V,+} - 1} + \dfrac{1 + \frac{p^r}{2\sqrt{\Delta_V}}}{\lambda^{p^m}_{V,-} - 1}\right), \dfrac{p^{s+m+\mathrm{ord}_p(d)}}{2\sqrt{\Delta_V}}\left(\dfrac{1}{\lambda^{p^m}_{V,+} - 1} - \dfrac{1}{\lambda^{p^m}_{V,-} - 1}\right)\right)$
- $\left(\dfrac{p^{s+m}}{2}\left(\dfrac{p^{r+\mathrm{ord}_p(t)}}{\lambda^{p^m}_{VI,+} - 1} + \dfrac{p^{r+\mathrm{ord}_p(t)}}{\lambda^{p^m}_{VI,-} - 1}\right),\ \dfrac{p^{s+m}}{2\sqrt{p^r t}}\left(\dfrac{p^{r+\mathrm{ord}_p(t)}}{\lambda^{p^m}_{VI,+} - 1} - \dfrac{p^{r+\mathrm{ord}_p(t)}}{\lambda^{p^m}_{VI,-} - 1}\right)\right)$

in Cases (II), (III), (IV), (V) and (VI) respectively.

(b) Likewise, we shall define a second pair $\left(\mathbf{e}_{\star,1}^{[2,m]}, \mathbf{e}_{\star,2}^{[2,m]}\right)$ *by setting it equal to*

- $\big(1, 0\big)$
- $\left(0\ ,\ \dfrac{p^{s+m}}{\lambda_{III,\pm}^{p^m} - 1}\right)$
- $\left(\dfrac{p^{s+m}\sqrt{d}}{2}\left(\dfrac{1}{\lambda_{IV,+}^{p^m} - 1} - \dfrac{1}{\lambda_{IV,-}^{p^m} - 1}\right),\ \dfrac{p^{s+m}}{2}\left(\dfrac{1}{\lambda_{IV,+}^{p^m} - 1} + \dfrac{1}{\lambda_{IV,-}^{p^m} - 1}\right)\right)$
- $\left(\dfrac{p^{s+m}d}{2\sqrt{\Delta_V}}\left(\dfrac{1}{\lambda_{V,+}^{p^m} - 1} - \dfrac{1}{\lambda_{V,-}^{p^m} - 1}\right),\ \dfrac{p^{s+m}}{2}\left(\dfrac{1 + \frac{p^r}{2\sqrt{\Delta_V}}}{\lambda_{V,+}^{p^m} - 1} + \dfrac{1 - \frac{p^r}{2\sqrt{\Delta_V}}}{\lambda_{V,-}^{p^m} - 1}\right)\right)$
- $\left(\dfrac{p^{s+m}\sqrt{p^r t}}{2}\left(\dfrac{1}{\lambda_{VI,+}^{p^m} - 1} - \dfrac{1}{\lambda_{VI,-}^{p^m} - 1}\right),\ \dfrac{p^{s+m}}{2}\left(\dfrac{1}{\lambda_{VI,+}^{p^m} - 1} + \dfrac{1}{\lambda_{VI,-}^{p^m} - 1}\right)\right)$

again in Cases (II), (III), (IV), (V) and (VI) respectively.

Proposition 32. *For integers $n \gg 0$, one has the character relations*

$$\tilde{\chi}_{1,N_{\star,1}^{(m)}} = \begin{cases} \chi_{1,n}^{0} \cdot \chi_{2,n}^{1} & \text{if } \star = II \\ \chi_{1,s+m}^{\mathbf{e}_{III,1}^{[1,m]}} \cdot \chi_{2,s+m}^{0} & \text{if } \star = III \\ \chi_{1,s+m}^{\mathbf{e}_{IV,1}^{[1,m]}} \cdot \chi_{2,s+m}^{\mathbf{e}_{IV,2}^{[1,m]}} & \text{if } \star = IV \\ \chi_{1,s+m+\mathrm{ord}_p(d)}^{\mathbf{e}_{V,1}^{[1,m]}} \cdot \chi_{2,s+m+\mathrm{ord}_p(d)}^{\mathbf{e}_{V,2}^{[1,m]}} & \text{if } \star = V \\ \chi_{1,s+m+r+\mathrm{ord}_p(t)}^{\mathbf{e}_{VI,1}^{[1,m]}} \cdot \chi_{2,s+m+r+\mathrm{ord}_p(t)}^{\mathbf{e}_{VI,2}^{[1,m]}} & \text{if } \star = VI \end{cases}$$

and

$$\tilde{\chi}_{2,N_{\star,2}^{(m)}} = \begin{cases} \chi_{1,s+m}^{1} \cdot \chi_{2,s+m}^{0} & \text{if } \star = II \\ \chi_{1,s+m}^{0} \cdot \chi_{2,s+m}^{\mathbf{e}_{III,2}^{[2,m]}} & \text{if } \star = III \\ \chi_{1,s+m}^{\mathbf{e}_{IV,1}^{[2,m]}} \cdot \chi_{2,s+m}^{\mathbf{e}_{IV,2}^{[2,m]}} & \text{if } \star = IV \\ \chi_{1,s+m}^{\mathbf{e}_{V,1}^{[2,m]}} \cdot \chi_{2,s+m}^{\mathbf{e}_{V,2}^{[2,m]}} & \text{if } \star = V \\ \chi_{1,s+m}^{\mathbf{e}_{VI,1}^{[2,m]}} \cdot \chi_{2,s+m}^{\mathbf{e}_{VI,2}^{[2,m]}} & \text{if } \star = VI. \end{cases}$$

Proof. The situation where $\star$ =II has already been dealt with in §3.2, cf. Equation (13). Let us instead suppose $\star \in \{$III,IV,V,VI$\}$. We first recall from Definition 16 that

- $\tilde{\chi}_{1,N_{\star,1}^{(m)}}\begin{pmatrix} x \\ y \end{pmatrix} = \chi_{1,N_{\star,1}^{(m)}}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathcal{T}_{\star,m,1} \begin{pmatrix} x \\ y \end{pmatrix}\right)$, and
- $\tilde{\chi}_{2,N_{\star,2}^{(m)}}\begin{pmatrix} x \\ y \end{pmatrix} = \chi_{2,N_{\star,2}^{(m)}}\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathcal{T}_{\star,m,2} \begin{pmatrix} x \\ y \end{pmatrix}\right)$

where $\mathcal{T}_{\star,m,j} := p^{N_{\star,j}^{(m)}}\left(\big(I_2 + M\big)^{p^m} - I_2\right)^{-1}$. Further, one can diagonalise the γ-action via

$$\big(I_2 + M\big)^{p^m} = P_\star\, D_\star^{p^m} P_\star^{-1} \quad \text{with } D_\star = \begin{pmatrix} \lambda_{\star,+} & 0 \\ 0 & \lambda_{\star,-} \end{pmatrix} \text{ and } P_\star \in \mathrm{GL}_2(\overline{\mathbb{Q}}_p).$$

The next objective is to calculate the matrices $\mathcal{T}_{\star,m,j}$ on an individual, case-by-case basis.

Case (III). Here $P_{III} = I_2$ and $N^{(m)}_{III,1} = N^{(m)}_{III,2} = s+m$, so that

$$p^{N^{(m)}_{III,j}}\Big((I_2+M)^{p^m} - I_2\Big)^{-1} = \begin{pmatrix} \frac{p^{s+m}}{(1+p^s)^{p^m}-1} & 0 \\ 0 & \frac{p^{s+m}}{(1+p^s)^{p^m}-1} \end{pmatrix}.$$

Case (IV). Here $P_{IV} = \begin{pmatrix} 1 & 1 \\ \sqrt{d} & -\sqrt{d} \end{pmatrix}$ and $N^{(m)}_{IV,1} = N^{(m)}_{IV,2} = s+m$, so that for each $j \in \{1,2\}$, the matrix $p^{N^{(m)}_{IV,j}}\Big((I_2+M)^{p^m} - I_2\Big)^{-1}$ equals

$$\frac{p^{s+m}}{2}\begin{pmatrix} \frac{1}{\lambda^{p^m}_{IV,+}-1} + \frac{1}{\lambda^{p^m}_{IV,-}-1} & \frac{1}{\sqrt{d}}\left(\frac{1}{\lambda^{p^m}_{IV,+}-1} - \frac{1}{\lambda^{p^m}_{IV,-}-1}\right) \\ \sqrt{d}\left(\frac{1}{\lambda^{p^m}_{IV,+}-1} - \frac{1}{\lambda^{p^m}_{IV,-}-1}\right) & \frac{1}{\lambda^{p^m}_{IV,+}-1} + \frac{1}{\lambda^{p^m}_{IV,-}-1} \end{pmatrix}.$$

Case (V). Assume that $n \geq s+m+\mathrm{ord}_p(d)$. Then $P_V = \begin{pmatrix} 1 & 1 \\ \frac{p^r}{2}+\sqrt{\Delta_V} & \frac{p^r}{2}-\sqrt{\Delta_V} \end{pmatrix}$ with $\Delta_V = d + p^{2r}/4 \in \mathbb{Z}_p$, while $N^{(m)}_{V,1} = s+m+\mathrm{ord}_p(d)$ and $N^{(m)}_{V,2} = s+m$; consequently for each choice $j \in \{1,2\}$, the matrix $p^{N^{(m)}_{V,j}}\Big((I_2+M)^{p^m} - I_2\Big)^{-1}$ equals

$$\frac{p^{N^{(m)}_{V,j}}}{2}\begin{pmatrix} \frac{1}{\lambda^{p^m}_{V,+}-1} + \frac{1}{\lambda^{p^m}_{V,-}-1} - \frac{p^r}{2\sqrt{\Delta_V}}\left(\frac{1}{\lambda^{p^m}_{V,+}-1} - \frac{1}{\lambda^{p^m}_{V,-}-1}\right) & \frac{1}{\sqrt{\Delta_V}}\left(\frac{1}{\lambda^{p^m}_{V,+}-1} - \frac{1}{\lambda^{p^m}_{V,-}-1}\right) \\ \frac{d}{\sqrt{\Delta_V}}\left(\frac{1}{\lambda^{p^m}_{V,+}-1} - \frac{1}{\lambda^{p^m}_{V,-}-1}\right) & \frac{1}{\lambda^{p^m}_{V,+}-1} + \frac{1}{\lambda^{p^m}_{V,-}-1} + \frac{p^r}{2\sqrt{\Delta_V}}\left(\frac{1}{\lambda^{p^m}_{V,+}-1} - \frac{1}{\lambda^{p^m}_{V,-}-1}\right) \end{pmatrix}.$$

Case (VI). Assume that $n \geq s+m+r+\mathrm{ord}_p(t)$. Then one has $P_{VI} = \begin{pmatrix} 1 & 1 \\ \sqrt{p^r t} & -\sqrt{p^r t} \end{pmatrix}$, while $N^{(m)}_{VI,1} = s+m+r+\mathrm{ord}_p(t)$ and $N^{(m)}_{VI,2} = s+m$; consequently, for each $j \in \{1,2\}$ the matrix $p^{N^{(m)}_{VI,j}}\Big((I_2+M)^{p^m} - I_2\Big)^{-1}$ equals

$$\frac{p^{N^{(m)}_{VI,j}}}{2}\begin{pmatrix} \frac{1}{\lambda^{p^m}_{VI,+}-1} + \frac{1}{\lambda^{p^m}_{VI,-}-1} & \frac{1}{\sqrt{p^r t}}\left(\frac{1}{\lambda^{p^m}_{VI,+}-1} - \frac{1}{\lambda^{p^m}_{VI,-}-1}\right) \\ \sqrt{p^r t}\left(\frac{1}{\lambda^{p^m}_{VI,+}-1} - \frac{1}{\lambda^{p^m}_{VI,-}-1}\right) & \frac{1}{\lambda^{p^m}_{VI,+}-1} + \frac{1}{\lambda^{p^m}_{VI,-}-1} \end{pmatrix}.$$

Since we know the form of each $\mathcal{T}_{\star,m,j}$, one now computes $\tilde{\chi}_{1,N^{(m)}_{\star,1}}\begin{pmatrix} x \\ y \end{pmatrix}$ and $\tilde{\chi}_{2,N^{(m)}_{\star,2}}\begin{pmatrix} x \\ y \end{pmatrix}$.

To illustrate the calculation, suppose we are in the last case $\star =$VI; then one obtains

$$\begin{aligned}
\tilde{\chi}_{1,N^{(m)}_{VI,1}}\begin{pmatrix} x \\ y \end{pmatrix} &= \chi_{1,N^{(m)}_{VI,1}}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathcal{T}_{VI,m,1}\begin{pmatrix} x \\ y \end{pmatrix}\right) \\
&= \chi_{1,s+m+r+\mathrm{ord}_p(t)}\left(\begin{pmatrix} \frac{p^{s+m+r+\mathrm{ord}_p(t)}}{2}\left(\frac{x+\frac{y}{\sqrt{p^r t}}}{\lambda_{VI,+}^{p^m}-1} + \frac{x-\frac{y}{\sqrt{p^r t}}}{\lambda_{VI,-}^{p^m}-1}\right) \\ 0 \end{pmatrix}\right) \\
&= \chi_{1,s+m+r+\mathrm{ord}_p(t)}\left(\begin{pmatrix} \frac{p^{s+m}}{2}\left(\frac{p^{r+\mathrm{ord}_p(t)}}{\lambda_{VI,+}^{p^m}-1} + \frac{p^{r+\mathrm{ord}_p(t)}}{\lambda_{VI,-}^{p^m}-1}\right)x \\ 0 \end{pmatrix}\right) \\
&\quad \cdot \chi_{2,s+m+r+\mathrm{ord}_p(t)}\left(\begin{pmatrix} 0 \\ \frac{p^{s+m}}{2\sqrt{p^r t}}\left(\frac{p^{r+\mathrm{ord}_p(t)}}{\lambda_{VI,+}^{p^m}-1} - \frac{p^{r+\mathrm{ord}_p(t)}}{\lambda_{VI,-}^{p^m}-1}\right)y \end{pmatrix}\right)
\end{aligned}$$

which equals $\chi_{1,s+m+r+\mathrm{ord}_p(t)}^{\mathbf{e}^{[1,m]}_{VI,1}}\begin{pmatrix} x \\ y \end{pmatrix} \cdot \chi_{2,s+m+r+\mathrm{ord}_p(t)}^{\mathbf{e}^{[1,m]}_{VI,2}}\begin{pmatrix} x \\ y \end{pmatrix}$. Likewise, one can show that

$$\begin{aligned}
\tilde{\chi}_{2,N^{(m)}_{VI,2}}\begin{pmatrix} x \\ y \end{pmatrix} &= \chi_{2,s+m}\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathcal{T}_{VI,m,2}\begin{pmatrix} x \\ y \end{pmatrix}\right) \\
= \chi_{1,s+m}&\left(\begin{pmatrix} \frac{p^{s+m}\sqrt{p^r t}}{2}\left(\frac{1}{\lambda_{VI,+}^{p^m}-1} - \frac{1}{\lambda_{VI,-}^{p^m}-1}\right)x \\ 0 \end{pmatrix}\right) \\
\cdot \chi_{2,s+m}&\left(\begin{pmatrix} 0 \\ \frac{p^{s+m}}{2}\left(\frac{1}{\lambda_{VI,+}^{p^m}-1} + \frac{1}{\lambda_{VI,-}^{p^m}-1}\right)y \end{pmatrix}\right) = \chi_{1,s+m}^{\mathbf{e}^{[2,m]}_{VI,1}}\begin{pmatrix} x \\ y \end{pmatrix} \cdot \chi_{2,s+m}^{\mathbf{e}^{[2,m]}_{VI,2}}\begin{pmatrix} x \\ y \end{pmatrix}.
\end{aligned}$$

The other remaining cases $\star =$III, $\star =$IV and $\star =$V follow in an analogous fashion. □

For Step 2, we introduce an equivalence relation '$\sim$' on ordered pairs of integers (a, b).

Definition 33. *(i) If $\star \in \{III, IV, V, VI\}$, then one sets*

$$\mathfrak{X}_{m,n} := \left\{(a,b) \in \left(\frac{\mathbb{Z}}{p^{N^{(m)}_{\star,1}}\mathbb{Z}} \times \frac{\mathbb{Z}}{p^{N^{(m)}_{\star,2}}\mathbb{Z}}\right) - p \cdot \left(\frac{\mathbb{Z}}{p^{N^{(m)}_{\star,1}}\mathbb{Z}} \times \frac{\mathbb{Z}}{p^{N^{(m)}_{\star,2}}\mathbb{Z}}\right)\right\} \Big/ \sim$$

where $(a,b) \sim (a',b')$, if and only if

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \equiv \begin{pmatrix} a' & 0 \\ 0 & b' \end{pmatrix} (I_2 + M)^j \mod \left((I_2+M)^{p^m} - I_2\right) \quad \textit{for some } j \in \mathbb{Z}/p^m\mathbb{Z}.$$

(ii) If $\star = II$, then one sets

$$\mathfrak{X}_{m,n} := \left\{(a,b) \in \frac{\mathbb{Z}}{p^n\mathbb{Z}} \times \left(\frac{\mathbb{Z}}{p^{s+m}\mathbb{Z}}\right)^{\times}\right\} \Big/ \sim$$

where $(a,b) \sim (a',b')$ if and only if $a \equiv a' \pmod{p^{n-m}}$.

The following result describes how to produce an explicit set of representatives for $\mathfrak{R}_{m,n}$. Again we assume that the integer $n \gg 0$ is chosen sufficiently large with respect to m.

Lemma 34. *(a) Up to isomorphism, the exact number of irreducible $\mathcal{G}_{\infty,n}$-representations $\rho_\chi = \mathrm{Ind}^{\mathcal{G}_{\infty,n}}_{\mathrm{Stab}_\Gamma(\chi)\ltimes\overline{\mathcal{H}}^{(m,n)}_\infty}(\chi)$ induced from primitive characters $\chi : \overline{\mathcal{H}}^{(m,n)}_\infty \to \mathbb{C}^\times$ equals*

$$\#\mathfrak{R}_{m,n} - \#\mathfrak{R}_{m-1,n} \;=\; \begin{cases} p^{n+s-1}\times(p-1) & \text{in Case (II)} \\ p^{2s+m-2}\times(p^2-1) & \text{in Cases (III) and (IV)} \\ p^{2s+m+\mathrm{ord}_p(d)-2}\times(p^2-1) & \text{in Case (V)} \\ p^{2s+m+r+\mathrm{ord}_p(t)-2}\times(p^2-1) & \text{in Case (VI).} \end{cases}$$

(b) If we define $\mathfrak{R}^{\mathrm{prim}}_{m,n} := \mathfrak{R}_{m,n} - \mathfrak{R}_{m-1,n}$ for every $m \in \{1,\dots,n-s\}$, then we can take as representatives for $\mathfrak{R}^{\mathrm{prim}}_{m,n}$ the set $\left\{\tilde{\chi}^a_{1,N^{(m)}_{\star,1}}\cdot\tilde{\chi}^b_{2,N^{(m)}_{\star,2}} \;\middle|\; (a,b)\in\mathfrak{X}_{m,n}\right\}$.

Proof. Part (a) follows (with $n \gg m$) on combining Proposition 10(iii) and Corollary 11. To show (b), first suppose that $\star \neq \mathrm{II}$. Then $\tilde{\chi}^a_{1,N^{(m)}_{\star,1}}\cdot\tilde{\chi}^b_{2,N^{(m)}_{\star,2}} = \gamma^j * \left(\tilde{\chi}^{a'}_{1,N^{(m)}_{\star,1}}\cdot\tilde{\chi}^{b'}_{2,N^{(m)}_{\star,2}}\right)$ if and only if $\tilde{\chi}_{1,N^{(m)}_{\star,1}}\begin{pmatrix} ax \\ ay\end{pmatrix}\cdot\tilde{\chi}_{2,N^{(m)}_{\star,2}}\begin{pmatrix} bx \\ by\end{pmatrix}$ equals

$$\tilde{\chi}_{1,N^{(m)}_{\star,1}}\left((I_2+M)^j\begin{pmatrix} a'x \\ a'y\end{pmatrix}\right)\cdot\tilde{\chi}_{2,N^{(m)}_{\star,2}}\left((I_2+M)^j\begin{pmatrix} b'x \\ b'y\end{pmatrix}\right) \quad \text{for all } x,y\in\mathbb{Z}_p.$$

This latter equality is equivalent to the pair of congruences

$$\begin{aligned} &\begin{pmatrix} p^{N^{(m)}_{\star,1}} & 0 \\ 0 & 0\end{pmatrix}\left((I_2+M)^{p^m}-I_2\right)^{-1}\begin{pmatrix} ax \\ ay\end{pmatrix} \\ \equiv\; &\begin{pmatrix} p^{N^{(m)}_{\star,1}} & 0 \\ 0 & 0\end{pmatrix}\left((I_2+M)^{p^m}-I_2\right)^{-1}(I_2+M)^j\begin{pmatrix} a'x \\ a'y\end{pmatrix} \mod p^{N^{(m)}_{\star,1}}\end{aligned}$$

and

$$\begin{aligned} &\begin{pmatrix} 0 & 0 \\ 0 & p^{N^{(m)}_{\star,2}}\end{pmatrix}\left((I_2+M)^{p^m}-I_2\right)^{-1}\begin{pmatrix} bx \\ by\end{pmatrix} \\ \equiv\; &\begin{pmatrix} 0 & 0 \\ 0 & p^{N^{(m)}_{\star,2}}\end{pmatrix}\left((I_2+M)^{p^m}-I_2\right)^{-1}(I_2+M)^j\begin{pmatrix} b'x \\ b'y\end{pmatrix} \mod p^{N^{(m)}_{\star,2}}\end{aligned}$$

holding for all $x,y\in\mathbb{Z}_p$; here we have exploited the construction of $\tilde{\chi}_{1,N^{(m)}_{\star,1}}$ and $\tilde{\chi}_{2,N^{(m)}_{\star,2}}$ given in Definition 16. Because $\left(I_2+M\right)^{p^m}-I_2$ and $(I_2+M)^j$ commute with each other, the above may be rewritten as a single congruence

$$\begin{aligned} &\begin{pmatrix} a & 0 \\ 0 & b\end{pmatrix}\left((I_2+M)^{p^m}-I_2\right)^{-1} \\ \equiv\; &\begin{pmatrix} a' & 0 \\ 0 & b'\end{pmatrix}(I_2+M)^j\left((I_2+M)^{p^m}-I_2\right)^{-1} \mod \mathrm{Mat}_{2\times2}(\mathbb{Z}_p).\end{aligned}$$

Note this congruence is satisfied for some $j\in\mathbb{Z}/p^m\mathbb{Z}$ precisely when $(a,b)\sim(a',b')$.

Let us instead suppose that $\star = \text{II}$. Then $\tilde{\chi}^{a}_{1,N^{(m)}_{\star,1}} \cdot \tilde{\chi}^{b}_{2,N^{(m)}_{\star,2}} = \gamma^{j} * \left(\tilde{\chi}^{a'}_{1,N^{(m)}_{\star,1}} \cdot \tilde{\chi}^{b'}_{2,N^{(m)}_{\star,2}}\right)$ if and only if

$$\tilde{\chi}_{1,N^{(m)}_{\star,1}}\begin{pmatrix} ax \\ ay \end{pmatrix} \cdot \tilde{\chi}_{2,N^{(m)}_{\star,2}}\begin{pmatrix} bx \\ by \end{pmatrix} = \tilde{\chi}_{1,N^{(m)}_{\star,1}}\begin{pmatrix} a'(x+p^{s}jy) \\ a'y \end{pmatrix} \cdot \tilde{\chi}_{2,N^{(m)}_{\star,2}}\begin{pmatrix} b'(x+p^{s}jy) \\ b'y \end{pmatrix}$$

at every $x, y \in \mathbb{Z}_p$. Again using Definition 16, we can rewrite this as

$$\zeta_{p^n}^{ay} \cdot \zeta_{p^{s+m}}^{bx} = \zeta_{p^n}^{a'y} \cdot \zeta_{p^{s+m}}^{b'(x+p^{s}jy)} \quad \text{for each } x, y \in \mathbb{Z}_p,$$

which is itself equivalent to the congruences

$$b \equiv b' \pmod{p^{s+m}} \qquad \text{and} \qquad a \equiv a' + jp^{n-m}b' \pmod{p^n} \text{ for some } j \in \mathbb{Z}/p^m\mathbb{Z}.$$

These last two congruences then reduce to $b \equiv b' \pmod{p^{s+m}}$ and $a \equiv a' \pmod{p^{n-m}}$.

Therefore in all possible cases $\star \in \{\text{II,III,IV,V,VI}\}$, one concludes that $\tilde{\chi}^{a}_{1,N^{(m)}_{\star,1}} \cdot \tilde{\chi}^{b}_{2,N^{(m)}_{\star,2}}$ and $\tilde{\chi}^{a'}_{1,N^{(m)}_{\star,1}} \cdot \tilde{\chi}^{b'}_{2,N^{(m)}_{\star,2}}$ lie in the same Γ-orbit if and only if $(a,b) \sim (a',b')$. □

Consequently Steps 1 and 2 have now been resolved, and it therefore only remains to complete Step 3. The latter task is covered by the next result, which enables us to compute both the size of ϖ and also the exponent $\mathbf{e}^{*}_{\chi,\varpi}$ occurring in Theorems 1 and 2, for each orbit ϖ and representative character $\chi \in \mathfrak{R}_{m,n}$.

Lemma 35. *(i) If $\varpi \in \text{orb}_{\Gamma}\big(\overline{\mathcal{H}}_{\infty}^{(m,n)}\big)$ contains an element $\overline{h} = \overline{h}_1^{x}\overline{h}_2^{y}$, then*

$$\varpi = \left\{ \overline{h}_1^{a}\overline{h}_2^{b} \text{ such that } \begin{pmatrix} a \\ b \end{pmatrix} \in \mathcal{Y}_{(x,y)} \bmod \left(\left(I_2+M\right)^{p^m} - I_2\right)\begin{pmatrix} \mathbb{Z}_p \\ \mathbb{Z}_p \end{pmatrix} + \begin{pmatrix} p^n\mathbb{Z}_p \\ p^n\mathbb{Z}_p \end{pmatrix}\right\}$$

where the set $\mathcal{Y}_{(x,y)}$ consists of the vectors $\left\{\left(I_2+M\right)^{j}\begin{pmatrix} x \\ y \end{pmatrix} \text{ with } j = 0, 1, \ldots, p^m - 1\right\}$.

(ii) For each character $\chi = \tilde{\chi}^{a}_{1,N^{(m)}_{\star,1}} \cdot \tilde{\chi}^{b}_{2,N^{(m)}_{\star,2}}$ on $\overline{\mathcal{H}}_{\infty}^{(m,n)}$, the number $\mathbf{e}^{*}_{\chi,\varpi} = \text{Tr}\big(\text{Ind}\chi^{*}\big)(\varpi)$ can be computed via the exponential sum formula

$$p^{\mathbf{m}_{\chi}-m} \cdot \sum_{j=0}^{p^m-1} \exp\left(-2\pi\sqrt{-1}\left(\left(\frac{a\mathbf{e}^{[1,m]}_{\star,1}}{p^{N^{(m)}_{\star,1}}} + \frac{b\mathbf{e}^{[2,m]}_{\star,1}}{p^{N^{(m)}_{\star,2}}}\right)x_j + \left(\frac{a\mathbf{e}^{[1,m]}_{\star,2}}{p^{N^{(m)}_{\star,1}}} + \frac{b\mathbf{e}^{[2,m]}_{\star,2}}{p^{N^{(m)}_{\star,2}}}\right)y_j\right)\right)$$

where the integer $\mathbf{m}_{\chi}$ is given in Proposition 5, and $\begin{pmatrix} x_j \\ y_j \end{pmatrix} := \left(I_2+M\right)^{j}\begin{pmatrix} x \\ y \end{pmatrix}$ for all j.

(iii) In particular, if ϖ consists of just the identity element, then $\mathbf{e}^{*}_{\chi,\varpi} = p^{\mathbf{m}_{\chi}} \in \mathbb{N}$.

Proof. To establish assertion (i), we remark that γ acts on the quotient group

$$\overline{\mathcal{H}}_{\infty}^{(m,n)} = \frac{\mathcal{H}_{\infty}/\mathcal{H}_{\infty}^{p^n}}{\left\langle \left[h_1^{x}h_2^{y} \bmod \mathcal{H}_{\infty}^{p^n},\ \gamma^{p^m}\right] \,\middle|\, x, y \in \mathbb{Z}_p\right\rangle} \cong \frac{\mathbb{Z}}{p^{N^{(m)}_{\star,1}}\mathbb{Z}} \times \frac{\mathbb{Z}}{p^{N^{(m)}_{\star,2}}\mathbb{Z}}$$

through the matrix $I_2 + M$, hence our description for the Γ-orbit follows immediately.

To show part (ii), by the definition of $\mathrm{Tr}\big(\mathrm{Ind}\chi^*\big)(\varpi)$ one calculates that

$$\begin{aligned}
\mathbf{e}^*_{\chi,\varpi} \;&=\; \frac{\#(\Gamma * \chi)}{p^m} \cdot \sum_{j=0}^{p^m-1} \chi^{-1}\left(\gamma^{-j}\overline{h}\gamma^j\right) \;=\; \frac{[\Gamma : \mathrm{Stab}_\Gamma(\chi)]}{[\Gamma : \Gamma^{p^m}]} \cdot \sum_{j=0}^{p^m-1} \chi^{-1}\left(\overline{h}_1^{x_j}\overline{h}_2^{y_j}\right) \\
&\overset{\text{by } 5}{=} p^{\mathbf{m}_\chi - m} \cdot \sum_{j=0}^{p^m-1} \tilde{\chi}_{1,N^{(m)}_{\star,1}}\left(\overline{h}_1^{x_j}\overline{h}_2^{y_j}\right)^{-a} \times \tilde{\chi}_{2,N^{(m)}_{\star,2}}\left(\overline{h}_1^{x_j}\overline{h}_2^{y_j}\right)^{-b} \\
&\overset{\text{by } 32}{=} p^{\mathbf{m}_\chi - m} \cdot \sum_{j=0}^{p^m-1} \chi_{1,N^{(m)}_{\star,1}}^{\mathbf{e}^{[1,m]}_{\star,1}} \cdot \chi_{2,N^{(m)}_{\star,1}}^{\mathbf{e}^{[1,m]}_{\star,2}}\left(\overline{h}_1^{x_j}\overline{h}_2^{y_j}\right)^{-a} \times \chi_{1,N^{(m)}_{\star,2}}^{\mathbf{e}^{[2,m]}_{\star,1}} \cdot \chi_{2,N^{(m)}_{\star,2}}^{\mathbf{e}^{[2,m]}_{\star,2}}\left(\overline{h}_1^{x_j}\overline{h}_2^{y_j}\right)^{-b} \\
&= p^{\mathbf{m}_\chi - m} \cdot \sum_{j=0}^{p^m-1} \chi_{1,N^{(m)}_{\star,1}}^{-a\mathbf{e}^{[1,m]}_{\star,1}} \cdot \chi_{1,N^{(m)}_{\star,2}}^{-b\mathbf{e}^{[2,m]}_{\star,1}}\left(\overline{h}_1^{x_j}\overline{h}_2^{y_j}\right) \times \chi_{2,N^{(m)}_{\star,1}}^{-a\mathbf{e}^{[1,m]}_{\star,2}} \cdot \chi_{2,N^{(m)}_{\star,2}}^{-b\mathbf{e}^{[2,m]}_{\star,2}}\left(\overline{h}_1^{x_j}\overline{h}_2^{y_j}\right)
\end{aligned}$$

and the last line is then equivalent to the stated formula.

Finally (iii) is a special case of (ii), corresponding to $x = y = 0$ and $x_j = y_j = 0$. $\square$

5.1 A worked example for Case (II)

We end by using Steps 1–3 to yield an explicit expression for the congruences in Case (II). Firstly by Lemma 34(b) and Definition 33(ii), if one takes $m \geq 1$ then

$$\mathfrak{R}^{\mathrm{prim}}_{m,n} \;=\; \left\{\chi^a_{2,n} \cdot \chi^b_{1,s+m} \;\middle|\; a \in \mathbb{Z}/p^{n-m}\mathbb{Z} \;\text{ and }\; b \in \left(\mathbb{Z}/p^{s+m}\mathbb{Z}\right)^\times\right\}$$

while $\mathfrak{R}_{0,n}$ coincides with $\left\{\chi^a_{2,n} \cdot \chi^b_{1,s} \;\middle|\; a \in \mathbb{Z}/p^n\mathbb{Z} \;\text{ and }\; b \in \mathbb{Z}/p^s\mathbb{Z}\right\}$. It follows that

$$\prod_{\chi \in \mathfrak{R}_{m,n}} \mathcal{N}_{\mathrm{Stab}_\Gamma(\chi)/\Gamma^{p^m}}\left(\cdots\right)^{\mathbf{e}^*_{\chi,\varpi}} \;=\; \prod_{m'=0}^{m} \prod_{a=1}^{p^{n-m'}} \prod_{\substack{b\,=\,1,\\ p\,\nmid\, b \text{ if } m' > 0}}^{p^{s+m'}} \mathcal{N}_{\mathbf{m}_\chi,m}\left(\cdots\right)^{\mathbf{e}^*_{\chi,\varpi}}\Bigg|_{\chi = \chi^a_{2,n}\cdot\chi^b_{1,s+m'}}.$$

Now suppose an orbit $\varpi_{\overline{h}} \in \mathrm{orb}_\Gamma\big(\overline{\mathcal{H}}^{(m,n)}_\infty\big)$ contains an element $\overline{h} = \overline{h}_1^x\overline{h}_2^y$. Then

$$\varpi_{\overline{h}} \;=\; \left\{\gamma^{-j}\overline{h}\gamma^j \;\middle|\; j \in \mathbb{Z}\right\} \;=\; \left\{\overline{h}_1^{x+jp^sy}\overline{h}_2^y \;\middle|\; j \in \mathbb{Z}\right\} \;=\; \overline{h} \cdot \left\{\overline{h}_1^{jp^sy} \;\middle|\; j = 1, \cdots, p^{m-\mathrm{ord}_p(y)}\right\}$$

in which case $\#\varpi_{\overline{h}} = p^{m-\mathrm{ord}_p(\tilde{y})}$, with $\tilde{y} \in \{1, \ldots, p^m\}$ chosen so that $\tilde{y} \equiv y \pmod{p^m}$.

Finally, if we consider a typical character $\chi = \chi^a_{2,n} \cdot \chi^b_{1,s+m'} = \chi^a_{2,n} \cdot \chi^{p^{m-m'}b}_{1,s+m}$ and the orbit $\varpi = \varpi_{\overline{h}}$ as above, then Lemma 35(ii) implies

$$\begin{aligned}
\mathbf{e}^*_{\chi,\varpi_{\overline{h}}} \;&=\; p^{\mathbf{m}_\chi - m} \cdot \sum_{j=0}^{p^m-1} \exp\left(-2\pi\sqrt{-1}\left(\left(\frac{p^{m-m'}b}{p^{s+m}}\right)(x + jp^sy) + \left(\frac{a}{p^n}\right)y\right)\right) \\
&= p^{\mathbf{m}_\chi - m} \cdot \exp\left(-2\pi\sqrt{-1}\left(\frac{bx}{p^{s+m'}} + \frac{ay}{p^n}\right)\right) \times \sum_{j=0}^{p^m-1} \exp\left(-2\pi\sqrt{-1}\left(\frac{bjy}{p^{m'}}\right)\right) \\
&= p^{\mathbf{m}_\chi - m} \cdot \exp\left(-2\pi\sqrt{-1}\left(\frac{bx}{p^{s+m'}} + \frac{ay}{p^n}\right)\right) \times \begin{cases} p^m & \text{if } p^{m'} \mid by \\ 0 & \text{if } p^{m'} \nmid by. \end{cases}
\end{aligned}$$

However the exponential term $\exp\left(-2\pi\sqrt{-1}\left(\frac{bx}{p^{s+m'}}+\frac{ay}{p^n}\right)\right)$ is then just equal to $\chi^{-1}(\overline{h})$. Because $\chi = \chi_{2,n}^{a}\cdot\chi_{1,s+m'}^{b}$ can be written as $\chi_{1,n}^{\mathbf{e}_1}\cdot\chi_{2,n}^{\mathbf{e}_2}$ with $\mathbf{e}_1 = p^{n-s-m'}b$ and $\mathbf{e}_2 = a$, one calculates via Proposition 5 that $\mathbf{m}_\chi = \max\{0, \tilde{\mathbf{m}}_\chi\}$ where

$$\tilde{\mathbf{m}}_\chi \overset{\text{by } 5}{=} n - s - \mathrm{ord}_p\big(p^{n-s-m'}b\big) \;=\; m' - \mathrm{ord}_p(b).$$

Consequently, if $\chi = \chi_{2,n}^{a}\cdot\chi_{1,s+m'}^{b}$ then $\mathbf{e}^*_{\chi,\varpi_{\overline{h}}} = \begin{cases}\chi^{-1}(\overline{h})\cdot p^{\max\{0,m'-\mathrm{ord}_p(b)\}} & \text{if } p^{m'} \mid by\\ 0 & \text{if } p^{m'}\nmid by.\end{cases}$

Corollary 36. *The congruences described in Equation (20) are equivalent to*

$$\prod_{m'=0}^{m}\prod_{a=1}^{p^{n-m'}}\prod_{\substack{b=1,\\ p\nmid b \text{ if } m'>0}}^{p^{s+m'}} \mathcal{N}_{\mathbf{m}_\chi,m}\left(\frac{\mathbf{y}_\chi}{\varphi(\mathbf{y}_{\chi^p})}\cdot\frac{\varphi\big(\mathcal{N}_{0,\mathbf{m}_\chi-1}(\mathbf{y_1})\big)}{\mathcal{N}_{0,\mathbf{m}_\chi}(\mathbf{y_1})}\right)^{\mathbf{e}^*_{\chi,\varpi_{\overline{h}}}}\Bigg|_{\chi=\chi_{2,n}^{a}\cdot\chi_{1,s+m'}^{b}}$$

$$\equiv\; 1 \mod p^{s+m+n+\mathrm{ord}_p(\tilde{y})}\cdot\mathbb{Z}_p\big[\big[\Gamma^{p^m}\big]\big]_{(p)}$$

for all integer pairs $m,n\geq 0$ with $m\leq n-s$, and at every choice of $\overline{h} = \overline{h}_1^{\tilde{x}}\overline{h}_2^{\tilde{y}} \in \overline{\mathcal{H}}_\infty^{(m,n)}$ with $\tilde{x}\in\{1,\ldots,p^n\}$ and $\tilde{y}\in\{1,\ldots,p^m\}$.

This completes the proof of Theorem 1, in the precise form stated in the Introduction.

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