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Number-Fact Knowledge and Mathematical Problem-solving of Five- to Seven-year-olds

Jenny Young-Loveridge[1)] **Brenda Bicknell**
The University of Waikato, New Zealand

Abstract

This paper examines children's number fact knowledge in relation to mathematics problem solving. These findings are derived from a study that set out to explore the impact on mathematics learning of using multiplication and division contexts with 84 five- to seven-year-old children from diverse cultural and linguistic backgrounds. After a series of focused lessons, children's knowledge of number facts, including single-digit addition, subtraction, and doubles improved substantially. However, children did not always apply this knowledge to relevant problem-solving situations. The difficulty level for recalling number facts was not directly related to the magnitude of the numbers, with certain salient facts such as 5+5 and 10+10 learned earlier than facts with smaller sums such as 2+3 and 1+4.

Keywords : multiplication & division, number facts, problem solving, counting, early childhood, mathematics

Corresponding author, [1)]educ2233@waikato.ac.nz

Introduction

In recent years, education systems worldwide have put a major emphasis on literacy and mathematics. Research shows that early mathematics skills are a strong predictor of later academic success and appear to be relatively stable for individuals over time (e.g., Bynner & Parsons, 2000; Claessens & Engel, 2013; Duncan et al., 2007; Wylie & Hodgen, 2011). Longitudinal research has shown that young children who started school with low levels of mathematics understanding relative to their peers tended to continue struggling with mathematics at later ages, left school early with few qualifications, and later experienced difficulties with employment, and with health (Bynner & Parsons, 1997; 2000; Parsons & Bynner, 2005).

Mathematics reform over the past few decades has resulted in the development of frameworks outlining progressions in number as students acquire increasingly sophisticated ways of thinking and reasoning (Bobiset al., 2005; Ministry of Education, 2008). Typically, at lower stages on these frameworks, children solve problems by using counting strategies (see Figure 1). Coming to appreciate the additive composition of numbers enables children to use strategies that involve partitioning and recombining quantities, using what they know to work out answers to new problems (sometimes referred to as derived-fact strategies, decomposition strategies, or part-whole thinking).

Many mathematics curricula focus initially on counting, then on addition and subtraction, before introducing other domains such as multiplication and division, and proportional reasoning (e.g., Australia: ACARA, 2011; England: Department for Education, 2013; Ireland: NCCA, 1999; New Zealand: Ministry of Education, 2007; United States: CCSSI, 2010). However, research has shown that young children have considerable knowledge of multiplication prior to formal instruction in this domain (e.g., Bakker, van den Heuval-Panhuizen, & Robitzsch, 2014). Similarly, children are able to work with division as a sharing process (e.g., Frydman & Bryant, 1988; Davis & Pitkethly, 1990; Squire & Bryant, 2003).

Stage	Description
0	***Emergent*** Cannot count
1	***One-to-One Counting*** Can count a small collection up to 10, but cannot use counting to add or subtract collections.
2.	***Counting from One on Materials*** Can add two collections by counting, but counts all the objects in both collections
3.	***Counting from One by Imaging*** Adds two collections by counting all, but counts mentally by imaging objects
4	***Advanced Counting*** Recognises that the last number in a counting sequence stands for all the objects in the collection, so counts on for the second collection
5.	***Early Additive Part-Whole Strategies*** Recognises that numbers are abstract units that can be partitioned (broken up) & recombined (part-whole thinking). Uses known number facts to derive answers
6.	***Advanced Additive Part-Whole Strategies*** Chooses from a range of different part-whole strategies to find answers to addition and subtraction problems
7.	***Advanced Multiplicative Part-Whole Strategies*** Chooses from a range of different part-whole strategies to find answers to multiplication and division problems
8.	***Advanced Proportional Part-Whole Strategies*** Chooses from a range of different part-whole strategies to find answers to problems involving fractions, proportions, and ratios

Figure 1. New Zealand Number Framework (from Ministry of Education, 2008)

Two concepts thought to be important for children's mathematical understanding are 'number as *hierarchical inclusion*' (i.e., each number includes all the numbers that come before it in the sequence of numbers), and 'number as *seriation*' (i.e., each number comes after, and is one more than, the previous number in the sequence of numbers) (see Sarama & Clements, 2009). These two concepts parallel the notions of *cardinality* (how many items altogether), and *ordinality* (what number comes before or after a particular number). A similar distinction has been made between *collections-based* and *counting-based* conceptions of number (Yackel, 2001). Experiences with counting build knowledge of number sequence and ordinality. A collections-based approach focuses on the composition of numbers in terms of groups (set-subset or part-whole relationships), drawing on understanding of hierarchical inclusion and cardinality (Sarama & Clements, 2009). Yackel's (2001, p. 24) view is that "having access to both conceptions provides much more flexibility for students … Therefore, [it is important] to foster the development of both conceptions".

Attention has be drawn to the *inherent contradiction* evident in the way that children from Western countries are initially encouraged to count by ones, thus constructing unitary *counting-based* number concepts with an emphasis on *ordinality*, but when place-value instruction begins, they are suddenly expected to reorganise their understanding into *collections-based* concepts involving units of ten and units of one, focusing on *cardinality* (Yang & Cobb, 1995). The Western emphasis on counting has been contrasted with the collections-based approach of Chinese mothers and teachers, who focus on groups (units) of ten. Yang and Cobb (1995) found that Chinese children showed more advanced mathematical understanding than American children, arguably because of the greater emphasis on groups of ten and the transparency of the decade-based structure of the counting words used in Chinese and other Asian languages (e.g., 11 as shíyī [ten-one], 12 as shíèr [ten-two], 20 as èrshí [two tens]).

The key characteristic of derived-fact or part-whole strategies is that numbers are partitioned and recombined using knowledge of number facts in order to solve problems, instead of a counting strategy such as *count all* or *count on*. For example, 9 and 4 can be solved by splitting 4 into 1 and 3, combining the 1 with 9 to make 10, leaving the 3 to be joined with 10 to make 13. Alternatively, 9 might be split into 3 and 6, so the 6 and 4 are combined to make 10, leaving the 3 added to 10 to make 13. A different approach might be to add 10 and 4 to make 14, reasoning that 9 and 4 must be one less; that is, 13. The focus on different ways of partitioning numbers fits with a more recent approach that recognises the importance for young children of developing an awareness of mathematical pattern and structure, particularly groups of ten (Mulligan, 2011; Mulligan & Mitchelmore, 2009).

An important distinction has been made between instrumental (procedural) and relational (conceptual) understanding (Skemp, 1978). Researchers interested in students' conceptual understanding have focused on developing children's awareness of number relationships. Components of number relationships include subitizing, part-whole relationships, and more and less relationships (e.g., Jung, Hartman, Smith, & Wallace, 2013). Gray and Tall (1994) distinguish between *procedures* as "things to do", and *concepts* as "things to know" (p. 117), introducing the term *procept* to refer to an amalgam of procedures/processes and concepts (e.g., 6 is the result of counting from 1 to 6, and as well as the result of adding 3+3). They propose a hierarchy whereby counting processes help to develop a concept of number, then counting on helps to develop a concept of sum, then experience with repeated addition helps to develop a concept of product. They argue that more capable children tend to develop flexible relational understanding of mathematics that enables them use proceptual thinking, whereas less able children persist in using counting procedures because they have more difficulty recognising the connections between concepts, so learning mathematics is far more challenging for them. As Mulligan (2011) has shown, drawing attention to connections by emphasizing the underlying pattern and structure of mathematics can lead to notable improvements in children's mathematics achievement.

A major emphasis in New Zealand Year 1 and 2 classrooms is on helping children learn to count objects in a group (Ministry of Education,n.d.). National Standards (Ministry of Education, 2009) specify that after one year at school, children should solve problems using (at the least) a 'count all' strategy, and one year later an advanced counting strategy such as *counting on* or *counting back*. After three years at school, children are expected to use knowledge of number facts and part-whole thinking strategies to solve problems. Initially children are expected to use part-whole thinking for problems involving addition with small numbers, but by the end of Year 5 (9-year-olds), children are expected to use this strategy with multi-digit subtraction as well as addition.

Being able to use number facts flexibly is vital for becoming a part-whole thinker. Derived facts are also essential for developing a range of mental strategies to solve number problems (Ministry of Education, 2008). For example, children might use their knowledge that 3+3=6 to work out that the answer to 4+3 is one more than 6. Although the New Zealand Number Framework positions part-whole thinking as a stage beyond *counting on* (see Figure 1), research has shown that the teaching of part-whole thinking (i.e., derived-fact strategies) need not wait until children regularly use *counting on* (Fischer, 1990; Henry & Brown, 2008; Steinberg, 1985). It appears that even children who use 'counting all' strategies can learn to use derived-fact strategies (part-whole thinking) to solve problems.

One of the challenges for children in learning to use part-whole thinking is in recalling number facts. Research shows that low achievers in mathematics have consistent difficulties in recalling number facts and using them to solve problems (Baroody, 2011). It has been suggested that one of the reasons for these difficulties is that these learners often continue to rely on counting strategies, which take a lot of energy and attention, meaning that number facts do not become known facts (Gray, 1991). Some researchers have highlighted the importance of learners developing automaticity in mathematics (Gray, 1991; Hattie & Yates, 2014; Hopkins & Lawson, 2002). It is suggested that learning to recall number facts and use them to solve new problems should be made more explicit (Baroody, 2011).

The project described here set out to explore the impact of using multiplication and division contexts with 5- to 7-year-olds on their emerging understanding of number, including number-fact knowledge, part-whole relationships, and problem-solving strategies.

Method

This study was set in an urban school (medium socio-economic status [SES]) in New Zealand. The participants were 84 5-7year-olds (42 girls & 42 boys) in four classes, two designated as Year 1, one Year 2, and one Year 3. The average age of the students at the beginning of the study was 6.3 years (range 5.0 to 7.9 years). The children were from a diverse range of ethnic backgrounds, with approximately one-third of European ancestry, one-third Māori (the indigenous people of New Zealand), one-fifth Asian, and the rest were Pasifika and African. One-fifth of the children had been identified as English Language Learners [ELL]. At the start of the study, the children were assessed individually using a diagnostic task-based interview designed to explore their number knowledge and problem-solving strategies (April). The assessment interview was completed again after two four-week teaching blocks (September). The assessment tasks included: addition, subtraction, multiplication, and division problems; recall of number facts; incrementing in tens; counting sequences; and place value.

Teaching using Multiplication and Division Contexts

Two series of 12 focused lessons were taught; the first phase was in May and the second in August. In these lessons, the children were introduced to groups of two, using familiar contexts such as pairs of socks, shoes, gumboots, and mittens. Multiplication and division was introduced using simple word problems, such as:

Three children each have a pair of shoes. How many shoes do the children have altogether?

Pene has 12 socks. He puts the socks into pairs. How many pairs of socks are there?

Once children were familiar in working with groups of two, groups of five were introduced using contexts such as gloves focusing on the number of fingers on each glove, and five candles on a cake, then groups of ten using the context of filling cartons with eggs. Although the emphasis of the study was on multiplication and division, the children also used addition and subtraction strategies to solve the problems or check their solutions. The focus in this article is specifically on children's knowledge of number facts and use of derived-fact strategies to solve problems.

There are 3 cakes on the table. Each cake has 5 candles. How many candles are there altogether?

We have 20 eggs. We put 10 eggs in each cartons. How many full cartons do we have?

A typical lesson began with all students completing a problem together on the mat, using materials to support the modelling process, and sharing ways of finding a solution. The teacher recorded children's problem-solving processes (including use of manipulatives) and their mathematical ideas in a large scrapbook (*modelling book*). The problem for the day was already written in the book and both drawings and number sentences were recorded, acknowledging individual children's contributions. The children then completed a problem in their own project books, choosing a similar or larger number, and/or selecting a new number. Materials were made available and children were encouraged to show their thinking using representations and to record matching equations.

Results

Children's performance on the tasks was examined to look for patterns and relationships. Items selected from the diagnostic task-based interview included recall of known facts and solutions to number problems. Children's correct responses on selected addition facts at the start and end of the project are shown in Table 1 as a percentage. By the end of the project, only four children did not know some of the number facts presented. The easiest number facts were the doubles and included: 5+5 (95%), 2+2 (94%), 3+3 (83%), 4+4 (81%), and 10+10 (86%). More children knew doubles such as 5+5 and 10+10 than knew plus-one (1+4: 65%) or doubles plus-one facts (2+3: 52%). This is despite the fact that the sum was 10 or 20 rather than five or smaller. This finding suggests that teachers need to be aware that certain key number facts may be learned early, even though they involve sums greater than five. There is a strong case to be made for children to be encouraged to recall the easiest facts, regardless of the number size. Performance on combinations for ten varied according to the distance between the two addends (apart from the easiest: 5+5). The most difficult fact was 4+6 (23%) whereas 1+9 was considerably easier (65%). More than half of the children were able to combine a multiple of ten (a *-teen* or a *-ty* number) with a single-digit quantity without using a counting strategy.

Table 2 presents the percentages of correct responses on selected problem-solving tasks and the use of particular types of strategies. Children improved markedly on multiplication tasks. For example, by the end of the project 90 per cent of the children were able to work out the answer for 6x2 (six small baskets each containing twoshells that were shown to the children before being hidden inside the baskets; see Figure 2), up from 55 per cent initially. For the problem showing four groups of 5 (Figure 3), the corresponding improvement was from 48 per cent to 81 per cent. More than one-third (38%) either recalled the fact or used their knowledge that two groups of 5 equal 10, so doubling 10 gives a total of 20. When shown an array of 30 cakes in three rows of 10 and asked how many altogether (Figure 4), the majority of children (79%) could work out the answer by the end of the project, an increase from 44%. Approximately one-fifth (19%) of these students used known or derived facts, and almost half (45%) used skip counting. Children were more successful on multiplication than on addition and subtraction, but also more of them used higher-level strategies to solve the problems, such

Table 1. *Percentages of correct responses on selected addition and subtraction facts*

Tasks	Y1 Initial	Y1 Final	*n=25* ***Diff***	Y2 Initial	Y2 Final	*n=24* ***Diff***	Y3 Initial	Y3 Final	*n=25* ***Diff***	Y1-3 Initial	Y1-3 Final	*n=84* ***Diff***
youngest in yrs	5.01	5.42		6.03	6.54		6.68	7.17		5.01	5.42	
oldest in yrs	5.82	6.34		6.85	7.35		7.88	8.36		7.28	7.76	
Average Age in yrs	*5.45*	5.94		6.43	6.95		7.28	7.76		6.27	6.77	
SD	*0.26*	0.27		0.27	0.27		0.29	0.30		0.82	0.82	
Doubles												
5 + 5	40	94	***54***	75	92	***17***	96	100	***4***	67	95	***29***
2 + 2	43	94	***51***	71	88	***17***	92	100	***8***	65	94	***29***
3 + 3	31	83	***51***	58	67	***8***	92	100	***8***	57	83	***26***
4 + 4	29	80	***51***	50	71	***21***	84	92	***8***	51	81	***30***
10 + 10	26	80	***54***	54	79	***25***	88	100	***12***	52	86	***33***
6 + 6	9	51	***43***	29	38	***8***	56	76	***20***	29	55	***26***
Doubles plus/minus one												
2 + 3	6	40	***34***	29	42	***13***	68	80	***12***	31	52	***21***
4 + 3	0	26	***26***	8	29	***21***	24	44	***20***	10	32	***23***
5 + 4	3	26	***23***	33	25	***-8***	60	60	***0***	29	36	***7***
5 + 6	0	14	***14***	8	13	***4***	28	40	***12***	11	21	***11***
Plus one/two												
2 + 1	31	71	***40***	46	75	***29***	84	100	***16***	51	81	***30***
3 + 1	20	60	***40***	42	63	***21***	76	100	***24***	43	73	***30***
1 + 4	20	51	***31***	46	63	***17***	72	88	***16***	43	65	***23***
1 + 9	17	54	***37***	46	58	***13***	76	88	***12***	43	65	***23***
5 + 2	3	46	***43***	21	58	***38***	60	80	***20***	25	60	***35***
Subtraction (half)												
4 - 2	0	29	***29***	21	42	***21***	68	76	***8***	26	46	***20***
8 - 4	0	29	***29***	21	42	***21***	44	60	***16***	19	42	***23***
10 - 5	3	40	***37***	33	54	***21***	72	84	***12***	32	57	***25***
Combinations making Ten												
4 + 6	0	11	***11***	8	21	***13***	24	40	***16***	10	23	***13***
7 + ? = 10	0	26	***26***	13	17	***4***	40	64	***24***	15	35	***19***
2 + 8	3	31	***29***	21	33	***13***	52	68	***16***	23	43	***20***
Place Value for -ty and -teen												
20 + 7	6	46	***40***	33	46	***13***	56	88	***32***	29	58	***30***
10 + 8	3	46	***43***	38	38	***0***	64	84	***20***	31	55	***24***

as skip counting or derived-fact strategies. For example, more than two-thirds of the children used a higher-level strategy (counting on/skip counting or known/derived facts) for 6x2, 4x5, and 3x10, whereas approximately half of them used one of these higher-level strategies to solve 4+3 or 8+5.

Figure 2. Six baskets each containing two shells (shown above the first basket)

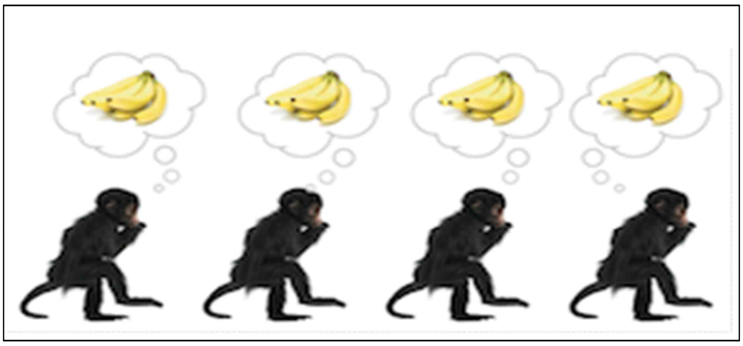

Figure 3. Picture of four monkeys each with five bananas

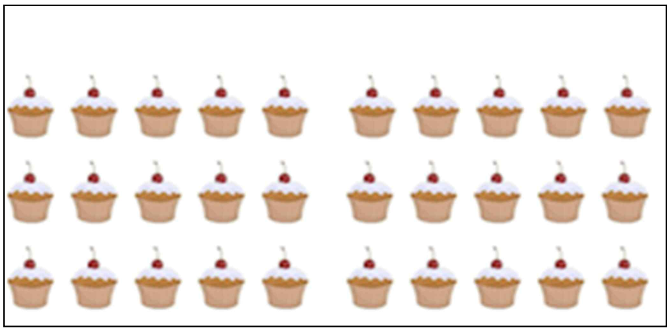

Figure 4. Picture showing three rows of 10 cupcakes with space between each group of five

Table 2. *Percentages of correct responses on selected problem-solving tasks*

Tasks	Y1 Initial	Y1 Final	*n=25* ***Diff***	Y2 Initial	Y2 Final	*n=24* ***Diff***	Y3 Initial	Y3 Final	*n=25* ***Diff***	Y1-3 Initial	Y1-3 Final	*n=84* ***Diff***
Addition/Subtraction												
3 + 4 (screened)	37	80	***43***	67	71	***4***	88	92	***4***	61	81	***20***
Counting All	17	37	***20***	33	25	***-8***	32	16	***-16***	26	27	***1***
Counting On	9	31	***23***	17	21	***4***	24	36	***12***	15	30	***14***
Known/Derived Facts	11	11	***0***	17	25	***8***	32	40	***8***	19	24	***5***
23 + 4	6	23	***17***	25	42	***17***	36	76	***40***	20	44	***24***
Counting On	6	14	***9***	21	17	***-4***	16	12	***-4***	13	14	***1***
Known/Derived Facts	0	9	***9***	4	25	***21***	20	64	***44***	7	30	***23***
8 + 5 (screened)	6	31	***26***	25	42	***17***	60	76	***16***	27	48	***20***
Counting All	0	9	***9***	4	4	***0***	8	0	***-8***	4	5	***1***
Counting On	6	23	***17***	13	33	***21***	36	40	***4***	17	31	***14***
Known/Derived Facts	0	0	***0***	8	4	***-4***	16	36	***20***	7	12	***5***
14 - 5 (screened)	6	11	***6***	25	29	***4***	40	76	***36***	21	36	***14***
Counting Back	3	11	***9***	17	13	***-4***	24	48	***24***	13	23	***10***
Known/Derived Facts	3	0	***-3***	8	17	***8***	16	28	***12***	8	13	***5***
Multiplication												
6 x 2 (screened)	23	86	***63***	71	88	***17***	84	100	***16***	55	90	***36***
Counting All	17	34	***17***	33	21	***-13***	24	4	***-20***	24	21	***-2***
Skip Counting	6	46	***40***	38	58	***21***	56	76	***20***	30	58	***29***
Known/Derived Facts	0	6	***6***	0	8	***8***	4	20	***16***	1	11	***10***
4 x 5 (picture)	23	63	***40***	54	88	***33***	76	100	***24***	48	81	***33***
Counting All	17	11	***-6***	21	25	***4***	12	4	***-8***	17	13	***-4***
Skip Counting	3	34	***31***	25	33	***8***	56	20	***-36***	25	30	***5***
Known/Derived Facts	3	17	***14***	8	29	***21***	8	76	***68***	6	38	***32***
3 x 10 (array)	31	66	***34***	46	83	***38***	60	92	***32***	44	79	***35***
Counting All	14	14	***0***	21	25	***4***	4	4	***0***	13	14	***1***
Skip Counting	17	43	***26***	17	42	***25***	36	52	***16***	23	45	***23***
Known/Derived Facts	0	9	***9***	8	17	***8***	16	36	***20***	7	19	***12***

Although children could recall many number facts, they did not appear to use them to solve problems involving operations. For example, by the end of the project, more than three-quarters of the children could recall 3+3 (83%) and 4+4 (81%), respectively. However, just under one-quarter (24%) used a derived-fact strategy to solve a problem involving 4+3 (counters hidden under cardboard). Instead 27 per cent used *counting all* and 30 per cent used *counting on*.

Discussion and Conclusion

The study showed that over the duration of the project, children developed a broader range of number facts but did not necessarily use that knowledge in solving problems. More children were more familiar with the fact: 5+5=10 than knew the plus-one facts for small sums (e.g., 2+1=3). The pattern of mastery (according to the size and types of facts) was contradictory to curriculum materials specifying progression in basic facts knowledge (e.g., Ministry of Education, 2008). However, the increase in those using known or derived facts supports work showing that young children can learn to use derived facts instead of relying on counting strategies (Fischer, 1990; Henry & Brown, 2008; Jung et al., 2013; Steinberg, 1985). Children need to be given the opportunity to work with multiplication and division problems and with larger numbers. They should also be encouraged to recognise the value of derived facts for arriving at solutions. This is consistent with the view that awareness of mathematical pattern and structure needs to be supported in young children (Mulligan, 2011; Mullligan & Mitchelmore, 2009). Awareness of pattern and structure could help to strengthen confidence, mastery, and automaticity of number fact knowledge (Hattie & Yates, 2014; Hopkins & Lawson, 2002; Wylie & Hodgen, 2011).

The introduction of multiplication and division contexts was clearly related to improvements in solving multiplication and division problems using more efficient strategies. However, its impact on addition and subtraction problem solving was less marked. In future iterations of the project, a more explicit emphasis on number relationships in the context of single-digit doubles (e.g., 4+3 is *one more than* 3+3) could provide further support for the use of derived-fact strategies for addition and subtraction, as well as strengthening children's understanding the idea of *two groups of* (i.e., doubles) within multiplication and division.

If an emphasis continues to be placed on the direct teaching of counting-on procedures (strategies) as happens in New Zealand schools, then the likely consequence is that less able children may become locked into counting as their only problem-solving strategy (Gray & Tall, 1994). This has been demonstrated in research with large cohorts of students showing that counting-on is the preferred strategy to solve addition and subtraction problems for between 14% and 8% of students in Years 7 to 9, respectively (12- to 14-year-olds; Young-Loveridge, 2010). The teaching of counting procedures can be likened to the traditional instrumental teaching that prevailed prior to mathematics education reform (e.g., Skemp, 1978). Gray and Tall's (1994) idea of a *procept*, an amalgam of process and concept, helps to underline the importance of teachers supporting children in moving from simple "count-on procedures" to developing procepts that reflect the reification of a conceptual entity. Gray and Tall's argument that it is more advanced mathematical thinkers who develop proceptual thinking then are able to use their known facts to derive new facts (part-whole thinking) suggests that being more explicit about connections could help those learners who don't notice them on their own. For example, three pairs of socks with skip counting (2, 4, 6) could help support the connection between 3 x 2 (three groups of two) and 2 x 3 (two groups of three). Going from 3 + 3 = 6, to 3 + 3 + 1 = 6 + 1 = 7 could hasten the development of proceptual thinking and hence part-whole thinking. Doubles and near-doubles could provide an important bridge between addition (3+3=6) and multiplication (2x3=6).

Teachers with lower levels of confidence and less competence in conceptual (relational) understanding of mathematics may not recognise the importance of helping young children move from a reliance on counting strategies to acquiring and applying number fact knowledge as part of encouraging proceptual thinking (Baroody, 2011; Skemp, 1978). New Zealand's National Standards for mathematics legitimate the teaching of *counting all* and *counting on*strategies during the first two years of schooling (Ministry of Education, 2009). The absence of a focus on multiple number relationships is further reinforced by the current curriculum (Ministry of Education, 2007). It is not until Level 3 [Years 5 and 6] that Ministry of Education information for teachers explicitly states that a key idea is that numbers can be represented in a variety of ways (Ministry of Education, n. d.). Unfortunately the omission of this vital piece of information at Levels 1 and 2 implies that multiple representations for (small) numbers need not be emphasized in the first four years of school.

The findings of this study show that young children can build a considerable repertoire of number facts when classroom instruction supports this knowledge acquisition. An explicit focus on using derived-fact strategies to solve addition and subtraction problems could be effective in deepening children's understanding of number relationships.

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