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SHEARED CORONAL ARCADES: AN EVALUATION OF RECENT STUDIES

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ABSTRACT

We show that the family of magnetic force-free equilibria obtained by Low using the generating function method is really a sequence of Gold-Hoyle flux tubes. This sequence is stable under a wide range of solar conditions since each member, specified by the shear parameter μ, is anchored to the photosphere along an axial slice. We go on to demonstrate that recent magnetic relaxation simulations by Klimchuk and Sturrock are fundamentally incapable of representing the unconnected helical field lines inherent in the high-shear ($\mu > 1$) Low solutions. Nonetheless, we believe that the numerical simulations are more likely to describe the equilibria of highly sheared arcades since they involve no change in topology with increasing shear. This view is reinforced by magnetic energy calculations which confirm that the Gold-Hoyle solutions are more energetic for $\mu > 1$ than the numerical equilibria of Klimchuk and Sturrock.

Subject headings: hydromagnetics — Sun: flares — Sun: magnetic fields

I. INTRODUCTION

It has often been suggested that a solar flare or prominence may develop from the photospheric shearing of a coronal magnetic arcade. Various authors (e.g., Low 1977*a*; Priest and Milne 1980) have developed analytic descriptions of sheared force-free fields and have argued for the onset of eruptive phenomena or "magnetic nonequilibrium" if the shear becomes too great. This approach has been summarized by Birn and Schindler (1981) who point out that equilibrium families of force-free solutions may provide information on the onset of a dynamic field eruption in much the same way that potential energy analysis can define the stability properties of simple mechanical systems.

In a detailed series of papers, Low (1977*a*, *b*, 1980) describes the changing character of a one-parameter family (μ) of analytic force-free solutions. As the shear parameter increases beyond a critical value ($\mu = 1$), the solutions are found to change topologically—spiral field lines develop which are unconnected to the underlying photosphere. Since topological changes cannot be effected by ideal fluid motions, Low suggests some eruptive resistive phenomenon—such as a solar flare—may be involved as the critical level of shear is exceeded. This view has recently been contested by Klimchuk and Sturrock (1989, hereafter KS) on the grounds that Low's high-shear boundary conditions can be satisfied by physically plausible magnetic arcades *that contain no disconnected field lines*. This suggests that the numerical construction of KS, rather than the spiral field solution of Low, is really the equilibrium state of the highly sheared arcade.

The physical relationship between the high-shear and low-shear branches of Low's solution family has also been explored by other workers (e.g., Hood and Priest 1980), but a compelling interpretation has not emerged. In this paper we present a simple interpretation of Low's solution that impacts both on the KS numerical study and the stability analysis of Hood and Priest (1980). We point out the following:

1. Low's family of solutions is really a series of Gold-Hoyle (GH) fields in disguise: the photospheric boundary is governed by the parameter μ which simply determines an axial slice of the cylindrical field and sets the level of twist.

2. Low's upper branch solutions are therefore stable provided only that the axis of the GH tube is not raised too high above the photosphere: the tube becomes unstable to the helical kink mode when photospheric line-tying can no longer influence the central core of the field.

3. The KS analysis is fundamentally incapable of representing the high-shear Low solutions: this follows from the fact that the GH field cannot, in general, be represented by single-valued Clebsch variables parameterized by the shear. We emphasise that this conclusion does not undermine the KS numerical results which we have verified independently to be substantially correct.

4. The high-shear boundary conditions of Low must generally admit two or more families of stable solutions. It appears that the GH branch simply has too much potential energy—unlike the KS solution—to represent the equilibrium of a slowly sheared magnetic arcade.

In § II we identify Low's equilibrium solution with the GH field and discuss the ramifications in terms of magnetic stability. In § III we discuss the numerical solutions of KS in terms of the Low analysis and clarify the interpretation of helical field lines in the Clebsch variable representation. In particular we show that the "disappearence" of regions of closed field lines in magnetic relaxation experiments (as in Yang, Sturrock, and Antiochos 1986) is to be expected on physical grounds and should not be confused with numerical diffusion. Our conclusions are summarized in § IV.

II. LOW'S SOLUTION AND THE GOLD HOYLE FIELD

a) The Equilibrium Model

A common model of a coronal arcade is a two-dimensional force-free magnetic field $\boldsymbol{B}(x, y)$, where the plane $y = 0$ represents the photosphere in which the emerging field lines are rigidly fixed. Low (1977*a*) proposed a model in which the x- and y-components of $\boldsymbol{B}$ are represented by a flux function

$$A = \ln\left[1 + k^2x^2 + k^2y^2 + \frac{2ky(1-\mu^2)}{1+\mu^2}\right], \tag{2.1}$$

where k and μ are constants. The force-free assumption then demands that B_z be a function of A, satisfying

$$\nabla^2 A + B_z \frac{dB_z}{dA} = 0 .$$

The parameter μ controls the photospheric shear—i.e., the relative z-displacement of the footpoints of the arched field lines. For $\mu < 1$, the field lines are all simple arches beginning and ending at the photosphere, but for $\mu > 1$, a region appears in which the field lines do not intersect the photosphere. These field lines are helices whose projections on the (x, y)-plane are circles. Low (1977*a*) suggested that this abrupt change in topology indicated a "loss of equilibrium"—that as the footpoint shear increased through $\mu = 1$, the field would be unable to adapt to the required new topology, and that reconnection and rapid release of magnetic energy would ensue. This view is challenged by KS who present numerical results which show that no such loss of equilibrium occurs, the field evolving smoothly through $\mu = 1$ to new equilibria without change of topology. Of course, these new equilibria cannot be represented in terms of Low's analytical formulae, which demonstrates that specification of magnetic flux and field line connectivity on the boundary does not guarantee a unique force-free solution (see also Sneyd 1990).

The interpretation of Low's field is sharpened when we realize that it is actually the well-studied Gold-Hoyle solution. Adding a suitable arbitrary constant to A, we write equation (2.1) in the form

$$A = \ln\left[1 + \frac{r^2}{a^2}\right], \quad a = \frac{2\mu}{k(\mu^2 + 1)}, \tag{2.2}$$

where (r, θ, z) are cylindrical polar coordinates with axis along the line

$$x = 0 , \quad y = \frac{\mu^2 - 1}{k(\mu^2 + 1)} = c \text{ (say)} .$$

Then the only component of $\boldsymbol{B}$ in the (x, y)-plane is the azimuthal component,

$$B_\theta(r) = \frac{-2r}{r^2 + a^2}\, \hat{\boldsymbol{\theta}} .$$

Furthermore, since the axial component $B_z(A)$ is a function of r only, it can be uniquely determined from the force-free equation. We therefore find

$$\boldsymbol{B} = \frac{2a}{r^2 + a^2}\, [(-r/a)\hat{\boldsymbol{\theta}} + \hat{\boldsymbol{z}}] .$$

This is the cylindrical Gold-Hoyle flux tube of twist $-1/a$ and field strength $2a/(r^2 + a^2)$ whose field lines are helices of constant pitch (e.g., Sneyd and Craig 1989; see also Hood and Priest 1980). For $\mu < 1$, $c < 0$ and the axis of the tube lies below the photosphere; for $\mu > 1$, $c > 0$, the axis is above the photosphere and a region $r < c$ of helical field lines unconnected to the photosphere appears.

b) *Stability of the Equilibria*

The stability of the Low field can readily be deduced from well-known results on Gold-Hoyle flux tubes (Hood and Priest 1981; Sneyd and Craig 1989; Foote and Craig 1990; Craig *et al.* 1990). Since field lines are fixed to the photosphere $y = 0$, the field must always be *more* stable than the corresponding isolated Gold-Hoyle flux tube, the class of permissible perturbations being more restricted. Low (1977*b*) has investigated the stability of the field to perturbations which are independent of the axial coordinate z. It is known that the Gold-Hoyle field is neutrally stable with respect to such perturbations, its only instabilities being helical kink modes (see, e.g., Craig *et al.* 1990). Thus Low finds that the field is stable for $\mu < 1$ and neutrally stable for $\mu = 1$. Indeed, it will also be neutrally stable for $\mu > 1$ since a displacement of the helical field line region $r < c$ in the z-direction leaves the field unchanged. Hood and Priest (1980) studied the stability of Low's field with respect to z-dependent perturbations and found that like the Gold-Hoyle field it is subject to helical kink modes, provided that the arcade is sufficiently long, and the axis $y = c$ far enough above the photosphere. These results could be anticipated from the work of Hood and Priest (1981) or Foote and Craig (1990) which show that a Gold-Hoyle tube is unstable if long enough, and far enough from lateral boundaries. For an infinitely long GH tube, lateral boundaries must be placed within at least 10 tube radii to stabilize the kink (e.g., Sneyd and Craig 1989).

III. NUMERICAL SOLUTIONS

a) *Limitations of the Clebsch Variable Representation*

The numerical scheme of Klimchuk and Sturrock (1989) is based on a Clebsch variable field representation

$$\boldsymbol{B} = \nabla\alpha \times \nabla\beta .$$

The function $\alpha(x, y)$ is a flux function for the x- and y-components (i.e., $\alpha = A$) and $\beta = z - \gamma(x, y)$. Thus since α and β are constant along field lines, γ represents the displacement of field line in the z-direction, or field-line shear.

For Low's field, we set $\alpha = A$ as given by equation (2.2) and $\gamma = a\theta$. This causes no problem when $\mu < 1$ since the field line projections on the (x, y)-plane are segments of circles whose centre lies below the photosphere. In this case, γ will be single-valued since θ varies by less than 2π as each field line is traversed. However, when $\mu > 1$ there exist helical field lines not intersecting the photosphere, whose projections are complete circles, and on which γ is multivalued. A branch cut would have to be introduced in the (x, y)-plane from the photosphere to $(0, c)$, and the code would have to undergo complex modifications to prevent differentiating γ across this cut.

We should point out that there exist alternative methods for expressing the Gold-Hoyle field in terms of smooth single-valued Clebsch variables. One possibility is to begin with a uniform field $B_0\,\hat{z}$ using Clebsch variables $\alpha = B_0\, x$, $\beta = y$. This field is then subjected to a radial compression and a uniform rotation through the angle $\theta = \tau z$ to produce a Gold-Hoyle field, where τ is the twist. Since α and β remain constant on any fluid particle throughout this transformation, their functional form may be determined. However, these Clebsch variables can no longer be interpreted in terms of flux and footpoint displacement, respectively.

For the $\mu > 1$ calculations, KS do not explain how α and γ were initialized on the internal mesh points. Suppose, for instance, that α was initialized by formula (2.1) and γ in some way consistent with footpoint shear. The code would then interpret γ as single-valued and so the region $r < c$, unconnected to the photosphere, would consist, not of helical field lines, but connected loops of magnetic field topologically equivalent to circles. In the next section, we show that, as a consequence of assuming zero plasma pressure, such a region would simply disappear during the calculation, leaving all field lines connected to the photosphere as in the case $\mu < 1$.

In summary, we conclude that the representation assumed by KS is incapable of modeling the upper branch Low solutions. The fact that KS obtain other solutions is then due, not to any defect or instability in Low's equilibria as might be suspected at first sight, but to the existence of a separate branch of high-shear photospherically connected stable solutions. Indeed, results from our own magnetofrictional code lend further weight to the KS solutions. Figure 1 shows the total magnetic energy as a function of the shear parameter μ for our relaxed solution, and for the field given by Low for the same boundary conditions. For $\mu < 1$, the energies of both solutions are in close agreement. However, for $\mu > 1$, the relaxed solution moves onto a lower energy branch, while the energy of Low's field continues to increase. In this sense KS are correct to assert that their solutions are the extension of Low's low-shear equilibria into the high-shear regime.

b) *The Magnetic Relaxation of Closed Field Lines*

Yang, Sturrock, and Antiochos (1986) and possibly KS have attempted to model regions of closed field lines using magnetic relaxation codes. Yang *et al.* point to the anomalous disappearance of closed field regions, implying that numerical diffusion at low resolution is the culprit. This is quite misleading—these structures should disappear at any resolution as we now demonstrate.

We consider the magnetic relaxation of a simple purely azimuthal field,

$$\boldsymbol{B} = f(r)\hat{\theta} ,$$

permeating a perfectly conducting pressureless plasma confined by a perfectly conducting circular cylinder at $r = a$. This

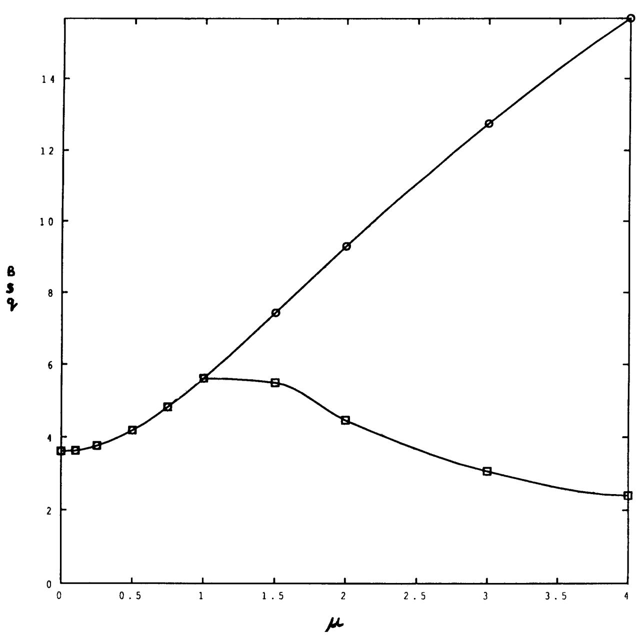

FIG. 1.—Magnetic energy against shear parameter μ for our numerically obtained relaxed solution (*squares*), and for the field given by Low (*circles*) for the same boundary conditions.

initial nonequilibrium field can be described in terms of Clebsch variables

$$\alpha = -\int_0^r f(\xi)d\xi = \alpha_0(r) \quad \text{say} , \quad \beta = z .$$

By symmetry the field will relax to equilibrium by purely radial displacements, and α will remain a function of r only. Since $\boldsymbol{J}$ is perpendicular to $\boldsymbol{B}$, equilibrium can be achieved only when one or other of these vectors vanishes at any given point. The equilibrium $\alpha(r)$ therefore satisfies Laplace's equation

$$\frac{d^2\alpha}{dr^2} + \frac{1}{r}\frac{d\alpha}{dr} = 0 ,$$

whose only solutions are $\alpha =$ constant, or $\alpha = k \ln(r)$, where k is a constant. It is not possible for the $\ln(r)$ solution to be valid at $r = 0$ since then one finds $\alpha \to -\pm\infty$ as $r \to 0$. This is impossible since in a perfectly conducting medium, the value of α is constant on a fluid particle and the range of values of α must remain finite. Thus the final equilibrium $\alpha(r)$ must be given by

$$\alpha = \begin{cases} \alpha_0(a)\ln(r/r_0)/\ln(a/r_0) , & r \geq r_0 ; \\ 0 , & r \leq r_0 . \end{cases} \tag{3.1}$$

In this solution the electric current vanishes everywhere except for a cylindrical current sheet at $r = r_0$.

The constant r_0 is determined by the precise path of relaxation to equilibrium. We have carried out numerical calculations for various $f(r)$ and invariably found $r_0 = 0$, so that (3.1) becomes

$$\alpha = \begin{cases} \alpha_0(a) , & r > 0 ; \\ 0 , & r = 0 , \end{cases} \tag{3.2}$$

the current and magnetic field now vanishing throughout. This is perhaps to be expected. In the solution (3.1), the region of plasma on which $\alpha = 0$ has undergone an infinite expansion from the axis $r = 0$ to the cylinder $r \leq r_0$. In the vicinity of $r = 0$ where both $\boldsymbol{B}$ and the magnetic pressure vanish, the Lorentz force is always directed radially inward, so it seems unlikely that such an expansion could occur. In equation (3.2), magnetic flux is preserved by being concentrated in a weak singularity at $r = 0$ where the field is infinite. Since there is zero current associated with this singularity, it is undetectable either physically or numerically.

We cannot be certain whether the initial KS field for the $\mu > 1$ calculations included regions of closed magnetic field lines, but if so, the above argument shows that they would disappear during the relaxation process.

IV. CONCLUSIONS

We have emphasised that the Low's family of solutions represents a sequence of Gold-Hoyle flux tubes which are anchored axially to the photosphere. This fact immediately implies the linear and global stability of the family under quite general circumstances (Craig and Sneyd 1990). Instabilities are possible, but only if the photospheric boundary is several length scales beneath the axial core of the loop.

It is something of an irony that Klimchuck and Sturrock (1989) are unable to represent a general GH loop in terms of Clebsch variables which define magnetic flux and footpoint displacement. This limitation implies that any attempt to model Low's upper branch solutions ($\mu > 1$) by magnetic relaxation experiments along the lines of KS is doomed to failure: in fact, the KS experiment is incapable of representing regions of field which are not connected to the photosphere; and if such regions were initially present they would vanish without trace (see § III*b*) during the relaxation.

It is important to emphasize that our findings in no way invalidate the central numerical results of KS. These authors doubtless exhibit new equilibria which are topologically distinct from Low's solution and which almost certainly represent the physical arcade behavior. The essential point is that the high-shear boundary conditions of Low admit at least two or more families of stable solutions. Our results indicate that the GH branch has considerably more magnetic energy than the branch delineated by K. S. Accordingly, it is hard to see how Low's upper branch solutions can be obtained by any experiment involving simple footpoint shear. Whether this conclusion holds for other force-free equilibria obtained by the generating function method is unclear. However, in the absence of compelling supporting evidence, it is clearly dangerous to speculate that the existence of topologically distinct branches of magnetic equilibria is an indicator of flarelike eruptions.

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