PHYSICS OF PLASMAS 23, 062901 (2016)

Comparison of the telegraph and hyperdiffusion approximations in cosmic-ray transport

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(Received 11 March 2016; accepted 26 May 2016; published online 9 June 2016)

The telegraph equation and its generalizations have been repeatedly considered in the models of diffusive cosmic-ray transport. Yet the telegraph model has well-known limitations, and analytical arguments suggest that a hyperdiffusion model should serve as a more accurate alternative to the telegraph model, especially on the timescale of a few scattering times. We present a detailed side-by-side comparison of an evolving particle density profile, predicted by the telegraph and hyperdiffusion models in the context of a simple but physically meaningful initial-value problem, compare the predictions with the solution based on the Fokker–Planck equation, and discuss the applicability of the telegraph and hyperdiffusion approximations to the description of strongly anisotropic particle distributions. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4953564]

I. INTRODUCTION

The Fokker–Planck equation for the particle distribution function plays a central role in the theory of cosmic-ray transport in turbulent astrophysical plasmas (see, e.g., Schlickeiser[1] for a systematic derivation of the equation and Schlickeiser[2] for a recent review of its history and applications). It is well known, however, that the complicated evolution of the particle distribution often can be described as a simpler diffusive process.[3,4] For example, the diffusion approximation applies when turbulent pitch-angle scattering is strong enough to ensure that the particle mean free path is significantly less than the scale of density variation and other relevant length scales of the transport problem.[5–9] The diffusion approximation provides a simple yet powerful tool for cosmic-ray modeling in a variety of applications.[8,10–14] Numerical studies both confirmed the accuracy of the approximation in a relevant parameter range[15] and illustrated its breakdown in the case of strong adiabatic focusing, caused by the mirror force in a nonuniform magnetic field.[16,17]

Because the signal propagation speed is infinite in the diffusion limit, attempts to obtain a more accurate description of cosmic-ray evolution are often made use of the telegraph equation and related hyperbolic partial differential equations, characterized by a finite signal propagation speed. Alternative derivations of a modified telegraph equation from the Fokker–Planck equation had been given, which employed different expansion schemes and explored parameter regimes appropriate in concrete applications.[18–26]

The derivation of the telegraph equation for particle density takes into account the higher-order terms in an expansion of the particle distribution function, leading to the expectations of higher accuracy of the telegraph approximation in comparison with the diffusion approximation.[21,24] Numerical solutions of the Fokker–Planck equation for focused particle transport, for instance, show that the profile of a propagating density pulse is characterized by a sharp propagating front, followed by a wake. These features, ultimately caused by a finite particle speed, cannot be reproduced in the diffusion approximation but are captured in the telegraph approximation.[16,17]

The hyperbolic nature of the telegraph equation denotes that the telegraph approximation typically yields solutions that contain singular components or sharp spikes at propagating diffusion fronts.[27] Although the δ-functional singularities of the analytical solutions cannot be resolved numerically, the solutions of boundary value problems are also predicted to contain discontinuities that are absent in the solutions of the underlying Fokker–Planck equation.[28,29] Malkov[30] and Malkov and Sagdeev[26] recently reiterated these unsatisfactory features of the telegraph approximation and advocated the use of an alternative—the hyperdiffusion approximation. Malkov and Sagdeev[26] argued that the hyperdiffusion approximation is superior to the telegraph approximation both for practical reasons, since the hyperdiffusion model is expected to have a broader validity range, and on general theoretical grounds, as an evolutionary self-contained theoretical description of the particle density.

The history of the telegraph approximation in cosmic-ray transport probably goes back to Axford[31] and Fisk and Axford,[32] and the criticism by Malkov and Sagdeev[26] deserves serious consideration. Unfortunately, Malkov and Sagdeev[26] had only presented an asymptotic analytical solution of the hyperdiffusion equation and they had not compared predictions of the hyperdiffusion model for a specific problem with those of either the telegraph equation or the underlying Fokker–Planck equation. Here, we do not question the validity of the asymptotic expansion presented by Malkov and Sagdeev[26] but rather use numerical solutions to illustrate the arguments given by Malkov and Sagdeev[26] and to investigate the validity of their claims in a concrete example. We consider a concrete physical model of energetic particle transport and compare the evolving particle density profiles, predicted by the Fokker–Planck equation and the three approximations, namely, the diffusion, telegraph, and hyperdiffusion approximations. We also demonstrate how anisotropic initial conditions influence the accuracy of the approximations.

II. THE FOKKER–PLANCK EQUATION AND ITS APPROXIMATIONS

We consider the Fokker–Planck equation for a cosmic-ray distribution function in a uniform constant magnetic field,[4,10] which incorporates the effects of turbulent pitch-angle scattering

$$\frac{\partial f_0}{\partial t} + \mu v \frac{\partial f_0}{\partial z} = \frac{\partial}{\partial \mu}\left(D_{\mu\mu}\frac{\partial f_0}{\partial \mu}\right). \quad (1)$$

Here, $f_0(z, \mu, v, t)$ is the distribution function of energetic particles (gyrotropic phase-space density), t is the time, μ is the cosine of the particle pitch angle, v is the particle speed, z is the distance along the mean magnetic field, and $D_{\mu\mu}(\mu)$ is the Fokker–Planck coefficient for pitch-angle scattering. In order not to obscure the essential points of the paper, we consider a simple, yet physically meaningful transport problem: we assume a medium at rest and ignore both the mirror forces due to a non-uniform magnetic field and energy losses (hence $v = \text{const}$). Finally, we assume for simplicity that $D_{\mu\mu}$ is an even function of μ, which is the case, for example, in the slab turbulence model if backward- and forward-propagating waves have equal intensities.

Although there seem to be "as many versions of telegrapher's equations in transport theory as there are authors who produce them,"[22] most derivations employ an expansion of the distribution function in terms of orthogonal eigenfunctions, say, the Legendre polynomials, to reduce the Fokker–Planck equation to an infinite set of coupled differential equations. Different assumptions used to truncate the infinite set yield different forms of the telegraph approximation. Simply neglecting higher-order terms in the eigenfunction expansion does not yield a rigorous derivation of either the hyperdiffusion or telegraph equation, because in general the higher-order terms or their derivatives cannot be neglected in the truncated expansion. Gombosi *et al.*[21] argued that the correct way of deriving the telegraph equation should rely on an asymptotic expansion in terms of a parameter proportional to the scattering time. The procedure yields the diffusion equation in the lowest order and the telegraph equation in the next order. Gombosi *et al.*[21] described the method in detail for the case of a uniform background magnetic field and isotropic scattering. Malkov and Sagdeev[26] appeared to use similar methods to derive the hyperdiffusion equation, which they claim to be based on "the least inaccurate out of all possible expansion schemes."[26]

To order to avoid extensively quoting the earlier papers and to keep our analysis self-contained, we give below a heuristic derivation of the alternative approximate models, which is based on an iterative solution of the Fokker–Planck equation.[25]

We express the distribution function in the form of a sum

$$f_0(z, \mu, t) = F_0(z, t) + g_0(z, \mu, t), \quad (2)$$

where the isotropic density F_0 is defined by

$$F_0(z, t) = \frac{1}{2}\int_{-1}^{1} f_0 d\mu, \quad (3)$$

and an anisotropic component g_0 satisfies

$$\int_{-1}^{1} g_0 d\mu = 0. \quad (4)$$

Litvinenko and Schlickeiser[25] solved Equation (1) by iterations to derive an approximation for diffusive transport in a weakly non-uniform mean magnetic field $B_0(z)$. Here, we use their results in the limit $L = \infty$, corresponding to $B_0 = \text{const}$ (see also Earl[33]). Note, for clarity that in this section, we use the same notation as by Litvinenko and Noble[16] and Effenberger and Litvinenko,[17] which differs slightly from the notation by Litvinenko and Schlickeiser.[25]

Integration of Equation (1) with respect to μ shows that the density F_0 satisfies

$$\frac{\partial F_0}{\partial t} + \frac{\partial S}{\partial z} = 0, \quad (5)$$

with the flux

$$S = \frac{v}{2}\int_{-1}^{1} \mu g_0 d\mu. \quad (6)$$

A diffusive limit corresponds to the evolution of the particle density when the pitch-angle scattering is strong enough to ensure that the scale of density variation significantly exceeds the particle mean free path. The accuracy of an explicit expression for S determines the accuracy of an approximate transport model. An iterative solution leads to

$$S = -\kappa_\parallel \frac{\partial F_0}{\partial z} + \tau\kappa_\parallel \frac{\partial^2 F_0}{\partial z \partial t}, \quad (7)$$

where $\kappa_\parallel$ and τ are defined by Equations (9), (10), and (14) in Litvinenko and Schlickeiser.[25]

Now, as long as the particle evolution is diffusive, the second term on the right-hand side of Equation (7) should be small compared with the first one. The diffusion approximation is obtained by neglecting the second term ($\tau = 0$) and substituting the resulting expression for S into Equation (5)

$$\frac{\partial F_0}{\partial t} = \kappa_\parallel \frac{\partial^2 F_0}{\partial z^2}. \quad (8)$$

Clearly, we can substitute Equation (7) into Equation (5) without assuming $\tau = 0$ and obtain a third-order equation for $F_0(z, t)$, which contains a mixed derivative. Since deviations from Equation (8) are assumed to be small, we can eliminate the mixed derivative by differentiating Equation (7) with respect to z and using Equation (8) to approximate the small τ-dependent term, which yields

$$\frac{\partial S}{\partial z} + \kappa_\parallel \frac{\partial^2 F_0}{\partial z^2} \approx \tau \frac{\partial^2 F_0}{\partial t^2} \approx \tau\kappa_\parallel^2 \frac{\partial^4 F_0}{\partial z^4}. \quad (9)$$

On substituting either of these approximate expressions into Equation (5) we obtain two seemingly equivalent higher-order models of diffusive cosmic-ray transport: the telegraph approximation,

$$\frac{\partial F_0}{\partial t} + \tau \frac{\partial^2 F_0}{\partial t^2} = \kappa_\parallel \frac{\partial^2 F_0}{\partial z^2} \tag{10}$$

and the hyperdiffusion approximation,

$$\frac{\partial F_0}{\partial t} = \kappa_\parallel \frac{\partial^2 F_0}{\partial z^2} - h \frac{\partial^4 F_0}{\partial z^4}, \tag{11}$$

where τ and the hyperdiffusion coefficient h are related by

$$h = \kappa_\parallel^2 \tau. \tag{12}$$

The diffusion equation is recovered in the limit $\tau = 0$. As an interesting aside, we note the formal analogy of these arguments with those used in the derivation of the continuum limit of a persistent random walk (see, for instance, Equations (3), (13), (14) by Rosenau[34]). More generally, an equation with a linear combination of the telegraph and hyperdiffusion terms could be considered, but we do not explore this possibility here.

The above derivation can be criticized for two reasons. First, the iterative solution, as well as the method of truncating an infinite system which is equivalent to the original Fokker–Planck equation, leads to an error in the coefficient τ and more generally to an equation which is not correctly ordered. A more rigorous second-order asymptotic expansion[21] results in a slightly different numerical value of τ. Second, perhaps more importantly, Malkov and Sagdeev[26] argued that the hyperdiffusion approximation is superior to the telegraph model because the latter leads to the solutions that contain unphysical singular components. Presumably then the hyperdiffusion model could offer an insight into the behavior of the particle density profile on a timescale of a few scattering times, that is until the simpler diffusion model becomes accurate enough.

Malkov and Sagdeev[26] considered an initial-value problem and gave an asymptotic analytical solution of the hyperdiffusion equation; yet, they did not directly compare the solution with that of either the diffusion equation, the telegraph equation, or the underlying Fokker–Planck equation. Motivated by that omission, in Section III, we consider two concrete initial-value problems and we present a detailed side-by-side comparison of evolving particle density profiles on a short timescale of a few scattering times, predicted by the diffusion, telegraph, and hyperdiffusion models. To evaluate the relative accuracy of the approximations, we compare the predictions with the corresponding solution of the Fokker–Planck equation. For simplicity, we consider the isotropic scattering in the following equation:[4]

$$D_{\mu\mu}(\mu) = D_0(1 - \mu^2), \tag{13}$$

where $D_0 = \text{const}$. We introduce dimensionless variables, by measuring distances in units of the scattering length $\lambda_0 = v/2D$, speeds in units of the constant particle speed v, and times in units of λ_0/v. The parallel diffusion coefficient $\kappa_\parallel$ is normalized by $\lambda_0 v$. On using Equations (10) and (14) in Litvinenko and Schlickeiser,[25] we obtain the dimensionless parameters $\kappa_\parallel = 1/3$, $\tau = 1$, and $h = 1/9$, corresponding to the isotropic scattering rate $D_{\mu\mu}$ given by Equation (13).

III. COMPARISON OF THE TELEGRAPH, HYPERDIFFUSION, AND DIFFUSION MODELS

In this section, we compare the evolving particle density profiles, predicted by the Fokker–Planck approach and the three approximations (telegraph, hyperdiffusion, and diffusion), for two initial-value problems in the context of the physical model of cosmic-ray transport, described in Section II. We focus our attention on the early propagation (a few scattering times) in order to compare the telegraph and hyperdiffusion models.

We determined the distribution function $f_0(z, \mu, t)$ numerically by solving a system of stochastic differential equations,

$$dz = \mu dt, \tag{14}$$

$$d\mu = -\mu dt + \sqrt{1 - \mu^2} dW, \tag{15}$$

where $W(t)$ represents a Wiener process with zero mean and variance t. The system contains the same information about the evolution of the particle distribution as the Fokker–Planck equation.[35] As in Ref. 16, we solved the stochastic differential equations using the Milstein approximation scheme[36] with 10^6 particles and time step $\Delta t = 1/1000$. Particles were reflected at $\mu = \pm 1$ to ensure particle number conservation. In order to obtain $F_0(z, t)$ from $f_0(z, \mu, t)$, we calculate the sum over all values of μ at a given location.

Both the telegraph and hyperdiffusion equations are linear partial differential equations, so it is straightforward to write down their formal solutions by Fourier transform. For simple initial conditions, the inverse transform in terms of Bessel functions is well known for the telegraph equation.[17,19] Malkov and Sagdeev[26] pointed out that the fundamental solution G_0 for a generalized telegraph model, derived by Litvinenko and Schlickeiser,[25] does not conserve the total particle number. Of course, this is a well-known property of the fundamental solution,[37] and the solution of an initial-value problem, based on G_0, does conserve the particle number.[17]

Although the inverse transform of the solution of the hyperdiffusion equation probably can be expressed in terms of standard special functions, we are only aware of an asymptotic analytical expression, given in Equation (41) by Malkov and Sagdeev[26] for $\kappa_\parallel = 0$ and $z^4 \gg 4ht$. Not unexpectedly, the asymptotic solution diverges at $z = 0$, whereas the Fourier solution given in Equation (40) by Malkov and Sagdeev[26] yields a finite expression for $z = 0$ and $t > 0$. Because of the limited validity of the asymptotic expression, we did not attempt to use it but integrated Equation (11) numerically. For consistency, we solved Equation (10) numerically as well and used its well-known analytical solution to verify the accuracy of the numerical solution.

We consider the evolution of particles, injected at $z = 0$, so that their initial distribution at $t = 0$ is given by

$$f_0(z, \mu, 0) = \delta(z)\phi(\mu), \tag{16}$$

where $\phi(\mu)$ specifies the initial angular distribution. Below we consider two cases: an isotropic initial distribution and a strongly anisotropic (beamed) distribution.

The isotropic initial angular distribution is described by

$$\phi(\mu) = \frac{1}{2} \tag{17}$$

and the corresponding initial conditions for $F_0(z,t)$ at $t=0$ are as follows:

$$F_0(z,0) = \frac{1}{2}\delta(z), \quad \frac{\partial F_0(z,0)}{\partial t} = 0, \tag{18}$$

where the second initial condition, required to solve Equation (10), follows from Equations (5) and (6). To represent $\delta(z)$ numerically, we use the normal distribution with mean zero and variance ϵ^2

$$N\left(z,0,\epsilon^2\right) = \frac{1}{\sqrt{2\pi}\epsilon} e^{-z^2/2\epsilon^2}. \tag{19}$$

We choose $\epsilon = 1/20$, so that it is sufficiently small to capture the characteristics of a narrow particle distribution but wide enough to resolve numerically and plot after a number of scattering times. Since the solution of the initial value problem of the telegraph equation with initial conditions (18) can be expressed in terms of Bessel functions,[19,20] we used the analytical result to verify that the numerical solution of Equation (10) is consistent with the analytical solution as we progressively reduce ϵ. The initial conditions in the stochastic simulations are given by a uniform distribution in $-1 < \mu < 1$ and the same normal distribution $N(z,0,\epsilon^2)$ to ensure the consistency of the initial conditions among the models.

Figures 1–5 show the particle density profiles at dimensionless times $t = 1/2$, 1, 5/2, 5, and 10 for the initial isotropic angular distribution. The plotted density distributions are normalized to unity. The black curves are the density histograms from the numerical solutions of the Fokker–Planck equation (1), the green curves are the numerical solutions of the telegraph equation (10), the blue curves are the numerical solutions of the hyperdiffusion equation (11), and the red curves are the solutions of the diffusion equation (8). This numerical solution clearly shows the unphysical spikes propagated by the telegraph equation, illustrating the points made by Malkov and Sagdeev.[26] By the time the spikes decay, the telegraph solution essentially coincides with both the diffusion solution and the density profile based on the solution of the Fokker–Planck equation.

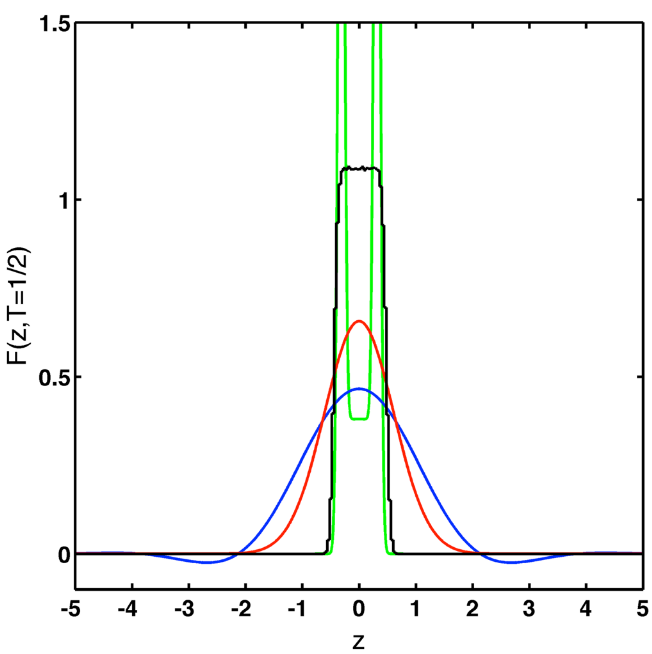

FIG. 1. Particle density profiles from solutions of the Fokker–Planck equation (black curve), the telegraph equation (green curve), the hyperdiffusion equation (blue curve), and the diffusion equation (red curve) at time $t = 1/2$. The initial angular distribution is isotropic, $\phi(\mu) = 1/2$. The plotted distribution $F = 2F_0$ is normalized to unity. The telegraph, hyperdiffusion, and diffusion equations are solved numerically with the same initial condition $F(z,0) = N(z,0,\epsilon^2)$ and $\epsilon = 1/20$. The solution of the telegraph equation also requires the second initial condition $\partial F(z,0)/\partial t = 0$. The figure is cropped at $F = 3/2$.

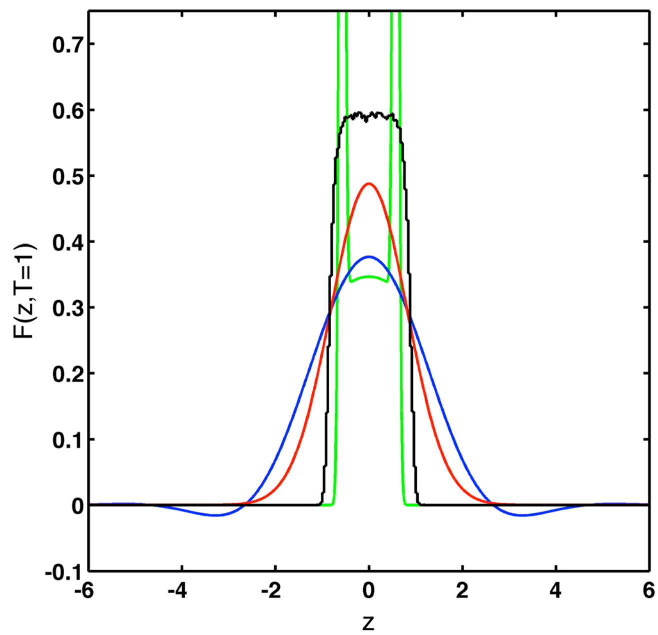

FIG. 2. Same as in Figure 1 at time $t = 1$. The figure is cropped at $F = 3/4$.

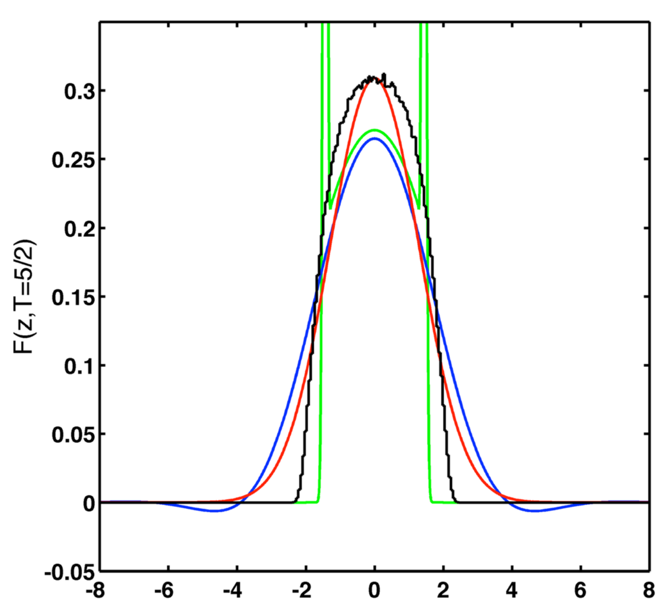

FIG. 3. Same as in Figure 1 at time $t = 5/2$. The figure is cropped at $F = 0.35$.

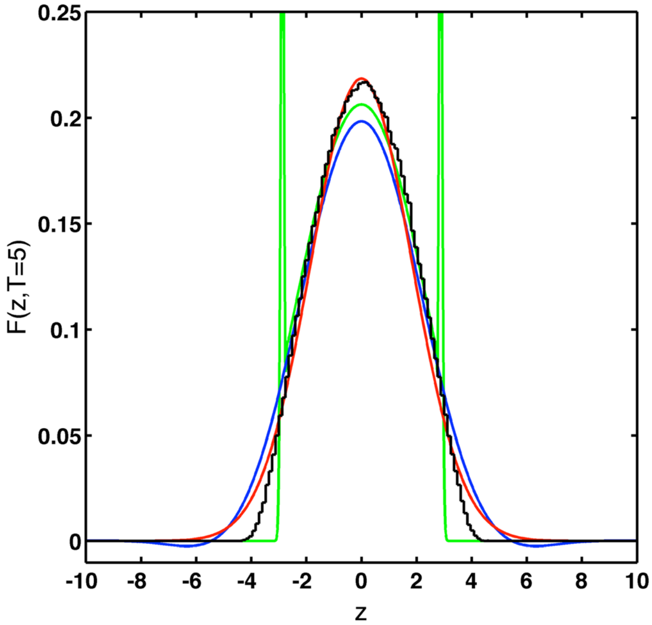

FIG. 4. Same as in Figure 1 at time $t = 5$. The figure is cropped at $F = 1/4$.

Somewhat unexpectedly, the hyperdiffusion model appears to offer virtually no improvement over the diffusion model. Consistent with the analytical asymptotic,[26] the hyperdiffusion model predicts a broader density profile and the unphysical negative densities some distance from the central peak at $z = 0$.

The strongly anisotropic initial angular distribution is described by

$$\phi(\mu) = \delta(\mu - 1) \tag{20}$$

and the corresponding initial conditions for $F_0(z, t)$ at $t = 0$ are as follows:

$$F_0(z, 0) = \frac{1}{2}\delta(z), \quad \frac{\partial F_0(z, 0)}{\partial t} = -\frac{1}{2}\delta'(z), \tag{21}$$

where the second initial condition, required to solve Equation (10), follows from Equations (5) and (6). Again, we solve the Fokker–Planck, telegraph, hyperdiffusion, and diffusion equations using $N(z, 0, \epsilon^2)$ with $\epsilon = 1/20$ to represent $\delta(z)$ numerically, and in the case of the second initial condition for the telegraph equation, using $-zN(z, 0, \epsilon^2)/\epsilon^2$ to represent $\delta'(z)$ numerically.

Figures 6–10 show the particle density profiles at dimensionless times $t = 1/2$, 1, 5/2, 5, and 10 for the strongly anisotropic initial distribution. The key difference from the isotropic case is that the initial particle beam propagates away from the injection point $z = 0$, while being spread out by scattering. As a result, the density profile maximum is shifted to the right of the injection location. The unphysical spikes are again present in the telegraph model, but after they decay the telegraph model shows a close agreement with the density profile predicted by the Fokker–Planck equation. This result follows from the fact that, for the hyperbolic telegraph equation, the solution of an initial-value problem requires the knowledge of the initial derivative $\partial F(z, 0)/\partial t$. By contrast, the density profiles, predicted by the diffusion and hyperdiffusion models, do not differ from those for the isotropic case, and so neither of the two models can distinguish between the isotropic and beamed initial conditions.

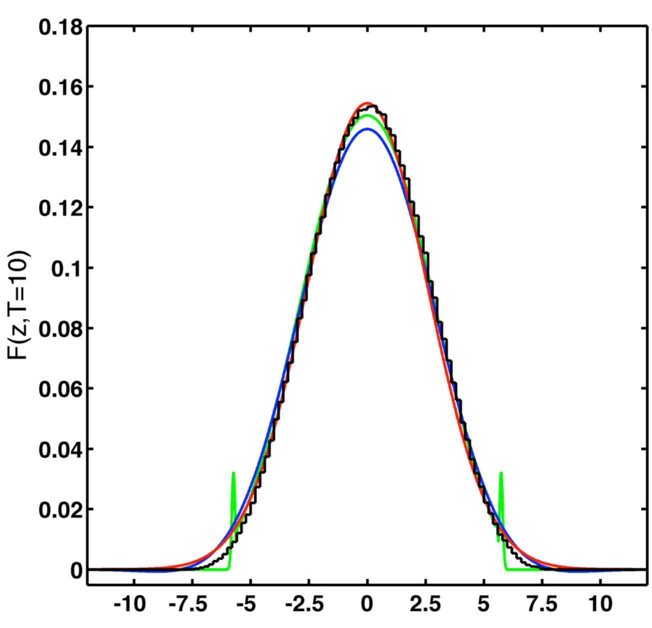

FIG. 5. Same as in Figure 1 at time $t = 10$.

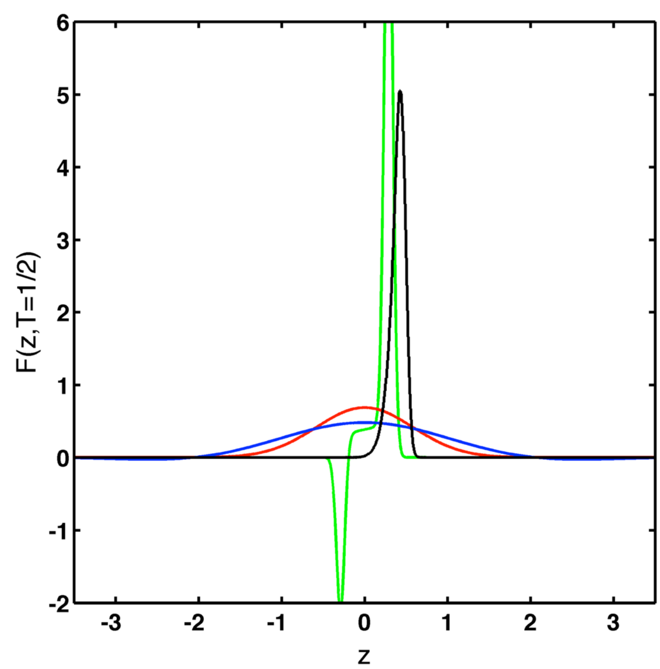

FIG. 6. Particle density profiles from solutions of the Fokker–Planck equation (black curve), the telegraph equation (green curve), the hyperdiffusion equation (blue curve), and the diffusion equation (red curve) at time $t = 1/2$. The initial angular distribution is strongly anisotropic, $\phi(\mu) = \delta(\mu - 1)$. The plotted distribution $F = 2F_0$ is normalized to unity. The telegraph, hyperdiffusion, and diffusion equations are solved numerically with the same initial condition $F(z, 0) = N(z, 0, \epsilon^2)$ and $\epsilon = 1/20$. The solution of the telegraph equation also requires the second initial condition $\partial F(z, 0)/\partial t = zN(z, 0, \epsilon^2)/\epsilon^2$. The figure is cropped at $F = -2$ and $F = 6$.

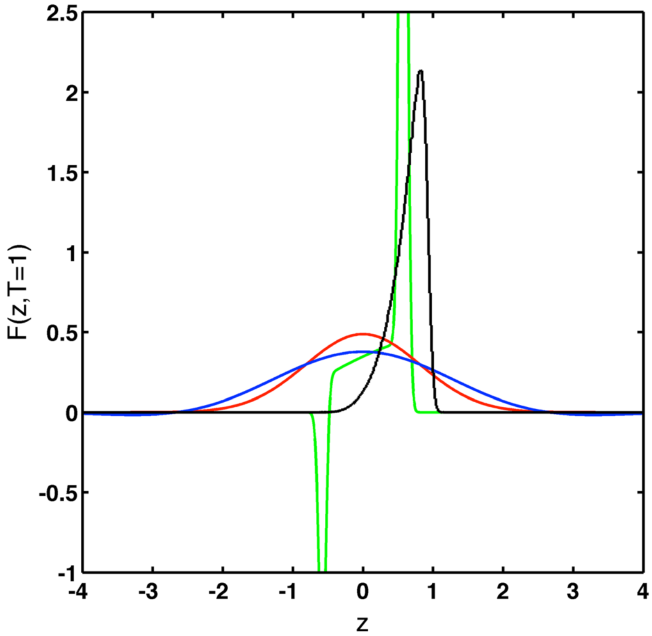

FIG. 7. Same as in Figure 6 at time $t = 1$. The figure is cropped at $F = -1$ and $F = 2.5$.

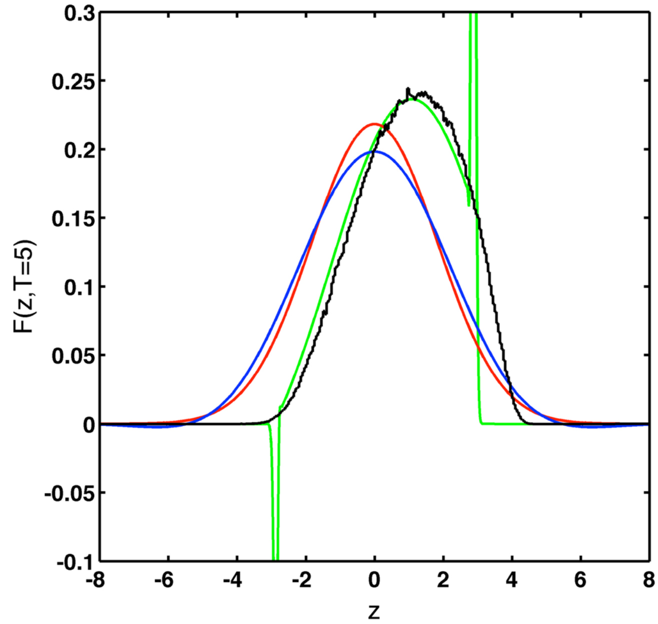

FIG. 9. Same as in Figure 6 at time $t = 5$. The figure is cropped at $F = -0.1$ and $F = 0.3$.

The differences between the two cases can be clarified by considering the scatter-free limit $D_0 = 0$ when complete analytical solutions of the Fokker–Planck Equation (1) are easy to obtain. For the initial condition given in Equation (16), the distribution function is as follows:

$$f_0(z, \mu, t) = \delta(z - \mu t)\phi(\mu). \tag{22}$$

Given the isotropic initial distribution of Equation (17), Equation (3) yields a broad, spatially symmetric distribution

$$F_0(z, t) = \frac{1}{4t} H(t - |z|), \tag{23}$$

where H denotes the Heaviside step function, which obviously does not contain the spikes of the telegraph solution. By contrast, the strongly anisotropic initial distribution of Equation (20) leads to a localized moving density peak,

$$F_0(z, t) = \frac{1}{2}\delta(z - t), \tag{24}$$

which cannot be reproduced by either diffusion or hyperdiffusion models. Perhaps, despite its theoretical shortcoming, the telegraph approximation might be useful in a concrete transport problem if $\partial F(z, 0)/\partial t$ could be estimated either observationally or on theoretical grounds. In practice, though

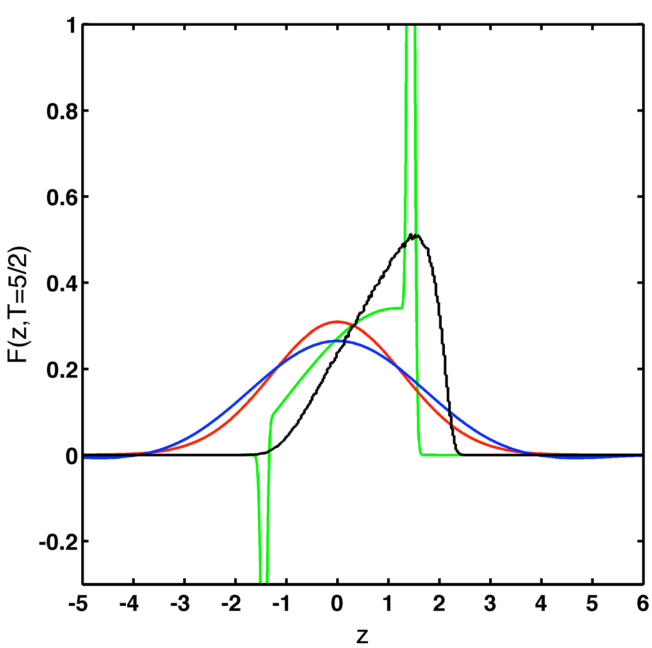

FIG. 8. Same as in Figure 6 at time $t = 5/2$. The figure is cropped at $F = -0.25$ and $F = 1$.

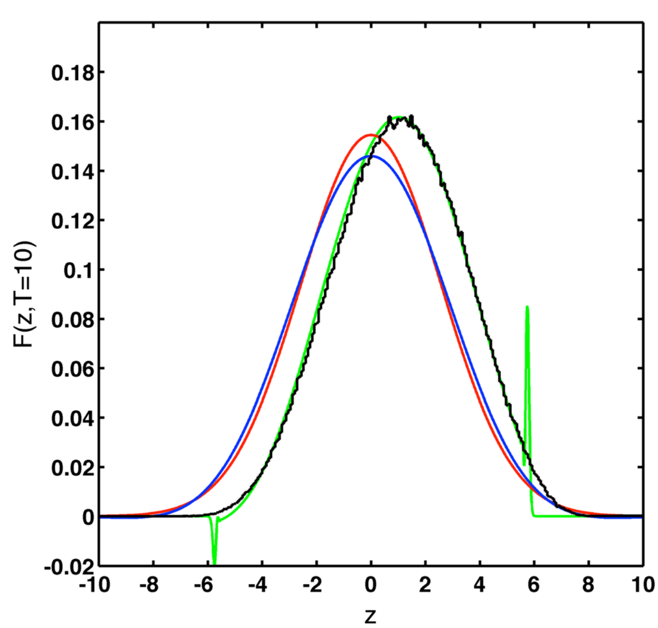

FIG. 10. Same as in Figure 6 at time $t = 10$. The figure is cropped at $F = -0.02$.

that would require some information on the distribution function $f_0(z, \mu, v, t)$ at some time t.

IV. DISCUSSION

The telegraph equation is often considered in modeling of diffusive transport problems.[27,38] An appealing feature of that hyperbolic equation is that its solutions are characterized by a finite signal propagation speed. Unfortunately, the solutions typically contain singular, δ-functional components or sharp peaks at propagating diffusion fronts. Such solutions may be relevant in persistent random walk problems,[39] and they may provide a heuristic description of spikes in solar energetic particle events, observed in the interplanetary medium.[25] However, as recently emphasized by Malkov and Sagdeev,[26] the telegraph model fundamentally disagrees with the Fokker–Planck description in the limit of vanishing scattering rate.

We compared the accuracy of the telegraph approximation and an alternative hyperdiffusion approximation, suggested by Malkov and Sagdeev,[26] for a finite scattering rate. Specifically, we compared the evolving particle density profiles, predicted by the diffusion, telegraph, and hyperdiffusion models, as well as by the underlying Fokker–Planck equation, for a simple but physically meaningful initial-value problem. We concentrated on short timescales of a few scattering times ($1/2 < t < 10$ in our dimensionless units), where the predictions in both the telegraph and hyperdiffusion models are expected to differ the most from the solution of the simpler diffusion equation.

For an initial isotropic distribution of energetic particles, we confirmed the presence of strong spikes at propagating fronts in the telegraph model (Figs. 1–5). The resulting evolving particle density profile differs qualitatively from that based on the solution of the Fokker–Planck equation. Although the peaks decay with time, they only become negligible when the solution is very close to the Gaussian profile of the diffusion equation ($t = 10$), and so, in agreement with the criticism by Malkov and Sagdeev,[26] the telegraph model does not lead to a more accurate description of particle transport in comparison with the simpler diffusion equation. The hyperdiffusion model, considered by Malkov and Sagdeev,[26] yields predictions that are similar to those of the diffusion model even for very short times—and both are rather different from the density profile based on the Fokker–Planck solution. In our side-by-side comparison, the hyperdiffusion model generally yields a broader density profile than does the diffusion model. One notable qualitative difference is that the hyperdiffusion model predicts unphysical negative densities some distance from a central peak. Paraphrasing the beginning of a classical novel, each inaccurate approximation is inaccurate in its own way.

The picture is more nuanced for a strongly anisotropic initial distribution of energetic particles (Figs. 6–10). Although the spikes are again present in the telegraph model, after they decay the solution approximates closely the density profile predicted by the Fokker–Planck equation. The key consequence of the initial beamed angular distribution is the shift of the density peak away from the particle injection location $z = 0$, and the telegraph model captures this effect quite accurately. This is of course a consequence of the fact that the solution of the hyperbolic telegraph equation requires the knowledge of the initial derivative $\partial F(z, 0)/\partial t$. By contrast, this additional information is not required to solve either the diffusion or hyperdiffusion equation, and so the hyperdiffusion model, just as the diffusion model, cannot distinguish between the isotropic and beamed initial conditions. Malkov and Sagdeev[26] expressed the viewpoint that this inability to incorporate the information on the initial angular distribution is actually an advantage of the hyperdiffusion model. From a theoretical point of view, a systematic self-contained model that does not incorporate any information on the angular distribution is preferable to a more heuristic model that does, notwithstanding the impression we may get from Fig. 10.

Our results also shed light on the recent application of the telegraph approximation to focused particle transport in a nonuniform mean magnetic field. The adiabatic focusing effect is in some sense analogous to an initial anisotropic distribution in which it introduces an asymmetry into the transport problem, causing the maximum of the particle density profile to shift from the injection location. Effenberger and Litvinenko[17] compared the diffusion and telegraph analytical solutions with the numerical solution of the Fokker–Planck equation for focused particle transport. Figures 2 and 3 by Effenberger and Litvinenko,[17] for instance, show that the telegraph model reproduces the shape of an evolving density pulse much better than does the diffusion model, especially when focusing is strong, even for times significantly exceeding the scattering time. While the overall shape of the density pulse is reproduced correctly, the plots by Effenberger and Litvinenko[17] are misleading to some extent since they omit the singular components of the analytical solution. In a numerical solution, the singular components would broaden into spikes, resembling those in our simulations.

It appears reasonable to conclude that, although the criticism of the telegraph approximation in cosmic-ray transport by Malkov and Sagdeev[26] is correct in principle, the telegraph model can be more accurate than either the diffusion or hyperdiffusion models in transport problems that are characterized by an asymmetry, caused for instance by the adiabatic focusing effect in a nonuniform magnetic field or by an anisotropic initial angular distribution or the cosmic-ray particles. The proviso is that the additional information on the asymmetry needs to be incorporated into the telegraph model and the model should only be applied after the unphysical spiky components decayed sufficiently. Finally, in our side-by-side comparison of particle transport on a short time scale after the injection, the hyperdiffusion model, proposed by Malkov and Sagdeev,[26] and the usual diffusion model yield essentially identical predictions for the evolving density profile, even for very short times.

Given the limitations of both the telegraph and hyperdiffusion models, it is worth pointing out that the traditional eigenfunction expansion method can be modified to yield an accurate description of particle transport even at small times. The key idea is to seek the distribution function as the sum of an unscattered part, representing particles that have not experienced scattering, and a scattered part associated with scattered particles. The method leads to an inhomogeneous telegraph equation whose solution does not contain the unphysical singular pulses; instead, the solution describes a well-defined wave-front, associated with the unscattered particles, which is followed by the diffusive population of the scattered particles.[40,41] The solution of the modified telegraph equation remains accurate for times much smaller than the scattering time. The surprisingly effective quasinumerical approach has been used to model the transport of interstellar pickup ions.[42,43]

ACKNOWLEDGMENTS

We thank Professor Reinhard Schlickeiser (Ruhr-Universität Bochum, Germany) for numerous useful discussions of the problem of diffusive cosmic-ray transport. We also thank an anonymous referee for useful comments and suggestions. Y.L.'s visit to Ruhr-Universität Bochum (RUB), where this work was completed, was supported by the RUB Research School PLUS Visiting International Professor Fellowship, funded by Germany's Excellence Initiative (DFG GSC 98/3).

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