

Quadratic algebra contractions and 2nd order superintegrable systems

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Abstract

Quadratic algebras are generalizations of Lie algebras; they include the symmetry algebras of 2nd order superintegrable systems in 2 dimensions as special cases. The superintegrable systems are exactly solvable physical systems in classical and quantum mechanics. For constant curvature spaces we show that the free quadratic algebras generated by the 1st and 2nd order elements in the enveloping algebras of their Euclidean and orthogonal symmetry algebras correspond one-to-one with the possible superintegrable systems with potential defined on these spaces. We describe a contraction theory for quadratic algebras and show that for constant curvature superintegrable systems, ordinary Lie algebra contractions induce contractions of the quadratic algebras of the superintegrable systems that correspond to geometrical pointwise limits of the physical systems. One consequence is that by contracting function space realizations of representations of the generic superintegrable quantum system on the 2-sphere (which give the structure equations for Racah/Wilson polynomials) to the other superintegrable systems one obtains the full Askey scheme of orthogonal hypergeometric polynomials.

Keywords: contractions; quadratic algebras; superintegrable systems; Askey scheme

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1 Introduction

In this special issue honoring Frank Olver, a paper devoted to algebraic issues for superintegrable systems might seem out of place. However, there are very close connections with Frank's interests. Quantum superintegrable systems are explicitly solvable problems with physical interest and special functions arise

through this association. Most special functions of mathematical physics, as listed in the Digital Library of Mathematical Functions, appear via separation of variables, determined by 2nd order symmetry operators of superintegrable systems. Most special functions that are solutions of 2nd order difference equations arise as function space realizations of representations of symmetry algebras of superintegrable systems. Orthogonal polynomials, continuous and discrete, appear naturally in this context. The structure theory of superintegrable systems provides a way of organizing special functions and relating their properties, an alternative approach to the DLMF.

For example, consider the following quantum superintegrable system: the generic 3-parameter potential on the 2-sphere [6]. The eigenvalue equation $H\Psi = E\Psi$ separates in spherical coordinates (in multiple ways) and in Lamé coordinates [11]. The spherical coordinate eigenfunctions are the orthogonal Prorior/Karlin-McGregor polynomials, orthogonal on a right triangle [10]. The corresponding eigenfunctions in 1-parameter function space realizations of the irreducible representations of the symmetry algebra are the Racah and Wilson polynomials, in full generality [3]. If we set two of the parameters in the potential equal to 0 so that the restricted system has axial symmetry, then the quantum system (the Higgs oscillator) still separates in two spherical coordinate systems. One set of eigenfunctions yields the Koschmieder polynomials, the other the Zerneke polynomials, orthogonal on the unit disk [10]. Corresponding function space realizations of the irreducible representations of the symmetry algebra yield Hahn and dual Hahn polynomials. Taking pointwise limits of this generic system we can contract it to a variety of quantum systems on flat space, with separable eigenfunctions expressed as products of Hermite, Laguerre and Jacobi polynomials for bound states, and with continuous spectra associated with hypergeometric, confluent hypergeometric and Bessel functions. Taking contractions of the irreducible function space realizations of the symmetry algebras and “saving a representation” in the sense of Wigner [2], we can recover the Askey scheme for hypergeometric orthogonal polynomials [13] and limit relations for more complicated functions, such as Lamé, Heun and Mathieu functions [14].

Given an n -dimensional Riemannian or pseudo-Riemannian manifold, real or complex, we define a quantum Hamiltonian in local coordinates x_i as $H = \Delta_n + V(\mathbf{x}) \equiv \frac{1}{\sqrt{g}} \sum_{j,k=1}^n \partial_j (g^{jk} \sqrt{g} \partial_k) + V(\mathbf{x})$ where Δ_n is the Laplace-Beltrami operator in these coordinates, $g^{jk}(\mathbf{x})$ is the contravariant metric tensor and g is the determinant of the covariant metric tensor. V is a scalar potential. The quantum system is (maximally) *superintegrable* if there are $2n - 1$ algebraically independent partial differential operators $L_1, \dots, L_{2n-2}, L_{2n-1} = H$ that commute with H . All functions of the coordinates are assumed locally analytic. Similarly a classical Hamiltonian $\mathcal{H} = \sum_{jk} g^{jk} p_j p_k + V(\mathbf{x})$ is *superintegrable* if there are $2n - 1$ functionally independent constants of the motion $\mathcal{L}_1, \dots, \mathcal{L}_{2n-2}, \mathcal{L}_{2n-1} = \mathcal{H}$ in involution with \mathcal{H} : $\{\mathcal{L}_\ell, \mathcal{H}\} = 0$, $\ell = 1, \dots, 2n - 1$, with respect to the Poisson bracket $\{\mathcal{F}(\mathbf{p}, \mathbf{x}), \mathcal{G}(\mathbf{p}, \mathbf{x})\} = \sum_{j=1}^n (\partial_{p_j} \mathcal{F} \partial_{x_j} \mathcal{G} - \partial_{p_j} \mathcal{G} \partial_{x_j} \mathcal{F})$. (Throughout the paper we use \mathcal{L} for constants of the motion and L for quantum symmetries.)

It is assumed that the \mathcal{L}_ℓ are polynomial functions of the momenta p_j and globally defined in the x_j except for possible singularities on lower dimensional manifolds. The maximum possible number of functionally independent con-

stants of the motion is $2n - 1$ and this maximum is rarely achieved. Superintegrability captures the properties of quantum Hamiltonian systems that allow the Schrödinger eigenvalue problem $H\Psi = E\Psi$ to be solved exactly, analytically and algebraically and the orbits of the classical superintegrable systems to be determined algebraically. For a more careful discussion of superintegrability and its applications, see [17].

The key to the connection between solvability and superintegrability lies in the symmetry algebra S produced from the generators L_ℓ by taking linear combinations, products and commutators. If a system is merely integrable with n commuting generators L_j then the algebra is abelian. However it is not possible to have more than n commuting independent operators, so for a superintegrable system the symmetry algebra is necessarily nonabelian. Since S maps each energy eigenspace of H into itself the eigenspaces are multiply degenerate, and the irreducible representations of S give the possible degeneracies and energy eigenvalues.

A quantum system is of order K if the maximum order of the symmetry operators, other than H , is K . (There is a similar definition for classical systems, based on the order of the symmetries as polynomials in the momenta.) Much of the recent excitement in superintegrability theory is due to the discovery of superintegrable systems for n and K arbitrarily large, e.g., [19, 21, 22], with no connection between these systems and group theory. However, for $n = 2$, $K = 1, 2$ a connection exists.

In [13] the concept of a contraction of the symmetry algebra of a 2D 2nd order superintegrable system was introduced and the Askey scheme as derived via contractions. However, it was unclear how the contractions were found; the procedure appeared complicated. Here we demonstrate that all of the limits are induced by Wigner-Inönü contractions of the Lie algebras $e(2, \mathbb{C})$ and $o(3, \mathbb{C})$, already classified. Further, all of the quadratic algebras of 2nd order 2D superintegrable systems correspond 1-1 to free quadratic algebras contained in the enveloping algebras of $e(2, \mathbb{C})$ and $o(3, \mathbb{C})$. Thus, though many of these systems admit no group symmetry, their structures are determined by the underlying Lie algebras.

2 2D 2nd order superintegrability

For $n = 2$, $K = 2$, a superintegrable system admits 3 symmetries and in this special case there is a 1 – 1 relation between quantum and classical symmetries, [9]. The potentials are the same and corresponding to a 2nd order classical constant of the motion $\mathcal{L} = \sum_{j,k=1}^2 L^{jk}(\mathbf{x})p_j p_k + W(\mathbf{x})$, L^{jk} a symmetric contravariant tensor, the quantum symmetry is $L = \frac{1}{\sqrt{g}} \sum_{i,j=1}^2 \partial_i(L^{ij}(\mathbf{x})\sqrt{g}\partial_j) + W(\mathbf{x})$. Here L is formally self-adjoint with respect to the bilinear form $\langle f_1, f_2 \rangle_g = \int f_1(\mathbf{x})f_2(\mathbf{x})\sqrt{g(\mathbf{x})} dx_1 dx_2$ on the manifold, [12]. The set $\{H, L_1, L_2\}$ of generating symmetries is required to be algebraically independent, i.e., there is no nontrivial polynomial $P(H, L_1, L_2)$, symmetric in L_1, L_2 such that $P \equiv 0$. For our treatment of 2nd order 2D quantum systems the values of the mass m and Planck's constant \hbar are immaterial, so we have normalized our Hamiltonians as given. Every 2D Riemannian space is conformally flat so there exist Cartesian-

like coordinates x_1, x_2 such that

$$H = \frac{1}{\lambda(\mathbf{x})}(\partial_{11} + \partial_{22}) + V(\mathbf{x}), \quad L_\ell = \frac{1}{\lambda} \sum_{j,k=1}^2 \partial_j \left(L_{(\ell)}^{jk} \lambda \right) \partial_k + W_{(\ell)}(\mathbf{x}), \quad k = 1, 2, \quad (1)$$

(A 1st order constant of the motion $\mathcal{X} = \sum_{j=1}^2 f_j(\mathbf{x}) p_j$ corresponds to the formally skew-adjoint symmetry operator $X = \sum_{j=1}^2 \left(f_j \partial_{x_j} + \frac{\partial_{x_j}(\lambda f_j)}{2\lambda} \right)$.) The symmetry relations $\{\mathcal{H}, \mathcal{L}_k\} = 0$, $k = 1, 2$, put conditions on the functions $W_{(1)}, W_{(2)}$. If we require that the symmetries are linearly functionally independent, i.e., that $g_1 L_1 + g_2 L_2 + g_3 H \equiv 0$ for functions g_j implies $g_1 \equiv g_2 \equiv g_3 \equiv 0$, we can solve for the partial derivatives $\partial_j W_{(k)}$ in terms of the function V and its 1st derivatives. The integrability conditions $\partial_1(\partial_2 W_{(k)}) = \partial_2(\partial_1 W_{(k)})$, the Bertrand-Darboux equations [7], lead to the necessary and sufficient condition that V must satisfy a pair of coupled linear equations of the form

$$V_{22} - V_{11} = A^{22} V_1 + B^{22} V_2, \quad V_{12} = A^{12} V_1 + B^{12} V_2, \quad (2)$$

for locally analytic functions $A^{ij}(\mathbf{x}), B^{ij}(\mathbf{x})$. Here $V_i = \partial_i V$, etc. We call these the *canonical equations*. If the integrability equations for (2) are satisfied identically then the solution space is 4-dimensional and we can always express the solution in the form $V(\mathbf{x}) = \sum_{j=1}^3 a_j V_{(j)}(\mathbf{x}) + a_4$ where a_4 is a trivial additive constant. In this case the potential is *nondegenerate* and 3-parameter. Another possibility is that the solution space is 2-dimensional with general solution $V(\mathbf{x}) = a_1 V_{(1)}(\mathbf{x}) + a_2$. Then the potential is *degenerate* and 1-parameter. Every degenerate potential can be obtained from some nondegenerate potential by parameter restriction, [4]. It is not *just* a restriction, however, because the symmetry algebra changes. A formally skew-adjoint 1st order symmetry appears and this induces a new 2nd order symmetry. A third possibility is that the integrability conditions are satisfied only by a constant potential. In that case we refer to the system as *free*; the free equation $H\Psi = E\Psi$ is just the Laplace-Beltrami eigenvalue equation. Note: Any 2-parameter potential extends to a 3-parameter potential, [4]. There is one remaining possibility: we can satisfy relations $[H, L_k] = 0$, but the symmetries L_1, L_2 are functionally linearly dependent. There is a single exceptional superintegrable system for which this is true, E_{15} in our listing [6]. All of the systems with nondegenerate potential (and E_{15}) have the remarkable property that the symmetry algebras generated by H, L_1, L_2 close polynomially under commutation, as follows. Define the 3rd order commutator R by $R = [L_1, L_2]$. Then the fourth order operators $[R, L_1], [R, L_2]$ are contained in the associative algebra of symmetrized products of the generators [7]:

$$[L_j, R] = \sum_{0 \leq e_1 + e_2 + e_3 \leq 2} M_{e_1, e_2, e_3}^{(j)} \{L_1^{e_1}, L_2^{e_2}\} H^{e_3}, \quad e_k \geq 0, \quad L_k^0 = I, \quad (3)$$

where $\{L_1, L_2\} = L_1 L_2 + L_2 L_1$ is the symmetrizer. Also the 6th order operator R^2 is contained in the algebra of symmetrized products up to 3rd order:

$$R^2 - \sum_{0 \leq e_1 + e_2 + e_3 \leq 3} N_{e_1, e_2, e_3} \{L_1^{e_1}, L_2^{e_2}\} H^{e_3} = 0. \quad (4)$$

In both equations the constants $M_{e_1, e_2, e_3}^{(j)}$ and N_{e_1, e_2, e_3} are polynomials in the parameters a_1, a_2, a_3 of degree $2 - e_1 - e_2 - e_3$ and $3 - e_1 - e_2 - e_3$, respectively.

For systems with one parameter potentials, [4], there are 4 generators: one 1st order \mathcal{X} and three 2nd order H, L_1, L_2 . The commutators $[X, L_1], [X, L_2]$ are 2nd order and expressed as

$$[X, L_j] = \sum_{0 \leq e_1 + e_2 + e_3 + e_4 \leq 1} P_{e_1, e_2, e_3, e_4}^{(j)} L_1^{e_1} L_2^{e_2} H^{e_3} X^{e_4}, \quad j = 1, 2. \quad (5)$$

The commutator $[L_1, L_2]$ is 3rd order, skew adjoint, and expressed as

$$[L_1, L_2] = \sum_{0 \leq e_1 + e_2 + e_3 + e_4 \leq 1} Q_{e_1, e_2, e_3, e_4} \{L_1^{e_1} L_2^{e_2}, X\} H^{e_3} X^{2e_4}. \quad (6)$$

Finally, there is a 4th order relation:

$$G \equiv \sum_{0 \leq e_1 + e_2 + e_3 + e_4 \leq 2} S_{e_1, e_2, e_3, e_4} \{L_1^{e_1}, L_2^{e_2}, X^{2e_4}\} H^{e_3} = 0, \quad X^0 = H^0 = I, \quad (7)$$

where $\{L_1^{e_1}, L_2^{e_2}, X^{2e_4}\}$ is the 6-term symmetrizer of three operators. The constants $P_{e_1, e_2, e_3, e_4}^{(j)}$, Q_{e_1, e_2, e_3, e_4} and S_{e_1, e_2, e_3, e_4} are polynomials in the parameter a_1 of degrees $1 - e_1 - e_2 - e_3 - e_4$, $1 - e_1 - e_2 - e_3 - e_4$ and $2 - e_1 - e_2 - e_3 - e_4$, respectively.

We call these symmetry algebras for degenerate and nondegenerate systems *quadratic algebras*, in the sense that the commutators of the generators are at most quadratic expansions in the generators. Usually, the generators for free systems form an algebra that doesn't close, not a quadratic algebra.

There is an analogous quadratic algebra structure for classical superintegrable systems in 2D. All these classical systems have the property that the symmetry algebras generated by $\mathcal{H}, \mathcal{L}_1, \mathcal{L}_2$ for nondegenerate potentials close under Poisson brackets. Define the 3rd order bracket \mathcal{R} by $\mathcal{R} = \{\mathcal{L}_1, \mathcal{L}_2\}$. Then the fourth order constants of the motion $\{\mathcal{L}_j, \mathcal{R}\}$ are can be expressed as, [7]:

$$\{\mathcal{L}_j, \mathcal{R}\} = \sum_{0 \leq e_1 + e_2 + e_3 \leq 2} M_{e_1, e_2, e_3}^{(j)} \mathcal{L}_1^{e_1} \mathcal{L}_2^{e_2} \mathcal{H}^{e_3}, \quad e_k \geq 0, \quad \mathcal{L}_k^0 = 1. \quad (8)$$

Also the 6th order constant of the motion \mathcal{R}^2 satisfies:

$$\mathcal{R}^2 - \sum_{0 \leq e_1 + e_2 + e_3 \leq 3} N_{e_1, e_2, e_3} \mathcal{L}_1^{e_1} \mathcal{L}_2^{e_2} \mathcal{H}^{e_3} = 0. \quad (9)$$

In both equations the constants $M_{e_1, e_2, e_3}^{(j)}$ and N_{e_1, e_2, e_3} are polynomials in the parameters a_1, a_2, a_3 of degree $2 - e_1 - e_2 - e_3$ and $3 - e_1 - e_2 - e_3$, respectively.

For one parameter potentials, [4], there are 4 generators: one 1st order in momenta \mathcal{X} and three 2nd order $\mathcal{H}, \mathcal{L}_1, \mathcal{L}_2$. The brackets $\{\mathcal{X}, \mathcal{L}_j\}$ are 2nd order:

$$\{\mathcal{X}, \mathcal{L}_j\} = \sum_{0 \leq e_1 + e_2 + e_3 + e_4 \leq 1} P_{e_1, e_2, e_3, e_4}^{(j)} \mathcal{L}_1^{e_1} \mathcal{L}_2^{e_2} \mathcal{H}^{e_3} \mathcal{X}^{e_4}, \quad j = 1, 2. \quad (10)$$

The bracket $\{\mathcal{L}_1, \mathcal{L}_2\}$ is 3rd order and expressed as

$$\{\mathcal{L}_1, \mathcal{L}_2\} = \sum_{0 \leq e_1 + e_2 + e_3 + e_4 \leq 1} Q_{e_1, e_2, e_3, e_4} \mathcal{L}_1^{e_1} \mathcal{L}_2^{e_2} \mathcal{X} \mathcal{H}^{e_3} \mathcal{X}^{2e_4}. \quad (11)$$

There is a 4th order relation obeyed by the generators:

$$\mathcal{G} \equiv \sum_{0 \leq e_1 + e_2 + e_3 + e_4 \leq 2} S_{e_1, e_2, e_3, e_4} \mathcal{L}_1^{e_1} \mathcal{L}_2^{e_2} \mathcal{X}^{2e_4} \mathcal{H}^{e_3} = 0, \quad \mathcal{X}^0 = \mathcal{H}^0 = 1. \quad (12)$$

The constants $P_{e_1, e_2, e_3, e_4}^{(j)}$, Q_{e_1, e_2, e_3, e_4} and S_{e_1, e_2, e_3, e_4} are polynomials in a_1 of degrees $1 - e_1 - e_2 - e_3 - e_4$, $1 - e_1 - e_2 - e_3 - e_4$ and $2 - e_1 - e_2 - e_3 - e_4$, respectively.

For free systems that do not admit a 1- or 3-parameter potential the algebra of the generators normally doesn't close, see §4. The structure equations for the quadratic algebras of associated classical and quantum systems are not identical, but they agree in the highest order terms. The differences are 1) quantum operators may not commute and for quantization, products of constants of the motion are replaced by operator symmetrizers, and 2) even order symmetry operators in the generating basis must be formally self-adjoint; odd order ones formally skew-adjoint.

We can study quadratic algebras in general, whether or not they arise as symmetry algebras of a superintegrable system. Thus, we define an abstract *nondegenerate (quantum) quadratic algebra* is a noncommutative associative algebra generated by linearly independent operators H, L_1, L_2 , with parameters a_1, a_2, a_3 , such that H is in the center and relations (3), (4) hold. Similarly we define an abstract *degenerate (quantum) quadratic algebra* is a noncommutative associative algebra generated by linearly independent operators X, H, L_1, L_2 , with parameter a_1 , such that H is in the center and relations (5),(6),(7) hold. We also consider systems where all of the parameters a_j are identically zero; these are *free nondegenerate* and *free degenerate* (quantum) quadratic algebras. Analogously, an abstract *nondegenerate (classical) quadratic algebra* is a Poisson algebra with functionally independent generators $\mathcal{H}, \mathcal{L}_1, \mathcal{L}_2$, and parameters a_1, a_2, a_3 , such that all generators are in involution with H and relations (8) and (9) hold. An abstract *degenerate (classical) quadratic algebra* is a Poisson algebra with linearly independent generators $\mathcal{X}, \mathcal{H}, \mathcal{L}_1, \mathcal{L}_2$, and parameter a_1 , such that all generators are in involution with H and relations (10),(11) and (12) hold. Systems with all a_j identically zero are *free nondegenerate* and *free degenerate* (classical) quadratic algebras.

2.1 Nondegenerate classical structure equations

Suppose the 2D classical second order superintegrable system with nondegenerate potential has 2nd order generators $\mathcal{L}_1, \mathcal{L}_2, \mathcal{H}$ with $\mathcal{R} = \{\mathcal{L}_1, \mathcal{L}_2\}$. The Casimir is $\mathcal{R}^2 - F(\mathcal{L}_1, \mathcal{L}_2, \mathcal{H}, a_1, a_2, a_3) = 0$ where the a_j are the parameters in the potential. It is easy to show that $\{\mathcal{L}_1, \mathcal{R}\} = \frac{1}{2} \frac{\partial F}{\partial \mathcal{L}_2}$, $\{\mathcal{L}_2, \mathcal{R}\} = -\frac{1}{2} \frac{\partial F}{\partial \mathcal{L}_1}$, so the Casimir contains within itself all of the structure equations. A similar, but more complicated result for nondegenerate quantum quadratic algebras will appear in a forthcoming paper.

2.2 Degenerate classical structure equations

Now suppose the 2D classical second order superintegrable system with degenerate (1-parameter) potential has generators \mathcal{X} (1st order), and $\mathcal{L}_1, \mathcal{L}_2, \mathcal{H}$

(2nd order) with Casimir $G(\mathcal{X}, \mathcal{L}_1, \mathcal{L}_2, \mathcal{H}, \alpha) = 0$, where the α is the parameter in the potential. Note that G is determined only to within a multiplicative constant. Now $0 = \{\mathcal{X}, G\} = \frac{\partial G}{\partial \mathcal{L}_1} \{\mathcal{X}, \mathcal{L}_1\} + \frac{\partial G}{\partial \mathcal{L}_2} \{\mathcal{X}, \mathcal{L}_2\} \implies \{\mathcal{X}, \mathcal{L}_1\} = K \frac{\partial G}{\partial \mathcal{L}_2}$, $\{\mathcal{X}, \mathcal{L}_2\} = -K \frac{\partial G}{\partial \mathcal{L}_1}$, for some constant K , since $\{\mathcal{X}, \mathcal{L}_j\}$, $\partial G / \partial \mathcal{L}_j$ are all 2nd order in the momenta. Further $0 = \{\mathcal{L}_1, G\} = \frac{\partial G}{\partial \mathcal{X}} \{\mathcal{L}_1, \mathcal{X}\} + \frac{\partial G}{\partial \mathcal{L}_2} \{\mathcal{L}_1, \mathcal{L}_2\}$, $0 = \{\mathcal{L}_2, G\} = \frac{\partial G}{\partial \mathcal{X}} \{\mathcal{L}_2, \mathcal{X}\} + \frac{\partial G}{\partial \mathcal{L}_1} \{\mathcal{L}_2, \mathcal{L}_1\}$. Assuming G depends nontrivially on at least one of $\mathcal{L}_1, \mathcal{L}_2$, we have

$$\{\mathcal{X}, \mathcal{L}_1\} = K \frac{\partial G}{\partial \mathcal{L}_2}, \quad \{\mathcal{X}, \mathcal{L}_2\} = -K \frac{\partial G}{\partial \mathcal{L}_1}, \quad \{\mathcal{L}_1, \mathcal{L}_2\} = K \frac{\partial G}{\partial \mathcal{X}}. \quad (13)$$

Thus the structure equations are determined by G to within a constant.

For a degenerate superintegrable system it would seem that it is possible that K is a rational constant of the motion; either 1) the ratio of two 2nd order polynomials in the momenta (necessarily two 2nd order constants of the motion) or 2) the ratio of two 1st order polynomials in the momenta (necessarily multiples of \mathcal{X}). However, in case 1) it is easy to see that this would imply 3 mutually involutive symmetries, impossible for a 2D Hamiltonian system and case 2) is trivially equivalent to a constant K . Thus for a 2D degenerate superintegrable system K is always a nonzero constant. However, for free superintegrable systems rational K can occur.

Example 1 For some functions $\mathcal{X}, \mathcal{L}_1, \mathcal{L}_2, \mathcal{H}$ satisfying a polynomial relation $G = 0$, K may be rational. For example, the flat space system $\mathcal{L}_1 = \mathcal{J}p_1$, $\mathcal{L}_2 = p_1^2$, $\mathcal{X} = \mathcal{J}$, $\mathcal{H} = p_1^2 + p_2^2$, with $\mathcal{J} = xp_2 - yp_1$, gives $G = -\mathcal{L}_1^2 + \mathcal{X}^2 \mathcal{L}_2 = 0$, $K = -p_2/\mathcal{X}$. However, this is not a degenerate superintegrable system. It is free.

Degenerate superintegrable systems are restrictions of the 3-parameter potentials to 1-parameter ones, such that new symmetries appear: We can take a particular basis of 2nd order generators $\mathcal{H}, \mathcal{L}_1, \mathcal{L}_2$, and parameters a_1, a_2, a_3 for the classical physical system with nondegenerate potential, such that for $a_2 = a_3 = 0$ the symmetry \mathcal{L}_1 becomes a perfect square: $\mathcal{L}_1|_{a_1=a_2=0} = \mathcal{X}^2$. Then \mathcal{X} will be a 1st order symmetry for $\mathcal{H}_0 = \mathcal{H}|_{a_1=a_2=0}$ with no potential term, i.e., a Killing vector. Noting the relation $\mathcal{R} = \{\mathcal{L}_1, \mathcal{L}_2\} = 2\mathcal{X}\{\mathcal{X}, \mathcal{L}_2\}$ upon restriction, we see that $\mathcal{L}_3 \equiv \{\mathcal{X}, \mathcal{L}_2\}$ is a 2nd order symmetry for \mathcal{H}_0 (usually linearly independent of the symmetries we already know). We can factor $2\mathcal{X}^2$ from each term of the restricted identity $\mathcal{R}^2 - \mathcal{F} = 0$ to obtain the Casimir $\mathcal{G} = 0$ for the contracted system, where $\mathcal{G} = \mathcal{L}_3^2 + \dots$. In the limit, (13) (with \mathcal{L}_1 replaced by \mathcal{L}_3) holds with constant K .

If however, \mathcal{L}_3 is a linear combination of $\mathcal{X}^2, \mathcal{L}_2, \mathcal{H}_0$ then the resulting expression is identically satisfied and we get no additional information about the degenerate structure algebra. By inspection one can verify that all Casimirs $G = 0$ can be obtained as limits of equations $\mathcal{R}^2 - \mathcal{F} = 0$ for some nondegenerate superintegrable system, except for degenerate systems Stäckel equivalent to E_4 or E_{13} , see below. For those systems the new 2nd order symmetries appear in a discontinuous manner. All these results have quantum analogies, as we shall show in a forthcoming paper.

3 Free 2D 2nd order superintegrable systems

As was shown in [8,9] the ‘free’ 2nd order superintegrable system obtained by setting all the parameters in a nondegenerate potential equal to zero retains all of the information needed to reconstruct the potential. Thus we can, in principle, restrict our attention to free systems. Here we explore this concept in more detail and extend it. First we review how the structure equations for 2D 2nd order nondegenerate classical superintegrable systems are determined. Such a system admits a symmetry $\mathcal{L} = \sum a^{ij}p_i p_j + W$ if and only if the Killing equations are satisfied

$$a_i^{ii} = -\frac{\lambda_1}{\lambda}a^{i1} - \frac{\lambda_2}{\lambda}a^{i2}, \quad i = 1, 2, \quad 2a_i^{ij} + a_j^{ii} = -\frac{\lambda_1}{\lambda}a^{j1} - \frac{\lambda_2}{\lambda}a^{j2}, \quad i, j = 1, 2, \quad i \neq j, \quad (14)$$

where $a_k^{ij} = \partial_{x_k} a^{ij}$, as well as $W_i = \lambda \sum_{j=1}^2 a^{ij} V_j$. Here $W_i = \partial_{x_i} W$ with a similar convention for subscripts on V . The equations for W can be solved provided the Bertrand-Darboux equation $\partial_{x_1} W_2 = \partial_{x_2} W_1$ holds. We can solve the two independent Bertrand-Darboux equations for the potential to obtain the canonical system (2) where the A^{ij}, B^{ij} are computable from the generating constants of the motion. For nondegenerate superintegrability, the integrability conditions for the canonical equations must be satisfied identically, so that V, V_1, V_2, V_{11} can be prescribed arbitrarily at a fixed regular point.

To obtain the integrability conditions for equations (2) we introduce the dependent variables $W^{(1)} = V_1, W^{(2)} = V_2, W^{(3)} = V_{11}$, and matrices

$$\mathbf{w} = \begin{pmatrix} W^{(1)} \\ W^{(2)} \\ W^{(3)} \end{pmatrix}, \quad \mathbf{A}^{(1)} = \begin{pmatrix} 0 & 0 & 1 \\ A^{12} & B^{12} & 0 \\ A^{13} & B^{13} & B^{12} - A^{22} \end{pmatrix}, \quad \mathbf{A}^{(2)} = \begin{pmatrix} A^{12} & B^{12} & 0 \\ A^{22} & B^{22} & 1 \\ A^{23} & B^{23} & A^{12} \end{pmatrix}, \quad (15)$$

$$\begin{aligned} A^{13} &= A_2^{12} - A_1^{22} + B^{12}A^{22} + A^{12}A^{12} - B^{22}A^{12} \\ B^{13} &= B_2^{12} - B_1^{22} + A^{12}B^{12}, \quad A^{23} = A_1^{12} + B^{12}A^{12}, \quad B^{23} = B_1^{12} + B^{12}B^{12}. \end{aligned}$$

Then the integrability conditions for system $\partial_{x_j} \mathbf{w} = \mathbf{A}^{(j)} \mathbf{w}$, $j = 1, 2$, must hold:

$$A_1^{(2)} - A_2^{(1)} = A^{(1)}A^{(2)} - A^{(2)}A^{(1)} \equiv [A^{(1)}, A^{(2)}]. \quad (16)$$

If and only if (16) holds, the system has a 4D vector space of solutions V .

From the conditions that \mathcal{L} is a constant of the motion and relations (2) we can solve for all of the first partial derivatives $\partial_i(a^{jk})$ to obtain

$$\begin{aligned} \partial_1 a^{11} &= -G_1 a^{11} - G_2 a^{12}, \quad \partial_2 a^{22} = -G_1 a^{12} - G_2 a^{22}, \\ 3\partial_2 a^{12} &= -3G_2 a^{12} + (a^{11} - a^{22})(-B^{12} - G_1) + a^{12}(-B^{22} + G_2), \\ 3\partial_1 a^{22} &= -3G_1 a^{22} + (a^{11} - a^{22})(2B^{12} - G_1) + a^{12}(2B^{22} + G_2), \\ 3\partial_1 a^{12} &= -3G_1 a^{12} + (a^{11} - a^{22})(A^{12} + G_2) + a^{12}(A^{22} + G_1), \\ 3\partial_2 a^{11} &= -3G_2 a^{11} + (a^{11} - a^{22})(-2A^{12} + G_2) + a^{12}(-2A^{22} + G_1), \end{aligned} \quad (17)$$

where $\lambda = \exp G$. This system closes, so the space of solutions is exactly 3 dimensional. Note that if $\mathcal{L}_1 = \sum_{k,j=1}^2 \ell^{kj}(x,y)p_k p_j + W_{(1)}(x,y)$, $\mathcal{L}_2 =$

$\sum_{k,j=1}^2 b^{kj}(x,y)p_k p_j + W_{(2)}(x,y)$, $\mathcal{L}_3 = \mathcal{H}$, is a basis for the symmetries then

$$A^{12} = -G_2 + \frac{D_{(2)}}{D}, \quad A^{22} = 2G_1 + \frac{D_{(3)}}{D}, \quad B^{12} = -G_1 - \frac{D_{(0)}}{D}, \quad B^{22} = -2G_2 - \frac{D_{(1)}}{D}, \quad (18)$$

$$D = \det \begin{pmatrix} \ell^{11} - \ell^{22} & \ell^{12} \\ b^{11} - b^{22} & b^{12} \end{pmatrix}, \quad D_{(0)} = \det \begin{pmatrix} 3\ell_2^{12} & -\ell^{12} \\ 3b_2^{12} & -b^{12} \end{pmatrix},$$

$$D_{(1)} = \det \begin{pmatrix} 3\ell_2^{12} & \ell^{11} - \ell^{22} \\ 3b_2^{12} & b^{11} - b^{22} \end{pmatrix}, \quad D_{(2)} = \det \begin{pmatrix} 3\ell_1^{12} & \ell^{12} \\ 3b_1^{12} & b^{12} \end{pmatrix}, \quad D_{(3)} = \det \begin{pmatrix} 3\ell_1^{12} & \ell^{11} - \ell^{22} \\ 3b_1^{12} & b^{11} - b^{22} \end{pmatrix}.$$

The functions $A^{22}, B^{22}, A^{12}, B^{12}$ are defined independent of the choice of basis for the 2nd order symmetries. To determine the integrability conditions for system (17) we define the vector-valued function $\mathbf{h}^{\text{tr}}(x,y,z) = (a^{11}, a^{12}, a^{22})$ and directly compute the 3×3 matrix functions $\mathcal{A}^{(j)}$ to get the first-order system $\partial_{x_j} \mathbf{h} = \mathcal{A}^{(j)} \mathbf{h}$, $j = 1, 2$, the integrability conditions for which are

$$\mathcal{A}_1^{(2)} - \mathcal{A}_2^{(1)} = \mathcal{A}^{(1)} \mathcal{A}^{(2)} - \mathcal{A}^{(2)} \mathcal{A}^{(1)} \equiv [\mathcal{A}^{(1)}, \mathcal{A}^{(2)}], \quad (19)$$

satisfied identically for a nondegenerate superintegrable system.

There is a similar analysis for a ‘‘free’’ 2nd order superintegrable system obtained by setting the parameter in a *degenerate* potential equal to zero, [4]: The free system retains all of the information needed to reconstruct the potential. All such degenerate superintegrable systems with potential are restrictions of nondegenerate systems obtained by restricting the parameters so that one 2nd order symmetry becomes a perfect square, e.g. $\mathcal{L}_1 = \mathcal{X}^2$. Then \mathcal{X} is a 1st order constant, necessarily of the form $\mathcal{X} = \xi_1 p_1 + \xi_2 p_2$, without a function term. Since the degenerate systems are obtained by restriction, the potential function must satisfy the equations (2) inherited from the nondegenerate system, with the same functions A^{ij}, B^{ij} . In addition the relation $\{\mathcal{X}, \mathcal{H}\} = 0$ imposes the condition $\xi_1 V_1 + \xi_2 V_2 = 0$. By relabeling the coordinates, we can always assume $\xi_2 \neq 0$ and write the system of equations for the potential in the form $V_2 = C^2 V_1$, $V_{22} = V_{11} + C^{22} V_1$, $V_{12} = C^{12} V_1$, where

$$C^2(x_1, x_2) = -\frac{\xi_1}{\xi_2}, \quad C^{22}(x_1, x_2) = A^{22} - \frac{\xi_1}{\xi_2} B^{22}, \quad C^{12}(x_1, x_2) = A^{12} - \frac{\xi_1}{\xi_2} B^{12}.$$

To find integrability conditions for these equations we introduce matrices

$$\mathbf{v} = \begin{pmatrix} V \\ V_1 \end{pmatrix}, \quad \mathbf{B}^{(1)} = \begin{pmatrix} 0 & 1 \\ 0 & \partial_2 C^2 + C^2 C^{12} - C^{22} \end{pmatrix}, \quad \mathbf{B}^{(2)} = \begin{pmatrix} 0 & C^2 \\ 0 & C^{12} \end{pmatrix}. \quad (20)$$

Then integrability conditions for system $\partial_{x_j} \mathbf{v} = \mathbf{B}^{(j)} \mathbf{v}$, $j = 1, 2$, must hold:

$$B_1^{(2)} - B_2^{(1)} = B^{(1)} B^{(2)} - B^{(2)} B^{(1)} \equiv [B^{(1)}, B^{(2)}]. \quad (21)$$

If and only (21) holds, the system has a 2Dl space of solutions V . Since $V = \text{constant}$ is always a solution, (21) is necessary and sufficient for the existence of a nonzero 1-parameter potential system. In this case we can prescribe the values V, V_2 at any regular point \mathbf{x}_0 ; there will exist a unique $V(\mathbf{x})$ taking these values.

3.1 Free triplets

A *2nd order classical free triplet* is a 2D system without potential, $\mathcal{H}_0 = \frac{p_1^2 + p_2^2}{\lambda(x,y)}$ and with a basis of 3 functionally independent second-order constants of the motion $\mathcal{L}_{(s)} = \sum_{i,j=1}^2 a_{(s)}^{ij} p_i p_j$, $a_{(s)}^{ij} = a_{(s)}^{ji}$, $s = 1, 2, 3$, $\mathcal{L}_{(3)} = \mathcal{H}_0$. Since the duals of these constants of the motion are 2nd order Killing tensors, the spaces associated with free triplets can be characterized as 2D manifolds that admit 3 functionally independent 2nd order Killing tensors. All such manifolds were classified by Koenigs [5, 15] who showed that the possibilities were constant curvature spaces [each admitting 3 linearly independent 1st order Killing vectors], 4 Darboux spaces, [each admitting a single Killing vector] and 11 Koenigs spaces [each admitting no Killing vectors]. Since the vectors $\{\mathbf{h}_{(s)}\}$, $\mathbf{h}_{(s)}^{\text{tr}}(x, y, z) = (a_{(s)}^{11}, a_{(s)}^{12}, a_{(s)}^{22})$ form a linearly independent set, there exist unique 3×3 matrices $\mathcal{C}^{(j)}$ such that $\partial_{x_j} \mathbf{h}_{(s)} = \mathcal{C}^{(j)} \mathbf{h}_{(s)}$, $j, s = 1, 2$. By linearity, any element $\mathcal{L} = \sum_{i,j=1}^2 a^{ij} p_i p_j$ of the space of 2nd order symmetries spanned by the basis triplet is characterized by matrix equations

$$\partial_{x_j} \mathbf{h} = \mathcal{C}^{(j)} \mathbf{h} \quad j = 1, 2, \quad \mathbf{h}^{\text{tr}}(x, y, z) = (a^{11}, a^{12}, a^{22}). \quad (22)$$

In particular, at any regular point \mathbf{x}_0 we can arbitrarily choose the value of the 3-vector \mathbf{h}_0 and solve (22) to find the unique symmetry \mathcal{L} of \mathcal{H}_0 such that $\mathbf{h}(\mathbf{x}_0) = \mathbf{h}_0$. A normalization condition for the $\mathcal{C}^{(j)}$: (22) is valid for $a^{11} = a^{22} = 1/\lambda, a^{12} = 0$, i.e., for \mathcal{H}_0 . Note that since the \mathcal{L} are Killing tensors, equations (19) must be compatible with the Killing equations (14). Also, integrability conditions hold:

$$\mathcal{C}_1^{(2)} - \mathcal{C}_2^{(1)} = \mathcal{C}^{(1)} \mathcal{C}^{(2)} - \mathcal{C}^{(2)} \mathcal{C}^{(1)} \equiv [\mathcal{C}^{(1)}, \mathcal{C}^{(2)}]. \quad (23)$$

It is clear from equations (17) that the restriction of a 2D 2nd order nondegenerate superintegrable system with all parameters equal to 0 is a free triplet. However the converse doesn't hold. We determine necessary and sufficient conditions that a free system extends to a system with nondegenerate potential.

A first step is a more detailed characterization of the matrices $\mathcal{C}^{(i)}$ for a free system. From the Killing equations (14) we obtain the conditions

$$\begin{aligned} \mathcal{C}_{11}^{(1)} &= -G_1, \quad \mathcal{C}_{12}^{(1)} = -G_2, \quad \mathcal{C}_{13}^{(1)} = 0, \quad \mathcal{C}_{31}^{(2)} = 0, \quad \mathcal{C}_{32}^{(2)} = -G_1, \quad \mathcal{C}_{33}^{(2)} = -G_2, \\ 2\mathcal{C}_{21}^{(1)} + \mathcal{C}_{11}^{(2)} &= 0, \quad 2\mathcal{C}_{22}^{(1)} + \mathcal{C}_{12}^{(2)} = -G_1, \quad 2\mathcal{C}_{23}^{(1)} + \mathcal{C}_{13}^{(2)} = -G_2, \\ 2\mathcal{C}_{21}^{(2)} + \mathcal{C}_{31}^{(1)} &= -G_1, \quad 2\mathcal{C}_{22}^{(2)} + \mathcal{C}_{32}^{(1)} = -G_2, \quad 2\mathcal{C}_{23}^{(2)} + \mathcal{C}_{33}^{(1)} = -G_1. \end{aligned}$$

From the requirement that \mathcal{H}_0 satisfies (22) we obtain the conditions

$$\begin{aligned} \mathcal{C}_{11}^{(1)} + \mathcal{C}_{13}^{(1)} &= -G_1, \quad \mathcal{C}_{21}^{(1)} + \mathcal{C}_{23}^{(1)} = 0, \quad \mathcal{C}_{31}^{(1)} + \mathcal{C}_{33}^{(1)} = -G_1, \\ \mathcal{C}_{11}^{(2)} + \mathcal{C}_{13}^{(2)} &= -G_2, \quad \mathcal{C}_{21}^{(2)} + \mathcal{C}_{23}^{(2)} = 0, \quad \mathcal{C}_{31}^{(2)} + \mathcal{C}_{33}^{(2)} = -G_2. \end{aligned}$$

Solving these equations we find

$$\mathcal{C}^{(1)} = \begin{pmatrix} -G_1, & -G_2, & 0 \\ -\frac{1}{2}\mathcal{C}_{11}^{(2)}, & -\frac{1}{2}G_1 - \frac{1}{2}\mathcal{C}_{12}^{(2)}, & \frac{1}{2}\mathcal{C}_{11}^{(2)} \\ -G_1 - 2\mathcal{C}_{21}^{(2)}, & -G_2 - 2\mathcal{C}_{22}^{(2)}, & 2\mathcal{C}_{21}^{(2)} \end{pmatrix}, \quad \mathcal{C}^{(2)} = \begin{pmatrix} \mathcal{C}_{11}^{(2)}, & \mathcal{C}_{12}^{(2)}, & -G_2 - \mathcal{C}_{11}^{(2)} \\ \mathcal{C}_{21}^{(2)}, & \mathcal{C}_{22}^{(2)}, & -\mathcal{C}_{21}^{(2)} \\ 0, & -G_1, & -G_2 \end{pmatrix},$$

with the 4 functions $C_{11}^{(2)}, C_{12}^{(2)}, C_{21}^{(2)}, C_{22}^{(2)}$ free. If we define the functions $A^{12}, B^{12}, A^{22}, B^{22}$ by the requirement

$$C_{11}^{(2)} = -\frac{2}{3}G_2 - \frac{2}{3}A^{12}, C_{12}^{(2)} = \frac{1}{3}G_1 - \frac{2}{3}A^{22}, C_{21}^{(2)} = -\frac{1}{3}G_1 - \frac{1}{3}B^{12}, C_{22}^{(2)} = -\frac{2}{3}G_2 - \frac{1}{3}B^{22},$$

then equations (22) agree with (17). Thus, for a free system there always exist unique functions A^{ij}, B^{ij} such that equations (17) hold. Then necessary and sufficient conditions for extension to a system with nondegenerate potential V satisfying equations (2) are that conditions (16) hold identically.

This analysis also extends, via restriction, to superintegrable systems with degenerate potential. A free triplet that corresponds to a degenerate superintegrable system is one that corresponds to a nondegenerate system but such that one of the free generators can be chosen as a perfect square. For these systems conditions (21) for the potential are satisfied identically.

Similarly, we define a *2nd order quantum free triplet* as a 2D quantum system without potential, $H_0 = \frac{1}{\lambda(\mathbf{x})}(\partial_{11} + \partial_{22})$, and with a basis of 3 algebraically independent second-order symmetry operators

$$L_k = \frac{1}{\lambda} \sum_{i,j=1}^2 \partial_i (\lambda a_{(k)}^{ij} \partial_j)(\mathbf{x}), \quad k = 1, 2, 3, \quad a_{(k)}^{ij} = a_{(k)}^{ji}, \quad L_3 = H_0$$

There is a 1-1 relationship between classical and quantum free triplets.

4 Superintegrable systems and enveloping algebras

Every 2D nondegenerate or degenerate superintegrable system is Stäckel equivalent to a superintegrable system on a constant curvature space [8]. Thus we study free triplets on flat space and the complex sphere, taking advantage of the fact that the symmetries can be identified with 2nd order elements in the enveloping algebras of $e(2, \mathbb{C})$ or $so(3, \mathbb{C})$. Then, conditions (23) are satisfied.

If we have a degenerate superintegrable system and turn off the potential then we have a free degenerate superintegrable system in the sense that the Poisson brackets of the free generators determine a degenerate quadratic algebra (without parameters). We will show, conversely, that every free triplet that forms degenerate quadratic algebra is the restriction of a superintegrable system with degenerate potential. We classify free triplet systems that are 2nd order in the enveloping algebras of $e(2, \mathbb{C})$ and $o(3, \mathbb{C})$ and which determine a degenerate quadratic algebra. In the classification we identify systems that are equivalent under the adjoint action of the corresponding Lie group. We will also identify each system with the superintegrable system with potential whose potential-free terms agree with it. For this we use the classification of constant curvature systems in [6] with $E_{3'}$ added in [16]. We start with flat space and consider free triplets in the $e(2, \mathbb{C})$ enveloping algebra.

4.1 Degenerate superintegrable systems from $e(2, \mathbb{C})$ (8 systems)

We use the classical realization for $e(2, \mathbb{C})$ with basis $p_1, p_2, \mathcal{J} = xp_2 - yp_1$, and Hamiltonian $\mathcal{H} = p_1^2 + p_2^2$. We classify all possible free degenerate superintegrable

systems in the enveloping algebra of $e(2, \mathbb{C})$, up to conjugacy, modulo \mathcal{H} . It turns out that each such system is the restriction of a degenerate flat space superintegrable system with potential; the relationship is 1-1. We write \tilde{E}_n as the free system that is the restriction of superintegrable system E_n . Up to conjugacy under the action of $e(2, \mathbb{C})$, the possible choices for the 1st order generator \mathcal{X} are: $\mathcal{X} = \mathcal{J}$, p_1 , $p_1 + ip_2$. We give some details for the first case and then just list the results.

We first choose $\mathcal{X} = \mathcal{J}$. We need to find 2nd order elements \mathcal{L}_1 , \mathcal{L}_2 of the enveloping algebra such that $\{\mathcal{X}^2, \mathcal{L}_1, \mathcal{L}_2, \mathcal{H}\}$ is linearly independent and such that $\{\mathcal{X}, \mathcal{L}_1, \mathcal{L}_2, \mathcal{H}\}$ define a degenerate quadratic algebra. The most general choice for \mathcal{L}_1 is $\mathcal{L}_1 = a_1 \mathcal{J} p_1 + a_2 \mathcal{J} p_2 + a_3 p_1^2 + a_4 p_1 p_2$. Case 1: suppose $a_1 \neq 0$ so we can take $a_1 = 1$. By a rotation, leaving \mathcal{J} fixed, we can assume that either $a_2 = 0$ or $a_2 = i$. We first consider: $a_2 = 0$. We can translate in x to achieve $a_4 = 0$ and in y to achieve $a_3 = 0$. Then for \mathcal{L}_2 we can take $\mathcal{L}_2 = b_1 \mathcal{J} p_2 + b_2 p_1^2 + b_3 p_1 p_2$. In order for these choices to generate a superintegrable system we must have

$$\{\mathcal{X}, \mathcal{L}_1\} = C_1 \mathcal{L}_1 + C_2 \mathcal{L}_2 + C_3 \mathcal{H} + C_4 \mathcal{X}^2 + C_5, \quad (24)$$

$$\{\mathcal{X}, \mathcal{L}_2\} = D_1 \mathcal{L}_1 + D_2 \mathcal{L}_2 + D_3 \mathcal{H} + D_4 \mathcal{X}^2 + D_5, \quad (25)$$

$$\{\mathcal{L}_1, \mathcal{L}_2\} = E_1 \mathcal{L}_1 \mathcal{X} + E_2 \mathcal{L}_2 \mathcal{X} + E_3 \mathcal{H} \mathcal{X} + E_4 \mathcal{X}^3 + E_5 \mathcal{X} \quad (26)$$

$$G = c_1 \mathcal{L}_1^2 + c_2 \mathcal{L}_2^2 + c_3 \mathcal{H}^2 + c_4 \mathcal{L}_1 \mathcal{L}_2 + c_5 \mathcal{H} \mathcal{L}_1 + c_6 \mathcal{H} \mathcal{L}_2 + c_7 \mathcal{X}^4 + c_8 \mathcal{X}^2 \mathcal{L}_1 + c_9 \mathcal{X}^2 \mathcal{L}_2 + c_{10} \mathcal{H} \mathcal{X}^2 + c_{11} \mathcal{L}_1 + c_{12} \mathcal{L}_2 + c_{13} \mathcal{H} + c_{14} \mathcal{X}^2 + c_{15} \equiv 0, \quad (27)$$

for some constants A_j, C_j, E_j, c_j where the c_j are not all 0. In \mathcal{L}_2 we assume first that $b_1 \neq 0$ and normalize to $B_1 = 1$. Then substituting into equation (24) and equating coefficients of powers of p_j , x and y on both sides of the identity. We get easily that $C_1 = C_3 = C_4 = C_5 = 0$, $C_2 = -1$, $b_2 = b_3 = 0$, so there is no solution unless $\mathcal{L}_1 = \mathcal{X} p_1$, $\mathcal{L}_2 = \mathcal{X} p_2$. All remaining conditions are satisfied. Now consider the case $b_1 = 0$ and assume $b_2 = 1$. This time equation (24) cannot be solved, so this case is impossible. Next we assume $b_1 = b_2 = 0$, $B_3 = 1$. Again, equation (24) cannot be solved, so this case is also impossible. Now we consider the possibility $a_1 = 1, a_2 = i$. By translating in y we can achieve $a_3 = 0$. Going step-by-step, we take $b_1 = 1$. Then we can satisfy (24) only if $a_4 = 0$, in which case we have $\{\mathcal{X}, \mathcal{L}_1\} = \mathcal{L}_1$. Going further we now substitute this result into equation (25) and equate coefficients. We find a solution only if $b_2 = b_3 = 0$, but now the space spanned by $\mathcal{L}_1, \mathcal{L}_2$ is the same as that spanned by $\mathcal{J} p_1, \mathcal{J} p_2$, already listed. This finishes Case 1. For Case 2 we can take $a_1 = 0, a_2 = 1$, and find no solutions. This finishes Case 2. For case 3 we take $a_1 = a_2 = 0, a_3 = 1$. Here there is a solution. Having demonstrated the step-by-step approach, we now merely list the results.

1. \tilde{E}_{18} : $\mathcal{H} = p_1^2 + p_2^2$, $\mathcal{X} = \mathcal{J}$, $\mathcal{L}_1 = \mathcal{J} p_1$, $\mathcal{L}_2 = \mathcal{J} p_2$,
Casimir : $-\frac{1}{2}(\mathcal{L}_1^2 + \mathcal{L}_2^2 - \mathcal{H} \mathcal{X}^2) = 0$, potential : $V = \frac{\alpha}{\sqrt{x^2 + y^2}}$,
2. \tilde{E}_3 : $\mathcal{H} = p_1^2 + p_2^2$, $\mathcal{X} = \mathcal{J}$, $\mathcal{L}_1 = p_1^2$, $\mathcal{L}_2 = p_1 p_2$,
Casimir : $-\mathcal{L}_2^2 - \mathcal{L}_1(\mathcal{L}_1 - \mathcal{H}) = 0$, potential : $V = \alpha(x^2 + y^2)$.
3. \tilde{E}_6 : $\mathcal{H} = p_1^2 + p_2^2$, $\mathcal{X} = p_1$, $\mathcal{L}_1 = \mathcal{J}^2$, $\mathcal{L}_2 = \mathcal{J} p_2$,
Casimir : $\mathcal{L}_1 \mathcal{X}^2 + \mathcal{L}_2^2 - \mathcal{H} \mathcal{L}_1 = 0$, potential : $V = \frac{\alpha}{x^2}$,

4. \tilde{E}_5 : $\mathcal{H} = p_1^2 + p_2^2$, $\mathcal{X} = p_1$, $\mathcal{L}_1 = \mathcal{J}p_1$, $\mathcal{L}_2 = p_2p_1$,
Casimir : $\frac{1}{2}(\mathcal{L}_2^2 + \mathcal{X}^4 - \mathcal{H}\mathcal{X}^2) = 0$, potential : $V = \alpha x$.
5. \tilde{E}_{12} : $\mathcal{H} = p_1^2 + p_2^2$, $\mathcal{X} = p_1 + ip_2$, $\mathcal{L}_1 = \mathcal{J}^2 + (p_1 - ip_2)^2$, $\mathcal{L}_2 = \mathcal{J}(p_1 + ip_2)$,
Casimir : $i(\mathcal{L}_1\mathcal{X}^2 - \mathcal{L}_2^2 - \mathcal{H}^2) = 0$, potential : $V = \frac{\alpha(x+iy)}{\sqrt{(x+iy)^2+c^2}}$,
6. \tilde{E}_{14} : $\mathcal{H} = p_1^2 + p_2^2$, $\mathcal{X} = p_1 + ip_2$, $\mathcal{L}_1 = \mathcal{J}^2$, $\mathcal{L}_2 = \mathcal{J}(p_1 + ip_2)$,
Casimir : $i(\mathcal{L}_1\mathcal{X}^2 - \mathcal{L}_2^2) = 0$, potential : $V = \frac{\alpha}{(x+iy)^2}$,
7. \tilde{E}_4 : $\mathcal{H} = p_1^2 + p_2^2$, $\mathcal{X} = p_1 + ip_2$, $\mathcal{L}_1 = \mathcal{J}(p_1 + ip_2)$, $\mathcal{L}_2 = (p_1 - ip_2)^2$,
Casimir : $-i(\mathcal{L}_2\mathcal{X}^2 - \mathcal{H}^2) = 0$, potential : $V = \alpha(x + iy)$,
8. \tilde{E}_{13} : $\mathcal{H} = p_1^2 + p_2^2$, $\mathcal{X} = p_1 + ip_2$, $\mathcal{L}_1 = \mathcal{J}(p_1 + ip_2)$, $\mathcal{L}_2 = (p_1 - ip_2)\mathcal{J}$,
Casimir : $i(\mathcal{L}_1\mathcal{H} - \mathcal{L}_2\mathcal{X}^2) = 0$, potential : $V = \frac{\alpha}{\sqrt{x+iy}}$.

4.2 Degenerate quadratic algebras from $o(3, \mathbb{C})$ (3 systems)

We use the classical realization for $o(3, \mathbb{C})$ with basis $\mathcal{J}_1 = yp_3 - zp_2$, $\mathcal{J}_2 = zp_1 - xp_3$, $\mathcal{J}_3 = xp_2 - yp_1$, and Hamiltonian $\mathcal{H} = \mathcal{J}_1^2 + \mathcal{J}_2^2 + \mathcal{J}_3^2$. We classify the possible systems up to conjugacy with respect to $O(3, \mathbb{C})$ group actions and modulo \mathcal{H} using the same step-by-step procedure as in Section 4.1, and merely list the results. Up to conjugacy, the choices for \mathcal{X} are \mathcal{J}_3 , $\mathcal{J}_1 + i\mathcal{J}_2$.

1. \tilde{S}_6 : $\mathcal{H} = \mathcal{J}_1^2 + \mathcal{J}_2^2 + \mathcal{J}_3^2$, $\mathcal{X} = \mathcal{J}_3$, $\mathcal{L}_1 = \mathcal{J}_3\mathcal{J}_1$, $\mathcal{L}_2 = \mathcal{J}_3\mathcal{J}_2$,
Casimir : $-\frac{1}{2}(\mathcal{L}_1^2 + \mathcal{L}_2^2 + \mathcal{X}^2(\mathcal{X}^2 - \mathcal{H})) = 0$, potential : $V = \frac{\alpha z}{\sqrt{x^2+y^2}}$,
2. \tilde{S}_3 : $\mathcal{H} = \mathcal{J}_1^2 + \mathcal{J}_2^2 + \mathcal{J}_3^2$, $\mathcal{X} = \mathcal{J}_3$, $\mathcal{L}_1 = (\mathcal{J}_1 + i\mathcal{J}_2)^2$, $\mathcal{L}_2 = (\mathcal{J}_1 - i\mathcal{J}_2)^2$,
Casimir : $-2i((\mathcal{H} - \mathcal{X}^2)^2 - \mathcal{L}_1\mathcal{L}_2) = 0$, potential : $V = \frac{\alpha}{z^2}$,
3. \tilde{S}_5 : $\mathcal{H} = \mathcal{J}_1^2 + \mathcal{J}_2^2 + \mathcal{J}_3^2$, $\mathcal{X} = \mathcal{J}_1 + i\mathcal{J}_2$, $\mathcal{L}_1 = \mathcal{J}_3^2$, $\mathcal{L}_2 = (\mathcal{J}_1 + i\mathcal{J}_2)\mathcal{J}_3$,
Casimir : $-i(\mathcal{L}_2^2 - \mathcal{X}^2\mathcal{L}_1) = 0$, potential : $V = \frac{\alpha}{(x+iy)^2}$.

4.3 Nondegenerate quadratic algebras from $e(2, \mathbb{C})$ (12 plus 1)

We use the realization for $e(2, \mathbb{C})$ with basis listed in Section 4.1. An alternate basis is $\mathcal{J}, p_1 + ip_2, p_1 - ip_2$. We classify systems, mod \mathcal{H} , up to conjugacy with respect to the group $E(2, \mathbb{C})$, including inversions and reflections. There are 8 conjugacy classes of 2nd order elements in the enveloping algebra, mod \mathcal{H} , with representatives

$$\begin{aligned} \mathcal{J}^2, \quad p_1^2, \quad (p_1 + ip_2)^2, \quad p_2\mathcal{J}, \quad (p_1 + ip_2)\mathcal{J}, \quad \mathcal{J}^2 + ap_1^2, \quad a \neq 0, \quad (28) \\ \mathcal{J}^2 + (p_1 + ip_2)^2, \quad 2(p_1 + ip_2)\mathcal{J} + (p_1 - ip_2)^2. \end{aligned}$$

A general 2nd order element in the enveloping algebra, mod \mathcal{H} , can be written as $a_1\mathcal{J}^2 + a_2p_1\mathcal{J} + a_3p_2\mathcal{J} + a_4p_1^2 + a_5p_1p_2$.

1st case: We choose $\mathcal{L}_1 = \mathcal{J}^2$ and try to determine the possibilities for \mathcal{L}_2 , up to conjugacy under $E(2, \mathbb{C})$, such that $\mathcal{L}_1, \mathcal{L}_2, \mathcal{H}$ generate a quadratic algebra. (As we go through the cases step-by-step, we ignore systems that have already been exhibited in earlier steps.) In general $\mathcal{L}_2 = a_2p_1\mathcal{J} + a_3p_2\mathcal{J} + a_4p_1^2 + a_5p_1p_2$

and a_2, a_3, a_4, a_5 are to be determined. Here $\mathcal{R} = \{\mathcal{L}_1, \mathcal{L}_2\} = -2a_2p_2\mathcal{J} + 2a_3p_1\mathcal{J} - 4a_4p_1p_2\mathcal{J} + 2a_5(2p_1^2 - \mathcal{H})\mathcal{J}$. We must require that $\mathcal{R}^2 = b_1\mathcal{L}_1^3 + b_2\mathcal{L}_2^3 + b_3\mathcal{H}^3 + b_4\mathcal{L}_1^2\mathcal{L}_2 + b_5\mathcal{L}_1\mathcal{L}_2^2 + b_6\mathcal{H}\mathcal{L}_1\mathcal{L}_2 + b_7\mathcal{H}\mathcal{L}_1^2 + b_8\mathcal{H}\mathcal{L}_2^2 + b_9\mathcal{H}^2\mathcal{L}_1 + b_{10}\mathcal{H}^2\mathcal{L}_2$, for some constants b_1, \dots, b_{10} . We substitute our expressions for \mathcal{L}_1 and \mathcal{L}_2 into \mathcal{R}^2 and equate coefficients of powers of p_1, p_2, x, y on both sides of the resulting equation. These yields a system of equations for the parameters a_j, b_k , polynomial in the a_j and linear in the b_k . The step-by-step procedure to solve for the parameters is similar to that demonstrated earlier for degenerate systems. Once a solution is obtained we check that it extends to a superintegrable system with potential by using the generators to compute the functions A^{ij}, B^{ij} and then verifying directly that these functions satisfy the integrability conditions (16). Then we identify the associated nondegenerate superintegrable system from the classification in [6]. We list the results, eliminating duplicates and exhibiting the 3-parameter potentials of the associated nonfree superintegrable systems.

1. \tilde{E}_{16} : $\mathcal{L}_1 = \mathcal{J}^2, \mathcal{L}_2 = p_1\mathcal{J}, \mathcal{R}^2 = 4\mathcal{L}_1(\mathcal{L}_1\mathcal{H} - \mathcal{L}_2^2)$.
 $V = \frac{1}{\sqrt{x^2+y^2}}\left(\alpha + \frac{\beta}{y+\sqrt{x^2+y^2}} + \frac{\gamma}{y-\sqrt{x^2+y^2}}\right)$,
2. \tilde{E}_{17} : $\mathcal{L}_1 = \mathcal{J}^2, \mathcal{L}_2 = (p_1 + ip_2)\mathcal{J}, \mathcal{R}^2 = -4\mathcal{L}_1\mathcal{L}_2^2$,
 $V = \frac{\alpha}{\sqrt{x^2+y^2}} + \frac{\beta}{(x+iy)^2} + \frac{\gamma}{(x+iy)\sqrt{x^2+y^2}}$,
3. \tilde{E}_1 : $\mathcal{L}_1 = \mathcal{J}^2, \mathcal{L}_2 = p_1^2, \mathcal{R}^2 = 16\mathcal{L}_1\mathcal{L}_2(\mathcal{H} - \mathcal{L}_2)$,
 $V = \alpha(x^2 + y^2) + \frac{\beta}{x^2} + \frac{\gamma}{y^2}$,
4. \tilde{E}_8 : $\mathcal{L}_1 = \mathcal{J}^2, \mathcal{L}_2 = (p_1 + ip_2)^2, \mathcal{R}^2 = -16\mathcal{L}_1\mathcal{L}_2^2$,
 $V = \frac{\alpha(x-iy)}{(x+iy)^3} + \frac{\beta}{(x+iy)^2} + \gamma(x^2 + y^2)$,
5. $\tilde{E}_{3'}$: $\mathcal{L}_1 = p_1^2, \mathcal{L}_2 = p_1p_2, \mathcal{R}^2 = 0$,
 $V = \alpha(x^2 + y^2) + \beta x + \gamma y$,
6. \tilde{E}_2 : $\mathcal{L}_1 = p_2^2, \mathcal{L}_2 = p_2\mathcal{J}, \mathcal{R}^2 = 4\mathcal{L}_1^2(\mathcal{H} - \mathcal{L}_1)$.
 $V = \alpha(4x^2 + y^2) + \beta x + \frac{\gamma}{y^2}$,
7. \tilde{E}_7 : $\mathcal{L}_1 = (p_1 + ip_2)^2, \mathcal{L}_2 = \mathcal{J}^2 + \frac{b}{2}(p_1 - ip_2)^2, b \neq 0, \mathcal{R}^2 = -16\mathcal{L}_1^2\mathcal{L}_2 + 16a\mathcal{L}_1\mathcal{H}^2$,
 $V = \frac{\alpha(x+iy)}{\sqrt{(x+iy)^2-b}} + \frac{\beta(x-iy)}{\sqrt{(x+iy)^2-b} \left(x+iy+\sqrt{(x+iy)^2-b}\right)^2} + \gamma(x^2 + y^2)$,
8. \tilde{E}_9 : $\mathcal{L}_1 = (p_1 + ip_2)^2, \mathcal{L}_2 = p_1\mathcal{J}, \mathcal{R}^2 = -2\mathcal{L}_1(2\mathcal{L}_1 + \mathcal{H})^2$,
 $V = \frac{\alpha}{\sqrt{x+iy}} + \beta y + \frac{\gamma(x+2iy)}{\sqrt{x+iy}}$,
9. \tilde{E}_{11} : $\mathcal{L}_1 = (p_1 + ip_2)^2, \mathcal{L}_2 = (p_1 - ip_2)\mathcal{J}, \mathcal{R}^2 = -4\mathcal{L}_1\mathcal{H}^2$,
 $V = \alpha(x - iy) + \frac{\beta(x-iy)}{\sqrt{x+iy}} + \frac{\gamma}{\sqrt{x+iy}}$,
10. \tilde{E}_{10} : $\mathcal{L}_1 = (p_1 - ip_2)^2, \mathcal{L}_2 = 4i(p_1 - ip_2)\mathcal{J} + (p_1 + ip_2)^2, \mathcal{R}^2 = 64\mathcal{L}_1^3$,
 $V = \alpha(x - iy) + \beta(x + iy - \frac{3}{2}(x - iy)^2) + \gamma(x^2 + y^2 - \frac{1}{2}(x - iy)^3)$,
11. \tilde{E}_{15} : $\mathcal{L}_1 = (p_1 - ip_2)^2, \mathcal{L}_2 = i(p_1 - ip_2)\mathcal{J}, \mathcal{R}^2 = 4\mathcal{L}_1^3$,
 $V = f(x - iy)$, where f is arbitrary. The exceptional case, characterized by the fact that generators $\mathcal{L}_1, \mathcal{L}_2, \mathcal{H}$ are functionally linearly dependent,

[6, 7]. This quadratic algebra is isomorphic, but not conjugate, to \tilde{E}_{10} and doesn't correspond to a nondegenerate superintegrable system.

12. \tilde{E}_{20} : $\mathcal{L}_1 = p_2\mathcal{J}$, $\mathcal{L}_2 = p_1\mathcal{J}$, $\mathcal{R}^2 = \mathcal{H}(\mathcal{L}_1^2 + \mathcal{L}_2^2)$,

$$V = \frac{1}{\sqrt{x^2+y^2}} \left(\alpha + \beta\sqrt{x + \sqrt{x^2+y^2}} + \gamma\sqrt{x - \sqrt{x^2+y^2}} \right),$$
13. \tilde{E}_{19} : $\mathcal{L}_1 = (p_1 + ip_2)\mathcal{J}$, $\mathcal{L}_2 = \mathcal{J}^2 + (p_1 - ip_2)^2$, $\mathcal{R}^2 = -4\mathcal{L}_2(\mathcal{L}_1^2 + \mathcal{H}^2)$,

$$V = \frac{\alpha(x+iy)}{\sqrt{(x+iy)^2-4}} + \frac{\beta}{\sqrt{(x-iy)(x+iy+2)}} + \frac{\gamma}{\sqrt{(x-iy)(x+iy-2)}}.$$

4.4 Nondegenerate quadratic algebras from $o(3, \mathbb{C})$ enveloping algebra (6 systems)

We make use of the classical realization for $o(3, \mathbb{C})$ given in Section 4.2. We classify the possible systems up to conjugacy with respect to $O(3, \mathbb{C})$ group actions and modulo \mathcal{H} . There are 5 conjugacy classes of 2nd order elements in the enveloping algebra, mod \mathcal{H} , with representatives

$$\begin{aligned} \mathcal{J}_3^2, \quad \mathcal{J}_1^2 + a\mathcal{J}_2^2, \quad (a \neq 0, \pm 1, |a| \leq 1), \quad (\mathcal{J}_1 + i\mathcal{J}_2)^2, \quad (29) \\ (\mathcal{J}_1 + i\mathcal{J}_2)^2 + \mathcal{J}_3^2, \quad \mathcal{J}_3(\mathcal{J}_1 + i\mathcal{J}_2). \end{aligned}$$

A general 2nd order element in the enveloping algebra, mod \mathcal{H} can be written as $a_1\mathcal{J}_1^2 + a_2\mathcal{J}_2^2 + a_3\mathcal{J}_1\mathcal{J}_2 + a_4\mathcal{J}_1\mathcal{J}_3 + a_5\mathcal{J}_2\mathcal{J}_3$. An alternate expression is $A_1(\mathcal{J}_1 + i\mathcal{J}_2)^2 + A_2(\mathcal{J}_1 - i\mathcal{J}_2)^2 + A_3\mathcal{J}_3^2 + A_4(\mathcal{J}_1 + i\mathcal{J}_2)\mathcal{J}_3 + A_5(\mathcal{J}_1 - i\mathcal{J}_2)\mathcal{J}_3$.

We list the results, eliminating duplicates and exhibiting the 3-parameter potentials of the associated nonfree superintegrable systems.

1. \tilde{S}_9 : $\mathcal{L}_1 = \mathcal{J}_3^2$, $\mathcal{L}_2 = \mathcal{J}_1^2$, $\mathcal{R}^2 = -16\mathcal{L}_1^2\mathcal{L}_2 - 16\mathcal{L}_1\mathcal{L}_2^2 + 16\mathcal{L}_1\mathcal{L}_2\mathcal{H}$,

$$V = \frac{\alpha}{x^2} + \frac{\beta}{y^2} + \frac{\gamma}{z^2},$$
2. \tilde{S}_4 : $\mathcal{L}_1 = \mathcal{J}_3^2$, $\mathcal{L}_2 = (\mathcal{J}_1 + i\mathcal{J}_2)\mathcal{J}_3$, $\mathcal{R}^2 = -4\mathcal{L}_1\mathcal{L}_2^2$,

$$V = \frac{\alpha}{(x+iy)^2} + \frac{\beta z}{\sqrt{x^2+y^2}} + \frac{\gamma}{(x+iy)\sqrt{x^2+y^2}},$$
3. \tilde{S}_7 : $\mathcal{L}_1 = \mathcal{J}_3^2$, $\mathcal{L}_2 = \mathcal{J}_1\mathcal{J}_3$, $\mathcal{R}^2 = -4\mathcal{L}_1^3 - 4\mathcal{L}_2^2\mathcal{L}_1 + 4\mathcal{L}_1^2\mathcal{H}$,

$$V = \frac{\alpha z}{\sqrt{x^2+y^2}} + \frac{\beta x}{y^2\sqrt{x^2+y^2}} + \frac{\gamma}{y^2},$$
4. \tilde{S}_8 : $\mathcal{L}_1 = \mathcal{J}_2(\mathcal{J}_2 + i\mathcal{J}_1)$, $\mathcal{L}_2 = \mathcal{J}_2\mathcal{J}_3$, $\mathcal{R}^2 = -2\mathcal{L}_1^3 + 2\mathcal{L}_1\mathcal{L}_2^2 + \mathcal{L}_1^2\mathcal{H} - \mathcal{L}_2^2\mathcal{H}$,

$$V = \frac{\alpha y}{\sqrt{x^2+z^2}} + \frac{\beta(y+ix+z)}{\sqrt{(y+ix)(z+ix)}} + \frac{\gamma(y+ix-z)}{\sqrt{(y+ix)(z-ix)}},$$
5. \tilde{S}_2 : $\mathcal{L}_1 = (\mathcal{J}_1 + i\mathcal{J}_2)^2$, $\mathcal{L}_2 = \mathcal{J}_3^2$, $\mathcal{R}^2 = -16\mathcal{L}_1^2\mathcal{L}_2$,

$$V = \frac{\alpha}{z^2} + \frac{\beta}{(x+iy)^2} + \frac{\gamma(x-iy)}{(x+iy)^3},$$
6. \tilde{S}_1 : $\mathcal{L}_1 = (\mathcal{J}_1 + i\mathcal{J}_2)\mathcal{J}_3$, $\mathcal{L}_2 = (\mathcal{J}_1 + i\mathcal{J}_2)^2$, $\mathcal{R}^2 = -4\mathcal{L}_2^3$,

$$V = \frac{\alpha}{(x+iy)^2} + \frac{\beta z}{(x+iy)^2} + \frac{\gamma(1-4z^2)}{(x+iy)^4},$$

4.5 The closure theorems

There are, up to conjugacy, 8 degenerate and 13 nondegenerate quadratic algebras in the enveloping algebra of $e(2, \mathbb{C})$, and these match 1-1 with the restrictions of the 8 degenerate, 12 nondegenerate and 1 exceptional superintegrable systems on complex flat space, also classified up to conjugacy. There are, up to conjugacy, 3 degenerate and 6 nondegenerate quadratic algebras in the enveloping algebra of $o(3, \mathbb{C})$, and these match 1-1 with the restrictions of the 3 degenerate and 6 nondegenerate superintegrable systems on the complex 2-sphere. Thus:

Theorem 1 *A classical free triplet on a constant curvature space extends to a superintegrable system if and only if it forms a free quadratic algebra, degenerate or nondegenerate.*

The main message that follows from this result is that we have found purely algebraic conditions on constant curvature spaces that replace the complicated analytic integrability conditions (16) or (23) for extension to a superintegrable system.

There is an analogous result for quantum free systems and quantum superintegrable systems. Indeed, If we have a nondegenerate quantum superintegrable system and turn off the potential then we will have a free nondegenerate superintegrable system in the sense that the commutators of the free generators will determine a nondegenerate quadratic algebra. Conversely, every quantum free triplet system for which the algebra formed from the generators closes to a nondegenerate quadratic algebra is the restriction of a superintegrable system with nondegenerate potential (or the exceptional case E_{15}). Indeed, since the highest order derivative terms in the commutator agree with the highest order polynomial terms in the Poisson bracket, every free quantum nondegenerate quadratic algebra uniquely determines a free classical nondegenerate quadratic algebra. The classical quadratic algebras correspond 1-1 with classical superintegrable systems and these in turn correspond 1-1 with quantum superintegrable systems. There is a similar correspondence for degenerate quadratic algebras. Thus we have

Theorem 2 *A quantum free triplet on a constant curvature space extends to a superintegrable system if and only if it forms a free quantum quadratic algebra.*

In a forthcoming paper we will show that these theorems extend to all 2D superintegrable systems, including those on Darboux and Koenig spaces.

4.6 Construction of superintegrable systems from free triplets

Suppose we have a classical free triplet with basis

$$\mathcal{L}_{(s)} = \sum_{i,j=1}^2 a_{(s)}^{ij} p_i p_j \quad a_{(s)}^{ij} = a_{(s)}^{ji}, \quad s = 1, 2, 3, \quad \mathcal{L}_{(3)} = \mathcal{H}_0 = \frac{p_1^2 + p_2^2}{\lambda(x, y)},$$

not \tilde{E}_{15} , that determines a free nondegenerate quadratic algebra, hence a free nondegenerate superintegrable system. Then the functions A^{ij}, B^{ij} , (18) expressed in terms of the Cartesian-like coordinates (x, y) , satisfy the integrability conditions (16) for the potential equations (2) and we are guaranteed a

4-dimensional vector space of solutions V . Further, these equations guarantee that the Bertrand-Darboux integrability conditions for equations $W_i^{(s)} = \lambda \sum_{j=1}^2 a_{(s)}^{ij} V_j$ are satisfied and we can compute the solutions $W^{(s)}$, $W^{(3)} = V$, unique up to additive constants, such that the constants of the motion $\mathcal{L}_{(s)} = \sum a_{(s)}^{ij} p_i p_j + W^{(s)}$ define a nondegenerate superintegrable system. This system is guaranteed to satisfy a nondegenerate quadratic algebra with potential whose highest order (potential-free) terms agree with the free quadratic algebra. Note that the functions A^{ij}, B^{ij} are defined independent of the basis chosen for the free triplet, although, of course, they do depend upon the particular coordinates chosen. Similarly, there is an associated quantum free triplet

$$L_s = \frac{1}{\lambda} \sum_{i,j=1}^2 \partial_i (\lambda a_{(s)}^{ij} \partial_j), \quad s = 1, 2, 3, \quad L_3 = H_0 = \frac{1}{\lambda(\mathbf{x})} (\partial_{11} + \partial_{22}),$$

that defines a free nondegenerate quantum quadratic algebra with potential. The functions $W^{(s)}$ are the same as before.

There is an analogous construction of degenerate superintegrable systems with potential from free triplets that generate a free quadratic algebras, but are such that one generator say, $\mathcal{L}_1 = \mathcal{X}^2$ is a perfect square. Then the system with its generator added determines a free degenerate quadratic algebra. The functions A^{ij}, B^{ij} are defined from the free triplet and $\mathcal{X} = \xi_1 p_1 + \xi_2 p_2$. The equations for the potential are

$$V_2 = C^2 V_1, \quad V_{22} = V_{11} + C^{22} V_1, \quad V_{12} = C^{12} V_1, \quad (30)$$

where $C^2(x_1, x_2) = -\frac{\xi_1}{\xi_2}$, $C^{22}(x_1, x_2) = A^{22} - \frac{\xi_1}{\xi_2} B^{22}$, $C^{12}(x_1, x_2) = A^{12} - \frac{\xi_1}{\xi_2} B^{12}$. Since the system determines a quadratic algebra, the integrability conditions for the potential equations (30) are satisfied identically and the solution space is 2-dimensional. The general solution takes the form $V = a_1 V^{(0)} + a_2$ where a_1, a_2 are constant coefficients. This defines the degenerate superintegrable system. The extension to the quantum case is obvious.

Example 2 E_1 : From §4.3 we have the classical free system $\mathcal{L}_1 = \mathcal{J}^2$, $\mathcal{L}_2 = p_1^2$, $\mathcal{R}^2 = 16\mathcal{L}_1\mathcal{L}_2(\mathcal{H} - \mathcal{L}_2)$, Using Cartesian coordinates $x_1 = x, x_2 = y$ we find $A^{12} = 0$, $A^{22} = \frac{3}{x}$, $B^{12} = 0$, $B^{22} = -\frac{3}{y}$. The general solution of the potential equations is $V = a_1(x^2 + y^2) + \frac{a_2}{x^2} + \frac{a_3}{y^2} + a_4$. Setting $a_4 = 0$ we find that the induced classical system is

$$\mathcal{H} = p_1^2 + p_2^2 + a_1(x^2 + y^2) + \frac{a_2}{x^2} + \frac{a_3}{y^2}, \quad (31)$$

$$\mathcal{L}_1 = (xp_2 - yp_1)^2 + a_2 y^2 / x^2 + a_3 x^2 / y^2, \quad \mathcal{L}_2 = p_1^2 + a_1 x^2 + a_2 / x^2.$$

The induced Casimir is $\mathcal{R}^2 = 16(\mathcal{L}_1\mathcal{L}_2\mathcal{H} - \mathcal{L}_1\mathcal{L}_2^2 - (a_2 + a_3)\mathcal{L}_2^2 - a_2\mathcal{H}^2 + 2a_2\mathcal{L}_2\mathcal{H} - a_1\mathcal{L}_1^2 + 4a_1a_2a_3)$. The quantum system is defined by

$$\begin{aligned} H &= \partial_x^2 + \partial_y^2 + a_1(x^2 + y^2) + \frac{a_2}{x^2} + a_3/y^2, \quad L_2 = \partial_x^2 + a_1 x^2 + a_2/x^2, \\ L_1 &= (x\partial_y - y\partial_x)^2 + a_2 y^2/x^2 + a_3 x^2/y^2, \end{aligned} \quad (32)$$

The induced Casimir is $R^2 = \frac{8}{3}(\{L_1, L_2, H\} - \{L_2, L_2, L_1\}) - (16a_2 + 12)H^2 - (\frac{176}{3} + 16a_2 + 16a_3)L_2^2 - 16a_1L_1^2 + (\frac{176}{3} + 32a_2)L_2H + \frac{176a_1}{3}L_1 - \frac{16a_1}{3}(12a_2a_3 + 9a_2 + 9a_3 + 2)$.

Example 3 S_9 : From §4.4 we have the classical free systems $\mathcal{L}_1 = \mathcal{J}_3^2$, $\mathcal{L}_2 = \mathcal{J}_1^2$, $\mathcal{R}^2 = -16\mathcal{L}_1^2\mathcal{L}_2 - 16\mathcal{L}_1\mathcal{L}_2^2 + 16\mathcal{L}_1\mathcal{L}_2\mathcal{H}$. The structure equations are more symmetrical if we choose a new basis symmetry $\mathcal{L}_3 = \mathcal{J}_2^2$ in place of \mathcal{H} . Using coordinates $x_1 = \psi, x_2 = \phi$ where $s_1 = \frac{\cos \phi}{\cosh \psi}$, $s_2 = \frac{\sin \phi}{\cosh \psi}$, $s_3 = \tanh \psi$, $s_1^2 + s_2^2 + s_3^2 = 1$, the Hamiltonian is $H = \cosh^2 \psi (p_\psi^2 + p_\phi^2)$, $\lambda = \frac{1}{\cosh^2 \psi}$,

$$A^{12} = 0, \quad A^{22} = \frac{3 \cosh^2 \psi - \sinh^2 \psi}{\sinh \psi \cosh \psi}, \quad B^{12} = 2 \frac{\sinh \psi}{\cosh \psi}, \quad B^{22} = -3 \frac{(\cos^2 \phi - \sin^2 \phi)}{\sin \phi \cos \phi}.$$

The general potential is $V = \frac{a_1}{s_1^2} + \frac{a_2}{s_2^2} + \frac{a_3}{s_3^2} + a_4$. Setting $a_4 = 0$ we find the classical symmetries. The induced classical S_9 system has a basis of symmetries

$$\mathcal{L}_2 = \mathcal{J}_1^2 + a_2 \frac{s_3^2}{s_2^2} + a_3 \frac{s_2^2}{s_3^2}, \quad \mathcal{L}_3 = \mathcal{J}_2^2 + a_3 \frac{s_1^2}{s_2^2} + a_1 \frac{s_3^2}{s_1^2}, \quad \mathcal{L}_1 = \mathcal{J}_3^2 + a_1 \frac{s_2^2}{s_1^2} + a_2 \frac{s_1^2}{s_2^2}, \quad (33)$$

where $\mathcal{H} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + a_1 + a_2 + a_3$. The classical Casimir is

$$\mathcal{R}^2 = 16\mathcal{L}_1\mathcal{L}_2\mathcal{L}_3 - 16a_2\mathcal{L}_3^2 - 16a_3\mathcal{L}_1^2 - 16a_1\mathcal{L}_2^2 + 64a_1a_2a_3.$$

The quantum superintegrable system is defined as

$$\begin{aligned} H &= J_1^2 + J_2^2 + J_3^2 + \frac{a_1}{s_1^2} + \frac{a_2}{s_2^2} + \frac{a_3}{s_3^2}, \quad L_1 = J_3^2 + \frac{a_2 s_1^2}{s_2^2} + \frac{a_1^2 s_2^2}{s_1^2}, \\ L_2 &= J_1^2 + \frac{a_3 s_2^2}{s_3^2} + \frac{a_2 s_3^2}{s_2^2}, \quad L_3 = J_2^2 + \frac{a_1 s_3^2}{s_1^2} + \frac{a_3 s_1^2}{s_3^2}, \end{aligned} \quad (34)$$

$H = L_1 + L_2 + L_3 + a_1 + a_2 + a_3$. The quantum Casimir is

$$\begin{aligned} R^2 &= \frac{8}{3} \{L_1, L_2, L_3\} - (16a_3 + 12)L_1^2 - (16a_1 + 12)L_2^2 - (16a_2 + 12)L_3^2 \\ &\quad + \frac{52}{3} (\{L_1, L_2\} + \{L_2, L_3\} + \{L_3, L_1\}) + \frac{1}{3} (16 + 176a_3)L_1 \\ &\quad + \frac{1}{3} (16 + 176a_1)L_2 + \frac{1}{3} (16 + 176a_2)L_3 + \frac{32}{3} (a_1 + a_2 + a_3) \\ &\quad + 48(a_1a_2 + a_2a_3 + a_3a_1) + 64a_1a_2a_3. \end{aligned}$$

Example 4 S_3 : This is a restriction of system \tilde{S}_9 in the preceding example and we use the same notation. We set $\mathcal{L}_1 = \mathcal{X}^2$, $\mathcal{X} = p_\phi$, $x_1 = \psi, x_2 = \phi$. We have $C^2 = 0$, $C^{22} = \frac{3 \cosh^2 \psi - \sinh^2 \psi}{\sinh \psi \cosh \psi}$, $C^{12} = 0$, so $V_2 = 0$, $V_{11} + \frac{3 \cosh^2 \psi - \sinh^2 \psi}{\sinh \psi \cosh \psi} V_1 = 0$. The general potential is $V = \frac{a_3}{s_3^2} + a_4$. The induced classical S_3 system has a basis of symmetries and Casimir relation

$$\begin{aligned} \mathcal{H}' &= \mathcal{J}_1^2 + \mathcal{J}_2^2 + \mathcal{J}_3^2 + \frac{a_3}{s_3^2}, \quad \mathcal{L}'_1 = \mathcal{J}_1^2 + a_3 \frac{s_2^2}{s_3^2}, \quad \mathcal{L}'_2 = \mathcal{J}_1\mathcal{J}_2 - a_3 \frac{s_1 s_2}{s_3^2}, \quad \mathcal{X} = \mathcal{J}_3, \\ \mathcal{L}'_1{}^2 + \mathcal{L}'_2{}^2 - \mathcal{L}'_1 \mathcal{H}' + \mathcal{L}'_1 \mathcal{X}^2 + a_3 \mathcal{X}^2 + a_3 \mathcal{L}_1 &= 0. \end{aligned} \quad (35)$$

The quantum superintegrable system is defined as

$$H = J_1^2 + J_2^2 + J_3^2 + \frac{a_3}{s_3^2}, \quad X = J_3, \quad L_1 = J_1^2 + \frac{a_3 s_2^2}{s_3^2}, \quad L_2 = \frac{1}{2} (J_1 J_2 + J_2 J_1) - \frac{a_3 s_1 s_2}{s_3^2}.$$

The Casimir is $\{L_1, X^2\} + 2L_1^2 + 2L_2^2 - 2L_1 H + \frac{5+4a_3}{2} X^2 - 2aL_1 - a_3 = 0$.

5 Contractions of superintegrable systems

Suppose we have a nondegenerate quantum superintegrable system with generators H, L_1, L_2 and structure equations (4), defining a quadratic algebra Q . If we make a change of basis to new generators $\tilde{H}, \tilde{L}_1, \tilde{L}_2$ and parameters $\tilde{a}_1, \tilde{a}_2, \tilde{a}_3$ such that

$$\begin{pmatrix} \tilde{L}_1 \\ \tilde{L}_2 \\ \tilde{H} \end{pmatrix} = \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \\ 0 & 0 & A_{3,3} \end{pmatrix} \begin{pmatrix} L_1 \\ L_2 \\ H \end{pmatrix} + \begin{pmatrix} B_{1,1} & B_{1,2} & B_{1,3} \\ B_{2,1} & B_{2,2} & B_{2,3} \\ B_{3,1} & B_{3,2} & B_{3,3} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad (36)$$

$$\begin{pmatrix} \tilde{a}_1 \\ \tilde{a}_2 \\ \tilde{a}_3 \end{pmatrix} = \begin{pmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

for some 3×3 constant matrices $A = (A_{i,j}), B, C$ such that $\det A \cdot \det C \neq 0$, we will have the same system with new structure equations of the form (4) for $\tilde{R} = [\tilde{L}_1, \tilde{L}_2], [\tilde{L}_j, \tilde{R}], \tilde{R}^2$, but with transformed structure constants. (Strictly speaking, since the space of potentials is 4-dimensional, we should have a term a_4 in the above expressions. However, normally, this term can be absorbed into H . Also, we could add constant terms to each of the symmetries \tilde{H}, \tilde{L}_j but we shall restrict ourselves to this class of basis changes here.) We choose a continuous 1-parameter family of basis transformation matrices $A(\epsilon), B(\epsilon), C(\epsilon)$, $0 < \epsilon \leq 1$ such that $A(1) = C(1)$ is the identity matrix, $B(1) = 0$ and $\det A(\epsilon) \neq 0$, $\det C(\epsilon) \neq 0$. Now suppose as $\epsilon \rightarrow 0$ the basis change becomes singular, (i.e., the limits of A, B, C either do not exist or, if they exist do not satisfy $\det A(0) \det C(0) \neq 0$) but the structure equations involving $A(\epsilon), B(\epsilon), C(\epsilon)$, go to a limit, defining a new quadratic algebra Q' . We call Q' a *contraction* of Q in analogy with Lie algebra contractions [2]. We can also define contractions of free superintegrable systems in an obvious manner from (36): Just set $a_1 = a_2 = a_3 = 0$ and $B = C = 0$.

For a degenerate superintegrable system with generators H, X, L_1, L_2 and structure equations (5),(7), defining a quadratic algebra Q , a change of basis to new generators $\tilde{H}, \tilde{X}, \tilde{L}_1, \tilde{L}_2$ and parameter \tilde{a} such that $\tilde{a} = Ca$, and

$$\begin{pmatrix} \tilde{L}_1 \\ \tilde{L}_2 \\ \tilde{H} \\ \tilde{X} \\ \tilde{X}^2 \end{pmatrix} = \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & 0 & A_1 \\ A_{2,1} & A_{2,2} & A_{2,3} & 0 & A_2 \\ 0 & 0 & A_{3,3} & 0 & 0 \\ 0 & 0 & 0 & A_{4,4} & 0 \\ 0 & 0 & 0 & 0 & A_{4,4}^2 \end{pmatrix} \begin{pmatrix} L_1 \\ L_2 \\ H \\ X \\ X^2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ 0 \\ 0 \end{pmatrix} a$$

for some 4×4 matrix $A = (A_{i,j})$, in the upper left-hand corner, with $\det A \neq 0$, complex 4-vectors $A' = (A_i)$, B and constant $C \neq 0$ yields the same superintegrable system with new structure equations of the form (5),(7) for $[\tilde{X}, \tilde{L}_j], [\tilde{L}_1, \tilde{L}_2]$, and $\tilde{G} = 0$, but with transformed structure constants. Suppose we choose a continuous 1-parameter family of basis transformation matrices $(A(\epsilon), A'(\epsilon), B(\epsilon), C(\epsilon))$, $0 < \epsilon \leq 1$ such that $A(1)$ is the identity matrix, $A'(1) = B(1) = 0$, $C(1) = 1$, and $\det A(\epsilon) \neq 0$, $C(\epsilon) \neq 0$. Now suppose as $\epsilon \rightarrow 0$ the basis change becomes singular but that the structure equations involving $A(\epsilon), A'(\epsilon), B(\epsilon), C(\epsilon)$ go to a finite limit, thus defining a new quadratic algebra Q' . We call Q' a *contraction* of Q . Contractions of free degenerate superintegrable systems are defined

in an analogous manner: Set $a = 0$, $B = 0$. There are analogous definitions of contractions for classical systems.

5.1 Lie algebra contractions of $o(3, \mathbb{C})$ and $e(2, \mathbb{C})$

In general, the classification of possible contractions of quadratic algebras is very complex, but for quadratic algebras associated with systems on constant curvature spaces, there is a class of contractions with important physical/geometrical significance that can easily be classified: contractions induced from Lie algebra contractions. In [2] Inönü and Wigner defined a family of contractions of Lie algebras, with special emphasis on the symmetry algebras of constant curvature spaces: the Wigner-Inönü contractions. Later a larger class of contractions was studied, so-called natural contractions, [20]. We recall the definition of natural (quantum) contraction. Let $(A; [\cdot; \cdot]_A)$, $(B; [\cdot; \cdot]_B)$ be two complex Lie algebras. We say B is a *contraction* of A if for every $\epsilon \in (0; 1]$ there exists a linear invertible map $t_\epsilon : B \rightarrow A$ such that for every $X, Y \in B$, $\lim_{\epsilon \rightarrow 0} t_\epsilon^{-1}[t_\epsilon X, t_\epsilon Y]_A = [X, Y]_B$. Thus, as $\epsilon \rightarrow 0$ the 1-parameter family of basis transformations can become nonsingular but the structure constants go to a finite limit. There is an analogous definition for classical contractions. For Lie algebras $e(2, \mathbb{C})$ and $so(3, \mathbb{C})$ the contractions have all been classified up to conjugacy, [1, 18, 23]. We first list these contractions and their physical implementations, then show that they induce contractions of free nondegenerate and degenerate classical quadratic algebras associated with constant curvature spaces and, ultimately, contractions of the nondegenerate and degenerate (classical and quantum) superintegrable systems with potential. We omit contractions to the abelian algebra and the identity contractions, irrelevant for our purposes.

We start with $e(2, \mathbb{C})$ and use the classical realization with basis p_1, p_2 , $\mathcal{J} = xp_2 - yp_1$ and Hamiltonian $\mathcal{H} = p_1^2 + p_2^2$.

Contractions of $e(2, \mathbb{C})$:

1. $\{\mathcal{J}', p'_1, p'_2\} = \{\mathcal{J}, \epsilon p_1, \epsilon p_2\} : e(2, \mathbb{C})$,
coordinate implementation $x' = \frac{x}{\epsilon}, y' = \frac{y}{\epsilon}$,
2. $\{\mathcal{J}', p'_1 + ip'_2, p'_1 - ip'_2\} = \{\mathcal{J}, \epsilon(p_1 + ip_2), p_1 - ip_2\} : e(2, \mathbb{C})$,
coordinate implementation $x' + iy' = x + iy, x' - iy' = \frac{x - iy}{\epsilon}$,
3. $\{\mathcal{J}', p'_1, p'_2\} = \{\mathcal{J} + \frac{p_1}{\epsilon}, p_1, p_2\} : e(2, \mathbb{C})$,
coordinate implementation $x' = x, y' = y - \frac{1}{\epsilon}$,
4. $\{\mathcal{J}', p'_1, p'_2\} = \{\mathcal{J} + \frac{p_1 + ip_2}{\epsilon}, p_1, p_2\} : e(2, \mathbb{C})$,
coordinate implementation $x' = x + \frac{i}{\epsilon}, y' = y - \frac{1}{\epsilon}$.

These last two contraction types can be combined, even including different powers of ϵ . A relevant example is

$$\{\mathcal{J}', p'_1, p'_2\} = \left\{ \mathcal{J} + \frac{p_1 + ip_2}{\epsilon} + \frac{p_1 - ip_2}{\sqrt{\epsilon}}, p_1, p_2 \right\} : e(2, \mathbb{C}), \quad (38)$$

$$\text{coordinate implementation } x' = x + \frac{i}{\epsilon} - \frac{i}{\sqrt{\epsilon}}, y' = y - \frac{1}{\epsilon} - \frac{1}{\sqrt{\epsilon}}.$$

5. $\{\mathcal{J}', p'_1, p'_2\} = \{\epsilon \mathcal{J}, p_1, \epsilon p_2\} : \text{Heisenberg algebra}$,
coordinate implementation $x' = x, y' = \frac{y}{\epsilon}, \mathcal{J}' = x' p'_2$,

We use the classical realization for $o(3, \mathbb{C})$ acting on the 2-sphere, with basis $\mathcal{J}_1 = s_2 p_3 - s_3 p_2$, $\mathcal{J}_2 = s_3 p_1 - s_1 p_3$, $\mathcal{J}_3 = s_1 p_2 - s_2 p_1$, Hamiltonian $\mathcal{H} = \mathcal{J}_1^2 + \mathcal{J}_2^2 + \mathcal{J}_3^2$. Here $\sum_{j=1}^3 s_j^2 = 1$ and restriction to the sphere gives $s_1 p_1 + s_2 p_2 + s_3 p_3 = 0$.

Contractions of $o(3, \mathbb{C})$: (39)

1. $\{\mathcal{J}'_1, \mathcal{J}'_2, \mathcal{J}'_3\} = \{\epsilon \mathcal{J}_1, \epsilon \mathcal{J}_2, \mathcal{J}_3\} : e(2, \mathbb{C})$,
coordinate implementation $x = s_1/\epsilon, y = s_2/\epsilon, s_3 \approx 1, \mathcal{J} = \mathcal{J}_3$,
2. $\{\mathcal{J}'_1 + i\mathcal{J}'_2, \mathcal{J}'_1 - i\mathcal{J}'_2, \mathcal{J}'_3\} = \{\mathcal{J}_1 + i\mathcal{J}_2, \epsilon(\mathcal{J}_1 - i\mathcal{J}_2), \mathcal{J}_3\} : e(2, \mathbb{C})$,
coordinate implementation $s_1 + is_2 = \epsilon z, s_1 - is_2 = \bar{z}, s_3 \approx 1$,
 $\mathcal{J}'_3 = i(zp_z - \bar{z}p_{\bar{z}}), \mathcal{J}'_1 + i\mathcal{J}'_2 = 2ip_{\bar{z}}, \mathcal{J}'_1 - i\mathcal{J}'_2 = -2ip_z$,
3. $\{\mathcal{J}'_1 + i\mathcal{J}'_2, \mathcal{J}'_1 - i\mathcal{J}'_2, \mathcal{J}'_3\} = \{\epsilon(\mathcal{J}_1 + i\mathcal{J}_2), \frac{\mathcal{J}_1 - i\mathcal{J}_2}{\epsilon}, \mathcal{J}_3\} : o(3, \mathbb{C})$,
coordinate implementation $s'_1 = \frac{\epsilon + \epsilon^{-1}}{2}s_1 + i\frac{\epsilon - \epsilon^{-1}}{2}s_2$,
 $s'_2 = -i\frac{\epsilon - \epsilon^{-1}}{2}s_1 + \frac{\epsilon + \epsilon^{-1}}{2}s_2, s'_3 = s_3$,
4. $\{\mathcal{J}'_1 + i\mathcal{J}'_2, \mathcal{J}'_1 - i\mathcal{J}'_2, \mathcal{J}'_3\} = \{\epsilon(\mathcal{J}_1 + i\mathcal{J}_2), \mathcal{J}_1 - i\mathcal{J}_2, \epsilon \mathcal{J}_3\} : \text{Heisenberg algebra}$,
coordinate implementation $s_1 = \frac{\cos \phi}{\cosh \psi}, s_2 = \frac{\sin \phi}{\cosh \psi}, s_3 = \frac{\sinh \psi}{\cosh \psi}$,
we set $\phi = \epsilon\theta - i \ln \sqrt{\epsilon}, \psi = \xi \sqrt{\epsilon}$, to get
 $\mathcal{J}'_3 = p_\theta, \mathcal{J}'_1 + i\mathcal{J}'_2 = \xi p_\theta - ip_\xi, \mathcal{J}'_1 - i\mathcal{J}'_2 = \xi p_\theta + ip_\xi$.

5.2 Quadratic enveloping algebra contractions from Lie algebra contractions

Note that once we choose a basis for a Lie algebra A , its enveloping algebra is uniquely determined by the structure constants. Structure relations in the enveloping algebra are continuous functions of the structure constants. Thus a contraction of one Lie algebra A to another, B induces a contraction of the corresponding enveloping algebras of A and B . In the case of $e(2, \mathbb{C}), o(3, \mathbb{C})$, free quadratic algebras constructed in the enveloping algebras will contract to free quadratic algebras generated by the target Lie algebras.

Consider only 4 contractions of $e(2, \mathbb{C})$ to itself and 1 to the Heisenberg algebra. Each of the first 4 when applied to a free nondegenerate or degenerate quadratic algebra \tilde{E}_j will contract to a quadratic algebra \tilde{E}_k where k may be distinct from j . The last contraction will also lead to a quadratic algebra which we call *singular* because the new Hamiltonian will be degenerate. We do not classify these singular systems but they are of physical and mathematical interest. Of the 4 nontrivial contractions of $o(3, \mathbb{C})$, 1 takes $o(3, \mathbb{C})$ to itself (so \tilde{S}_j to \tilde{S}_k), 2 take it to $e(2, \mathbb{C})$ (so \tilde{S}_j to \tilde{E}_k and 1 to the Heisenberg algebra (so \tilde{S}_j to a singular system).

Example 5 $\tilde{E}_1 \rightarrow \tilde{E}_8$: Use $\{\mathcal{J}', p'_1 + ip'_2, p'_1 - ip'_2\} = \{\mathcal{J}, \epsilon(p_1 + ip_2), p_1 - ip_2\}$.
 $\mathcal{H}' = \epsilon \mathcal{H} = (p'_1 + ip'_2)(p'_1 - ip'_2), \mathcal{L}'_1 = (\mathcal{J}')^2 = \mathcal{L}_1, \mathcal{L}'_2 = 4\epsilon^2 \mathcal{L}_2 = (p'_1 + ip'_2)^2$.

Example 6 $\tilde{E}_1 \rightarrow \tilde{E}_1$. Use $\{\mathcal{J}', p'_1, p'_2\} = \{\mathcal{J}, \epsilon p_1, \epsilon p_2\}$. $\mathcal{L}'_1 = \mathcal{J}^2 = \mathcal{L}_1$,
 $\mathcal{L}'_2 = \epsilon^2 p_1^2 = \epsilon^2 \mathcal{L}_2, \mathcal{H}' = \epsilon^2 (p_1^2 + p_2^2) = \epsilon^2 \mathcal{H}$.

Example 7 $\tilde{E}_1 \rightarrow \text{Heisenberg}$. Use $\{\mathcal{J}', p'_1, p'_2\} = \{\epsilon \mathcal{J}, p_1, \epsilon p_2\}$. $\mathcal{L}'_1 = \epsilon^2 \mathcal{L}_1 = \mathcal{J}'^2 = x'^2 p_2'^2, \mathcal{L}'_2 = \mathcal{L}_2 = p_1'^2, \mathcal{H}' = \epsilon^2 (\mathcal{H} - \mathcal{L}_2) = p_2'^2$. Structure relations:
 $\mathcal{R} = \{\mathcal{L}'_1, \mathcal{L}'_2\}, \mathcal{R}^2 = 4p_1'^2 p_2'^4 = 4\mathcal{L}'_1 \mathcal{H}'^2$.

Example 8 $\tilde{S}_3 \rightarrow \tilde{S}_3$. $\{\mathcal{J}'_1+i\mathcal{J}'_2, \mathcal{J}'_1-i\mathcal{J}'_2, \mathcal{J}'_3\} = \{\epsilon(\mathcal{J}_1+i\mathcal{J}_2), (\mathcal{J}_1-i\mathcal{J}_2)/\epsilon, \mathcal{J}_3\}$.
 $\mathcal{X}' = \mathcal{X} = \mathcal{J}_3$, $\mathcal{H}' = \mathcal{J}'_1{}^2 + \mathcal{J}'_2{}^2 + \mathcal{J}'_3{}^2 = \mathcal{H}$, $\mathcal{L}'_1 = (\mathcal{J}'_1 + i\mathcal{J}'_2)^2 = 4i\epsilon^2\mathcal{L}_2$,
 $\mathcal{L}'_2 = (\mathcal{J}'_1 - i\mathcal{J}'_2)^2 = \frac{2}{\epsilon^2}(\mathcal{L}_1 - i\mathcal{L}_2 - \frac{1}{2}\mathcal{H} + \frac{1}{2}\mathcal{X}^2)$.

We list the contractions in tables. For $e(2, \mathbb{C})$ the relevant contractions are (37):

$e(2, \mathbb{C})$		Contractions (degenerate)		
$e(2) \rightarrow e(2), \mathbf{1}$	$e(2) \rightarrow e(2), \mathbf{2}$	$e(2) \rightarrow e(2), \mathbf{3}$	$e(2) \rightarrow e(2), \mathbf{4}$	$e(2) \rightarrow$ Heisenberg
\tilde{E}_3 : \tilde{E}_3	\tilde{E}_3	\tilde{E}_5	\tilde{E}_4	$\mathcal{L}'_1\mathcal{H}' = \mathcal{L}'_2{}^2$
\tilde{E}_4 : \tilde{E}_4	\tilde{E}_4	\tilde{E}_4	\tilde{E}_4	$\mathcal{H}' = \mathcal{X}'^2$
\tilde{E}_5 : \tilde{E}_5	\tilde{E}_4	\tilde{E}_5	\tilde{E}_5	$\mathcal{L}'^2 = \mathcal{X}'^2\mathcal{H}'$
\tilde{E}_6 : \tilde{E}_6	\tilde{E}_{14}	\tilde{E}_5	\tilde{E}_5	$\mathcal{L}'_2{}^2 = \mathcal{L}'_1\mathcal{H}'$
\tilde{E}_{12} : \tilde{E}_4	\tilde{E}_{12}	\tilde{E}_4	\tilde{E}_{12}	$\mathcal{L}'_2{}^2 = \mathcal{L}'_1\mathcal{H}' + \mathcal{H}'^2$
\tilde{E}_{13} : \tilde{E}_{13}	\tilde{E}_{13}	\tilde{E}_4	\tilde{E}_{13}	$\mathcal{L}'_1\mathcal{L}'_3 = \mathcal{H}\mathcal{L}'_2$
\tilde{E}_{14} : \tilde{E}_{14}	\tilde{E}_{14}	\tilde{E}_4	\tilde{E}_{14}	$\mathcal{L}'_2{}^2 = \mathcal{L}'_1\mathcal{H}'$
\tilde{E}_{18} : \tilde{E}_{18}	\tilde{E}_{18}	\tilde{E}_5	\tilde{E}_{13}	$\mathcal{L}'_2{}^2 = \mathcal{X}'^2\mathcal{H}'$
(40)				

$e(2, \mathbb{C})$		Contractions (nondegenerate)		
$e(2) \rightarrow e(2), \mathbf{1}$	$e(2) \rightarrow e(2), \mathbf{2}$	$e(2) \rightarrow e(2), \mathbf{3}$	$e(2) \rightarrow e(2), \mathbf{4}$	$e(2) \rightarrow$ Heisenberg
\tilde{E}_1 : \tilde{E}_1	\tilde{E}_8	\tilde{E}_2	$\tilde{E}_{3'}$	$\mathcal{R}'^2 = \mathcal{L}'_1\mathcal{H}'^2$
\tilde{E}_8 : \tilde{E}_8	\tilde{E}_8	\tilde{E}'_3	$\tilde{E}_{15}, \tilde{E}_{10}$	$\mathcal{R}'^2 = \mathcal{L}'_1\mathcal{H}'^2$
\tilde{E}_{10} : $\tilde{E}_{3'}$	$\tilde{E}_{3'}$	$\tilde{E}_{3'}$	\tilde{E}_{10}	$\mathcal{R}'^2 = \mathcal{H}'^3$
\tilde{E}_2 : \tilde{E}_2	\tilde{E}_{15}	\tilde{E}_2	$\tilde{E}_{3'}$	$\mathcal{R}'^2 = \mathcal{L}'_1\mathcal{H}'^2$
$\tilde{E}_{3'}$: $\tilde{E}_{3'}$	$\tilde{E}_{3'}$	$\tilde{E}_{3'}$	$\tilde{E}_{3'}$	$\mathcal{R}'^2 = 0$
\tilde{E}_{16} : \tilde{E}_{16}	\tilde{E}_{17}	\tilde{E}_2	$\tilde{E}_{15}, \tilde{E}_{10}$	$\mathcal{R}'^2 = 4\mathcal{L}'_1{}^2\mathcal{H}'$
\tilde{E}_7 : $\tilde{E}_{3'}$	\tilde{E}_7	$\tilde{E}_{3'}$	$\tilde{E}_{15}, \tilde{E}_{10}$	$\mathcal{R}'^2 = 4(\mathcal{L}'_1 + a\mathcal{H}')\mathcal{H}'^2$
\tilde{E}_{17} : \tilde{E}_{17}	\tilde{E}_{17}	$\tilde{E}_{3'}$	$\tilde{E}_{15}, \tilde{E}_{10}$	$\mathcal{R}'^2 = 0$
\tilde{E}_{19} : \tilde{E}_{11}	\tilde{E}_{19}	$\tilde{E}_{3'}$	$\tilde{E}_{15}, \tilde{E}_{10}$	$\mathcal{R}'^2 = 0$
\tilde{E}_{11} : \tilde{E}_{11}	\tilde{E}_{11}	\tilde{E}_{11}	\tilde{E}_{11}	$\mathcal{R}'^2 = \mathcal{H}'^3$
\tilde{E}_9 : \tilde{E}_9	$\tilde{E}_{15}, \tilde{E}_{11}$	$\tilde{E}_{3'}$	\tilde{E}_9	$\mathcal{R}'^2 = \mathcal{L}'_1{}^2\mathcal{H}'$
\tilde{E}_{20} : \tilde{E}_{20}	\tilde{E}_{20}	$\tilde{E}_{3'}$	\tilde{E}_{11}	$\mathcal{R}'^2 = \mathcal{L}'_1{}^2\mathcal{H}'$
(41)				

Note: For the $\tilde{E}_7 \rightarrow \tilde{E}_{10}$, $\tilde{E}_8 \rightarrow \tilde{E}_{10}$, $\tilde{E}_{16} \rightarrow \tilde{E}_{10}$, $\tilde{E}_{17} \rightarrow \tilde{E}_{10}$ and $\tilde{E}_{19} \rightarrow \tilde{E}_{10}$ contractions we use (38). For $\tilde{E}_{10} \rightarrow \tilde{E}_{3'}$, case **3** we use the composite contraction $\{\mathcal{J}', p'_1, p'_2\} = \{\mathcal{J} + \frac{p_1+p_2}{\epsilon}, p_1, p_2\}$.

The relevant $o(3, \mathbb{C})$ contractions are, (39):

$o(3, \mathbb{C})$		Contractions (degenerate)			
		$o(3) \rightarrow e(2), \mathbf{1}$	$o(3) \rightarrow e(2), \mathbf{2}$	$o(3) \rightarrow o(3), \mathbf{3}$	$o(3) \rightarrow$ Heisenberg
\tilde{S}_3	: \tilde{E}_3	\tilde{E}_3	\tilde{S}_3		$\mathcal{X}'^2 \mathcal{H}' = \mathcal{L}'_2{}^2$
\tilde{S}_5	: \tilde{E}_{14}	\tilde{E}_{14}	\tilde{S}_5		$\mathcal{X}'^2 \mathcal{H}' = \mathcal{L}'_2{}^2$
\tilde{S}_6	: \tilde{E}_{18}	\tilde{E}_{18}	\tilde{S}_6		$\mathcal{X}'^2 = \mathcal{H}'$

(42)

$o(3, \mathbb{C})$		Contractions (nondegenerate)			
		$o(3) \rightarrow e(2), \mathbf{1}$	$o(3) \rightarrow e(2), \mathbf{2}$	$o(3) \rightarrow o(3), \mathbf{3}$	$o(3) \rightarrow$ Heisenberg
\tilde{S}_9	: \tilde{E}_1	\tilde{E}_8	\tilde{S}_2		$\mathcal{R}'^2 = -\mathcal{L}'_2{}^2 \mathcal{H}'$
\tilde{S}_4	: \tilde{E}_{17}	\tilde{E}_{17}	\tilde{S}_4		$\mathcal{R}'^2 = -4\mathcal{L}'_2{}^2 \mathcal{H}'$
\tilde{S}_7	: \tilde{E}_{16}	\tilde{E}_{17}	\tilde{S}_4		$\mathcal{R}'^2 = -4\mathcal{L}'_2{}^2 \mathcal{H}'$
\tilde{S}_8	: \tilde{E}_9	\tilde{E}_{11}	\tilde{S}_4		$\mathcal{R}'^2 = -4\mathcal{L}'_2{}^2 \mathcal{H}'$
\tilde{S}_2	: \tilde{E}_8	\tilde{E}_8	\tilde{S}_2		$\mathcal{R}'^2 = -16\mathcal{L}'_2{}^2 \mathcal{H}'$
\tilde{S}_1	: \tilde{E}_{11}	\tilde{E}_{11}	\tilde{S}_1		$\mathcal{R}'^2 = 0$

(43)

5.3 Contractions/restrictions of free nondegenerate systems to free degenerate ones

These are not contractions in the standard sense. As we have shown in Section 2.2, they arise through the following mechanism. Suppose we take a particular basis of 2nd order generators $\mathcal{H}, \mathcal{L}_1, \mathcal{L}_2$ for the classical nondegenerate free system such that the symmetry \mathcal{L}_1 is a perfect square: $\mathcal{L}_1 = \mathcal{X}^2$. Then \mathcal{X} will be a 1st order symmetry for \mathcal{H} , i.e., a Killing vector. From the relation $\mathcal{R} = \{\mathcal{L}_1, \mathcal{L}_2\} = 2\mathcal{X}\{\mathcal{X}, \mathcal{L}_2\}$, we see that $\mathcal{L}_3 = \{\mathcal{X}, \mathcal{L}_2\}$ is a 2nd order symmetry for \mathcal{H} (which in most case turns out to be linearly independent of the symmetries $\mathcal{H}, \mathcal{L}_1, \mathcal{L}_2$ we already know). Then we can factor $4\mathcal{X}^2$ from each term of the identity $\mathcal{R}^2 - \mathcal{F} = 0$ to obtain the Casimir $\mathcal{G} = 0$ for the contracted system, where $\mathcal{G} = \mathcal{L}_3^2 + \dots$. In any case, we are guaranteed by theory that a 2nd order symmetry \mathcal{L}_3 exists such that $\mathcal{X}, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{H}$ define a unique free degenerate quadratic algebra. We give some examples:

1. $\tilde{S}_9 \rightarrow \tilde{S}_3$: In system \tilde{S}_9 we have $\mathcal{L}_1 = \mathcal{J}_3^2, \mathcal{L}_2 = \mathcal{J}_1^2$. We note that $\mathcal{X} = \mathcal{J}_3$, a Killing vector. The Casimir for the original system is

$$\mathcal{R}^2 = -16\mathcal{L}_1^2 \mathcal{L}_2 - 16\mathcal{L}_1 \mathcal{L}_2^2 + 16\mathcal{L}_1 \mathcal{L}_2 \mathcal{H} \quad (44)$$

where $\mathcal{R} = \{\mathcal{L}_1, \mathcal{L}_2\}$. In the contracted system we take $\mathcal{L}_3 = \{\mathcal{X}, \mathcal{L}_2\}$. Setting $\mathcal{L}_3 = 2\mathcal{L}'_2, \mathcal{L}_2 = \mathcal{L}'_1$ we see that (44) reduces to the Casimir $(\mathcal{L}'_1)^2 + (\mathcal{L}'_2)^2 - \mathcal{L}'_1 \mathcal{H} + \mathcal{L}'_1 \mathcal{X}^2 = 0$, which can be identified with \tilde{S}_3 .

2. $\tilde{E}_1 \rightarrow \tilde{E}_3$: In system \tilde{E}_1 : $\mathcal{L}_1 = \mathcal{J}^2, \mathcal{L}_2 = p_1^2, \mathcal{R}^2 = 16\mathcal{L}_1 \mathcal{L}_2 (\mathcal{H} - \mathcal{L}_2)$, let $\mathcal{X} = \mathcal{J}$. Setting $\{\mathcal{X}, \mathcal{L}_2\} = 2\mathcal{L}'_2, \mathcal{L}_2 = \mathcal{L}'_1$, we see that the Casimir for \tilde{E}_1 reduces to $\mathcal{L}'_1{}^2 + \mathcal{L}'_2{}^2 - \mathcal{L}'_1 \mathcal{H}' = 0$, the structure equation for \tilde{E}_3 .

3. $\tilde{E}_8 \rightarrow \tilde{E}_{14}$: In system \tilde{E}_8 : $\mathcal{L}_1 = \mathcal{J}^2$, $\mathcal{L}_2 = (p_1 + ip_2)^2$, $\mathcal{R}^2 = -16\mathcal{L}_1\mathcal{L}_2^2$, let $\mathcal{X} = p_1 + ip_2$. Setting $\mathcal{L}_1 = \mathcal{L}'_1$, $\{\mathcal{X}, \mathcal{L}'_1\} = 2i\mathcal{L}'_2$ we find the Casimir $-\mathcal{L}'_2{}^2 + \mathcal{X}^2\mathcal{L}'_1 = 0$ for \tilde{E}_{14} .
4. $\tilde{E}_{10} \rightarrow \tilde{E}_4$: This case is less obvious.. In system \tilde{E}_{10} : $\mathcal{L}_1 = (p_1 - ip_2)^2$, $\mathcal{L}_2 = 4i(p_1 - ip_2)\mathcal{J} + (p_1 + ip_2)^2$, $\mathcal{R}^2 = 64\mathcal{L}_1^3$, we set $\mathcal{X} = p_1 - ip_2$, $\mathcal{L}_2 = \mathcal{L}'_1$. Now the Casimir restricts to $4\mathcal{X}^2\{\mathcal{X}, \mathcal{L}'_1\}^2 = 64\mathcal{X}^6$, or $\{\mathcal{X}, \mathcal{L}'_1\} = \pm 4\mathcal{X}^2$. We take the plus sign to be definite. It appears that this system closes on itself and doesn't give us a 4th generator. However, it follows from the analysis in §4.1 that there is a *unique* free degenerate quadratic algebra E_4 containing the algebra generated by $\mathcal{X}, \mathcal{H}, \mathcal{L}'_1$, namely the one generated by $\mathcal{X}, \mathcal{H}, \mathcal{L}'_1, \mathcal{L}'_2$ where $\mathcal{L}'_2 = (p_1 + ip_2)^2$.

Contractions of free nondegenerate $e(2, \mathbb{C})$ systems to degenerate systems:

$e(2, \mathbb{C})$	Contractions :	nondegenerate	\rightarrow	degenerate
	\mathcal{J}	p_1		$p_1 + ip_2$
\tilde{E}_1 :	\tilde{E}_3	\tilde{E}_6		—
\tilde{E}_8 :	\tilde{E}_3	—		\tilde{E}_{14}
\tilde{E}_{10} :	—	—		\tilde{E}_{14}
\tilde{E}_2 :	—	\tilde{E}_6, \tilde{E}_5		—
$\tilde{E}_{3'}$:	\tilde{E}_3	\tilde{E}_5		—
\tilde{E}_{16} :	\tilde{E}_{18}	—		—
\tilde{E}_7 :	\tilde{E}_3	—		\tilde{E}_{12}
\tilde{E}_{17} :	\tilde{E}_{18}	—		—
\tilde{E}_{19} :	—	—		—
\tilde{E}_{11} :	—	—		\tilde{E}_4
\tilde{E}_9 :	—	—		\tilde{E}_{13}
\tilde{E}_{20} :	—	—		—

Contractions of free nondegenerate $o(3, \mathbb{C})$ systems to degenerate systems:

$o(3, \mathbb{C})$	Contractions :	nondegenerate	\rightarrow	degenerate
	\mathcal{J}_3	$\mathcal{J}_1 + i\mathcal{J}_2$		
\tilde{S}_9 :	\tilde{S}_3	—		
\tilde{S}_4 :	\tilde{S}_6	—		
\tilde{S}_7 :	\tilde{S}_6	—		
\tilde{S}_8 :	—	—		
\tilde{S}_2 :	\tilde{S}_3	\tilde{S}_5		
\tilde{S}_1 :	—	\tilde{S}_5		

5.4 Contractions of superintegrable systems with potential induced by free quadratic algebra contractions

Suppose we have a classical free triplet $\mathcal{H}^{(0)}, \mathcal{L}_1^{(0)}, \mathcal{L}_2^{(0)}$ that determines a nondegenerate quadratic algebra $Q^{(0)}$ and structure functions $A^{ij}(\mathbf{x}), B^{ij}(\mathbf{x})$ in some

set of Cartesian-like coordinates (x_1, x_2) . Further, suppose this system contracts to another nondegenerate system $\mathcal{H}'^{(0)}, \mathcal{L}'_1^{(0)}, \mathcal{L}'_2^{(0)}$ with quadratic algebra $Q'^{(0)}$ via the mechanism described in the preceding sections. We show here that this contraction induces a contraction of the associated nondegenerate superintegrable system $\mathcal{H} = \mathcal{H}^{(0)} + V$, $\mathcal{L}_1 = \mathcal{L}_1^{(0)} + W^{(1)}$, $\mathcal{L}_2 = \mathcal{L}_2^{(0)} + W^{(2)}$, Q to $\mathcal{H}' = \mathcal{H}'^{(0)} + V'$, $\mathcal{L}'_1 = \mathcal{L}'_1^{(0)} + W^{(1)'}$, $\mathcal{L}'_2 = \mathcal{L}'_2^{(0)} + W^{(2)'}$, Q' . The point is that in the contraction process the symmetries $\mathcal{H}'^{(0)}(\epsilon)$, $\mathcal{L}'_1^{(0)}(\epsilon)$, $\mathcal{L}'_2^{(0)}(\epsilon)$ remain continuous functions of ϵ , linearly independent as quadratic forms, and $\lim_{\epsilon \rightarrow 0} \mathcal{H}'^{(0)}(\epsilon) = \mathcal{H}'^{(0)}$, $\lim_{\epsilon \rightarrow 0} \mathcal{L}'_j^{(0)}(\epsilon) = \mathcal{L}'_j^{(0)}$. Thus the associated functions $A^{ij}(\epsilon), B^{ij}(\epsilon)$ will also be continuous functions of ϵ and $\lim_{\epsilon \rightarrow 0} A^{ij}(\epsilon) = A'^{ij}$, $\lim_{\epsilon \rightarrow 0} B^{ij}(\epsilon) = B'^{ij}$. Similarly, the integrability conditions for the potential equations

$$\begin{aligned} V_{22}^{(\epsilon)} &= V_{11}^{(\epsilon)} + A^{22}(\epsilon)V_1^{(\epsilon)} + B^{22}(\epsilon)V_2^{(\epsilon)}, \\ V_{12}^{(\epsilon)} &= A^{12}(\epsilon)V_1^{(\epsilon)} + B^{12}(\epsilon)V_2^{(\epsilon)}, \end{aligned} \quad (47)$$

will hold for each ϵ and in the limit. This means that the 4-dimensional solution space for the potentials V will deform continuously into the 4-dimensional solution space for the potentials V' . Thus the target space of solutions V' is uniquely determined by the free quadratic algebra contraction.

A similar argument using the functions C^2, C^{22}, C^{12} where

$$V_2 = C^2 V_1, \quad V_{22} = V_{11} + C^{22} V_1, \quad V_{12} = C^{12} V_1, \quad (48)$$

applies to contractions of free degenerate quadratic algebras. Again the 2-dimensional space of source potentials deforms continuously to the target space.

Theorem 3 *A Lie algebra contraction of the free quadratic algebra of a free triplet system to another such system induces a unique contraction relating the associated superintegrable systems with potential.*

There is an apparent lack of uniqueness in this procedure, since for a nondegenerate superintegrable system one typically chooses a basis $V^{(j)}$, $j = 1, \dots, 4$ for the potential space and expresses a general potential as $V = \sum_{j=1}^4 a_j V^{(j)}$. Of course the choice of basis for the source system is arbitrary, as is the choice for the target system. Thus the structure equations for the quadratic algebras and the dependence $a_j(\epsilon)$ of the contraction constants on ϵ will vary depending on these choices. However, all such possibilities are related by a basis change matrix.

Example 9 *We describe how a Lie algebra contraction induces the contraction of E_1 to E_2 , including the potential terms. Recall for E_1 in Cartesian coordinates $x_1 = x, x_2 = y$ we have $H = p_x^2 + p_y^2 + V$,*

$$A^{12} = 0, \quad A^{22} = \frac{3}{x}, \quad B^{12} = 0, \quad B^{22} = -\frac{3}{y}. \quad (49)$$

The general potential is $V = a_1(x^2 + y^2) + \frac{a_2}{x^2} + \frac{a_3}{y^2} + a_4$. For E_2 and using Cartesian coordinates $x_1 = x', x_2 = y'$ we have $A'^{12} = 0$, $A'^{22} = 0$, $B'^{12} = 0$, $B'^{22} = -\frac{3}{y'}$. Thus, the general potential for E_2 is $V' = b_1(4x'^2 + y'^2) + b_2x' + \frac{b_3}{y'^2} + b_4$. In terms of these coordinates the contraction is defined by

$x = x' + \frac{1}{\epsilon}$, $y = y'$. Substituting these values in (49) and going to the limit as $\epsilon \rightarrow 0$ we get $A'^{12} = A'^{22} = B'^{12} = 0$, $B'^{22} = -\frac{3}{y'}$, the canonical equations for E_2 . In the limit the 4 dimensional space of potentials for E_1 must go to the 4 dimensional vector space for E_2 . However the chosen basis functions for the E_1 potential, $x^2 + y^2$, $\frac{1}{x^2}$, $\frac{1}{y^2}$, 1 will not go to a new basis in the limit; 1 basis function blows up and 1 basis function goes to 0. One of the simplest choices of basis that avoids this problem is $V^{(1)}(\epsilon) = x^2 + y^2 + \frac{1}{\epsilon^4 x^2} - \frac{1}{\epsilon^2} \rightarrow 4x'^2 + y'^2$, $V^{(2)}(\epsilon) = \frac{-1}{2\epsilon}(\frac{1}{\epsilon^2 x^2} - 1) \rightarrow x'$, $V^{(3)}(\epsilon) = \frac{1}{y^2} \rightarrow \frac{1}{y'^2}$, $V^{(4)}(\epsilon) = 1 \rightarrow 1$. Thus if we set $V = \sum_{j=1}^4 b_j V^{(j)}(1)$ then the coefficients would stay fixed under the contraction. However, in terms of the original chosen basis the coefficients would transform as $a_1 = b_1$, $a_2 = \frac{b_1}{\epsilon^4} - \frac{b_2}{2\epsilon^3}$, $a_3 = b_3$, $a_4 = -\frac{b_1}{\epsilon^2} + \frac{b_2}{2\epsilon} + b_4$.

5.5 Contractions to the Heisenberg algebra

For contractions to nondegenerate or degenerate superintegrable systems formed from the Heisenberg algebra, our theorems concerning the potential do not apply, since the Heisenberg Hamiltonian is singular. In a paper to follow we will describe their forms. However, it is not difficult to work out each individual case and see that the induced contractions always exist.

Example 10 $S_9 \rightarrow$ Heisenberg algebra. We use $\{\mathcal{J}'_1 + i\mathcal{J}'_2, \mathcal{J}'_1 - i\mathcal{J}'_2, \mathcal{J}'_3\} = \{\epsilon(\mathcal{J}_1 + i\mathcal{J}_2), \mathcal{J}_1 - i\mathcal{J}_2, \epsilon\mathcal{J}_3\}$. with coordinate implementation $s_1 = \frac{\cos \phi}{\cosh \psi}$, $s_2 = \frac{\sin \phi}{\cosh \psi}$, $s_3 = \frac{\sinh \psi}{\cosh \psi}$, and substitutions $\phi = i\epsilon\alpha - i \ln \sqrt{\epsilon}$, $\psi = \xi\sqrt{\epsilon}$, to get $\mathcal{J}'_3 = -ip_\alpha$, $\mathcal{J}'_1 + i\mathcal{J}'_2 = -i(\xi p_\alpha + p_\xi)$, $\mathcal{J}'_1 - i\mathcal{J}'_2 = i(-\xi p_\alpha + p_\xi)$. The contraction from S_9 is $\mathcal{H}' = -p_\alpha^2$, $-\mathcal{L}'_1 = (\xi p_\alpha + p_\xi)^2 + c_1 \xi^2 + \frac{a_3}{\xi^2}$, $\mathcal{L}'_2 = p_\xi^2 - \xi^2 p_\alpha^2 + c_2 \xi^2 + \frac{a_3}{\xi^2}$, where the potential parameters a_j of S_9 contract as $a_1 = -c_1/18\epsilon^4 - c_2/8\epsilon^3$, $a_2 = -c_1/18\epsilon^4 + c_2/8\epsilon^3$. Note that there is no nonconstant potential V but there are potential-like terms in the remaining symmetry generators. The contracted system is exactly the same as one obtains from the ansatz

$$\mathcal{H}' = -p_\alpha^2 + V(\alpha, \xi), \quad \mathcal{L}'_1 = -(\xi p_\alpha + p_\xi)^2 + W_1(\alpha, \xi), \quad \mathcal{L}'_2 = p_\xi^2 - \xi^2 p_\alpha^2 + W_2(\alpha, \xi),$$

by requiring a nondegenerate quadratic algebra. The structure relation is $\mathcal{R}'^2 = 16 \left(\mathcal{L}'_1{}^2 \mathcal{H}' - (c_2 \mathcal{L}'_1 + c_1 \mathcal{L}'_2)(\mathcal{L}'_1 + \mathcal{L}'_2) - 4c_1 a_3 \mathcal{H}' + a_3(c_2 - c_1) \right)$.

6 Conclusions and discussion

The principal results obtained in this paper are as follows:

1. We showed that there is a one-to-one correspondence between conjugacy classes of quadratic algebras in the enveloping algebras of $\epsilon(2, \mathbb{C})$ and $\mathfrak{o}(3, \mathbb{C})$, and isomorphism classes of 2nd order superintegrable systems with potential on constant curvature spaces. In effect, these Lie algebras “know” the classical and quantum superintegrable systems they can produce. Thus, the associated classical orbits and quantum special functions and their properties are derivable from the Lie algebras, even though the superintegrable systems may exhibit no group symmetry whatsoever. Part of the proof was based on a classification of all quadratic algebras up to conjugacy, and we expect to find a more compact, direct proof in the future.

2. We showed that Lie algebra contractions of $e(2, \mathbb{C})$ and $o(3, \mathbb{C})$, which are few in number and have long since been classified, induce contractions of free quadratic algebras, and these in turn induce contractions of the corresponding classical and quantum superintegrable systems with potential. These algebraic contractions correspond to geometrical pointwise limiting processes in the physical models. The procedure is rigid and deterministic. As shown in [13], one of the consequences of contracting between superintegrable systems is a series of limiting relations between special functions associated with the superintegrable systems, a special case of which is the Askey scheme for hypergeometric orthogonal polynomials. Again, part of the conclusions are based on step-by-step classification, which we expect to replace with a more compact proof.

In follow-up papers we will extend these results to all 2nd order 2D superintegrable systems, including those on Darboux and Koenig spaces. We shall also classify abstract quadratic algebras and their contractions, including those not induced from Lie algebras, and study their relations with superintegrable systems.

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