Critical sets of 2-balanced Latin rectangles

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Abstract

An $(m, n, 2)$-balanced Latin rectangle is an $m \times n$ array on symbols 0 and 1 such that each symbol occurs n times in each row and m times in each column, with each cell containing either two 0's, two 1's or both 0 and 1. We completely determine the structure of all critical sets of the full $(m, n, 2)$-balanced Latin rectangle (which contains 0 and 1 in each cell). If $m, n \geq 2$, the minimum size for such a structure is shown to be $(m-1)(n-1)+1$. Such critical sets in turn determine defining sets for $(0,1)$-matrices.

Keywords: Full design, critical set, $(0,1)$-matrix, balanced Latin rectangle, Latin square.

1 Introduction

Since this paper deals with multisets, we first clarify our use of set theory notation. If we denote the multiplicity of an element s in a multiset A by $\nu_A(s)$, then some of the multiset notations are defined by the following *multiplicity functions*:

- $\nu_{A\cap B}(s) = \min\{\nu_A(s), \nu_B(s)\}$,
- $\nu_{A\cup B}(s) = \max\{\nu_A(s), \nu_B(s)\}$,
- $\nu_{A\setminus B}(s) = \max\{0, \nu_A(s) - \nu_B(s)\}$,

- $\nu_{A \uplus B}(s) = \nu_A(s) + \nu_B(s)$,

where $A \uplus B$ is the *multiset sum* of the multisets A and B. Finally, the size of a multiset is the sum of multiplicities of its elements.

For each natural number n, $[n]$ denotes the set $\{0, 1, 2, \dots, n-1\}$. An (m, n, t)-*balanced Latin rectangle* is a (possibly multi-)set of ordered triples $(r, c, s) \in [m] \times [n] \times [t]$, such that:

- for each $r \in [m]$ and $c \in [n]$, there are t triples of the form (r, c, s);
- for each $r \in [m]$ and $s \in [t]$, there are n triples of the form (r, c, s);
- for each $c \in [n]$ and $s \in [t]$, there are m triples of the form (r, c, s).

We represent such a structure in two ways. Firstly, given an (m, n, t)-balanced Latin rectangle R, we may construct an $m \times n$ array of multisets, with the set in cell (r, c) containing λ occurrences of element s if and only if the triple (r, c, s) has multiplicity λ in R. Thus we may think of an (m, n, t)-balanced Latin rectangle as an $m \times n$ array of multisets, each of size t, with each element from $[t]$ occurring m times in each column and n times in each row.

The second representation will be via an edge-coloured bipartite graph. Given an (m, n, t)-balanced Latin rectangle R, such a graph B_R has partite sets $[m]$ and $[n]$, with λ edges of colour s between vertices r and c whenever the triple (r, c, s) has multiplicity λ in R. We will switch freely between these equivalences in this paper, using whichever form makes proofs easier to follow.

We may trivially construct an (m, n, t)-balanced Latin rectangle for any $m, n, t \geq 1$ by placing the set $[t]$ in each cell of an $m \times n$ array. We call such a structure (which is equal to $[m] \times [n] \times [t]$) the *full* (m, n, t)-balanced Latin rectangle.

Henceforth in this paper we focus on the case $t = 2$ and we denote the full $(m, n, 2)$-balanced Latin rectangle by $F_{m,n}$. A *defining set* for an $(m, n, 2)$-balanced Latin rectangle R is some $D \subset R$ such that if R' is an $(m, n, 2)$-balanced Latin rectangle and $D \subset R'$, then $R' = R$. In other words, a defining set is some partially filled-in array with unique completion to an $(m, n, 2)$-balanced Latin rectangle. A *critical set* is a minimal defining set; i.e. deleting any element from D allows at least two completions. Our main goal in this paper is to completely determine the structure of any critical set of the full $(m, n, 2)$-balanced Latin rectangle. The precise structure is described in the next section.

We emphasize that deleting any element of a critical set of $F_{m,n}$ allows a completion to an $(m, n, 2)$-balanced Latin rectangle which is not equal to

$F_{m,n}$. We illustrate this example below: on the left is a critical set C of $F_{3,4}$; we give two completions of $C \setminus \{(0,3,1)\}$, one of which is $F_{3,4}$ and the other a $(3,4,2)$-balanced Latin rectangle not equal to $F_{3,4}$.

	1	1	1
	1	1	1
0			

0,1	0,1	0,1	0,1
0,1	0,1	0,1	0,1
0,1	0,1	0,1	0,1

1,1	0,1	0,1	0,0
0,1	0,1	0,1	0,1
0,0	0,1	0,1	1,1

This paper is motivated by the analogous concept of full designs (see [1, 8, 9, 10, 12, 13]). For block size k, a full design simply consists of all the possible subsets of size k from a foundation set $[v]$. In [9], it is shown that any minimal defining set for a design is the result of an intersection of the design with a minimal defining set of the full design of the same order.

The following similar result was shown in [6].

Theorem 1.1. *Let C be a defining set of the full (n,n,n)-Latin rectangle and let L be any Latin square of order n. Then $L \cap C$ is a defining set for L.*

By similar arguments, the critical sets in this paper can be used to identify defining sets for $(0,1)$-*matrices*. Given integral vectors R and S of orders m and n respectively, $\mathcal{A}(R,S)$ is the set of all $m \times n$ matrices with entries either 0 or 1 with row and column sums prescribed by R and S. We may think of a $(0,1)$-matrix as a set of ordered triples as above. A defining set for a $(0,1)$-matrix $A \in \mathcal{A}(R,S)$ is thus a subset $D \subseteq A$ such that if $D \subseteq A' \in \mathcal{A}(R,S)$ then $A = A'$.

Theorem 1.2. *Let C be a defining set of $F_{m,n}$ and let $A \in \mathcal{A}(R,S)$ be a $(0,1)$-matrix with R and S integral vectors of orders m and n, respectively. Then $A \cap C$ is a defining set for A.*

Proof. Suppose that $A \cap C$ is not a defining set for A. Then there exists a $(0,1)$-matrix $A' \in \mathcal{A}(R,S)$ such that $A' \neq A$ and $A \cap C \subset A'$. Let $T = A \setminus A'$ and $T' = A' \setminus A$. Since $A \cap C \subset A \cap A'$, $T \cap C = \emptyset$.

Now T and T' are partially filled-in, disjoint arrays with the same set of occupied cells, with 0 and 1 occurring the same number of times in each row and column. Consider the array given by $G_{m,n} := (F_{m,n} \setminus T) \uplus T'$. From the above properties of T and T', 0 and 1 occur n times in each row of $G_{m,n}$ and m times in each column of $G_{m,n}$. Thus $G_{m,n}$ is an $(m,n,2)$-balanced Latin rectangle. Furthermore since $T \cap C = \emptyset$, $C \subset G_{m,n}$. Since $G_{m,n} \neq F_{m,n}$, C is not a defining set for $F_{m,n}$, a contradiction. $\square$

The results in this paper thus have the potential to tell us much about defining sets in $(0,1)$-matrices.

In Section 2 we describe sets of partial $m \times n$ arrays $A[\boldsymbol{a}, \boldsymbol{b}]$ where $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors of non-negative integers satisfying certain properties. Using the theory of trades developed in Section 3, in Section 4 we show that each such array is a critical set of the full $(m,n,2)$-balanced rectangle $F_{m,n}$. In Section 5 we show that in fact all critical sets of $F_{m,n}$ are thus described (up to a reordering of rows and columns), thus completing the proof of our main result Theorem 2.1. Having completed our classification, in Section 6 we show that the size of the smallest critical set of $F_{m,n}$ is $(m-1)(n-1)+1$ whenever $m, n \geq 2$.

Section 2 of [6] determines the structure of any *saturated* critical set C of the full (m,n,t)-balanced rectangle for $t \geq 2$; here saturated means that each cell of C is either empty or contains $[t]$.

Theorem 1.3. ([6]) *A set C is a saturated critical set for the full (m,n,t)-balanced Latin rectangle if and only if the bipartite graph B is a tree, where B is on partite sets $[m]$ and $[n]$ with $i \in [m]$ and $j \in [n]$ adjacent if and only if cell (i,j) is empty in C.*

The results in this paper generalize this result in the case $t = 2$.

2 The general structure of a critical set

In this section, we describe arrays which we will ultimately classify all critical sets of the full $(m,n,2)$-balanced rectangle $F_{m,n}$ (up to a reordering of the rows and columns).

Henceforth, $(\boldsymbol{a}, \boldsymbol{b}) = ((a_1, a_2, \dots, a_k), (b_1, b_2, \dots, b_k))$ is always a pair of integral non-negative vectors such that $\sum_{i=1}^{k} a_i = m$ and $\sum_{i=1}^{k} b_i = n$. We use the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ to define partitions of the sets $[m]$ and $[n]$. For each $I \in [k]$, let $R_I = [\sum_{i=1}^{I} a_i] \setminus [\sum_{i=1}^{I-1} a_i]$ and $C_I = [\sum_{i=1}^{I} b_i] \setminus [\sum_{j=1}^{I-1} b_i]$. Note that if $a_I = 0$ ($b_I = 0$) then R_I (respectively, C_I) is empty.

Definition 2.1. *We define $A[\boldsymbol{a}, \boldsymbol{b}]$ to be the set of all $m \times n$ arrays A with the following structure. Let $r \in R_I$ and $c \in C_J$ where $I, J \in [k]$.*

- *If $I > J$, cell (r,c) of A contains entry 0.*
- *If $I < J$, cell (r,c) of A contains entry 1.*
- *If $I = J$, cell (r,c) of A is either empty or contains $\{0,1\}$, subject to the following. Let B_I be a bipartite graph with partite sets given by R_I and*

C_I, with edge $\{r,c\}$, $r \in R_I$, $c \in C_I$ existing if and only if cell (r,c) is empty in A. Then B_I is a tree.

In general there is more than one array in $A[\boldsymbol{a},\boldsymbol{b}]$ (since there are typically many choices for B_I). For example, the following are two elements of $A[(3,2,1),(3,1,3)]$:

0,1	0,1		1	1	1	1
0,1	0,1		1	1	1	1
			1	1	1	1
0	0	0		1	1	1
0	0	0		1	1	1
0	0	0	0			

0,1		0,1	1	1	1	1
0,1			1	1	1	1
		0,1	1	1	1	1
0	0	0		1	1	1
0	0	0		1	1	1
0	0	0	0			

We sometimes describe an array A in $A[\boldsymbol{a},\boldsymbol{b}]$ in terms of *blocks*. For each $I, J \in [k]$, the block $A_{I,J}$ is the subarray of A induced by the rows R_I and columns C_J. That is, $A_{I,J} = \{(r,c,s) \mid r \in R_I, c \in C_J, (r,c,s) \in A\}$. The blocks of the form $A_{I,I}$ are said to form the *main diagonal blocks*. Thus all cells below the main diagonal blocks contain 0 and all those above contain 1.

Definition 2.2. *We say that a pair of vectors $(\boldsymbol{a},\boldsymbol{b})$ is* good *if:*

(C1) *for each $i \in [k]$, $a_i > 0$ or $b_i > 0$;*

(C2) *if $a_i = 0$ then $a_{i-1} \geq 2$, $b_{i-1} \geq 1$ (if $i > 1$) and $a_{i+1} \geq 2$, $b_{i+1} \geq 1$ (if $i < k$); and*

(C3) *if $b_i = 0$ then $b_{i-1} \geq 2$, $a_{i-1} \geq 1$ (if $i > 1$) and $b_{i+1} \geq 2$, $a_{i+1} \geq 1$ (if $i < k$).*

For a good pair of vectors, observe that for each I at least one of R_I or C_I is non-empty.

The following theorem, the significance of which is that the critical sets of $F_{m,n}$ are precisely classified, is the main result of this paper. Its proof is given in Sections 4 and 5.

Theorem 2.1. *Let $(\boldsymbol{a},\boldsymbol{b})$ be a pair of good vectors. Then any element of $A[\boldsymbol{a},\boldsymbol{b}]$ is a critical set of the full $(m,n,2)$-balanced Latin rectangle $F_{m,n}$. Conversely, up to a reordering of the rows and columns, any critical set of $F_{m,n}$ is an element of $A[\boldsymbol{a},\boldsymbol{b}]$ for some pair $(\boldsymbol{a},\boldsymbol{b})$ of good vectors.*

In the example below we exhibit elements of $A[(2,0,2),(3,1,2)]$, $A[(3,1,0),(2,1,2)]$ and $A[(3,1),(2,3)]$, respectively. The pair of vectors in the centre array is not in fact good since $a_2 = 1$ and $a_3 = 0$. However it is a superset of the array on the right. We give the general version of this idea in the next lemma, which will be an important step in our classification.

0,1	0,1		1	1	1
			1	1	1
0	0	0	0		0,1
0	0	0	0		

0,1		1	1	1
		1	1	1
	0,1	1	1	1
0	0		1	1

0,1		1	1	1
		1	1	1
	0,1	1	1	1
0	0			

In the following lemma, $(\boldsymbol{a}, \boldsymbol{b})_i$ denotes the pair of vectors obtained by adding the ith element to the $(i+1)$th element (in each vector), decreasing the order of each vector by 1. Formally,

$$(\boldsymbol{a}, \boldsymbol{b})_i = ((a'_1, a'_2, \ldots, a'_{k-1}), (b'_1, b'_2, \ldots, b'_{k-1}))$$

where $a'_x = a_x$, $b'_x = b_x$ (if $x < i$); $a'_i = a_i + a_{i+1}$, $b'_i = b_i + b_{i+1}$; and $a'_x = a_{x+1}$, $b'_x = b_{x+1}$ (if $k > x > i$).

Lemma 2.1. *Suppose that there exist a pair of vectors $(\boldsymbol{a}, \boldsymbol{b})$ such that A is an array in $A[\boldsymbol{a}, \boldsymbol{b}]$. Then there exists a pair of good vectors $(\boldsymbol{a}', \boldsymbol{b}')$ and an array A' (obtained by deleting entries from A if needed) such that A' is an array in $A[\boldsymbol{a}', \boldsymbol{b}']$.*

Proof. Suppose that A is an array in $A[\boldsymbol{a}, \boldsymbol{b}]$, where the pair $(\boldsymbol{a}, \boldsymbol{b})$ is not good. We first make adjustments to the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

Satisfying (C1) is fairly trivial; simply delete from $(\boldsymbol{a}, \boldsymbol{b})$ any pairs $(a_i, b_i) = (0, 0)$.

We next show that consecutive 0's in either vector are essentially redundant. Formally, if $a_i = a_{i+1} = 0$ or $b_i = b_{i+1} = 0$ for some i, we replace $(\boldsymbol{a}, \boldsymbol{b})$ with $(\boldsymbol{a}, \boldsymbol{b})_i$. Repeat this step for each pair of consecutive 0's in either vector. The resultant vectors still describe A as in this section, however contain no consecutive 0's.

Next, suppose that $a_i = 0$ and either $a_{i-1} = 1$ or $b_{i-1} = 0$. The cells in row $\sum_{i=1}^{i-1} a_x - 1$ and columns from C_i each contain the entry 1 only. Delete 1 from each of these cells, making them empty. If $a_{i-1} = 1$, replace $(\boldsymbol{a}, \boldsymbol{b})$ (the latest pair of vectors) with $(\boldsymbol{a}, \boldsymbol{b})_{i-1}$. Otherwise $a_{i-1} > 1$ and $b_{i-1} = 0$; adjust $\boldsymbol{a}$ by decreasing a_{i-1} by 1 and incrementing a_i by 1, leaving $\boldsymbol{b}$ unchanged.

Next, suppose that $a_i = 0$ and either $a_{i+1} = 1$ or $b_{i+1} = 0$. The cells in rows $\sum_{i=1}^{i} a_x$ and columns from C_i each contain entry 0 only. Delete 0 from each of these cells, making them empty. If $a_{i+1} = 1$, replace $(\boldsymbol{a}, \boldsymbol{b})$ (the latest pair of vectors) with $(\boldsymbol{a}, \boldsymbol{b})_i$. Otherwise $a_{i+1} > 1$ and $b_{i+1} = 0$; adjust $\boldsymbol{a}$ by decreasing a_{i+1} by 1 and incrementing a_i by 1, leaving $\boldsymbol{b}$ unchanged.

The case when $b_i = 0$ is similar to above. Repeat this process for each violation of conditions (C2) and (C3). (Since whenever we delete an element from a cell we increase the dimensions of a block, the above algorithms terminate.) □

3 Trades in $(m, n, 2)$-balanced Latin rectangles

Understanding the structure of a critical set or defining set of any kind of combinatorial design typically involves an analysis of the *trades* in that combinatorial design (see, for example, [4]). Informally, a trade in an $(m, n, 2)$-balanced Latin rectangle R is a subset T which can be removed and replaced with a (disjoint) subset T' to create a distinct $(m, n, 2)$-balanced Latin rectangle R'.

Definition 3.1. *A* trade *in $F_{m,n}$ is some non-empty $T \subset F_{m,n}$ such that there exists a* disjoint mate *T' where $T' \cap T = \emptyset$ and $(F_{m,n} \setminus T) \uplus T'$ is an $m \times n$ balanced 2-rectangle (which is clearly not full). A trade T is said to be* minimal *if there does not exist a trade U such that $U \subset T$.*

As with other combinatorial designs, trades are intrinsically related to defining sets and critical sets.

Lemma 3.1. *The set D is a defining set of $F_{m,n}$ if and only if $D \subseteq F_{m,n}$ and D intersects every trade within $F_{m,n}$. A defining set D of $F_{m,n}$ is in turn a critical set if and only if, for each $(r, c, e) \in D$, there is a trade T in $F_{m,n}$ such that $T \cap D = \{(r, c, e)\}$.*

Proof. Suppose that $D \subset F_{m,n}$ and T is a trade within $F_{m,n}$ with disjoint mate T' such that $D \cap T = \emptyset$. Then $D \subset (F_{m,n} \setminus T) \uplus T'$, so D is not a defining set. Conversely, if D is not a defining set of $F_{m,n}$, there exists an $(m, n, 2)$-balanced Latin rectangle R not equal to $F_{m,n}$ such that $D \subset R$. Then $T = F_{m,n} \setminus R$ is a trade in $F_{m,n}$ with disjoint mate given by $R \setminus F_{m,n}$. Moreover T does not intersect D. The second statement in the lemma follows from the fact that a critical set is a minimal defining set. □

Let $B_{m,n}$ denote the bipartite edge-coloured graph corresponding to $F_{m,n}$ (see the Introduction). This can be thought of as a bipartite graph with partite sets $[m]$ and $[n]$ with one red edge and one blue edge (corresponding to entries 0 and 1, respectively) between each pair of vertices from different partite sets.

Consider the subgraphs B_T and $B_{T'}$ of $B_{m,n}$ equivalent to a trade T with disjoint mate T' (respectively) in $F_{m,n}$. Since $B_{T'}$ is obtained from B_T by switching the colours on each edge of B_T, it follows that each vertex in T is incident with the same number of blue and red edges. Suppose that there exists vertices v and w such that there exists both a red edge and a blue edge in T of the form $\{v, w\}$. Then there also exists an edge of each colour on $\{v, w\}$ in T', contradicting the fact that $T \cap T' = \emptyset$. It follows that B_T is the union of properly edge-coloured cycles (an even cycle is said to be *properly edge-coloured* if its edges are coloured alternately red and blue (i.e. each vertex in the cycle is adjacent to one red edge and one blue edge)). We call a properly edge-coloured even cycle (and the corresponding array) a *trade cycle* and we have proven the following.

Lemma 3.2. *Any trade within $F_{m,n}$ is the union of cell-disjoint trade cycles.*

Corollary 3.1. *Any minimal trade is a trade cycle.*

We next need the following graph theoretic result. To this end, two edges in a multigraph are *parallel* if they are distinct yet share the same pair of vertices. A *doubled cycle* refers to the multigraph obtained by replacing each edge in a cycle with a parallel pair of edges.

The following lemma is a generalization of the elementary result from graph theory that if two non-equal cycles share an edge, there is a cycle not including that edge. Note that cycles in this paper are simple graphs and thus a bipartite cycle has at least 4 edges.

Lemma 3.3. *Let G be a bipartite multigraph with each edge coloured either red or blue, satisfying the following properties.*

1. *There exist properly edge-coloured cycles C_1 and C_2 (each of even size) and edges e_1 and e_2 (from cycles C_1 and C_2, respectively) such that e_1 and e_2 are parallel.*

2. *Every edge of G belongs to either C_1 or C_2 (possibly both).*

3. *Any two parallel edges are coloured differently.*

Then either there exists a doubled cycle including the edges e_1 and e_2 or a properly edge-coloured cycle including neither e_1 nor e_2.

Proof. Suppose first that every edge belongs to a parallel pair. Since every edge belongs to a cycle and a cycle cannot contain parallel edges, for each edge $\{v, w\}$ in G, there is a red edge $\{v, w\}$ in C_i and a blue edge $\{v, w\}$ in C_{2-i}, where $i \in \{1, 2\}$. Therefore if we replaced each parallel pair with a single edge, we would obtain a cycle. Thus the entire graph G is a doubled cycle. Otherwise delete all parallel pairs of edges from G to create a non-empty bipartite graph G'. Since G is bipartite, it suffices to show that each vertex v of degree at least 1 in G' is incident with at least one red edge and at least one blue edge. Suppose, for the sake of contradiction, that this is false.

Without loss of generality, v is adjacent to a blue edge but not a red edge in G'. Then, since every edge of G belongs to a properly edge-coloured cycle, there is a pair of parallel edges in G including v. Since the maximum degree in G is 4, there is at most one such pair of edges. The blue edge in this parallel pair cannot belong to a cycle in G, a contradiction. □

Lemma 3.4. *Let D be a critical set of $F_{m,n}$. Suppose that cell (r, c) of D contains $\{0, 1\}$. Let $D' = D \setminus \{(r, c, 0), (r, c, 1)\}$. Let B_0 be the bipartite graph with partite sets given by $\{r_1, r_2, \ldots, r_m\}$ and $\{c_1, c_2, \ldots, c_n\}$ with an edge $\{r_i, c_j\}$ if and only cell (r_i, c_j) is empty in D'. Then B_0 contains a cycle which includes the edge $\{r, c\}$.*

Proof. By Lemma 3.1, since D is a critical set, each entry 0 and 1 in cell (r, c) belongs to a trade of $F_{m,n}$ which intersects D in only one element; let such trades be T_1 and T_2, respectively. From Lemma 3.2 and Corollary 3.1, we may assume T_1 and T_2 are trade cycles.

Construct a bipartite multigraph G (with edges coloured either red or blue) with partite sets $\{r_1, r_2, \ldots, r_m\}$ and $\{c_1, c_2, \ldots, c_n\}$, with an edge $\{r_i, c_j\}$ coloured red whenever $(r_i, c_j, 0) \in T_1 \cup T_2$ and an edge $\{r_i, c_j\}$ coloured blue whenever $(r_i, c_j, 1) \in T_1 \cup T_2$. Then G satisfies the conditions of Lemma 3.3, where cycles C_1 and C_2 correspond to T_1 and T_2, respectively, with e_1 and e_2 each on the vertices r and c.

If there exists a properly edge-coloured cycle in G including neither e_1 nor e_2, this corresponds to a trade in $F_{m,n}$ which does not include D, contradicting the fact that D is a defining set. Thus, by Lemma 3.3, there is a doubled cycle in G including the edge $\{r, c\}$. This corresponds to a cycle in B_0 which includes the edge $\{r, c\}$. □

4 Existence of critical sets

In this section we show that any element of $A[\boldsymbol{a}, \boldsymbol{b}]$ is indeed a critical set of the full $(m, n, 2)$-balanced Latin rectangle, where $(\boldsymbol{a}, \boldsymbol{b})$ is a good pair of vectors.

This proves the first part of Theorem 2.1. We make use of the theory of trades developed in the previous section.

Lemma 4.1. *Let $(\boldsymbol{a}, \boldsymbol{b})$ be a good pair of vectors. Let A be an array in $A[\boldsymbol{a}, \boldsymbol{b}]$ and suppose that cell (r, c) of A contains 0 only. Then there exists a row $r' < r$ and a column $c' > c$ satisfying one of the following cases:*

1. *$(r', c, 0) \in A$ and $(r, c'), (r', c')$ are each empty;*
2. *$(r, c', 0) \in A$ and $(r', c), (r', c')$ are each empty;*
3. *$(r', c', 1) \in A$ and $(r, c'), (r', c)$ are each empty; or*
4. *(r', c') is empty and $(r', c, 0), (r, c', 0) \in A$.*

Proof. We illustrate the above cases below.

$$\begin{array}{c|c|c|} & c & c' \\ \cline{2-3} r' & 0 & \\ \cline{2-3} r & 0 & \\ \cline{2-3} \end{array} \qquad \begin{array}{c|c|c|} & c & c' \\ \cline{2-3} r' & & \\ \cline{2-3} r & 0 & 0 \\ \cline{2-3} \end{array} \qquad \begin{array}{c|c|c|} & c & c' \\ \cline{2-3} r' & & 1 \\ \cline{2-3} r & 0 & \\ \cline{2-3} \end{array} \qquad \begin{array}{c|c|c|} & c & c' \\ \cline{2-3} r' & 0 & \\ \cline{2-3} r & 0 & 0 \\ \cline{2-3} \end{array}.$$

Let $r \in R_I$ and $c \in C_J$; by definition $I > J$ (see Section 2). First, suppose that both $|R_J| \geq 1$ and $|C_I| \geq 1$. Then selecting $r' \in R_J$ and $c' \in C_I$ such that (r', c) and (r, c') are empty, cell (r', c') must contain 1. This results in Case 3 above.

Next, suppose that $|R_J| = 0$. Since $I > J$, $J < k$, so since $(\boldsymbol{a}, \boldsymbol{b})$ is a good pair of vectors, $|R_{J+1}| \geq 2$ and $|C_{J+1}| \geq 1$. If $I = J + 1$, since B_{J+1} is a tree and is thus connected, there exists $r' \in R_{J+1}$ and $c' \in C_{J+1}$ such that (r', c') and (r, c') are each empty. Then cell (r', c) lies in block $A_{I,J}$ and thus contains entry 0 only. This results in Case 1. If $|R_J| = 0$ and $I > J + 1$, then let (r', c') be an empty cell in block $A_{J+1,J+1}$. Then (r, c') and (r', c) are in blocks $A_{I,J+1}$ and $A_{J+1,J}$, each of which contain only 0's. Case 4 follows.

Otherwise $|C_I| = 0$. Then $|C_{I-1}| \geq 2$ and $|R_{I-1}| \geq 1$. Block $A_{I,I}$ is empty and each other block of the form $A_{I,J'}$ with $J' < I$ contains only 0's. Suppose that $J = I - 1$. Thus, since B_{I-1} is a tree, there is a row $r' \in R_{I-1}$ and a column $c' \neq c$ in C_{I-1} such that (r', c') and (r', c) are empty. Since (r, c') contains only entry 0 we have Case 2. If $J < I - 1$, let $c' \in C_{I-1}$ and let $r' \in R_{I-1}$ where (r', c') is empty. Then (r, c') and (r', c) each only contain entry 0, so we have Case 4. □

The following theorem proves one part of Theorem 2.1.

Theorem 4.1. *Let A be an element of $A[\boldsymbol{a}, \boldsymbol{b}]$. Then A is a critical set of the full $(m, n, 2)$-balanced Latin rectangle.*

Proof. Suppose first that A is not a defining set of the full $(m, n, 2)$-balanced Latin rectangle. Then there exists an $(m, n, 2)$-balanced Latin rectangle $F' \neq F_{m,n}$ such that $A \subset F'$. Let $T = F_{m,n} \setminus F'$ and $T' = F' \setminus F_{m,n}$. Observe that the non-empty cells of T (and T') are precisely the cells of F' which contain either $\{0, 0\}$ or $\{1, 1\}$.

Without loss of generality, let (i, j) be a cell of F' containing $\{0, 0\}$. Then, since the total number of 0's and 1's in each row and column is fixed, there exists a column $j' \neq j$ such that (i, j') is a cell of F' containing $\{1, 1\}$. Similarly, there exists a row $i' \neq i$ such that (i', j') is a cell of F' containing $\{0, 0\}$.

By finiteness there exists a list of distinct cells

$$(i(1), j(1)), (i(1), j(2)), (i(2), j(2)), \ldots, (i(\mu), j(\mu)), (i(\mu), j(\mu + 1) = j(1))$$

where $(i(a), j(a))$ contains $\{0, 0\}$ (in F') and $(i(a), j(a+1))$ contains $\{1, 1\}$ (in F') for each $a \in N(\mu)$.

Because of the structure of A, for each $a \geq 1$ either $i(a) \in R_I$ and $i(a+1) \in R_J$ for some I and J with $J > I$ *or* cells $(i(a), j(a+1))$ and $(i(a+1), j(a+1))$ belong to the same main diagonal block. Moreover, either $j(a) \in C_I$ and $j(a + 1) \in C_J$ for some $J > I$ *or* cells $(i(a), j(a))$ and $(i(a), j(a + 1))$ belong to the same main diagonal block. Since $\mu > 1$, either $i(\mu) \in R_J$ and $i(1) \in R_I$ with $I < J$ or the entire trade is contained within a main diagonal block. If the former holds, cell $(i(\mu), j(1))$ contains 0 only (in A), a contradiction, and if the latter holds there is a cycle in the graph B_I (for some I), also a contradiction.

We next show that A is a minimal defining set. That is, we remove each entry from A and show that it is no longer a defining set.

Case 1: The cell (r, c) contains one element. By symmetry we may assume without loss of generality, that $(r, c, 0) \in A$ and let $A' = A \setminus \{(r, c, 0)\}$. In each of the Cases 1 to 4 given in Lemma 4.1, $F_{m,n} \setminus A'$ contains a trade cycle (on the four cells given by that lemma). So by Lemma 3.1, A' is not a critical set of $F_{m,n}$.

Case 2: The cell (r, c) contains two elements. Thus (r, c) belongs to a block of the form $A_{I,I}$. Since B_I is a tree, there is a trade cycle using either $(r, c, 0)$ or $(r, c, 1)$ and entries in cells which are empty in A.

By Lemma 3.1, A is a critical set of $F_{m,n}$.

□

5 A classification of critical sets

In Theorem 4.1 we showed that any element of $A[\boldsymbol{a},\boldsymbol{b}]$ is a critical set of $F_{m,n}$, where $(\boldsymbol{a},\boldsymbol{b})$ is a pair of good vectors. In this section we will show that *any* critical set of $F_{m,n}$ is thus described, up to a reordering of rows and columns. We will thus complete the proof of Theorem 2.1.

In what follows, let D be a critical set of $F_{m,n}$. Let B_0 be the bipartite graph with partite sets $V_1 = \{1, 2, \ldots, m\}$ and $V_2 = \{1, 2, \ldots, n\}$ with an edge from $r \in V_1$ to $c \in V_2$ if and only if cell (r, c) is empty in D. Then any cycle in B_0 gives rise to a trade cycle in $F_{m,n}$ which does not intersect D (see Section 3), contradicting Lemma 3.1. It follows that B_0 is a forest.

Suppose there are ℓ components (i.e. trees) of the graph B_0, where $\ell \geq 0$. Partition the rows and columns of D into $\ell + 1$ sets $R_1, R_2, \ldots, R_{l+1}$ and $C_1, C_2, \ldots, C_{l+1}$ so that the i^{th} tree is on vertex set $R_i \cup C_i$, for each i. Thus R_{l+1} and C_{l+1} contain any vertices which do not belong to trees.

Let $A_{I,I}$ be the block formed by the intersection of rows R_I and columns C_I, where $1 \leq I \leq l$. We shall call these blocks the *main blocks*. The definition of a tree implies that if (r, c) is a non-empty cell within a main block, adding the edge $\{r, c\}$ creates a cycle within B_0. Thus if cell (r, c) contains only one entry, there is a trade cycle which does not intersect D; it follows that each non-empty cell of a main block contains $\{0, 1\}$. Thus in each main block, each cell either contains $\{0, 1\}$ or is empty, with the empty cells forming a tree.

Next, Lemma 3.4 implies that there are no cells containing $\{0, 1\}$ outside of the main blocks. Thus any cell outside of the main blocks has exactly one entry (0 or 1). We strengthen this result in the following lemma.

Lemma 5.1. *Let $1 \leq I \leq l$. Let $r, r' \in R_I$ and $c \notin C_I$. Then cells (r, c) and (r', c) contain the same entry in D. Let $c, c' \in C_I$ and $r \notin R_I$. Then cells (r, c) and (r, c') contain the same entry in D.*

Proof. Suppose, for the sake of contradiction, that $(r, c, 0), (r', c, 1) \in D$ where r, r' are distinct rows in R_I and $c \notin C_I$. Then there is a unique path P in B_0 from r to r'. Adding the edges $\{r, c\}$ and $\{r', c\}$ to this path creates a cycle. In particular there is a trade cycle in $F_{m,n} \setminus D$ (including $(r, c, 1)$ and $(r', c, 0)$), contradicting the fact that D is a defining set. The second observation is proven similarly. □

The following lemma has a very similar proof which we omit.

Lemma 5.2. *Let (r, c), (r', c') belong to distinct main blocks. Then cells (r, c') and (r', c) contain distinct entries. Let (r, c) belong to a main block but suppose* *that cells (r,c'), (r',c) and (r',c') do not belong to a main block, with (r',c) and (r,c') containing the same entry $e \in \{0,1\}$. Then (r',c') also contains e.*

For the next step, we exploit some known results on $(0,1)$-matrices. To this end, we transform D to two $(0,1)$-matrices D_0 and D_1, as follows. For each cell belonging to a main block, replace its contents (whether full or empty) with the entry e to create D_e, where $e \in \{0,1\}$.

A *trade* in a $(0,1)$-matrix M is a subset T such that there exists a disjoint mate T' such that $(M \setminus T) \cup T'$ is a $(0,1)$-matrix with the same column and row sums as M. Observe that any $(0,1)$-matrix $m \times n$ is in fact a subset of $F_{m,n}$. Indeed, any trade in a $(0,1)$-matrix is a disjoint union of trade cycles ([3]; an equivalent result is also shown by Lemma 3.2.1 of [2]).

We once again turn to a coloured edge bipartite representation to make our proof easier to explain.

Lemma 5.3. *Let G be a bipartite multigraph with each edge coloured either red or blue. Let W be a closed walk in G such that consecutive edges in the walk have different colours. Suppose there exists a pair of vertices $\{v,w\}$ such that there is a unique edge in W on these vertices. Then W contains a properly 2-coloured cycle.*

Proof. Recursively delete any closed sub-walks from W not containing the edge which occurs uniquely. What remains must be a properly 2-coloured cycle. □

Lemma 5.4. *Let $e \in \{0,1\}$. If there is a trade in D_e there is a trade in $F_{m,n} \setminus D$.*

Proof. Suppose there is a trade T in D_e. Then T contains a trade cycle C. Since T contains occurrences of both entry 0 and entry 1 and cells in a main block inside D_e contain entry e, there is at least one cell of T outside a main block (containing entry $1-e$). $(*)$

List the non-empty cells of C in a list L so that consecutive elements in the list share either a row or column. Now consider the list of cells L with respect to D. We adjust the list L to create a new list L' as follows. Each cell in the list L either remains unchanged or is replaced by a sublist. Firstly, any cells outside of the main blocks of D remain unchanged. Whenever cell $(r,c) \in L$ lies within a main block, if that cell is empty leave it unchanged. Otherwise, since B_0 is a forest, there is a unique path P on edges in B_0 from vertex r to c which forms a cycle if edge $\{c,r\}$ is appended. Replace (r,c) in the list L with the cells corresponding to P.

Observe that the resultant list of $2k$ cells $L' = l_1, l_2, \ldots, l_{2k}$ has the following properties. Firstly, any consecutive cells, in particular l_1 and l_2, either lie in the same row or column. Assume the former without loss of generality. From $(*)$, we may also assume without loss of generality that cell l_1 contains the entry $1-e$ only. Thus, if i is even and $i \geq 2$ then l_i lies in the same row as l_{i-1}. If i is odd and $i \geq 3$ then l_i lies in the same column as l_{i-1}. We also have that l_1 lies in the same column as l_{2k}. Finally, if i is even then cell l_i is either empty or contains e and if i is odd then cell l_i is either empty or contains $1-e$. $(**)$.

We now construct a closed walk W with edge sequence $w_1, w_2, \ldots, w_{2k}$, based on the above list of cells. If $l_i = (r_i, c_i)$, then edge $w_i = \{r_i, c_i\}$. Colour the edges of W alternately with two colours. From $(*)$, there exists an edge not repeated in W. Thus from the above lemma, W contains a properly 2-coloured cycle. From $(**)$, there is a trade in the corresponding cells of $F_{m,n} \setminus D$, a contradiction. □

From the previous lemma we may conclude that D_e has no trades, for each $e \in \{0, 1\}$. This is useful because it is well-known that a $(0,1)$-matrix has no trades (and is thus the unique member of its class $\mathcal{A}(R, S)$) if and only if its rows and columns can be rearranged so that a line of non-increasing gradient can be drawn with all the 0's below and the 1's above. This statement of the Gale-Ryser theorem (see [2]) is given as Lemma 3 in [3].

Since D_0 contains no trades, we may rearrange the rows and columns of D_0 so that there exist a line l_0 of non-increasing gradient such that each cell below line l_0 contains 0 and each cell above contains 1. Apply the same rearrangement to the rows and columns of D.

From this property of l_0, for each pair of rows $\{r_1, r_2\}$ and each pair of columns $\{c_1, c_2\}$ such that $r_1 < r_2$ and $c_1 < c_2$, we must have one of the following 2×2 subarrays in D_0:

$$
\begin{array}{c|c|c|}
 & c_1 & c_2 \\ \hline
r_1 & 0 & 0 \\ \hline
r_2 & 0 & 0 \\ \hline
\end{array}
\quad
\begin{array}{c|c|c|}
 & c_1 & c_2 \\ \hline
r_1 & 0 & 1 \\ \hline
r_2 & 0 & 0 \\ \hline
\end{array}
\quad
\begin{array}{c|c|c|}
 & c_1 & c_2 \\ \hline
r_1 & 0 & 1 \\ \hline
r_2 & 0 & 1 \\ \hline
\end{array}
\quad
\begin{array}{c|c|c|}
 & c_1 & c_2 \\ \hline
r_1 & 1 & 1 \\ \hline
r_2 & 0 & 0 \\ \hline
\end{array}
\quad
\begin{array}{c|c|c|}
 & c_1 & c_2 \\ \hline
r_1 & 1 & 1 \\ \hline
r_2 & 0 & 1 \\ \hline
\end{array}
\quad
\begin{array}{c|c|c|}
 & c_1 & c_2 \\ \hline
r_1 & 1 & 1 \\ \hline
r_2 & 1 & 1 \\ \hline
\end{array}.
$$

We next show that we can further rearrange the rows and columns so there are no "gaps" between the main blocks and the line l_0. We express this formally in the following claim.

Claim: If (r, c) is a cell from a main block and (r, c') contains 0 in D where $c' > c$, we can swap columns c and c' in D and D_0, preserving the properties of the line l_0. Similarly, if (r, c) is a cell from a main block and (r', c) contains 0 in D where $r' < r$, we can swap columns r and r' in D and D_0, preserving the properties of the line l_0.

So if our claim holds, we may assume that the rows and columns within each main block are contiguous and that the line l_0 includes the upper edge and right-hand-side edge of each main block. It follows that there exist a pair of vectors $(\boldsymbol{a}, \boldsymbol{b})$ such that $D \in A[\boldsymbol{a}, \boldsymbol{b}]$. Lemma 2.1 implies that D is the superset of a critical set $D' \in A[\boldsymbol{a}', \boldsymbol{b}']$ where $\boldsymbol{a}'$ and $\boldsymbol{b}'$ are good vectors. This completes the proof of Theorem 2.1.

So it remains to prove the above claim. We prove only the first part of the claim as the second follows by a similar (transpose) argument. To this end, let (r, c) be a cell from a main block and suppose there exists a column $c' > c$ such that (r, c') does not belong to a main block and (r, c') is below the line l_0 (thus $(r, c', 0) \in D \cap D_0$ and $(r, c, 0) \in D_0$). Our aim is to show that for each $r \neq r'$, either (i) (r', c) belongs to a main block (indeed the same main block as (r, c) and $(r', c, 0) \in D$ *or* (ii) both (r', c) and (r', c') contain the same entry in D. The claim then follows.

Let $r' \neq r$. First suppose that (r', c) belongs to a main block. If (r', c') also belongs to a main block, then so does (r, c'), a contradiction. Otherwise (r', c') does not belong to a main block and $(r', c', 0) \in D$ (by Lemma 5.1). This proves (i).

Otherwise (r', c) does not belong to a main block. If (r', c') belongs to a main block, Lemma 5.2 forces (r', c) to contain 1. Since (r, c), (r, c') and (r', c') each contain 0 in D_0 and $c < c'$, we cannot have one of the six possible 2×2 configurations above, a contradiction. If (r', c) contains 1 and (r', c') contains 0 then we again cannot have one of the six possible 2×2 configurations above. If (r', c) contains 0 and (r', c') contains 1 then Lemma 5.2 is contradicted. This proves (ii).

6 The smallest critical set

Having now classified the structure of any critical set in the full $(m, n, 2)$-balanced Latin rectangle $F_{m,n}$, we can now determine the smallest possible size of such a structure.

To this end, for $m, n > 1$ we define $R^1_{m,n}$ and $R^2_{m,n}$ be the unique elements of $A[(m-1, 1), (1, n-1)]$ and $A[(1, m-1), (n-1, 1)]$, respectively. Below is the partial array $R^1_{3,4}$:

	1	1	1
	1	1	1
0			

From Theorem 4.1, both $R^1_{m,n}$ and $R^2_{m,n}$ are critical sets of $F_{m,n}$. Observe that they each have size $(m-1)(n-1)+1$. We next show that $R^1_{m,n}$ and $R^2_{m,n}$ are critical sets of $F_{m,n}$ of minimum size and are unique in this property.

Lemma 6.1. *If $m, n > 1$ then the size of the smallest critical set of the full $(m, n, 2)$-balanced Latin rectangle is $(m-1)(n-1)+1$. Up to a reordering of rows and columns, $R^1_{m,n}$ and $R^2_{m,n}$ are the unique critical sets with this property (unless $m = n = 2$).*

Proof. Let C be a critical set of the full $(m, n, 2)$-balanced Latin rectangle and let e be the number of empty cells in C. From Theorem 2.1, the graph B_0 corresponding to the empty cells forms a forest, so $e \leq m+n-1$. If $e = m+n-1$, then B_0 is a tree on $m+n$ vertices. Thus each non-empty cell contains 2 elements and $|C| \geq 2(mn-m-n+1) \geq (m-1)(n-1)+1$, with equality only possible in the case $m = n = 2$. Otherwise, $e \leq m+n-2$ and $|C| \geq mn-m-n-2 = (m-1)(n-1)+1$, with equality only possible if B_0 has two components and no cells containing $\{0, 1\}$.

Next, suppose that C is a critical set of $F_{m,n}$ of size $(m-1)(n-1)+1$ where $(m, n) \neq (2, 2)$. From above, B_0 has precisely two components, each of which is a complete bipartite graph and thus a star. So $C \in A[(a_1, a_2), (b_1, b_2)]$ and either $a_1 = b_2 = 1$ or $a_2 = b_1 = 1$. □

Observe that in the case $m = n = 2$, any element of $A[(2), (2)]$ also gives a critical set of minimum possible size.

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