Using quotitive division problems to promote
place-value understanding

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A robust understanding of place value is essential. Using a problem-based approach set within meaningful contexts, students' attention may be drawn to the multiplicative structure of place value. By using quotitive division problems through a concrete–representational–abstract lesson structure, this study showed a powerful strengthening of Year 3 students' conceptual understanding of place value.

Place value is commonly introduced to young children using 'bundling into tens' activities. In this article, we share a problem-based approach and a lesson structure implemented with a class of seven-year-olds that focused on the multiplicative structure of place value. Children were given a variety of quotitive division problems set in meaningful contexts and encouraged to use multiple representations to support their learning. Quotitive division problems involving groups of ten provided opportunities for students to make connections between the concrete and symbolic representations for two- and three-digit numbers.

Background

Part-whole thinking, or partitioning, is a key component of mathematical reasoning reflected in the content of current curriculum documents. It is a "big idea" identified by Baroody (2004, p. 199) and is foundational for understanding concepts such as place value. In order to develop the concept of place value, a student must have an understanding of part-whole relationships and four key properties: (1) positional (the quantity is represented by the position of a digit within a multi-digit number); (2) base-ten (numbers increase in powers of ten from right to left); (3) multiplicative (the value of each digit is its place value multiplied by its face value); and (4) additive (the total is represented by the sum of the values of the individual digits) (Ross, 1989). Children's early reasoning relies primarily on additive properties but it becomes important that they develop understanding of part-whole relationships using multiplicative structures and ideas about grouping, equal composing, and decomposing into equal-sized parts (Baroody, 2004).

Introducing multiplication and division

Multiplication and division can be introduced to students from a younger age than is often the case in practice. Children are provided with many opportunities at pre-school and early in their schooling to work with units of one, and to partition numbers or decompose quantities into parts, and to solve addition and subtraction problems using counting strategies. However, a focus on multiplication and division problem-solving provides an opportunity to move from counting by ones to concepts of the group, equivalence, and the unit (Sophian, 2012). For example, Sophian explains that children may see a collection of six shoes as six discrete items or as three pairs (groups of two as composite units). She argues that although the multiplicative relationship is more complex, "young children are capable of reasoning both additively and multiplicatively about relations between quantities" (p. 170). In order to understand multiplication, children need to appreciate how the whole can be composed from, or decomposed into, equal-sized parts (Baroody, 2004). Likewise, decomposing a whole into equal-size groups (equal partitioning) is the conceptual basis of division. Multiplicative thinking is also important for developing mathematical understanding in the domains of proportio and ratio, measurement, and place value.

A clear implication from the literature is that children's mathematics learning can be supported by the introduction of multiplication and division early

in their schooling. There is substantial evidence that children prior to school age are able to work with equal-group multiplication and fair-share division (see, for example, Baroody, Lai, & Mix, 2006; Wright et al., 2014). Hence, it makes sense to capitalise on that prior knowledge in the mathematics classroom. It is also reasonable to expect that the experience of working with units greater than one may help children to develop part-whole thinking sooner than may be otherwise the case.

There are several different types of multiplication and division situations in the real world that are accessible to children as young as five years of age. These everyday experiences of composite units include pairs of eyes, shoes, gloves, and wheels on bicycles (twos), legs on animals and tables (fours), fingers on hands and toes on feet (fives), chocolates or biscuits in trays, and eggs in cartons (tens). These contexts can be used for reading, writing, talking about, and solving problems. Also, provided the language is simple, the numbers are small, and resources are readily available to model the problems, young children are capable of solving multiplication and division problems (Fuson, 2004). Students can solve the problems by acting them out, modelling them with objects, or by drawing simple pictures and diagrams. They can then be supported and encouraged to solve and represent the problems using drawings that show the quantities in a more abstract way, and moving, for example, from tally marks to a ten-frame and then to the written equation. Additionally, they can be given opportunities to solve problems using larger numbers.

In order to develop place-value understanding, students need to be familiar with the concept of the unit and recognise the difference between units of ten and units of one. This signals a move from thinking about numbers as units of one to composite units (e.g., groups of ten) and is a key feature of place-value understanding. To think about numbers in this way, students need to be part-whole thinkers (Ross 1989). Developing an understanding of multi-digit structures is a gradual process and is commonly approached through the use of 'bundling into tens' activities and structured base-ten materials. However, like the Cognitively Guided Instruction [CGI] model (Carpenter, Fennema, & Franke, 1996), the teaching of place value, reported in this article, uses a conceptual problem-based approach.

The study

In this article, we share some work extracted from a two-year study that explored the use of multiplication and division problem-solving contexts to develop part-whole thinking with young children. Whole-number division problems can be classified as either partitive (equal sharing) or quotitive (measurement) division. When young children solve a partitive division problem, they share out a quantity often using singletons with the action of "one for you, one for you" etc. whereas in quotitive division, groups of a specified quantity (the divisor) are extracted from the initial total (the dividend). Lessons in the larger study focused on solving multiplication problems with groups of two, five, and ten, and quotitive division problems with two, five, and ten as the divisor. The lessons from which the following ideas were extracted focused on quotitive division problems that involved students making 'groups of ten'.

The lesson structure

We provided a culturally diverse class of Year 3 students (seven year-olds), from a provincial city in New Zealand, with a series of two-digit quotitive division problems over a four-week period (12 lessons altogether). A common lesson structure began with the teacher providing a number knowledge warm-up activity on the interactive whiteboard. For example, the Numbers to Forty on the Ten Frame, Grade One number sense and equivalence activity (http://www.dreambox.com/teachertools), was used alongside individual laminated sets of ten-frames, where students practised combining tens and ones to form a two-digit number. A number between 21 and 40 was quickly revealed and then hidden on the screen. Students were directed to represent the number they had seen using their individual sets of ten-frames. The screen image then revealed the number represented in ten-frames and the students checked that their representation was a match. Using this particular activity (with individual sets of ten-frames) ensured all students participated in the task. This was followed by discussion about differences in answers and the organisation of the ten-frames. The activity supported students' recognition of a quantity and their ability to hold a given number in their mind.

The warm-up activity was followed by the teacher posing one problem for the whole class to work on. The students were provided with manipulatives to help solve the problem and then they collectively shared their strategies. Prior to the lesson, the teacher had entered the problem on the interactive whiteboard. The ways in which the students had modelled the solution were recorded by the children and/or the teacher on the board. The teacher then encouraged and supported the students in providing a range of equations that reflected their thinking (Figure 1).
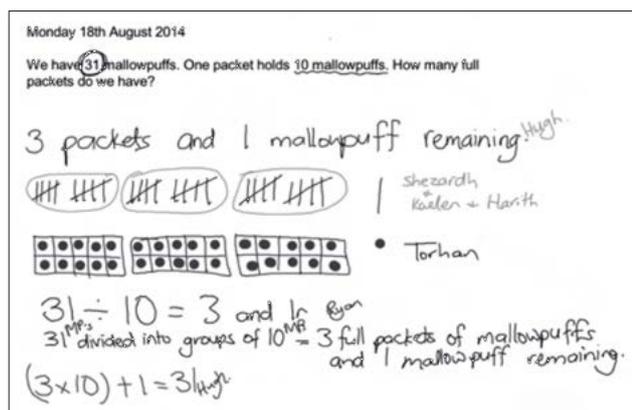
Figure 1. The class problem and students' solutions recorded on the interactive whiteboard.

The students then worked on parallel word problems that had been copied and pasted into individual student books. The follow-up problem used the same context as the previous whole-class problem, but the students were given a choice of numbers. Students were expected to model the problem with manipulatives (if needed) or to draw it, and then record their problem-solving strategies and equations using multiple representations. At the end of the lesson, the teacher used a reflective circle where the students shared what they had enjoyed doing in the lesson and talked about the numbers they had used in their problems. This 'circle time' also gave an opportunity for students to share their 'aha' moments, or what had helped them solve their problems, especially when they had found it challenging. They could also show some of their thinking with drawings or equations recorded on the whiteboard. These were retained by the teacher for future reference and formative assessment.

The problems, contexts, and manipulatives

The problems for groups of ten were based around meaningful real-life contexts such as the eggs in a carton (with ten compartments), chocolates in a tray, and biscuits in a packet. We began the lessons with problems that used multiples of ten and then added quantities that had remainders. The students could choose the quantity for their independent problems from a choice of three; the last choice used empty brackets for self-selection (Figure 2). To model the problem, they could use Unifix cubes in egg cartons or counters on a ten-frame (Figure 3). Some of the students moved to a greater level of abstraction when they no longer needed to use discrete objects to represent each 'one' in a group of ten, but instead used an

empty egg carton to represent ten. For example, when working on a problem where the student began with a 3-digit number, he set out the egg cartons using a row of ten cartons to represent each 'one hundred'.

Class problem (multiples of ten)
There are 30 eggs. Each carton holds 10 eggs. How many full cartons are there?
Student problem (multiples of ten)
There are 20 [50] [ ] eggs. Each carton holds 10 eggs. How many full cartons are there?
Class problem (multiples of ten with remainder)
There are 24 chocolates. Each tray holds 10 chocolates. How many full trays are there?
Student problem (multiples of ten with remainder
There are 13 [27] [ ] chocolates. Each tray holds 10 chocolates. How many full trays are there

Figure 2. Quotitive division problems provided for the whole class and follow-up independent problems for students.

Figure 3. Digit labels are used to show the number of full egg cartons (units of ten) and remainder (units of one) for 25 ÷ 10.

Student representations

Representations play an important role in the development of mathematical concepts (Goldin & Shteingold, 2001). Representations may include drawings, graphs, diagrams, equations, and computer-generated images. The students' representations in this study included modelling, drawings, and equations, and were made in individual project books or at 'circle time' on the interactive whiteboard. The teacher would usually

place the student names next to their recorded contributions. These contributions could be referred back to, as examples for the students and as evidence of individual student input to class problem-solving and student strategies. The students showed their 'groups of ten' in different ways. Some students used tally marks whereas others showed less sense of structure and drew a set of discrete items that they then circled into groups of ten. Several children drew ten-frames, either horizontally or vertically, and placed circles in each compartment. One student shared a 'smart' strategy for creating a ten-frame using tally marks. He constructed a rectangle (longer side placed horizontally) and then divided it up with four vertical lines and finally one horizontal line. In earlier lessons, a few students had struggled when drawing a ten-frame as they tended to place five vertical lines in the rectangle followed by a horizontal line, so creating a 12-frame.

The students were encouraged to "work like a mathematician" and record equations using symbols. The idea of remainder had been introduced earlier when they were working with quotitive division problems with groups of two, and so they were familiar with recording using the symbol 'r'. This helped them to appreciate the difference between even and odd numbers, with the latter quantity resulting in a 'left over' unpaired item. Acknowledging the remainder was most salient for these quotitive-division-by-ten problems. We drew the students' attention to the total number of tens and ones by reiterating: "How many full cartons are there?" and "How many eggs are left over?" We wanted the students to make the connections between the digit standing for the tens value and the cartons of ten eggs, and likewise, the 'ones' digit with the left over eggs standing for the remainder.

Other connections were made between division, repeated subtraction, repeated addition, and multiplication. Students were encouraged to check their solutions by recording equations derived from their solution (Figure 4). For example, if they had five full egg cartons, how many eggs were there altogether? They then added on the remainders (left-overs) and checked for a match with the original total or dividend.

Results and insights

A pre- and post-diagnostic interview assessment was used to measure student learning in the domain of number knowledge, including sequencing, subitising, and number fact recall. We also assessed the operations of addition, subtraction, multiplication, and division problem-solving, as well as pattern and structure. The students made notable improvements in multiplication facts for groups of 5 and 10 (2×5, 2×10, 3×10, 4×10) and division into groups of 10 ($60 \div 10$, $80 \div 10$). This improvement was also evident in the task asking how many 'bunches' of 10 sticks could be made from a bag containing 60 sticks, and how many $10 notes would be needed to buy a toy costing $80. There was substantial improvement in division of 23 by 10 (division with remainder), and considerable improvement in the number of children connecting the '2' in '24' with two groups of 10 objects, and making $31 using $10 notes and $1 coins. These tasks were used to measure 'tens awareness' as an indication of early place-value understanding.

We collected all student workbooks and had video recorded all lessons, so together these multiple sources of data meant we had powerful evidence to support student sense-making and development of place-value understanding. The process of student self-selection of numbers showed that they were keen to work with two-digit numbers, and a few students extended to three-digit numbers.
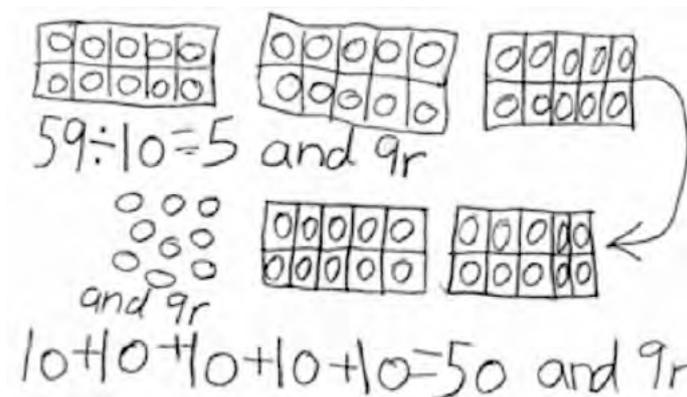

Figure 4. Student representation for solving the problem: There are 59 eggs. Each carton holds 10 eggs. How many full cartons are there?

This study showed how problem solving using meaningful contexts, and particularly quotitive division problems with ten as the divisor, was very powerful for strengthening students' conceptual understanding of place value. The process of solving the problems using manipulatives meant that the focus was placed on the 'groups of ten', and for each group of ten, the single units could still be seen. However, student recognition of a new unit, ten, was established and for many students this became an abstraction. When students recorded the problems and solutions with symbols, place-value connections were made between each digit (face value), its place, and total value. Likewise, recording the division equations followed by related multiplication equations helped to develop understanding of the inverse operation.

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Acknowledgements