Generalised domain and E-inverse semigroups

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Abstract

A generalised D-semigroup is here defined to be a left E-semiabundant semigroup S in which the $\overline{\mathcal{R}}_E$-class of every $x \in S$ contains a unique element $D(x)$ of E, made into a unary semigroup. Two-sided versions are defined in the obvious way in terms of $\overline{\mathcal{R}}_E$ and $\overline{\mathcal{L}}_E$. The resulting class of unary (bi-unary) semigroups is shown to be a finitely based variety, properly containing the variety of D-semigroups (defined in an order-theoretic way in Communications in Algebra, 3979–4007, 2014). Important subclasses associated with the regularity and abundance properties are considered. The full transformation semigroup T_X can be made into a generalised D-semigroup in many natural ways, and an embedding theorem is given. A generalisation of inverse semigroups in which inverses are defined relative to a set of idempotents arises as a special case, and a finite equational axiomatisation of the resulting unary semigroups is given.

Keywords. E-semiabundant semigroup, D-semigroup, regular semigroup.

1 Introduction

1.1 Preliminaries

In what follows, S is a semigroup and $E(S)$ its set of idempotents. The *natural right quasiorder* on $E(S)$ (and hence on any subset E of $E(S)$) is given by $e \leq_r f$ if and only if $e = ef$, and the *natural left quasiorder* on $E(S)$ is given by $e \leq_l f$ if and only if $e = fe$. Denote by $\sim_r$ and $\sim_l$ the respective induced equivalence relations. The *natural order* $\leq$ on $E(S)$ is the intersection of $\leq_l$ and $\leq_r$, so that $e \leq f$ if and only if $e = ef = fe$, and is a partial order on $E(S)$.

Let S be a semigroup with $E \subseteq E(S)$. If the elements of E commute with one-another we say E *commutes*; if also E is closed under multiplication then E is a semilattice under the semigroup operation, with the associated partial order being the natural order. E is said to be *right reduced* if $e = ef$ implies $e = fe$, *left reduced* if $e = fe$ implies $e = ef$, and *reduced* if it is both. So E is right reduced if $\leq_r \subseteq \leq_l$, left reduced if the opposite inclusion holds, and reduced if the two quasiorders are equal. If E commutes then it is reduced. E is *right pre-reduced* if $\leq_r$ is partial order (so that $e = ef$ and $f = fe$ imply $e = f$), *left*

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pre-reduced if $\leq_l$ is partial order, and *pre-reduced* it is both. Obviously, if E is (left/right) reduced, then it is (left/right) pre-reduced (although the converses fail).

A $*$-semigroup is a semigroup with involution $x \mapsto x^*$, meaning that the laws $(xy)^* = y^*x^*$ and $x^{**} = x$ hold. A *projection* in a $*$-semigroup S is $e \in E(S)$ for which $e^* = e$; denote by $E^*(S)$ the set of all projections in S, a reduced set of idempotents as is easily seen.

1.2 The approach via order

In [11], the authors studied unary (and sometimes bi-unary) semigroups S equipped with a domain-like (and sometimes also a range-like) operation D (and possibly R) such that $D(s)$ lies in a multiplicative semilattice $E \subseteq E(S)$ for all $s \in S$ and is the smallest (under the natural order) $e \in E$ for which $es = s$ (and dually for $R(s)$ if it is defined); it then follows easily that $D(S) = \{D(s) \mid s \in S\} = E$. The resulting unary (bi-unary) semigroups form a finitely based variety, called the class of C-semigroups (two-sided C-semigroups). This concept appeared in earlier work [3] where the unary semigroups were called type SL γ-semigroups, and subsequently in [8] where the term left E-semiadequate semigroups was used, and in [13] where they were called guarded semigroups. A particular two-sided case was considered in some detail in [12]. A ring-theoretic analog was considered in [7]. Examples are many and varied, but notably include the semigroup of partial transformations P_X of a set X, with E chosen to be all restrictions of the identity map, and then $D(f)$ is the restriction of the identity map defined only on the domain of f. Conversely, the class of unary semigroups embeddable in such examples is the class of left restriction semigroups, a class by now considered by many authors.

But there is reason to go further. In many naturally occurring semigroups, there is a set of idempotents E of a semigroup S which is is not a semilattice nor even closed under multiplication, yet for which, for each $s \in S$, there is a smallest $e \in E$ under the natural order, say $D(s)$, for which $es = s$; when this happens it again follows that $D(S) = E$. Abstract unary semigroups of this sort are called D-semigroups in [16]. Important examples come from the multiplicative semigroups of rings arising in functional analysis, such as Rickart $*$-rings, where the elements of the form $D(s)$ are precisely the projections. In any D-semigroup S, the set of idempotents $D(S)$ is left reduced; indeed in a Rickart $*$-ring, $D(S) = E^*(S)$, which is reduced. In [16], a finite equational axiomatisation of D-semigroups as a class of unary semigroups is given, generalising the commuting case of C-semigroups given in [11]. two-sided versions of all these concepts are also considered in [16].

1.3 The approach via left E-semiabundance

A seemingly unrelated approach to defining an idempotent-valued "domain-like" unary operation on a semigroup comes from consideration of certain generalised Green's relations. For a semigroup S and non-empty $E \subseteq E(S)$, the equivalence relation $\overline{\mathcal{R}}_E$ on S is obtained by setting

$$(x, y) \in \overline{\mathcal{R}}_E \text{ if for all } e \in E,\ ex = x \ \Leftrightarrow\ ey = y.$$

This relation may be viewed as a generalisation of Green's relation $\mathcal{R}$ and indeed of $\mathcal{R}^*$ (defined in detail shortly), and in fact $\mathcal{R} \subseteq \mathcal{R}^* \subseteq \overline{\mathcal{R}}_E$ on any semigroup, with all three coinciding if the semigroup is regular. The semigroup S is said to be *left E-semiabundant* if each $\overline{\mathcal{R}}_E$-class in S contains an element of E. This terminology was introduced in [16], as a one-sided version of the concept considered by Lawson in [12]. Left E-semiabundance generalises regularity and indeed (left) abundance. One can define $\overline{\mathcal{L}}_E$ and right E-semiabundance in a dual manner, with $\mathcal{L} \subseteq \mathcal{L}^* \subseteq \overline{\mathcal{L}}_E$ on any semigroup.

Earlier, Fountain in [5] considered the case of right adequate semigroups (those in which every $\mathcal{L}^*$-class contains an idempotent) in the case that $E(S)$ is a semilattice. If S is right adequate, then $\mathcal{L}^* = \overline{\mathcal{L}}_{E(S)}$, and the fact that $E(S)$ is a semilattice forces uniquness of the idempotent in each $\overline{\mathcal{L}}_{E(S)}$-class. It was noted in [11] that a right C-semigroup S is nothing but a right E-semiabundant semigroup in which E is assumed to be a semilattice, in which case each $s \in S$ has a unique $D(s) \in E$ in its $\overline{\mathcal{R}}_E$-class. In [8], left E-semiabundance is called weak left E-abundance, and the generalised left restriction semigroups considered there are nothing but left E-semiabundant semigroups in which E is assumed to be a band and every $\overline{\mathcal{R}}_E$-class contains a unique element of E. (So left C-semigroups are exactly generalised left restriction semigroups in which E is a semilattice.)

It was also noted in [16] that a D-semigroup is simply a left E-semiabundant semigroup in which E is left reduced, a condition that ensures that the $\overline{\mathcal{R}}_E$ class of $s \in S$ contains a unique element of E, which happens to be $D(s)$ as defined earlier. In fact Gould's generalised left restriction semigroups are special cases of these, since for bands, the property of being left reduced is equivalent to that of it being left regular ($efe = ef$ for all e, f), a property of $D(S)$ in a generalised left restriction semigroup that was observed in [8].

As noted in [8], when E is a band, the left regularity (equivalently for bands, the left reduced) property of E is necessary and sufficient for uniqueness of elements of E in each $\overline{\mathcal{R}}_E$-class of a left E-semiabundant semigroup. However, if E is not a band, the left reduced property is stronger than necessary for uniqueness; see Corollary 2.2. If we simply require that each $\overline{\mathcal{R}}_E$-class in a left E-semiabundant semigroup S contains a unique element of E, and define the unary operation D based on this, we call the resulting unary semigroup a *generalised D-semigroup.* If also each $\overline{\mathcal{L}}_E$-class contains a unique element of E and we define the unary operation R accordingly, the result is a *generalised DR-semigroup.* These turn out to be properly more general than the D-semigroups and DR-semigroups considered in [16].

One of the main topics we consider is *E-inverse semigroups*, defined here for the first time. These are regular semigroups having a distinguished set of idempotents E such that each s has a unique inverse s' with respect to the constraint that both $ss' \in E$ and $s's \in E$. These arise in various settings; for example if $E = E(S)$, we recover the definition of inverse semigroups, and more generally if S is a $*$-semigroup and $D(S) = E^*(S)$, we recover the $*$-regular semigroups in the sense of Drazin [4]. E-inverse semigroups turn out to furnish examples of generalised DR-semigroups (that are not necessarily DR-semigroups) if one defines $D(s) = ss'$ and $R(s) = s's$, and then $D(S) = E$, so we can make use of results for generalised DR-semigroups to study them. Indeed such E-inverse semigroups provided much of the motivation for the author to define and study the generalisation of D-semigroups and DR-semigroups considered here.

1.4 Content of the paper

In the next section, we begin by assembling some needed facts about the (generalised) Green's relations used (namely $\mathcal{R}$, $\mathcal{R}^*$, $\overline{\mathcal{R}}_E$ and their left-sided versions) and left (and two-sided) E-semiabundant semigroups, including the special cases of left E-abundant and left E-regular semigroups. We then show that the class of generalised D-semigroups forms a finitely based variety, inside which the variety of D-semigroups properly sits, and we give a (quasi)order-theoretic characterisation generalising the definition of D-semigroups as in [16]. Various conditions forcing a generalised D-semigroup to be a D-semigroup are considered, and conversely it is shown that there are semigroups that can be generalised D-semigroups but not D-semigroups.

In Section 3, the regularity and left abundance conditions are considered in the generalised D-semigroup setting: S is D-regular if $\mathcal{R} = \overline{\mathcal{R}}_{D(S)}$ (equivalently, D is such that $D(s)$ is the unique element of $D(S)$ that is $\mathcal{R}$-related to s) and D-abundant if $\mathcal{R}^* = \overline{\mathcal{R}}_{D(S)}$ (replace $\mathcal{R}$ by $\mathcal{R}^*$ in the previous comment). The class of D-abundant generalised D-semigroups is shown to be a proper quasivariety of unary semigroups. Under any choice of idempotent from each $\mathcal{R}$-class, the full transformation semigroup T_X becomes a D-regular (and hence D-abundant) generalised D-semigroup, and conversely we show that every D-abundant generalised D-semigroup embeds in T_X (for some choice of X), viewed as a generalised D-semigroup in such a way.

In Section 4, the special case of E-inverse semigroups, defined above, is considered. These are shown to form a variety of I-semigroups (unary semigroups with a generalised inversion operation satisfying $xx'x = x$ and $x'' = x$), properly containing the variety of inverse semigroups.

Throughout this paper, unless otherwise stated, S is a semigroup and $E \subseteq E(S)$ a non-empty set of idempotents.

2 Generalised D-semigroups

Throughout this section, S is a semigroup with $E \subseteq E(S)$ a non-empty set of idempotents.

2.1 Generalised Green's relations

Recall Green's relation $\mathcal{R}$ given by $(x, y) \in \mathcal{R}$ providing $xS^1 = yS^1$. Here, S^1 denotes the monoid obtained by adjoining an identity element to the semigroup S. The principal left ideal generated by $x \in S$ is $xS^1 = \{xs \mid s \in S\} \cup \{x\}$.

The generalised Green's relation $\mathcal{R}^*$ was introduced in [6]. The original definition of this relation on a semigroup S is that $(x, y) \in \mathcal{R}^*$ if and only if there is some oversemigroup of S in which $(x, y) \in \mathcal{R}$ on that semigroup. But as is well-known, $\mathcal{R}^*$ may equivalently be defined as follows: $(x, y) \in \mathcal{R}^*$ providing

$$\text{for all } a, b \in S^1,\ ax = bx \text{ if and only if } ay = by.$$

As mentioned previously, the equivalence relation $\overline{\mathcal{R}}_E$ on S is defined by setting

$$(x, y) \in \overline{\mathcal{R}}_E \text{ if for all } e \in E,\ ex = x \ \Leftrightarrow\ ey = y.$$

As is easy to see and has been observed elsewhere, $\mathcal{R} \subseteq \mathcal{R}^* \subseteq \overline{\mathcal{R}}_E$.

The relations $\mathcal{L}$, $\mathcal{L}^*$ and $\overline{\mathcal{L}}_E$ are defined dually to $\mathcal{R}$, $\mathcal{R}^*$ and $\overline{\mathcal{R}}_E$ respectively, and so $\mathcal{L} \subseteq \mathcal{L}^* \subseteq \overline{\mathcal{L}}_E$.

For idempotent elements, these relations take on the same simple form, as the following elementary result shows (see Corollary 2.7 in [8] for example).

Proposition 2.1 *Suppose $e, f \subseteq E(S)$. The following are equivalent:*

1. $(e, f) \in \mathcal{R}$;
2. $(e, f) \in \mathcal{R}^*$;
3. $(e, f) \in \overline{\mathcal{R}}_E$ *for any E containing e, f;*
4. $e \sim_l f$.

Corollary 2.2 *If θ is any of the relations just considered, then each θ-class of S has at most one member of E if and only if E is left pre-reduced.*

The following fact has been observed previously, for example as Lemma 2.9 in [8].

Proposition 2.3 *For all $s \in S$ and $e \in E$, $(s, e) \in \overline{\mathcal{R}}_E$ if and only if $es = s$ and if $fs = s$ for some $f \in E$ then $e \leq_l f$.*

There is a largest $H \subseteq E(S)$ for which $\overline{\mathcal{R}}_E = \overline{\mathcal{R}}_H$: just take the union of all those $F \subseteq E(S)$ for which $\overline{\mathcal{R}}_E = \overline{\mathcal{R}}_F$. Call this set $\overline{E}$.

Proposition 2.4 $\overline{E} = \{e \in E(S) \mid e \sim_l f$ *for some* $f \in E\}$.

Proof. Let $H = \{e \in E(S) \mid e \sim_l f$ for some $f \in E\}$. Clearly $E \subseteq H$, so $\overline{\mathcal{R}}_H \subseteq \overline{\mathcal{R}}_E$. Conversely, suppose $(x, y) \in \overline{\mathcal{R}}_E$. If $ex = x$ for some $e \in H$, then there exists $f \in E$ such that $e \sim_l f$, so $ef = f$ and $fe = e$, and hence $fx = f(ex) = (fe)x = ex = x$, so $fy = y$ since $(x, y) \in \overline{\mathcal{R}}_E$, Hence $ey = e(fy) = (ef)y = fy = y$. By symmetry, if $ey = y$ for some $e \in H$ then $ex = x$. Hence $(x, y) \in \overline{\mathcal{R}}_H$. So $\overline{\mathcal{R}}_E = \overline{\mathcal{R}}_H$ and so $H \subseteq \overline{E}$.

Conversely, pick $e \in \overline{E}$. Then there is $f \in E$ such that $(e, f) \in \overline{\mathcal{R}}_E = \overline{\mathcal{R}}_{\overline{E}}$, so because $e, f \in \overline{E}$, we have that $e \sim_l f \in E$ by Proposition 2.1. So $\overline{E} \subseteq H$. □

2.2 Regularity, abundance and their generalisations

We say S is *left E-abundant* if every $\mathcal{R}^*$-class of S contains an element of E. E-abundance is defined in the obvious two-sided way. If $E = E(S)$, we recover the familar definitions of left and two-sided abundance. Similarly, we define *left E-regularity* and *E-regularity* in the analogous ways in terms of $\mathcal{R}$ and $\mathcal{L}$. (Note that E-regularity is defined in different and element-wise way in [9], a matter we return to in Section 4.) Again, we recover regularity when $E = E(S)$, and in this case left = right = two-sided regularity (though this does not work for all choices of E as we observe later).

Following [12], S was defined in [16] to be *left E-semiabundant* if each $\overline{\mathcal{R}}_E$-class contains an element of E. The term *left semiabundance* applied to S refers to the case $E = E(S)$, and we often write $\overline{\mathcal{R}}$ rather than $\overline{\mathcal{R}}_{E(S)}$. There are obvious notions of right E-semiabundance and E-semiabundance. (This property is called weak left E-abundance in [8].)

The following facts are well-known in case $E = E(S)$.

Proposition 2.5 *If the semigroup S is left E-abundant then $\overline{\mathcal{R}}_E = \mathcal{R}^*$, and if S is left E-regular then $\overline{\mathcal{R}}_E = \mathcal{R}^* = \mathcal{R}$.*

Proof. Generally, $\mathcal{R}^* \subseteq \overline{\mathcal{R}}_E$. Conversely, if S is left E-abundant and $(x, y) \in \overline{\mathcal{R}}_E$, then there are $e, f \in E$ for which $(x, e), (y, f) \in \mathcal{R}^*$. So $(x, e), (y, f) \in \overline{\mathcal{R}}_E$, yet $(x, y) \in \overline{\mathcal{R}}_E$, so $(e, f) \in \overline{\mathcal{R}}_E$, and so $(e, f) \in \mathcal{R}^*$ by Proposition 2.1. So $(x, e), (e, f), (f, y) \in \mathcal{R}^*$, and so $(x, y) \in \mathcal{R}^*$. The argument for the left E-regular case is very similar. □

The semigroup $S = \{0, a, 1\}$, in which 0 is a zero, 1 is an identity element and $a^2 = 0$, has $E(S) = \{0, 1\}$. It is commutative, so left=right for all our generalised Green's relations (whatever choice of E is made). It is easy to see that S is not abundant since the $\mathcal{L}^*$-classes are singletons yet a is not idempotent, so nor is S regular. However, S is semiabundant, with $\overline{\mathcal{R}} = \overline{\mathcal{L}}$ giving the partition $\{a, 1\}, \{0\}$. S is therefore *semiamiable* in the sense of [2], meaning that every $\overline{\mathcal{L}}$ and every $\overline{\mathcal{R}}$-class contains a unique element of $E(S)$; indeed $E(S)$ commutes so S is ($E(S)$-)semiadequate.

For convenience, we now list out the main definitions used in the remainder of the paper, that have just been discussed. Thus if S is a semigroup with $E \subseteq E(S)$, S is:

- *left E-semiabundant* if every $\overline{\mathcal{R}}_E$-class in S contains a member of E;
- *left E-semiadequate* if it is left E-semiabundant and E is a semilattice;
- *left E-abundant* if every $\mathcal{R}^*$-class in S contains a member of E;
- *left E-regular* if every $\mathcal{R}$-class contains a member of E.

There are right-sided versions of each definition above, and if "E" is omitted, it means that E is taken to be $E(S)$.

2.3 Defining generalised D-semigroups

We now turn to the generalisations of D-semigroups that are the main concern of this paper. From Corollary 2.2, we have the following.

Corollary 2.6 *If S is left E-semiabundant, then each $\overline{\mathcal{R}}_E$-class contains a unique element of E if and only if E is left pre-reduced.*

So, if S is a left E-semiabundant semigroup, we may define a unary operation D which for each $s \in S$ picks out the unique $e \in E$ in its class if and only if E is left pre-reduced. In this case we say S is a *generalised D-semigroup*, and we write $D(S) = \{D(s) \mid s \in S\} = E$. The reason for the name is straightforward: the D-semigroups considered in [16] were shown in Proposition 1.3 there to be nothing but left E-semiabundant semigroups in which E is left reduced, a stronger property in general than being left pre-reduced.

All of our facts about left E-semiabundant semigroups have dual versions for right E-semiabundant semigroups. Likewise, we may define *generalised R-semigroups* dually to generalised D-semigroups, and then we use the symbol R for the unary operation ("range" rather than "domain"), and define $R(S)$ as expected. All results to follow have dualised versions.

A given semigroup S may simultaneously be both a generalised D-semigroup and a generalised R-semigroup with respect to the same choice of $E \subseteq E(S)$; if so, then we say S is a *generalised DR-semigroup.*

Not every left E-semiabundant semigroup has a compatible D-semigroup structure (as we show shortly). However, we do have the following.

Proposition 2.7 *Suppose S is left E-semiabundant. There exists a generalised D-semigroup structure on S which is such that $D(S) \subseteq E$, and $\overline{\mathcal{R}}_{D(S)} = \overline{\mathcal{R}}_E$, so that $(x,y) \in \overline{\mathcal{R}}_E$ if and only if $D(x) = D(y)$, and the posets $(D(S), \leq_l)$ and $(E, \leq_l)/\sim_l$ are isomorphic. Namely, select precisely one element of E from each $\overline{\mathcal{R}}_E$-class to form $D(S)$ and then define $D(x) = e \in D(S)$ whenever $(x,e) \in \overline{\mathcal{R}}_E$.*

Proof. For each $\overline{\mathcal{R}}_E$-class, form F by picking (using the Axiom of Choice) exactly one element of E from each $\overline{\mathcal{R}}_E$-class. By Proposition 2.1, those elements of E deleted in this process are necessarily all $\sim_l$-related to elements of F, so $F \subseteq E \subseteq \overline{F}$ and so $\overline{\mathcal{R}}_{\overline{F}} \subseteq \overline{\mathcal{R}}_E \subseteq \overline{\mathcal{R}}_F = \overline{\mathcal{R}}_{\overline{F}}$, so in particular $\overline{\mathcal{R}}_E = \overline{\mathcal{R}}_F$. Now make S a generalised D-semigroup by defining $D(x)$ be the unique $e \in F$ in the $\overline{\mathcal{R}}_F$-class of $x \in S$. The rest is clear. □

There is an order-theoretic characterisation of generalised D-semigroups, generalising the definition of D-semigroups as in [16].

Proposition 2.8 *If for all $s \in S$ there is a smallest $e \in E$ under $\leq_l$ for which $es = s$, then S is a generalised D-semigroup in which $D(S) = E$. Conversely, if S is a generalised D-semigroup then $D(S)$ is such that for all $s \in S$, $D(s)$ is the smallest $e \in D(S)$ under $\leq_l$ for which $es = s$. In this case $D(S)$ is left pre-reduced.*

Proof. Suppose that for all $s \in S$ there is a smallest $e \in E$ under $\leq_l$ for which $es = s$; call it e_s. By Proposition 2.3, e_s is the unique element of E in the $\overline{\mathcal{R}}_E$-class containing s. Hence S is a generalised D-semigroup in which $D(S) = E$ if we define $D(s) = e_s$ for each $s \in S$.

Conversely, if S is a generalised D-semigroup then for all $x \in S$, there is a unique $e \in D(S)$ such that $(x,e) \in \overline{\mathcal{R}}_E$, namely $D(x)$, which has the given description by Proposition 2.3.

If $e, f \in D(S)$ and $ef = f$ and $fe = e$ then $e \sim_l f$ so by Proposition 2.1, $(e,f) \in \overline{\mathcal{R}}_E$, and so $e = f$. Hence $D(S)$ is left pre-reduced. □

If S is a D-semigroup, $D(S)$ is left reduced; equivalently, $\leq_l$ coincides with $\leq$ on $D(S)$. So $D(s)$ is the smallest $e \in D(S)$ under the natural order for which $es = s$; indeed this was how D-semigroups were defined in [16].

A finite equational axiomatisation of D-semigroups exists (see Propositions 1.1 and 1.2 in [16]). From the previous result, it is relatively easy to write down a quasi-equational axiomatisation for generalised D-semigroups. But we can do better.

Proposition 2.9 *Let S be a unary semigroup with unary operation D. Then S is a generalised D-semigroup if and only if D satisfies the following equational laws: for all $x, y \in S$, and $e, f \in D(S) = \{D(s) \mid s \in S\}$.*

1. $D(x)x = x$;
2. $D(x)D(xy) = D(xy)$;
3. $D(e) = e$;
4. $D(D(ef)e) = D(ef)$.

Proof. First, assume S is a generalised D-semigroup. Then for all $x \in S$, $D(x)x = x$, so $D(x)(xy) = xy$ for any $y \in S$, so $D(xy) \leq_l D(x)$, giving the second law. Since $D(e)$ is the unique element of $D(S)$ in the $\overline{\mathcal{R}}_E$-class containing $e \in D(S)$, we have $D(e) = e$. For the final law, note that $D(ef)(D(ef)e) = D(ef)e$, so $D(D(ef)e) \leq_l D(ef)$ by Proposition 2.8, and conversely,

$$D(D(ef)e)ef = D(D(ef)e)(D(ef)ef) = D(D(ef)e)(D(ef)e)f = D(ef)ef = ef,$$

so $D(ef) \leq_l D(D(ef)e)$ by Proposition 2.8. Since $\leq_l$ is a partial order on $D(S)$, we obtain the final law.

Now assume the laws. From the first law, we obtain $D(D(x))D(x) = D(x)$, so from the third, $D(x)^2 = D(x)$. Hence $D(S) = \{D(s) \mid s \in S\} \subseteq E(S)$. If $e \in D(S)$ is such that $ex = x$, then $eD(x) = D(e)D(ex) = D(ex) = D(x)$ by the second law, so $D(x) \leq_l e$. To show it is unique with this property, it suffices to prove the left pre-reduced property of $D(S)$ since then $\leq_l$ will be a partial order. Assume $e, f \in D(S)$ are such that $ef = f$ and $fe = e$. But then using the fourth law, we obtain $f = D(f) = D(ef) = D(D(ef)e) = D(fe) = D(e) = e$. □

The variety of D-semigroups as in [16] satisfies the additional law $D(xy)D(x) = D(x)$ which ensures that $D(S)$ is left reduced, and then the final law above is redundant. Generalised DR-semigroups are axiomatized by the above laws, together with the obvious dual laws involving the operation R, and connecting laws to ensure $D(S) = R(S)$, namely, $D(R(x)) = R(x)$, $R(D(x)) = D(x)$. The generalised left restriction semigroups as in [8] are precisely generalised D-semigroups in which $D(S)$ is a band, and so may be axiomatized by the above laws plus the law $D(D(x)D(y)) = D(x)D(y)$ (although a different axiomatisation is given in [8]).

We note that in [14], the author considers semigroups satisfying a "superabundance" property, in which there is a distinguished set of idempotents E such that every equivalence class determined by the relation $\overline{\mathcal{L}}_E \cap \overline{\mathcal{R}}_E$ contains a (necessarily unique) element of E. The unique idempotent x^+ in the class containing x is then a "smallest two-sided identity" in E for x: $x^+x = xx^+ = x$. This and a finite number of other laws are shown to axiomatize unary semigroups equipped with this operation. The generalised D-semigroups and DR-semigroups considered here are different, because the smallest left and right identities of an element can differ, and uniqueness is not forced by either left E-semiabundance or right E-semiabundance. A given left E-semiabundant semigroup can in general have many different generalised D-semigroup structures associated with it, uniqueness only being forced when an assumption about E is made (for example, the assumption that E is a semilattice).

In summary, we have three equivalent ways of viewing a generalised D-semigroup:

- as a semigroup S with some set of idempotents E for which, for all $x \in S$, the $\overline{\mathcal{R}}_E$-class containing x contains a unique element $D(x)$ of E (equivalently, S is left E-semiabundant and E is left pre-reduced), and then $E = \{D(s) \mid s \in S\}$;
- as a semigroup S equipped with a distinguished set of idempotents E such that for all $x \in S$ there exists $D(x) \in E$ for which $D(x)x = x$ and which is smaller under $\leq_l$ than all other $f \in E$ for which $fx = x$, and then $E = \{D(s) \mid s \in S\}$;
- as a unary semigroup satisfying the laws in Proposition 2.9.

Generalised D-semigroups also arise as left E-semiabundant semigroups in which we (randomly) choose one element of E from each $\overline{\mathcal{R}}_E$-class and define D accordingly; E is not completely determined by the generalised D-semigroup structure in this case, although $\overline{E}$ as in Proposition 2.4 obviously is, and so is the structure of the poset $(D(S), \leq_l)$.

Example 2.10 *Let S be the band with the following multiplication table.*

$\cdot$	a	b	d	e	f
a	a	d	d	a	a
b	e	b	b	e	b
d	a	d	d	a	d
e	e	b	b	e	e
f	a	d	d	a	f

It is routine if tedious to verify that $(S, \cdot)$ is a semigroup. The Green's relation $\mathcal{R}$ gives the partition $\{a, d\}, \{b, e\}, \{f\}$, while $\mathcal{L}$ gives $\{a, e\}, \{b, d\}, \{f\}$. So letting $E = \{d, e, f\}$, E is left pre-reduced and indeed left reduced as is easily seen, so $\leq_l$ is the natural order on E, with $d < f$. Hence S is a D-semigroup in which $D(S) = E$, with $D(a) = d$ and $D(b) = e$. Also, E is right pre-reduced but not right reduced, and $\leq_r$ is a partial order with $d, e <_r f$. So S is also a generalised R-semigroup (but not *an R-semigroup) with respect to E, in which $R(a) = e$ and $R(b) = d$. Hence S is a generalised DR-semigroup with $D(S) = R(S) = E$. This shows that the notion of generalised D-semigroup (DR-semigroup) is properly more general than that of D-semigroup (DR-semigroup).*

2.4 When generalised D-semigroups are D-semigroups

In a D-semigroup S, $D(S)$ is left reduced, and in a DR-semigroup it is reduced. This property of being (left) reduced is fairly common: for example, if $D(S)$ is a semilattice this holds, or if S has an involution and $D(S) \subseteq E^*(S)$, as is the case in many of the ring-theoretic examples considered in [16].

We next consider two other cases in which the left pre-reduced set E is automatically left reduced.

Proposition 2.11 *Assume that E is left pre-reduced. If $E = E(S)$ or E is a band, then it is left reduced.*

Proof. Suppose E is left pre-reduced with $E = E(S)$ or E is a band. Suppose $e, f \in E$ with $ef = f$. Then $f(fe) = fe$ and $(fe)f = ff = f$.

If $E = E(S)$, then $fefe = ffe = fe$, so $fe \in E(S)$, and so $fe = f$ by the left pre-reduced property of E. If E is a band, then $fe \in E$, and the left pre-reduced property of E again implies that $fe = f$.

Hence E is left reduced. □

Dualising, we obtain the following.

Corollary 2.12 *Assume that E is pre-reduced. If $E = E(S)$ or E is a band, then it is reduced.*

We have the following easy consequence for generalised D-semigroups (and their two-sided versions, generalised DR-semigroups).

Corollary 2.13 *Suppose S is a generalised D-semigroup (generalised DR-semigroup) in which $D(S) = E(S)$ or $D(S)$ is a band. Then S is a D-semigroup (DR-semigroup).*

We define the class of *D-semiamiable* (*DR-semiamiable*) generalised D-semigroups (generalised DR-semigroups) to be those for which $D(S) = E(S)$ ($D(S) = R(S) = E(S)$). These were called *full* in the D-semigroup case considered in [16]. By the previous corollary, these are always D-semigroups (DR-semigroups).

As already noted, if $D(S)$ is a band then we recover the generalised left restriction semigroups considered by Gould in [8], which are therefore always D-semigroups. It follows from Proposition 3.3 in [8] that a band is left reduced if and only if it is left regular.

Example 2.14 *Recall that if a semigroup S is abundant, then $\mathcal{L}^* = \overline{\mathcal{L}} = \overline{\mathcal{L}}_{E(S)}$ and $\mathcal{R}^* = \overline{\mathcal{R}} = \overline{\mathcal{R}}_{E(S)}$. In [1], the authors give the following example of an abundant semigroup in which the idempotents are unique in each $\mathcal{L}^*$-class and $\mathcal{R}^*$-class (the author calls such a semigroup* amiable*) but do not commute: $S = \{a, b, c, d\}$, where*

$\cdot$	a	b	c	d
a	a	c	c	c
b	d	b	c	d
c	c	c	c	c
d	d	c	c	c

In this example (which is not regular, as otherwise $\mathcal{L} = \mathcal{L}^*$ and $\mathcal{R} = \mathcal{R}^*$ and so would be inverse and the idempotents would commute), $E(S) = \{a, b, c\}$, with $\mathcal{R}^*$ giving the classes $\{b, d\}$, $\{a\}$ and $\{c\}$, while $\mathcal{L}^*$ gives $\{a, d\}$, $\{b\}$ and $\{c\}$. Corollary 2.13 "explains" why $E(S) = \{a, b, c\}$ is reduced: this will be true of any amiable semigroup, or indeed of any semiamiable semigroup. For in such cases, $D(S) = R(S) = E(S)$, which is therefore pre-reduced and hence reduced by the corollary. If $E(S)$ is a band in an amiable semigroup, then it is a semilattice since it is both left and right regular.

In a generalised D-semigroup, $D(S)$ is left pre-reduced but need not be left reduced, as we have seen. However, a given generalised D-semigroup structure on a semigroup S may be equivalent to a D-semigroup structure on S (meaning that each gives rise to the same $\overline{\mathcal{R}}_E$). Indeed (the dual of) this happens with the semigroup S as in Example 2.10: recall that S there is a generalised R-semigroup with respect to $R(S) = E = \{d, e, f\}$, but note that $E' = \{a, b, f\}$ is such that $\overline{\mathcal{L}}_E = \overline{\mathcal{L}}_{E'}$ where E is as defined there, and E' is right reduced (in contrast to E).

In general we have the following.

Proposition 2.15 *Suppose $e, f \in E(S)$ with $e \leq_l f$. Then there are $e', f' \in E(S)$ such that $e' \sim_l e$, $f' \sim_l f$ and for which $e' \leq f'$.*

Proof. Let $e' = ef$ and $f' = f$. □

This suggests that one might be able to convert a generalised D-semigroup into an equivalent D-semigroup by replacing comparable pairs of elements of $D(S)$ under $\leq_l$ by pairs comparable under $\leq$; certainly if $|D(S)| = 2$, it is possible to do this. However, problems can arise if a given element of $D(S)$ has two elements above it under $\leq_l$.

Example 2.16 *Let $S = \{0, 0', e, f\}$ be the band with the following multiplication table.*

$\cdot$	0	$0'$	e	f
0	0	$0'$	0	$0'$
$0'$	0	$0'$	0	$0'$
e	0	$0'$	e	$0'$
f	0	$0'$	0	f

This is evidently a generalised D-semigroup in only two possible ways: $D(S) = \{0, e, f\}$ or $\{0', e, f\}$. Moreover, these are equivalent since $0 \sim_l 0'$ and so $\overline{D(S)} = S$ for both by Proposition 2.4. However, neither is left reduced, since $f0 = 0$ yet $0f = 0'$, and similarly $e0' = 0'$ yet $0'e = 0$. So there is no possible D-semigroup structure on S, let alone one equivalent to these generalised D-semigroup structures. This shows that the class of semigroups admitting a generalised D-semigroup structure is properly larger than the class admitting a D-semigroup structure.

3 D-regularity, D-abundance and D-semiabundance

3.1 Basic properties

We say a generalised D-semigroup S is *D-regular* if it is left $D(S)$-regular; obviously any such S is regular. and in view of Proposition 2.1, the unique element of $D(S)$ in the $\mathcal{R}$-class containing $x \in S$ must be $D(x)$, and so $\mathcal{R} = \overline{\mathcal{R}}_{D(S)}$.

Proposition 3.1 *The following are equivalent.*

1. *S is left E-regular and E is left pre-reduced.*
2. *S is a D-regular generalised D-semigroup in which $D(S) = E$.*

3. *S is a generalised D-semigroup such that for all $x \in S$ there exists $y \in S$ for which $xy = D(x)$.*

Proof. (1) $\Leftrightarrow$ (2). If S is left E-regular then $\overline{\mathcal{R}}_E = \mathcal{R}$ by Proposition 2.5. Since E is left pre-reduced, then by Corollary 2.2, S becomes a generalised D-semigroup if we define $D(x)$ to be the unique $e \in E$ in the $\overline{\mathcal{R}}_E$-class containing $x \in S$, with $D(S) = E$, and it is D-regular since $\mathcal{R} = \overline{\mathcal{R}}_{D(S)}$. The converse is immediate.

(1), (2) $\Leftrightarrow$ (3). If S is D-regular then $(x, D(x)) \in \mathcal{R}$, so $xS^1 = D(x)S^1$, so in particular, $xy = D(x)$ for some $y \in S$. Conversely, if (3) holds and $x \in S$, then there is $y \in S$ for which $xy = D(x)$, so $D(x)S^1 \subseteq xS^1 = (D(x)x)S^1 \subseteq D(x)S^1$, and so $(x, D(x)) \in \mathcal{R}$, so S is D-regular. □

The third condition above spells out that D-regular generalised D-semigroups are, in the case of D-semigroups, nothing but *strong* as in [16] and prior to that in [11].

In the two-sided case, we say the DR-semigroup S is *DR-regular* if every $\mathcal{R}$-class and every $\mathcal{L}$-class contains an element of $D(S) = R(S)$. Note that although one can always select exactly one element of $E(S)$ in each $\mathcal{R}$-class of a regular semigroup S to give a D-regular generalised D-semigroup structure on S, there may be no way to ensure this choice gives exactly one in each $\mathcal{L}$-class as well. For instance, the regular semigroup in Example 2.16, which is D-regular since $\overline{\mathcal{R}}_{D(S)} = \mathcal{R}$, has differing numbers of $\mathcal{L}$-classes and $\mathcal{R}$-classes, so there is no DR-regular generalised DR-semigroup structure on it.

Similarly, we say the generalised D-semigroup S is *D-abundant* if it is left $D(S)$-abundant. The next result generalises Corollary 3.7 in [8], and indeed the more general Proposition 2.2 of [16].

Proposition 3.2 *The following are equivalent.*

1. *S is left E-abundant and E is left pre-reduced.*
2. *S is a D-abundant generalised D-semigroup in which $D(S) = E$.*
3. *S is a generalised D-semigroup such that, for all $x \in S$ and $a, b \in S^1$, if $ax = bx$ then $aD(x) = bD(x)$.*

Proof. The argument for (1) $\Leftrightarrow$ (2) is very similar to the corresponding part of the proof Proposition 3.1.

(1), (2) $\Leftrightarrow$ (3). If S is D-abundant with $x \in S$, then $(x, D(x)) \in \mathcal{R}^*$, so if $a, b \in S^1$ with $ax = bx$ then $aD(x) = bD(x)$. Conversely, if (3) holds and $x \in S$, then for all $a, b \in S^1$, $ax = bx$ if and only if $aD(x) = bD(x)$ (since $D(x)x = x$), so $(x, D(x)) \in \mathcal{R}^*$, so S is D-abundant. □

Proposition 3.3 *Every D-abundant generalised D-semigroup embeds in a D-regular one.*

Proof. Let S be a D-abundant generalised D-semigroup. By definition, $\mathcal{R}^*$ on S is the restriction to S of $\mathcal{R}$ on some oversemigroup T of S. Now define D on T as follows. If $x \in T$ is such that $(x, s) \in \mathcal{R}$ for some $s \in S$, then let $D(x) = D(s)$; this is well-defined because $D(s_1) = D(s_2)$ for all $s_1, s_2 \in S$ for which $(s_1, s_2) \in \mathcal{R}$ in T. This defines D on all $\mathcal{R}$-classes of T having non-empty intersection with S. For the other $\mathcal{R}$-classes, just pick one element of $E(S)$ from each and define D accordingly. □

In fact the laws for D-abundant generalised D-semigroups can be simplified, with some of the laws as in Proposition 2.9 becoming redundant. (Note that the fourth law there is equivalent to $D(S)$ being left pre-reduced in the presence of the others.)

Proposition 3.4 *Suppose S is a unary semigroup satisfying the following laws:*

1. $D(x)x = x$;
2. *for all $a, b \in S^1$, if $ax = bx$ then $aD(x) = bD(x)$;*
3. $D(S) = \{D(s) \mid s \in S\}$ *is left pre-reduced.*

Then S is a D-abundant generalised D-semigroup.

Proof. Suppose the above laws are satisfied. Then applying the second law to the first twice gives $D(x)D(x) = D(x)$ and $D(x)D(D(x)) = D(D(x))$, so because $D(D(x))D(x) = D(x)$ by the first law, applying the third gives that $D(x) = D(D(x))$. If $D(x)y = y$ then the second law gives that $D(x)D(y) = D(y)$, so $D(y) \leq_l D(x)$. Since $D(S)$ is left pre-reduced, $\leq_l$ is a partial order, and so S is a generalised D-semigroup, which is D-abundant by Proposition 3.2. □

A weakened version of the D-abundance property is the so-called *left congruence condition*, considered in various settings by previous authors in the commuting case, and given by the law $D(xy) = D(xD(y))$. This law is necessary and sufficient for the equivalence relation $\overline{\mathcal{R}}_{D(S)}$ to be a left congruence on S.

A *D-semiabundant* generalised D-semigroup is one for which each $\overline{\mathcal{R}}$-class of S contains an element of $D(S)$. (We cannot simply specify that S be left $D(S)$-semiabundant since for this, the generalised Green's relation is itself determined by $D(S)$. In general, "every $\overline{\mathcal{R}}$-class contains an element of E" is not the same as "every $\overline{\mathcal{R}}_E$-class contains an element of E".)

Proposition 3.5 *The following are equivalent.*

1. *every $\overline{\mathcal{R}}$-class contains an element of E, and E is left pre-reduced.*
2. *S is a D-semiabundant generalised D-semigroup in which $D(S) = E$.*
3. *S is a generalised D-semigroup such that, for all $e \in E(S)$, $eD(e) = D(e)$.*

Proof. The argument for (1) $\Leftrightarrow$ (2) is again very similar to previous cases.

$(1), (2) \Leftrightarrow (3)$. If S is D-semiabundant and $e \in E(S)$ then $(e, D(e)) \in \overline{\mathcal{R}}$, so because $ee = e$, it must be that $eD(e) = D(e)$. Conversely, if (3) holds and $x \in S$, and $ex = x$ for some $e \in E(S)$ then $D(e)x = D(e)(ex) = ex = x$, so $D(x) \leq_l D(e)$, and so

$$eD(x) = e(D(e)D(x)) = (eD(e)D(x)) = D(e)D(x) = D(x),$$

from which it follows easily that $(x, D(x)) \in \overline{\mathcal{R}}$, so (1) holds. □

The definitions of DR-abundant and DR-semiabundant DR-semigroups are very analogous to those of DR-regular DR-semigroups, and the same general comment applies as applied there.

Proposition 3.6 *Let S be a generalised D-semigroup that is left abundant as a semigroup. Then S is D-abundant if and only if it is D-semiabundant.*

Proof. Suppose S is left abundant. If S is D-abundant, then it is clearly also D-semiabundant by Propositions 3.2 and 3.5. Conversely, if S is D-semiabundant, then it is left semiabundant and $\overline{\mathcal{R}}_{D(S)} = \overline{\mathcal{R}}$. But S is left abundant, so $\overline{\mathcal{R}} = \mathcal{R}^*$ by Proposition 2.5. So $\overline{\mathcal{R}}_{D(S)} = \mathcal{R}^*$ and so S is D-abundant by definition. □

Corollary 3.7 *Let S be a generalised D-semigroup that is regular as a semigroup. The following are equivalent.*

- *S is D-regular;*
- *S is D-abundant;*
- *S is D-semiabundant.*

Proof. Even without regularity of S, we have (1) $\Rightarrow$ (2) $\Rightarrow$ (3) because $\mathcal{R} \subseteq \mathcal{R}^* \subseteq \overline{\mathcal{R}}$. But if S is regular then $\mathcal{R} = \overline{\mathcal{R}}$, and so (3) $\Rightarrow$ (1). □

Corollary 3.8 *Let S be a generalised DR-semigroup that is regular as a semigroup. Then S is DR-regular if and only if it is DR-abundant, if and only if it is DR-semiabundant.*

It follows from Proposition 3.2 that the class of D-abundant (DR-abundant) generalised D-semigroups (DR-semigroups) is a quasivariety of unary semigroups. In fact it is a proper quasivariety, a fact that contaminates the various special cases we have considered previously (in which $D(S)$ is a band, in which it commutes, in which $D(S) = E(S)$), as the following example shows.

Example 3.9 *Let $S = \{e, a, b, c, f, 0\}$ be the semigroup in which 0 is a zero and otherwise the multiplication table is as follows:*

$\cdot$	e	a	b	f	c
e	e	a	b	0	c
a	0	0	0	a	0
b	b	c	e	0	a
f	0	0	0	f	0
c	0	0	0	c	0

(This example was discovered with the assistance of Mace4 *so must indeed satisfy associativity.) Note that $E(S) = \{0, e, f\}$ is a subsemilattice of S, with $ef = fe = 0$, while*

$\mathcal{R}^*$ is given by the partition $\{e,a,b,c\},\{0\},\{f\}$, and $\mathcal{L}^*$ by $\{e,b\},\{a,f,c\},\{0\}$. Clearly S is amiable, and so S is a DR-abundant DR-semigroup, in which $D(S)$ commutes. Now $D(a)=D(c)$ and $R(a)=R(c)$, so the semigroup congruence θ that collapses a,c together but collapses nothing else is also a DR-semigroup congruence. The quotient DR-semigroup is not D-abundant though, because although $a\ \theta\ c=ba$, $D(a)=e$ while $bD(a)=be=b$, yet $(b,e)\notin\theta$. In particular then, the class of amiable semigroups is not closed even under quotients that separate idempotents (let alone those preserving D,R), so is not a variety of bi-unary semigroups. Likewise for the class of D-abundant generalised D-semigroups as well as their two-sided versions, and all versions in which $D(S)$ commute or else is at least a band: all are proper quasivarieties.

3.2 Generalised D-abundant D-semigroups and T_X

Proposition 3.3 shows that D-abundant generalised D-semigroups embed in D-regular ones. In fact, all can be embedded in T_X viewed as a D-regular generalised D-semigroup, as we soon show.

First some observations about idempotent elements of the full transformation semigroup $S=T_X$. It is well-known and easy to see that $E(S)$ consists of projections onto subsets of X: any $e\in E(S)$ maps everything in each $ker(e)$-class to some distinguished element in that class. For $x\in X$, let f_x denote the constant projection, taking every $z\in X$ to x.

Proposition 3.10 *For $s,t\in T_X$, $(s,t)\in\mathcal{R}$ if and only if $ker(s)=ker(t)$.*

Proof. In this proof, we write functions on the right of their arguments (so xf rather than $f(x)$).

For $e,f\in E(S)$, $e\leq_l f$ asserts that $fe=e$, which is easily seen to be equivalent to $ker(f)\subseteq ker(e)$, and so $e\sim_l f$ (or equivalently, $(e,f)\in\mathcal{R}=\mathcal{R}^*$ by Proposition 2.1) asserts that $ker(f)=ker(e)$.

Since T_X is regular, $\mathcal{R}=\mathcal{R}^*$, so for $s,t\in T_X$, $(s,t)\in\mathcal{R}$ says that for all $a,b\in T_X$ (which is a monoid), $as=bs$ if and only if $at=bt$.

Suppose $(s,t)\in\mathcal{R}$. Then the following are equivalent: $(x,y)\in ker(s)$; $xs=ys$; $f_xs=f_ys$, $f_xt=f_yt$; $xt=yt$; $(x,y)\in ker(t)$. Hence $ker(s)=ker(t)$.

Conversely, suppose $ker(s)=ker(t)$. For $a,b\in T_X$, the following are equivalent: $as=bs$; for all $x\in X$, $xas=xbs$; for all $x\in X$, $(xa,xb)\in ker(s)$; for all $x\in X$, $(xa,xb)\in ker(t)$; for all $x\in X$, $xat=xbt$; $at=bt$. So $(s,t)\in\mathcal{R}^*=\mathcal{R}$. □

It is now clear how to make T_X into a D-regular (hence D-abundant) generalised D-semigroup: for each equivalence relation on X, select exactly one projection whose kernel is that equivalence relation, to form a set E of idempotents of T_X, and define $D(s)$ to be the projection in E with kernel agreeing with that of $s\in T_X$. Indeed this exhausts the possible ways to achieve the goal, as an easy argument involving a right regular representation shows.

Theorem 3.11 *Every D-abundant generalised D-semigroup embeds in the D-regular generalised D-semigroup T_X for some choice of X, made into a D-regular generalised D-semigroup in one of the possible ways described above.*

Proof. Let S be a D-abundant generalised D-semigroup. For each $s \in S$, define $\psi_s : S^1 \to S^1$ by setting $x\psi_s = xs$ for all $x \in S^1$, and then define $\theta : S \to T_X$ (where $X = S^1$) by setting $s\theta = \psi_s$ for all $s \in S$. This determines a semigroup embedding of S into T_X. For all $x, y \in S^1$ and $s \in S$ we have $xs = ys$ if and only if $xD(s) = yD(s)$, so $ker(\psi_s) = ker(\psi_{D(s)})$, and so $\psi_{D(s)}$ is a projection whose kernel agrees with that of ψ_s.

Now let $F = \{e\theta \mid e \in D(S)\} \subseteq E(T_X)$. Since θ is a semigroup embedding, it follows easily that the left pre-reduced property of $D(S)$ is passed on to F. Hence, each $\mathcal{R}$-class of T_X contains at most one element of F by Proposition 2.1. Enlarge F to E by including precisely one element of $E(T_X)$ for all those $\mathcal{R}$-classes not containing an element of F, and define D on T_X using E. The result is a D-regular generalised D-semigroup structure on T_X for which $D(T_X) = E$.

Now for $s \in S$, $D(s)\theta = \psi_{D(s)}$, which as we have seen is a projection in E whose kernel agrees with that of ψ_s, which by Proposition 3.10 is $D(\psi_s) = D(s\theta)$, the unique element of E in the $\mathcal{R}$-class containing ψ_s. Hence θ respects D as well. □

4 E-regular and E-inverse semigroups

4.1 E-regular semigroups

In [9], the authors use the term "E-regularity" to refer to elements of a semigroup: $x \in S$ is E-regular if there exists an inverse y (so that $xyx = x, yxy = y$) for which $xy, yx \in E$. Let us call y an *E-inverse* of the E-regular element a when this happens. Shortly we relate this to our definition of E-regularity for entire semigroups.

A seemingly more general property of elements is as follows. We might say $x \in S$ is *weakly E-regular* if x has a pseudoinverse y (so that $xyx = x$) for which $xy, yx \in E$. However, in this case let $z = yxy$. Then as is well-known, z is an inverse of x, but also $xz = x(yxy) = (xy)^2 = xy \in E$, and $zx = (zxz)z = yx \in E$, so z is an E-inverse of x. So an element is weakly E-regular if and only if it is E-regular!

Proposition 4.1 *The following are equivalent.*

1. *Every element of S is E-regular.*

2. *Every $\mathcal{L}$-class and $\mathcal{R}$-class contains an element of E.*

Proof. (1) $\Rightarrow$ (2). Suppose $x \in S$; then there exists $y \in S$ for which $xyx = x$ with $xy, yx \in E$, from which it follows easily that $xS^1 = (xy)S^1$, and so $(x, xy) \in \mathcal{R}$. Similarly, $(x, yx) \in \mathcal{L}$.

(2) $\Rightarrow$ (1). Pick $x \in S$. Suppose $(x, e) \in \mathcal{R}$ and $(x, f) \in \mathcal{L}$, where $e, f \in E$. Then $xS^1 = eS^1$, so $x = ey$ for some $y \in S^1$, so $x = ex$. Also, $e = xz$ for some $z \in S^1$, and if $z = 1$ then $x = e$ so instead let $z = e \in S$. So $e = xz$ for some $z \in S$. Dually, we also have $x = xf$ with $f = wx$ for some $w \in S$. Let $u = wxz$. Then $xux = xwxzx = xzx = x$, with $xu = xwxz = xz \in E$ and $ux = wxzx = wx \in E$. □

This shows that our concept of E-regularity for a semigroup is consistent with that defined in terms of elements in [9], and we freely move between these two equivalent views in what follows.

4.2 E-inverse semigroups

Let us say an E-regular semigroup is *E-inverse* if every element has a unique E-inverse. In particular, an $E(S)$-inverse semigroup is nothing but an inverse semigroup, and in this case $E(S)$ is a semilattice.

More generally, in a $*$-regular semigroup in the sense of Drazin [4], it is well-known that each element has a unique $E^*(S)$-inverse. Note that $E^*(S)$ is reduced.

In fact it is not hard to identify the most general property on E to ensure uniqueness of E-inverses in an E-regular semigroup.

Proposition 4.2 *The E-regular semigroup S is E-inverse if and only if E is pre-reduced.*

Proof. Suppose S has at most one E-inverse for each element. For $e \in E$, $eee = e$ and $e^2 = e \in E$, so E is its own E-inverse. So if $e, f \in E$ are such that $efe = e$ and $fef = f$ with the idempotents ef, fe in E, it must be that $e = f$.

Now suppose $e, f \in E$, with $ef = f$ and $fe = e$. Then of course $efe = ee = e$ and $fef = ff = f$, with $ef = f$ and $fe = e$ both in E, so $e = f$. Similarly, if $ef = e, fe = f$ then $e = f$. So E is pre-reduced.

Conversely, suppose E is pre-reduced. Suppose x has inverses y, z, so that $xyx = x, yxy = y, xzx = x$ and $zxz = z$, with $xy, yx, xz, zx \in E$. Then $xyxz = xz$ and $xzxy = xy$, so $xy = xz$; similarly, $yx = zx$ (using the other implication holding on E). Hence $y = yxy = yxz = zxz = z$. □

Recall that a generalised DR-semigroup S is DR-regular if every $\mathcal{L}$-class and every $\mathcal{R}$-class contains an element of $D(S)$. Earlier we noted that these did not arise from general E-regular semigroups. This is highlighted by the following.

Proposition 4.3 *The following are equivalent.*

1. *S is a generalised D-semigroup in which $D(S) = E$, which is pre-reduced, and for all $x \in S$ there exists $x' \in S$ for which $D(x) = xx'$ and $D(x') = x'x$.*
2. *S is an E-inverse semigroup.*
3. *S is a DR-regular generalised DR-semigroup in which $D(S) = E$.*

In these equivalent cases, x' in (1) is the unique E-inverse of x, and in (3), $R(x) = x'x$.

Proof. (1) $\Rightarrow$ (2). For all $x \in S$, we have that $xx'x = D(x)x = x$ and $x'xx' = D(x')x' = x'$, with $xx', x'x \in D(S) = E$ so S is E-regular. Because $E = D(S)$ is pre-reduced, S is E-inverse by Proposition 4.2 with x' the unique E-inverse of x.

(2) $\Rightarrow$ (1), (3). For $x \in S$, let x' be the E-inverse of x and define D, R as in terms of it, via $D(x) = xx'$ and $R(x) = x'x$. Then for any $x \in S$, $D(x) \in E$, $D(x)x = (xx')x = x$, and if $ex = x$ for some $e \in E$, then $exx' = xx'$, so $eD(x) = D(x)$. Now E is pre-reduced by Proposition 4.2, so $D(x)$ is the smallest $e \in E$ with respect to $\leq_l$ such that $ex = x$, and so S is a generalised D-semigroup in which $D(S) = E$. A dualised argument shows that S is also a generalised R-semigroup, in which $R(x) = x'x$, and since $D(S) = R(S) = E$,

it is a generalised DR-semigroup, establishing (3). Now since S is E-inverse, $x'' = x$, so $D(x') = x'x'' = x'x$, so (1) holds also.

(3) $\Rightarrow$ (2). S is E-regular, and since E is pre-reduced, it is E-inverse by Proposition 4.2. □

As noted in Section 3.1, the band in Example 2.16 has no DR-regular generalised DR-semigroup structure on it. Hence, not every regular semigroup is E-inverse for some choice of E.

If S is regular and $E(S)$ commutes, then S is an inverse semigroup. However, by Proposition 4.2, $E(S)$ being pre-reduced is necessary and sufficient for this.

In an E-inverse semigroup S, uniqueness of E-inverses means we can define a unary operation $'$ of E-inversion, and it is immediate that $(S, \cdot, ')$ is an *I-semigroup*, meaning that for all $x \in S$, $x'' = x$, $xx'x = x$ (and hence also $x'xx' = x'$).

Let us say the I-semigroup $(S, \cdot, ')$ is *normal* if it satisfies the following two quasi-identities:

- $xx'yy' = yy'$ & $yy'xx' = xx' \Rightarrow xx' = yy'$
- $xx'yy' = xx'$ & $yy'xx' = yy' \Rightarrow xx' = yy'$

Proposition 4.4 *If S is an E-inverse semigroup, then $(S, \cdot, ')$ is a normal I-semigroup where x' is the E-inverse of $x \in S$, and $E = \{xx' \mid x \in S\}$. Conversely, if $(S, \cdot, ')$ is a normal I-semigroup then S is an E-inverse semigroup, where $E = \{xx' \mid x \in S\}$ and x' is the E-inverse of $x \in S$.*

Proof. Suppose S is an E-inverse semigroup, with x' the E-inverse of $x \in S$. Then $(S, \cdot, ')$ is an I-semigroup. Moreover, for $e \in E$, because $eee = e$, $e' = e$, and since $xx' \in E$ for all $x \in S$, we have $E = \{xx' \mid x \in S\}$. Normality of $(S, \cdot, ')$ now follows from the pre-reduced property of E (Proposition 4.2).

Conversely, suppose $(S, \cdot, ')$ is a normal I-semigroup. Let $E = \{xx' \mid x \in S\}$. Then E is pre-reduced by normality. Also, $xx'x = x$ where $xx' \in E$, and $x'x = x'x'' \in E$, so S is E-regular, indeed E-inverse since E is pre-reduced (again by Proposition 4.2). □

The class of normal I-semigroups is evidently a quasivariety. But since every normal I-semigroup is a generalised DR-semigroup as in Proposition 4.3, it is also a finitely based variety, given by the DR-semigroup laws with $D(x)$ and $R(x)$ replaced by xx' and $x'x$ in all cases, along with a small number of further laws to ensure that the equivalent conditions of Proposition 4.3 hold; for example, the law $x'' = x$ is enough. These laws generalise those of inverse semigroups (viewed as unary semigroups), which are given by the identities: $xx'x = x$. $x'' = x$, $(xy)' = y'x'$ and $xx'yy' = yy'xx'$ (see [10]).

To summarise, the following are essentially the same:

- E-inverse semigroups;
- DR-regular generalised DR-semigroups;
- normal I-semigroups.

It turns out that the notion of morphism does not depend on which of these viewpoints is adopted, so the associated categories are isomorphic. (For E-inverse semigroups S, T, the natural notion of morphism is a semigroup homomorphism $S \to T$ that maps E into its counterpart in R.)

Proposition 4.5 *Let S, T be normal I-semigroups, viewed as DR-regular generalised DR-semigroups as above, with $f : S \to T$ a semigroup homomorphism. The following are equivalent:*

1. *$f(D(S)) \subseteq D(T)$;*
2. *f is an I-semigroup homomorphism;*
3. *f is a generalised DR-semigroup homomorphism;*
4. *f is a generalised D-semigroup homomorphism.*

Proof. (1) ⇒ (2). Suppose $f(D(S)) \subseteq D(T)$. Now for all $x \in S$, $f(x) = f(xx'x) = f(x)f(x')f(x)$, and similarly $f(x') = f(x')f(x)f(x')$. Also, $f(x)f(x') = f(xx') \in D(T)$ by assumption (since $xx' \in D(S)$), and similarly $f(x')f(x) = f(x'x) \in D(T)$. So by definition, $f(x')$ is a $D(T)$-inverse of $f(x) \in T$. Since T is a $D(T)$-inverse semigroup, it follows that $f(x') = f(x)'$.

(2) ⇒ (3) is obvious since D, R are defined in terms of $'$, and (3) ⇒ (4) is immediate. (4) ⇒ (1) is trivial since $D(x) \in D(S)$ for all $x \in S$, and $D(y) \in D(T)$ for all $y \in T$. □

An example of an E-inverse semigroup that is not inverse is furnished by Example 2.10. The band S there is a generalised DR-semigroup which is D-regular since $ab = D(a)$, $ba = D(b)$, hence is E-inverse by Proposition 4.3. Recall that $D(S)$ is left reduced but not right reduced and hence not reduced, let alone commuting. This shows that the law $(xx')(yy') = (yy')(xx')$ for inverse semigroups need not hold for E-inverse semigroups. The inverse semigroup law $(xy)' = y'x'$ also need not hold in an E-inverse semigroup, as the same exapmple shows: $(ef)' = e' = e$, while $f'e' = fe = a$. In this example, E is not reduced. However, it is well-known that even in $*$-regular semigroups in the sense of Drazin [4], in which $E = E^*(S)$ is reduced, this law need not hold.

$E(S)$-inverse semigroups are exactly inverse semigroups, and in this case, $E = E(S)$ commutes. Conversely, we have the following.

Proposition 4.6 *If S is an E-inverse semigroup in which E commutes, then $E = E(S)$ and so S is inverse.*

Proof. If S is an E-inverse semigroup in which E commutes, then from Proposition 4.3, it is a generalised D-semigroup such that for all $x \in S$, there is $x' \in S$ for which $D(x) = xx'$ and $D(x') = x'x$; indeed it is a left C-semigroup in the sense of [11], and x' is nothing but a "true inverse" of s in the sense used in [11] (strictly, it satisfies the left-sided dual of the definition used there). It follows from Corollary 2.13 of [11] that S is an inverse semigroup with $E(S) = D(S) = E$. □

Finally, recall that Clifford semigroups are inverse semigroups satisfying the law $xx' = x'x$. If this law is added to those of E-inverse semigroups (viewed as unary semigroups), the author has shown using the *Prover9* software that the law $xx'yy' = yy'xx'$ also holds; in other words, E commutes. It then follows from Proposition 4.6 that S is an inverse semigroup, and hence is a Clifford semigroup. Unfortunately, the machine-generated proof is very long, and a human-readable proof awaits further work.

5 Open questions

In contrast to the case of amiable semigroups, we do not know whether the class of semiamiable semigroups, viewed as a class of bi-unary semigroups (namely, DR-semiamiable generalised DR-semigroups) is a variety. (The quotient in Example 3.9 is semiamiable, hence DR-semiamiable.) Similarly for the obvious notion of right semiamiable semigroups (D-semiamiable generalised D-semigroups). These are quasivarieties, in which the law $x^2 = x \Rightarrow D(x) = x$ is added to the generalised DR-semigroup (or D-semigroup) laws. Likewise, the class of D-semiabundant generalised D-semigroups is a quasivariety by Proposition 3.5, but it is not known whether it is proper.

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