Constructing $(0, 1)$-matrices with large minimal defining sets

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Abstract

If D is a partially filled-in $(0, 1)$-matrix with a unique completion to a $(0, 1)$-matrix M (with prescribed row and column sums), we say that D is a *defining set* for M. Let $\mathcal{A}_{2m,m}$ be the set of $(0, 1)$-matrices of dimensions $2m \times 2m$ with uniform row and column sum m. It is shown in (Cavenagh, 2013) that the smallest possible size for a defining set of an element of $\mathcal{A}_{2m,m}$ is precisely m^2. In this note when m is a power of two we construct an element of $\mathcal{A}_{2m,m}$ which has no defining set of size less than $2m^2 - o(m^2)$. Given that it is easy to show any $\mathcal{A}_{2m,m}$ has a defining set of size at most $2m^2$, this construction is asympotically optimal. Our construction is based on a array, defined using linear algebra, in which any subarray has asymptotically the same number of 0's and 1's.

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1 Introduction

Where convenient, we keep notation consistent with [1].

Let $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ be vectors of non-negative integers such that $\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} s_j$. Then $\mathcal{A}(R,S)$ is defined to be the set of all $m \times n$ $(0,1)$-matrices with r_i 1's in row i and s_j 1's in column j, where $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$. If $M \in \mathcal{A}(R,S)$, we refer to R and S as the row sum and column sum vectors for M, respectively.

With R and S as above, we next define $\mathcal{A}'(R,S)$ to be the set of all $m \times n$ $(0,1,\star)$-matrices with:

1. at most r_i 1's in row i,
2. at most $n - r_i$ 0's in row i,
3. at most s_j 1's in column j,
4. at most $m - s_j$ 0's in column j.

We call a matrix $M \in \mathcal{A}'(R,S)$ a *partial* $(0,1)$-matrix with row sum vector R and column sum vector S. If $M_{ij} = \star$ we say that position (i,j) is *empty*. Indeed, $\mathcal{A}(R,S) \subseteq \mathcal{A}'(R,S)$ and a partial $(0,1)$-matrix is a $(0,1)$-matrix if and only if it none of its positions are empty (equal to $\star$). (Note that our definition of a partial $(0,1)$-matrix allows for the possibility of a matrix $M \in \mathcal{A}'(R,S)$ which has *no* completion to a a $(0,1)$-matrix in $\mathcal{A}(R,S)$.)

We sometimes consider a partial $(0,1)$-matrix M as a set of triples

$$M = \{(i, j, M_{ij}) \mid 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n, M_{ij} \in \{0,1\}\}.$$

Note that we naturally omit here the empty positions of M. It is usually clear by context whether we are considering a partial $(0,1)$-matrix as a matrix or as a set of ordered triples. For example, if we say that $M_1 \subseteq M_2$ or write $M_1 \setminus M_2$ for two partial $(0,1)$-matrices M_1 and M_2, we are considering M_1 and M_2 as sets. Similarly, the *size* of a partial $(0,1)$-matrix M refers to $|M|$ where M is a set (i.e. the number of 0's and 1's).

With this in mind, suppose that $M \in \mathcal{A}(R,S)$ and $D \in \mathcal{A}'(R,S)$. We say that D is a *defining set* for M if M is the unique member of $\mathcal{A}(R,S)$ such that $D \subseteq M$.

This is analogous to the usual definition of defining sets of Latin squares and other combinatorial designs ([4, 8]).

Given a $(0,1)$-matrix M, the size of the smallest defining set in M is denoted by $\mathrm{sds}(M)$. For given row column sum vectors R and S, $\mathrm{sds}(\mathcal{A}(R,S))$

is the size of the smallest defining set for all members of the set $\mathcal{A}(R,S)$. More precisely,

$$\text{sds}(\mathcal{A}(R,S)) = \min\{\text{sds}(M) \mid M \in \mathcal{A}(R,S)\}.$$

We also define

$$\text{maxsds}(\mathcal{A}(R,S)) = \max\{\text{sds}(M) \mid M \in \mathcal{A}(R,S)\}.$$

This last definition is the focus of this paper, in the case when row and column sums are constant. To this end, we define $R_{n,x}$ to be the row sum vector with dimension n and constant row sum $r_1 = r_2 = \cdots = r_n = x \leqslant n$. The column sum vector $S_{n,x}$ is defined similarly. Then, $\mathcal{A}_{n,x} = \mathcal{A}(R_{n,x}, S_{n,x})$ is the set of $n \times n$ $(0,1)$-matrices with constant row and column sum x.

In fact, elements of $\mathcal{A}_{n,x}$ may also be thought of as *frequency squares* (sometimes *F-squares*). Let $n, \alpha, \lambda_1, \lambda_2, \ldots, \lambda_\alpha \in \mathbb{N}$ and $\sum_{i=1}^{\alpha} \lambda_i = n$. A *frequency square* or *F-square* $F(n; \lambda_1, \lambda_2, \ldots, \lambda_\alpha)$ of *order* n is an $n \times n$ array on symbol set $\{s_1, s_2, \ldots, s_\alpha\}$ such that each cell contains one symbol and symbol s_i occurs precisely λ_i times in each row and λ_i times in each column. Thus if we let $\alpha = 2$, $s_1 = 1$ and $s_2 = 0$, the frequency square $F(n; x, n-x)$ is in effect an element of $\mathcal{A}_{n,x}$.

Critical and defining sets of frequency squares have previously been studied in [5]. The following results are directly implied by Theorems 2, 3, 4, 5 of [5].

Theorem 1. ([5]) $\text{sds}(\mathcal{A}_{n,1}) = n-1$. $\text{sds}(\mathcal{A}_{n,2}) \leqslant 2n-3$ *and* $\text{sds}(\mathcal{A}_{n,2}) = 2n-4$ *if n is even.* $\text{sds}(\mathcal{A}_{n,x}) \leqslant xn - x^2$ *if x divides n and $x < n$. If $x \leqslant k$ then* $\text{sds}(\mathcal{A}_{xk+1,x}) \leqslant (k-1)x^2 + x(x+1)/2$.

In particular, observe that the above results imply that $\text{sds}(\mathcal{A}_{2m,m}) \leqslant m^2$, for each integer m. In [3] some lower bounds for $\text{sds}(\mathcal{A}_{n,x})$ are given, showing in particular that $\text{sds}(\mathcal{A}_{2m,m})$ is in fact equal to m^2.

Theorem 2. ([3]) *Any defining set D in a matrix from $\mathcal{A}_{n,x}$ has size at least min*$\{x^2, (n-x)^2\}$.

Corollary 3. ([3]) $\text{sds}(\mathcal{A}_{2m,m}) = m^2$.

In this paper we will ultimately prove the following.

Theorem 4. *If m is a power of two,* $\text{maxsds}(\mathcal{A}_{2m,m}) = 2m^2 - O(m^{7/4})$.

Since taking every occurrence of the symbol 1 in a $(0,1)$-matrix always forms a defining set, each element of $\mathcal{A}_{2m,m}$ has a defining set of size $2m^2$. In fact, fixing a cell containing 1, it is not hard to show that each 1 in the same row and column can be omitted, retaining the property of being a defining set. Thus $\text{maxsds}(\mathcal{A}_{2m,m}) \leqslant 2m^2 - 2m + 1$. It follows that, at least when m is a power of 2, $\text{maxsds}(\mathcal{A}_{2m,m}) = 2m^2 - o(m^2)$.

The analogous question has been considered for Latin squares in [7], where it is shown that every Latin square of order n has a defining set of size at most $n^2 - \frac{\sqrt{\pi}}{2}n^{9/6}$ and that for each n there exists a Latin square L with no defining set of size less than $n^2 - (e + o(1))n^{10/6}$. In contrast to the proof in this paper, the latter result is non-constructive.

2 Theory on trades and defining sets in $(0,1)$-matrices

In this section we develop the theory from [3] which is relevant to our paper. The results on trades in this section are a restatement of theory in [1]; however results on defining sets are new. We define a *trade* to be a non-empty partial $(0,1)$-matrix T such that there exists a *disjoint mate* T' such that:

- T_{ij} is empty if and only if T'_{ij} is empty;
- if T_{ij} is non-empty, then $T_{ij} \neq T'_{ij}$;
- if 1 appears precisely k times in a row r (column c) of T, then 1 also appears k times in row r (column c) of T';
- if 0 appears precisely k times in a row r (column c) of T, then 0 also appears k times in row r (column c) of T';

Lemma 5. ([3]) *A partial $(0,1)$-matrix D is a defining set for a $(0,1)$-matrix M if and only if $D \subseteq M$ and $|D \cap T| \geqslant 1$ for every trade $T \subseteq M$.*

Thus we can study the properties of defining sets of $(0,1)$-matrices through an analysis of the trade structure of $(0,1)$-matrices. We say that a trade T is a *cycle* if each row and each column of T contains either 0 or 2 non-empty positions.

The notions of cycle and intercalate are very similar to the notions of *minimal balanced matrix* and *interchange* (respectively) given in [1]; however

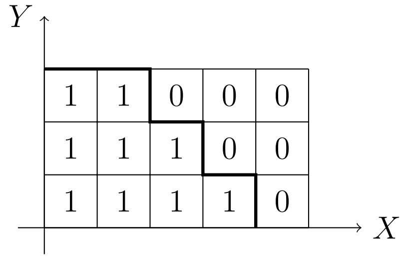

Figure 1: The unique member of $\mathcal{A}((2,3,4),(0,0,1,2,3))$.

in [1] these matrices are formed as (0 ± 1)-matrices (with the 0's denoting "empty" cells) rather than $(0,1,\star)$-matrices. For our purposes it is helpful to define trades as subsets of $(0,1)$-matrices, hence our choice of definitions.

By similar reasoning to 3.2 of [1] however, the following can be shown.

Theorem 6. (Lemma 3.2.1 of [1]) *Any trade T in a $(0,1)$-matrix is a union of disjoint cycles.*

Lemma 7. [3] *Let P be a finite, non-empty partial $(0,1)$-matrix such that every non-empty row or column contains at least one 0 and at least one 1. Then P contains a trade.*

We say that matrix $M \in \mathcal{A}'(R,S)$ is in *good form* if whenever $(i,j,0)$, $(i,j',1) \in M$, then $j < j'$ and whenever $(i,j,0)$, $(i',j,1) \in M$, then $i < i'$. Somewhat informally, a partial matrix $M \in \mathcal{A}'(R,S)$ is in good form if and only if a South-East walk $\mathcal{C}$ exists with only 1's below the line and only 0's above the line.

The following theorem is equivalent to the Gale-Ryser theorem [6, 9] and Theorem 3.2.4 ([2, 9, 10]) in [2].

Theorem 8. *A matrix $M \in \mathcal{A}(R,S)$ is the unique member of $\mathcal{A}(R,S)$ if and only if its rows and columns can be rearranged so that it is in good form.*

In Figure 1, the unique member of $\mathcal{A}((2,3,4),(3,3,2,1,0))$ is given, with the South-East walk $\mathcal{C}$ shown as a thick line.

We next give a new classification of defining sets in $(0,1)$-matrices that will be useful for our purposes.

Theorem 9. *The set D is a defining set of a $(0,1)$-matrix M if and only if $D \subset M$ and the rows and columns of $M \setminus D$ can be rearranged so that $M \setminus D$ is in good form.*

Proof. Suppose first that D is a subset of M such that the rows and columns of $M \setminus D$ can be arranged so that a South-East walk $\mathcal{C}$ exists such that there are only 1's below $\mathcal{C}$ and only 0's above $\mathcal{C}$. If there exists a trade T which is a subset of $M \setminus D$, then T is also a subset of any superset of $M \setminus D$. In particular, T is a subset of the $(0,1)$-matrix M' created by placing a 1 in each cell below $\mathcal{C}$ and a 0 in each cell above $\mathcal{C}$. But M' has no trades by Theorem 8, a contradiction.

Conversely, let D be a defining set of a $(0,1)$-matrix M. Consider $M_0 := M \setminus D$. By Lemma 5, M_0 contains no trades. We obtain a non-increasing sequence $M_0, M_1, \dots$ via an iterative process. Given M_k, where $k \geqslant 0$, rearrange the rows of M_k so that rows contaning only 0 are contiguously the first rows and that any rows of M_k contaning only 1 are contiguously the last rows. Next, rearrange the columns of M_k so that the columns contaning only 0 are contiguously the last columns and that any columns of M_k contaning only 1 are contiguously the last rows. Let M_{k+1} be the partial $(0,1)$-matrix obtained by deleting from M_k the above rows and columns (i.e. any rows or columns of M_k that do not contain both 0 and 1).

If $M_{k+1} = M_k$ and is not empty, then every row and column of M_k contains both 0 and 1. Thus M_k contains a trade by Lemma 7, a contradiciton. Thus we have a sequence $M_0, M_1, \dots, M_K$ where M_K is empty. Next, reconstruct $M \setminus D$ via nesting the above matrices in reverse; observe that via this process we have rearranged the rows and columns of $M \setminus D$ so that it is in good form. □

3 A matrix in which 0's and 1's are closely balanced

In this section we show the existence of a $(0,1)$-matrix with the property that within any rectangular subarray the difference between the number of 1's and 0's is small.

Let $k \geqslant 2$. Let $V = (v_1, v_2, \dots, v_{2^k-1})$ be a fixed vector with entries from $\mathbb{Z}_2$. Construct a $(2^k-1) \times 2^k$ $(0,1)$-matrix $M := M(V)$ as follows.

Let $W := \{W_1, W_2, \dots, W_{2^k-1}\}$ be some ordering of the set of all non-zero column vectors of dimension k over $\mathbb{Z}_2$. Label the columns of M with the elements of $W \cup \{W_0\}$ where W_0 is the zero vector. Row i of M corresponds to equation over $\mathbb{Z}_2$ of the form

$$(x_1, x_2, \dots, x_k) \cdot W_i \quad = \quad v_i. \tag{1}$$

Then, place a 1 in row i of column W_j if and only if W_j is a solution to the equation corresponding to row i; otherwise place a 0.

For each j, let y_j be the number of 1's in column W_j and z_j be the number of 0's in column W_j, with $\Delta_j := y_j - z_j$. Finally, define

$$\Delta := \sum_{j=0}^{2^k-1} |\Delta_j|,$$

noting that Δ is a function of V.

Example 10. *Let $k = 3$ and $V = (v_1, v_2, v_3, v_4, v_5, v_6, v_7) = (0, 1, 1, 1, 0, 1, 1)$. Then $M(V)$ is given below, where the columns are labelled $W_0, W_1, \dots, W_7$ and each row i, $1 \leqslant i \leqslant 7$, is labelled as in Equation* (1) *above. Observe that*

$$(\Delta_1, \Delta_2, \dots, \Delta_7) = (-3, -3, 5, 1, -3, 1, 1, 1)$$

and $\Delta = 18$.

	0	1	0	0	1	1	0	1
	0	0	1	0	1	0	1	1
	0	0	0	1	0	1	1	1
$x_1 = 0$	1	0	1	1	0	0	1	0
$x_2 = 1$	0	0	1	0	1	0	1	1
$x_3 = 1$	0	0	0	1	0	1	1	1
$x_1 + x_2 = 1$	0	1	1	0	0	1	1	0
$x_1 + x_3 = 0$	1	0	1	0	0	1	0	1
$x_2 + x_3 = 1$	0	0	1	1	1	1	0	0
$x_1 + x_2 + x_3 = 1$	0	1	1	1	0	0	0	1

Theorem 11. *For each choice of V, $\Delta(V) \leqslant m^{3/2}$, where $m = 2^k$.*

Proof. Our proof uses elementary probability theory. We fix V but choose a column W_j uniformly at random, for each row i, let $A_i := 1$ if the entry of M in row i and column j is 1; otherwise $A_i := -1$.

Observe that A_i is a random variable with $\Pr\{A_i = 1\} = \Pr\{A_i = -1\} = 1/2$, $\mathrm{E}[A_i] = 0$ and $\mathrm{Var}(A_i) = \mathrm{E}[A_i^2] = 1$. In fact, for each $i \neq j$, A_i and A_j are independent events. To see this, the equations corresponding to A_i and A_j each have 2^{k-1} solutions; whereas the linear system corresponding to A_i and A_j has 2^{k-2} solutions (since the equations corresponding to each pair of rows are linearly independent). Thus $\Pr\{A_i = 1, A_j = 1\} = \Pr\{A_i = 1\}\Pr\{A_j = 1\} = 1/4$. As an aside, it is not always true that three or more of these random variables are independent as a subset (in particular if the subset of rows is inconsistent as a linear system); but pairwise independence is enough for our purposes.

Next, $D_j := \sum_{i=1}^{2^k-1} A_i$. From above, $\mathrm{E}[D_j] = 0$ and, from pairwise independence, $\mathrm{Var}(D_j) = 2^k - 1$. However, we can also calcuate the variance of D_j by considering all possible columns:

$$\mathrm{Var}(D_j) = \mathrm{E}[D_j^2] = \sum_{j=1}^{2^k} \Delta_j^2/2^k.$$

Thus

$$\sum_{j=1}^{2^k} \Delta_j^2 = m(m-1) < m^2.$$

The result follows. □

Let $m = 2^k$ and $n = 2m$ where $k \geqslant 2$. We now define an $n \times n$ $(0,1)$-matrix $B \in \mathcal{A}_{2m,m}$ which we will in turn show cannot have a small defining set. Let $Y := \{Y_1, Y_2, \dots, Y_{2^{k+1}}\}$ be some ordering of all the vectors of dimension $k+1$ over $\mathbb{Z}_2$. Label the columns of B with the elements of Y. The rows of B are labelled with all equations over $\mathbb{Z}_2$ of the form

$$(x_1, x_2, \dots, x_k) \cdot W' = a,$$

where $W' \in W$ and $a = \{0, 1\}$. This defines $2(2^k - 1) = 2^{k+1} - 2$ rows; the remaining two rows correspond to $x_{2^{k+1}} = 0$ and $x_{2^{k+1}} = 1$. As above, we place a 1 in column j and row i whenever the vector corresponding to column j is a solution to the equation corresponding to row i. We immediately have that each row of B has precisely m 0's and m 1's and each column of B has precisely m 0's and m 1's. It is also immediate that each row of B has a *complement* row; formed by replacing each entry $e \in \{0, 1\}$ with $1-e \in \{1, 0\}$.

Example 12. *Let $k = 2$. Then B is given below.*

	0	1	0	0	1	1	0	1
	0	0	1	0	1	0	1	1
	0	0	0	1	0	1	1	1
$x_1 = 0$	1	0	1	1	0	0	1	0
$x_2 = 1$	0	0	1	0	1	0	1	1
$x_1 + x_2 = 1$	0	1	1	0	0	1	1	0
$x_3 = 1$	0	0	0	1	0	1	1	1
$x_1 = 1$	0	1	0	0	1	1	0	1
$x_2 = 0$	1	1	0	1	0	1	0	0
$x_1 + x_2 = 0$	1	0	0	1	1	0	0	1
$x_3 = 0$	1	1	1	0	1	0	0	0

Given any subset R of the rows of B and subset C of the columns, the subarray $B(R, C)$ is the $|R| \times |C|$ array of B induced by R and C and $\delta(B(R, C))$ is given by the total number of 1's in $B(R, C)$ minus the total number of 0's in $B(R, C)$.

We are now ready to apply Theorem 11.

Lemma 13. *Let R and C be any subsets of the rows and columns, respectively, of B. Then $|\delta(B(R, C)| \leqslant 2m^{3/2} + m$.*

Proof. Without loss of generality assume that $\delta(B(R, C)) \geqslant 0$.

If $|R| > m$, there are at least $|R| - m$ distinct {row,complement row} pairs in R. For each such pair, delete the row with more 0's than 1's (or either row if they each have the same number of 0's and 1's). The resultant $B(R', C)$ clearly has the property $\delta(B(R', C)) \geqslant \delta(B(R, C))$.

Next, if $|R| < m$, there are at least $m - |R|$ distinct {row,complement row} pairs *not* in R. For each such pair, add the row with more 1's than 0's (or either row if they each have the same number of 1's and 0's). The resultant $B(R', C)$ has the property $\delta(B(R', C)) \geqslant \delta(B, (R, C))$.

Thus we may assume that $|R| = m$. If there exists a row in R whose complement row is also in R, there exists another {row, complement} pair not in R. Again, we may replace a row from R with one not in R so that $\delta(B(R, C))$ is not decreased. Repeat this until R intersects every {row,complement} pair.

Next, remove the unique row from $B(R, C)$ corresponding to the equation containing x^{2k+1}. We now have a matrix which is based on a matrix $M(V)$ for some V, where each column of $M(V)$ is included either once, twice, or not at all. (The vector V is determined by the constant terms in each of the equations corresponding to rows of R.) The result then follows from Theorem 11. $\square$

Finally, Theorem 4 is implied by the following theorem.

Theorem 14. *Let D be a defining set in the $n \times n$ $(0,1)$-matrix B. Then $|D| \geqslant n^2/2 - O(n^{7/4})$.*

Proof. Let $n = 2^{k+1}$. Since we are obtaining a lower bound for the size of D, we may assume that D is a minimal defining set. From Theorem 9, the rows and columns of D (and B in correspondence) can be arranged so that a South-East walk $\mathcal{C}$ can be drawn in $B \setminus D$ with only 1's below $\mathcal{C}$ and only 0's above $\mathcal{C}$. Indeed since D is minimal, $B \setminus D$ contains every occurence of 1 from B below $\mathcal{C}$ and every occurrence of 0 from B above $\mathcal{C}$.

Let α_0 and α_1 be the number of 0's and 1's (respectively) in B below $\mathcal{C}$, with β_0 and β_1 the number of 0's and 1's (respectively) in B above $\mathcal{C}$. Then:

$$\alpha_0 + \beta_0 = \alpha_1 + \beta_1 = 2m^2$$

and $|D| = \alpha_0 + \beta_1$.

Our next aim is to find an upper bound for $|\alpha_1 - \alpha_0|$. Let $K = \lceil (k+1)/4 \rceil$. Create partitions $\mathcal{R} = \{R_1, R_2, \ldots, R_{2^K}\}$ and $\mathcal{C} = \{C_1, C_2, \ldots, C_{2^K}\}$ of the rows and columns so that each subset is contiguous and of size $2^{k+1-K} \leqslant 2^{3K}$. Each $R_i \in \mathcal{R}$ and $C_j \in \mathcal{C}$ induces a *block*; that is a subarray created by the intersection of the rows from R_i and the columns from C_j. Observe that at most 2^{K+1} blocks contain both 0 and 1 within $B \setminus D$.

Let S be the set of blocks which contain only 1 in $B \setminus D$. Our aim is to show that S can be partitioned into at most 2^K rectangles (a rectangle here is a set of contiguous blocks forming a rectangle shape). We first remove the largest such rectangle possible from S contained in the last $n/2$ rows. Next, we remove the largest such rectangle from the last $n/4$ rows and from rows $n/4+1$ to $n/2$. At step i, we remove 2^{i-1} rectangles, each contained within a set of $n/2^i$ rows, specifically the sets of rows:

$$\{\frac{nj}{2^{i-1}} + \frac{n}{2^i} + k \mid 1 \leqslant k \leqslant \frac{n}{2^i}\}$$

where $0 \leqslant j \leqslant 2^{i-1} - 1$. At the final step, $i = K$. In total we have removed at most $\sum_{i=1}^{K} 2^{i-1} = 2^K - 1$ rectangles.

Moreover these rectangles include every block strictly below $\mathcal{C}$. From Corollary 13, the difference between the number of 1's and 0's in B within each such rectangle is at most $2m^{3/2} + m < n^{3/2}$. Thus over all of the rectangles we have an upper bound ifor the difference between the number of 1's and 0's of $O(n^{7/4})$. The difference between the the number of 1's and 0's in an individual block in B is bounded by the size of that block, so for the blocks intersecting $\mathcal{C}$ we have a net upper bound for the difference between the number of 1's and 0's of $2^{K+1} \times (2^{3K})^2 = O(n^{7/4})$. Thus

$$|\alpha_1 - \alpha_0| = |\beta_1 - \beta_0| = O(n^{7/4}).$$

It follows that

$$\alpha_0 - 2m^2 = -\beta_0 \geqslant -\beta_1 - O(n^{7/4})$$

and $|D| = \alpha_0 + \beta_1 \geqslant n^2 - O(n^{7/4})$. □

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