Series solutions for the Dirac equation in Kerr–Newman space-time

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The Dirac equation is solved for an electron in a Kerr–Newman geometry using an adaptation of the procedure of Chandrasekhar. The corresponding eigenfunctions obtained can be represented as series of Jacobi polynomials. The spectrum of eigenvalues can be calculated using continued fraction techniques. Representations for the eigenvalues and eigenfunctions are obtained for various ranges of the parameters appearing in the Kerr–Newman metric. Some comments concerning the bag model of nucleons are made.

I. INTRODUCTION AND DIRAC EQUATION SEPARATION

The Kerr–Newman space-time represents the external gravitational field of a charged rotating black hole. The Kerr space-time has the remarkable property that many of the equations of mathematical physics are solvable by means of a separation-of-variables-type ansatz. This property allows one to study linear gravitational perturbations, for example, in the neighborhood of a Kerr space-time solution. Solutions of Maxwell's equations and the Dirac equation can also be obtained by this method. The separable functions that arise have been studied by a number of authors. In this article, after rederiving the separability of the Dirac equation in the Kerr–Newman space-time background, we develop new methods for solving the resulting equations. In particular we compute the spectrum of the separation parameter for small values of the parameter $a$, as well as for large values. Using symmetry properties of the equations in the angular variable $\theta$ we reduce the problem to one involving a three-term recurrence relation. This enables the spectrum to be computed in terms of continued fractions. In addition to developing properties of the $\theta$-dependent separation functions we derive a three-term matrix recurrence relation for the separated $r$-dependent equations. Representations of these solutions for large $a$ and $r$ are then developed. For the particular case of flat space, expansion theorems are given for the $r$-dependent functions. Finally we comment on the applicability of these functions to bag models of nucleons.

We use consistently the spinor notation of Penrose and Rindler and, in particular, the null tetrad formalism. The Kerr–Newman solution of Einstein's equations has the line element

$$ds^2 = \left(1 - \frac{2Mr}{\rho^2}\right)dt^2 - \rho^2 \left(\frac{dr^2}{\Delta} + d\theta^2\right) - \left((r^2 + a^2) + \frac{2a^2Mr \sin^2 \theta}{\rho^2}\right)\sin^2 \theta \, d\phi^2$$

$$+ \frac{4aMr \sin^2 \theta}{\rho^2} \, dt \, d\phi,$$

where $\Delta = r^2 + a^2 + e^2 - 2Mr, \rho^2 = r^2 + a^2 \cos^2 \theta$, and $\rho = r + ia \cos \theta$. The electromagnetic field (vector potential) due to the charge of this solution is $(A_\rho A_\varphi A_\sigma A_\phi) = (er/\rho^2, 0, 0, er \sin^2 \theta/\rho^2)$. Specifically, we adopt the Kinnersley null tetrad of vectors with components

$$l^i = (1/\sqrt{2\Delta}) (r^2 + a^2, \Delta, 0, a),$$

$$n^i = (1/\sqrt{2\rho^2}) (r^2 + a^2, -\Delta, 0, a),$$

$$m^i = (1/\sqrt{2\rho}) (ia \sin \theta, 0, 1, i \sec \theta),$$

$$\bar{m}^i = (1/\sqrt{2\rho}) (-ia \sin \theta, 0, 1, -i \sec \theta).$$

In tetrad components the vector potential is

$$A_{00} = -er/\sqrt{2\Delta},$$

$$A_{11} = -er/\sqrt{2\rho^2},$$

$$A_{01} = A_{10} = 0.$$

The Dirac equation for spin $\frac{1}{2}$ particles in an electromagnetic field is, in spinor form,
\((\nabla_B^{B^r} - ieA_B^{B^r})\psi_B = (im/\sqrt{2})\chi_B^r,\)  
\((\nabla_B^{B^r} - ieA_B^{B^r})\chi_B^r = -(im/\sqrt{2})\psi_B.\)

In terms of the modified field components \(\psi_0 = \phi_0 e^{i(m \phi + \alpha t)}, \phi^* \psi_1 = \phi_1 e^{i(m \phi + \alpha t)}, \chi_0' = X_0 e^{i(m \phi + \alpha t)},\)
\(\phi^* \chi_1' = X_1 e^{i(m \phi + \alpha t)},\) these equations assume the form

\[- L_{1/2} \phi_0 + (D_0 - ier/\sqrt{2} \Delta) \phi_1 = - im_r (r - ia \cos \theta) X_0,\]
\[(\Delta D_{1/2}^{1/2} + ier/\sqrt{2}) \phi_0 + L_{1/2} \phi_1 = - im_r (r - ia \cos \theta) X_1,\]
\[- L_{1/2} X_0 + (D_0 - ier/\sqrt{2} \Delta) X_1 = im_r (r + ia \cos \theta) \phi_0,\]
\[(\Delta D_{1/2}^{1/2} + ier/\sqrt{2}) X_0 + L_{1/2} X_1 = im_r (r + ia \cos \theta) \phi_1.\]

These equations can be solved by the usual ansatz
\[\phi_0 = R_{1/2} S_{1/2}, \quad \psi_1 = R_{-1/2} S_{1/2},\]
\[X_0 = - R_{1/2} S_{-1/2}, \quad X_1 = R_{-1/2} S_{-1/2},\]

to give the coupled equations
\[L_{1/2} S_{1/2} = (\lambda - am \cos \theta) S_{-1/2},\]
\[L_{1/2} S_{-1/2} = - (\lambda + am \cos \theta) S_{1/2},\]
\[(D_0 - ier/\sqrt{2} \Delta) R_{-1/2} = (\lambda + irm) R_{1/2},\]
\[(\Delta D_{1/2}^{1/2} + ier/\sqrt{2}) R_{1/2} = (\lambda - irm) R_{-1/2}.\]

where
\[Q = a \sigma \sin \theta + m \csc \theta, \quad K = (r^2 + a^2) \sigma + am.\]

The first two of these equations are the same coupled equations that are obtained in the case of Kerr space-time (and even flat space \(M = 0\)). The last two equations are generalizations of the \(r\)-dependent equations obtained for the electron. The main problem is to determine the eigenvalues \(\lambda\). Looking for series solutions of the form
\[S_{1/2} = \sum_{n=-N}^{\infty} a_n \mu_{m,n}^{1/2},\]
\[S_{-1/2} = \sum_{n=-N}^{\infty} b_n \mu_{m,n}^{1/2},\]
where \(\mu_{m,n}\) (Ref. 11) are the matrix elements of the rotation group and \(N = \min\{\lfloor m/2 \rfloor\},\) we obtain the recurrence formulas
\[
\begin{align*}
- \lambda + \frac{amm_1}{(2r - 1)r} & b_r + am e^{i(2r - 1)/2} a_r + i am e^{-i(r - 1)/2} a_{r - 1} + \frac{\sqrt{(r - m^2)(r - 1)} b_r}{(2r - 1)(r - 1)} + \frac{\sqrt{(r + 1)^2 - m^2)(r + 1)} (2r + 3)(r + 1)} b_{r + 1} \\
& = - i \left( r + \frac{1}{2} \right) \frac{i am e^{-i(r - 1)/2} a_{r - 1} + \sqrt{(r - m^2)(r - 1)} a_r + \sqrt{(r + 1)^2 - m^2)(r + 1)} b_{r + 1}}{(r + 1)(2r + 3)} \right),
\end{align*}
\]
\[
\begin{align*}
- \lambda + \frac{amm_1}{(2r + 1)r} & a_r + am e^{(2r + 1)/2} a_r + i am e^{-(r + 1)/2} a_{r + 1} + \frac{\sqrt{(r - m^2)(r - 1)} a_r}{(2r - 1)(r - 1)} + \frac{\sqrt{(r + 1)^2 - m^2)(r + 1)} (2r + 3)(r + 1)} a_{r + 1} \\
& = - i \left( r + \frac{1}{2} \right) \frac{i am e^{(r + 1)/2} b_r - iam e^{-(r + 1)/2} b_{r - 1}}{r(2r + 3)} + \frac{\sqrt{(r - m^2)(r - 1)} b_r}{(r + 1)(2r + 3)} + \frac{\sqrt{(r + 1)^2 - m^2)(r + 1)} (2r + 3)} b_{r + 1} \right).
\end{align*}
\]
These two relations can be written in the form

$$a \alpha C_{r-1} + \beta \beta C_r + \gamma C_{r+1} = 0,$$

where

$$C_r = \begin{bmatrix} u_r \\ b_r \end{bmatrix}.$$ 

In order to compute the spectrum of $\lambda$, we could suitably redefine the vector $C_r = \Omega_r D_r$ such that the three-term vector recurrence relation has the form $-\delta D_{r-1} + D_r + \epsilon D_{r+1} = 0, D_0 + \epsilon D_1 = 0$. The corresponding spectrum for $\lambda$ can then be calculated from the determinant of the matrix infinite continued fraction:

$$\text{det}(I + \epsilon I + \epsilon I + \cdots \delta I)^{-1}\delta I) = 0.$$ 

(1.13)

To obtain a three-term recurrence relation, we observe that the equations admit the discrete symmetry obtained by the transformation

$$\begin{align*}
P: \theta &\rightarrow \pi - \theta, \\
PS_{1/2}(\theta) &= S_{1/2}(\pi - \theta) = eS_{-1/2}(\theta), \\
PS_{-1/2}(\theta) &= S_{-1/2}(\pi - \theta) = eS_{1/2}(\theta).
\end{align*}$$

(1.14)

Therefore we can impose the symmetry requirements that

$$\begin{align*}
S_{1/2}(\pi - \theta) &= eS_{-1/2}(\theta), \\
S_{-1/2}(\pi - \theta) &= eS_{1/2}(\theta),
\end{align*}$$

(1.15)

where $e = \pm 1$.

Using the relation $u^{1/2}(\pi-\theta) = (-1)^{j-m} u^{1/2}(\cos \theta)$ and requiring that our solutions be eigenfunctions of $P$ with eigenvalue $e$, we see that $a_r = i e b_r (-1)^{j+1-m}$. Consequently the three-term matrix recurrence relations satisfied by the vector $C_r$ become a single three-term recurrence relation:

$$\begin{align*}
a(\epsilon \sigma - m_\epsilon) \frac{h(r,m)}{r(2r-1)} b_{r-1} \\
- a(\epsilon \sigma + m_\epsilon) \frac{h(r+1,m)}{(r+1)(2r+3)} b_{r+1} \\
+ \left[ \lambda - \frac{a m e}{2r(r+1)} \right] b_r = 0,
\end{align*}$$

(1.16)

where $h(j,m) = \sqrt{(r^2 - m^2)(r^2 - 1/4)}$ and $e_r = (-1)^j e_r$.

The expressions for the coefficients $b_r$ can be calculated iteratively using the recurrence formula, expressing the eigenvalue $\lambda$ in the form $\lambda = \Sigma_{r=0}^\infty a_r \epsilon_r$ and using the expansion $b_{j=m}/b_j = \Sigma_{k=j}^\infty A_r a_j k, j = 0, 1, 2, 3, \ldots$. The first few terms in these series are

$$\begin{align*}
\lambda_0 &= -\frac{e(2j+1)}{2}, \\
\lambda_1 &= -\frac{m(\sigma(2j+1) - m_j)}{2j(j+1)}, \\
\lambda_2 &= -\frac{2e}{(2j+1)^2} \left[ (\epsilon \sigma - m_\epsilon)^2 h^2(j,m) \\
+ \frac{(\epsilon \sigma + m_\epsilon)^2 h(j+1,m)^2}{(j+1)^2(2j+3)} \right], \\
\lambda_3 &= -\frac{4(e \sigma - m_\epsilon)^2 h^2(j,m)}{(2j-1)(2j+1)^3} \left[ (j-1)j(j+1) \\
+ \frac{2e a m j}{(j-1)(j+1)} \right] \\
- \frac{4(e \sigma + m_\epsilon)^2 h(j+1,m)}{(j+1)^2(2j+3)(2j+1)^3} \\
\times \left[ - \frac{m m_\epsilon + \sigma e}{j(j+1)(j+2)} + \frac{2e a m(j+1)}{j(j+2)} \right],
\end{align*}$$

(1.17)

for the expansion of the eigenvalue $\lambda$, and

$$\begin{align*}
\mathcal{A}_r &= -\frac{2e(m_\epsilon - \sigma e) h(j+1,m)}{(2j+1)^2(j+1)}, \\
\mathcal{A}_r^{-1} &= \frac{h(j,m)(e \sigma - m_\epsilon)}{2ej(j+1)}.
\end{align*}$$

where \( \epsilon = ( - 1)^{j+1} \).

To justify the preceding perturbation computation we note that the eigenvalue equation can be written in the form \( L\psi = \lambda\psi, L = L_0 + aV \), where

\[
L_0 = \begin{bmatrix}
0 & \frac{m}{\cos \theta} - \frac{\cot \theta}{2} \\
\frac{m}{\sin \theta} + \cot \theta & 0
\end{bmatrix},
\]

\( V = \begin{bmatrix}
-m_e \cos \theta - \epsilon \sin \theta \\
-\sigma \sin \theta + m_e \cos \theta
\end{bmatrix},
\]

\( S = \begin{bmatrix}
S_{-1/2} \\
S_{+1/2}
\end{bmatrix}, \quad \sigma^2 - m_e^2 > 0.
\]

The boundary conditions are (1.15). The inner product is

\[
(I, S) = \int_0^\pi (T^{+}_{1/2}S_{1/2} + T^{-}_{1/2}S_{-1/2}) \sin \theta \, d\theta.
\]

We restrict the argument that follows to the case when \( \epsilon = ( - 1)^{j+1} \); the case when \( \epsilon = ( - 1)^{j} \) can be treated similarly. It is easy to verify that \( L \) is formally self-adjoint with respect to the inner product. Moreover, the operator \( L_0 \) on this space is self-adjoint with discrete spectrum

\[
L_0 S_{j} = \mu S_{j}, \quad \mu_j = -j - \frac{\sigma}{2}, \quad j = 0, 1, 2, \ldots.
\]

Thus the eigenvalue decomposition for the resolvent operator \( (L_0 - \lambda I)^{-1} \) takes the form

\[
(L_0 - \lambda I)^{-1} S_{j} = \frac{-1}{j + \lambda + \frac{\sigma}{2}} S_{j}, \quad j = 0, 1, 2, \ldots.
\]

It follows that \( (L_0 - \lambda I)^{-1} \) is a compact self-adjoint operator for real \( \lambda \) not in the spectrum of \( L_0 \). The perturbing operator \( V \) is bounded and self-adjoint with operator norm \( \| V \| = \sigma \). From the identity

\[
(L - \lambda I)^{-1} = (L_0 - \lambda I)^{-1} - a(L_0 - \lambda I)^{-1}V(L_0 - \lambda I)^{-1}
\]

and the facts that (i) the product of a compact operator and a bounded operator is compact, and (ii) the sum of two compact operators is compact, it follows that the resolvent operator \( (L - \lambda I)^{-1} \) is self-adjoint and compact for real \( \lambda \) not in the spectrum. Thus \( L \) can be defined uniquely as a self-adjoint operator with discrete spectrum. It is easy to check that the spectrum is of multiplicity 1. It follows from Chap. VII of Ref. 13 that \( L = L_0 + aV \) is a so-called "self-adjoint holomorphic family of type (A)" defined for all real \( a \). The radius of convergence for each power series expansion of the perturbed eigenvalues and eigenfunctions about \( a = 0 \) is at least \( 1/(2\sigma) \) in the complex \( a \) plane. Moreover, the eigenfunctions found from the perturbation process from a Hilbert space basis for all real \( a \).

For large \( a \) the asymptotic form of the eigenvalues and eigenfunctions can be computed. In order to do this we look for solutions of the form

\[
\psi = \psi_{1/2}(\theta) + a \psi_{3/2}(\theta) + \cdots,
\]

\( \lambda = a\mu + \lambda_0 + \frac{\lambda_1}{a} + \frac{\lambda_2}{a^2} + \cdots. \)

Evaluating powers of \( a \) we find the leading order condition

\[
\phi^2 + \mu^2 - \sigma^2 + (\sigma^2 - m_e^2) \cos^2 \theta = 0.
\]

In order to obtain a single-valued expression for \( \phi \) we choose \( \mu = m_e \) for which \( \phi = \pm \sqrt{\sigma^2 - m_e^2} \cos \theta \) and the condition

\[
m_e(1 + \cos \theta) \psi_{1/2} = (\beta - \sigma) \sin \theta \psi_{-1/2}.
\]

Further conditions can be solved to give

\[
\psi_{-1/2} = m_e(\sin(\theta/2))^{-n}(-1)^{n}(\cos(\theta/2))^n, \quad \psi_{1/2} = (\beta - \sigma)(\sin(\theta/2))^{-n}(-1)^{n}(\cos(\theta/2))^n.
\]
where the integer \( n = j - \frac{2l}{m} + \frac{1}{2} |m - \frac{1}{2}| + |m + \frac{1}{2}| \).

The functions \( f_1^\pm \) are given by
\[
f_1^- = A \frac{\sin(\theta/2)}{\cos^2(\theta/2)} + B \tan \frac{\theta}{2} + C \sec^2 \frac{\theta}{2} + D \csc^2 \frac{\theta}{2}
+ E \cot \theta + F \csc \theta + G \sec \frac{\theta}{2} + H \sec \theta,
\]

where
\[
A = -\frac{m_n(2m-1)}{6\beta}, \quad B = -\frac{2m_n(m+1)}{3\beta},
\]
\[
C = -\frac{\lambda_0(\sigma - \beta)(n + \frac{1}{2})}{8\beta m_e} - \frac{\lambda_0(\sigma + \beta)(m - \frac{1}{2})}{4\beta m_e} + \frac{\lambda_0^2}{4\beta}
\]

The next term in the asymptotic series for the eigenvalue \( \lambda \) is given by
\[
\lambda m_e = -\frac{\lambda_0}{4\beta m_e} \left[ \frac{7}{2} + 2n + 2m \right] \sigma
+ \left( \frac{3}{2} - 2m \right) \beta \left( \frac{n^2 + n/2 - 1 - 3mn}{16} \right).
\]

The higher-order terms in the asymptotic series become increasingly complex. The next term is given by

\[
\lambda_2 m_e = \left[ -\frac{\lambda_1(2m+2n+1)\beta}{4m_e} + \frac{\lambda_0^2(\sigma + \beta)(2m+2n+1)}{16m_e^2} + \frac{\lambda_0(\beta - \sigma)}{8m_e^2} + \frac{(3n-1)(2m+2n+1)}{32m_e} 
- \frac{\lambda_0(\sigma - \beta)(n + \frac{1}{2})}{4m_e} \right] + \frac{1}{32} \left[ 2n(n+2) + \left( m + \frac{1}{2} \right)(n+1) \right]
\]
\[
\times \left[ \frac{\lambda_0(\sigma + \beta)(2m+n+1)}{\beta m_e} \right] + \frac{\lambda_1 m_3}{\beta} + \frac{\lambda_0(\beta - \sigma)}{64} \left( 10m + 2n + 1 \right) \right]
\]
\[
\times \left[ \frac{\lambda_0(\sigma + \beta)(2m+2n+1)}{\beta m_e} \right] + \frac{\lambda_1 m_3}{\beta m_e} + \frac{1}{32\beta} \left( 2n + 6m - 1 \right)
\]
\[
\times \left( 2n - 2m - 1 \right) \right] - \frac{\left( m + \frac{1}{2} \right)(n-3)}{8} \left[ \frac{\lambda_0(\sigma + \beta)(2m+2n+1)}{8m_e} \right] + \frac{\lambda_1 m_e}{\beta} + \frac{\lambda_0(\beta - \sigma)}{4\beta m_e}
\]
\[
- \frac{1}{8\beta} \left( 2m - 2n + 1 \right) \left( n + 2m - 1 \right).
\]
To obtain the connection with the functions that are eigenfunctions of $P$ we need only take suitable combinations of the two independent functions whose asymptotic properties we have computed here. The second solution can be obtained by taking the transformation $\theta \to \pi - \theta$.

II. THE FUNCTIONS $R_{+1/2}$ AND THEIR PROPERTIES

The coupled equations for the functions $R_{\pm 1/2}$ can be solved by methods similar to those adopted for the $\theta$-dependent functions $S_{\pm 1/2}$. Choosing new functions $V_{\pm}$ defined according to

$$V_+ = \Lambda^{1/2} R_{+1/2}, \quad V_- = R_{-1/2}$$

and a new variable $A$ defined by

$$r - M = \sqrt{\alpha^2 + \beta^2 - M^2} \sinh A$$

we can write these equations as

$$\frac{\partial}{\partial A} \left[ R \cosh A + \left( S - \frac{1}{2} \right) \tanh A + \frac{T}{\cosh A} \right] V_- = (\lambda + im\epsilon(M + \sqrt{\alpha^2 + \beta^2 - M^2} \sinh A)) V_+, \quad (2.3)$$

$$\frac{\partial}{\partial A} \left[ R \cosh A - \left( S - \frac{1}{2} \right) \tanh A - \frac{T}{\cosh A} \right] V_+ = (\lambda - im\epsilon(M + \sqrt{\alpha^2 + \beta^2 - M^2} \sinh A)) V_-,$$

where $R = i\alpha \sqrt{\alpha^2 + \beta^2 - M^2}$, $S = 2i\alpha M - ie/\sqrt{2}$, and

$$\frac{\partial}{\partial A} \left[ \frac{m\epsilon \sqrt{\alpha^2 + \beta^2 - M^2} \rho(\eta - \frac{1}{2})}{(\nu + r)(\nu + r + 1)} \right] A_r - m\epsilon \sqrt{\alpha^2 + \beta^2 - M^2} \frac{\sqrt{[(\nu + r + 1)^2 - \rho^2][(\nu + r + 1)^2 - (\eta - \frac{1}{2})^2]}}{(\nu + r + 1)(2(\nu + r) + 3)} A_{r+1}.$$
\[
\lambda - im\varepsilon M + m_e \sqrt{\alpha^2 + \beta^2 - M^2 \rho (\eta + \frac{1}{2})} \left[ B_r + m_e \sqrt{\alpha^2 + \beta^2 - M^2} \right] \left[ \frac{\sqrt{((v + r)^2 - \rho^2)}[(v + r)^2 - (\eta + \frac{1}{2})^2]}{(v + r)(2(v + r) - 1)} B_{r-1} \right]
\]
\[
+ \frac{\sqrt{((v + r + 1)^2 - \rho^2)}[(v + r + 1)^2 - (\eta + \frac{1}{2})^2]}{(v + r + 1)(2(v + r) + 3)} B_{r+1} \right]
\]
\[
= - \left[ \sqrt{\left( \frac{v + r + 1}{2} \right)^2 - \eta^2 + iR \frac{(v + r + 1)^2 - \eta^2}{(v + r)(v + r + 1)} A_{r-1} \right] \frac{\sqrt{((v + r + \eta)^2 - \frac{1}{4})[(v + r)^2 - \rho^2]}}{(v + r)(2(v + r) - 1)} A_r \left[ \frac{\sqrt{((v + r - \eta + 1)^2 - \frac{1}{4})[(v + r)^2 - \rho^2]}}{(v + r)(2(v + r) + 1)} A_{r-1} \right].
\]
\]

These relations are of the same type as derived for the functions \( S_{+1/2} \) with recurrence relations of the form
\[
\zeta_r Z_{r-1} + \eta_r Z_r + \omega_r Z_{r+1} = 0,
\]
where
\[
Z_r = \begin{bmatrix} A_r \\ B_r \end{bmatrix}.
\]

For large \( a \) representations of the solutions can be achieved as follows. The expansion of the eigenvalue \( \lambda \) takes the form
\[
\lambda = am_e \left( 1 + \frac{\lambda_1}{\alpha} + \frac{\lambda_2}{\alpha^2} + \frac{\lambda_3}{\alpha^3} + \cdots \right).
\]
For large \( a \) we seek solutions
\[
V_\pm = \psi_\pm e^{\mp r} \left[ 1 + \frac{f_1^+(r)}{\alpha} + \frac{f_2^+(r)}{\alpha^2} + \frac{f_3^+(r)}{\alpha^3} + \cdots \right].
\]
For solutions of this type the constants \( \psi_\pm \) must satisfy
\[
i(\beta + \sigma) \psi_- = m_\phi \psi_+.
\]
where \( \beta = \sqrt{\sigma^2 - m_\phi^2} \).
Without loss of generality we can assume \( f_n^+(r) = \sum_{n=0}^{\infty} a_n^+ \) with \( a_{n0}^+ = 0, n = 0, 1, 2, \ldots \). The first few terms in this expansion are
\[
v_1^- = -M - i\lambda/m_e
\]
In the case of flat space, i.e., \( e = M = 0 \), it is possible to expand one set of complete eigenfunctions of the Dirac equation in Kerr-Newman space-time.
equation in terms of another. For the Dirac equation this is achieved using the Majorana representation of the Dirac gamma matrices, viz.,

\[
\gamma = \begin{bmatrix} 0 & -\sigma \\ \sigma & 0 \end{bmatrix}, \quad \gamma^2 = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}, \quad \gamma^0 = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.
\]

(2.17)

From our knowledge of the separation of variables in oblate coordinates (i.e., inserting the conditions \( e = M = 0 \)) the separable solutions are characterized as eigenfunctions of the operator

\[
Q = \gamma^2 \nu \cdot L + \gamma^1 + \sigma \left( \gamma^2 \nu^1 \frac{\partial}{\partial x^2} + \gamma^2 \nu^0 \frac{\partial}{\partial x^1} \right).
\]

(2.18)

where \( L_i = e_{\mu \nu} \frac{\partial}{\partial x^\nu} \), \( i = 1, 2, 3 \).

We now look for solutions of the eigenvalue equation \( Q\psi = \lambda \psi \) that are also solutions of the Dirac equation:

\[
i \gamma^\mu \frac{\partial}{\partial x^\mu} \psi = m_\epsilon \psi.
\]

(2.19)

Here \( x^\mu, \mu = 0, 1, 2, 3, \) are Cartesian coordinates in Minkowski space-time. If a standard choice of spherical coordinates is made, i.e., \( x^0 = t, x^1 = \omega \sin \alpha \cos \gamma, x^2 = \omega \sin \alpha \sin \gamma, x^3 = -\omega \cos \alpha \), and a formal Fourier transform

\[
\Psi = \int \exp(ik^0 x^0 + ik \cdot x) \psi dz^0 dz \nonumber
\]

taken with \( k^0 = \sigma \).

\[
k = (k^1, k^2, k^3) = \sqrt{\sigma^2 - m_\epsilon^2} (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta),
\]

and \( dx = \sin \alpha \, d\alpha \, d\gamma \), these two conditions are equivalent to the four equations

\[
\left[ \frac{\partial}{\partial \theta} + i m \frac{\partial}{\sin \theta \partial \varphi} + \cot \theta + \frac{\sigma \alpha \sin \theta}{2} \right] \psi_4 = -(\lambda - am_\epsilon \cos \theta) \psi_4,
\]

\[
\left[ \frac{\partial}{\partial \theta} + i m \frac{\partial}{\sin \theta \partial \varphi} + \cot \theta - \frac{\sigma \alpha \sin \theta}{2} \right] \psi_2 = (\lambda - am_\epsilon \cos \theta) \psi_2,
\]

(2.20)

where \( \psi = R_1(- \theta) R_3(- \varphi) \bar{\psi} \) and \( R_1, R_3 \) are rotations about the indicated coordinate axes in three-space.

The solutions of the eigenvalue equations in the transformed space of Dirac spinors \( \bar{\psi} \) are of the form

\[
\bar{\psi}_1 = (\beta - i \alpha) S_{-1/2}(\theta) e^{i(\alpha t + m \varphi)},
\]

\[
\bar{\psi}_2 = ime S_{1/2}(\theta) e^{i(\alpha t + m \varphi)},
\]

\[
\bar{\psi}_3 = ime S_{-1/2}(\theta) e^{i(\alpha t + m \varphi)},
\]

\[
\bar{\psi}_4 = (\beta - i \alpha) S_{1/2}(\theta) e^{i(\alpha t + m \varphi)}.
\]

(2.21)

If the solutions are also eigenfunctions of the discrete transformation \( P \) then the functions \( S_{\pm 1/2} \) appearing in these expressions are just those we have already studied. The basic idea is the following. From the expressions for \( \bar{\psi} \) recover the expressions for \( \bar{\psi} \). Using the expansion of the function \( e^{i k x} \) in terms of spherical Bessel functions, as for instance found in Ref. 14, the form of \( \psi \) is recovered. The Dirac spinors that result are eigenfunctions of \( Q \) and \( P \), are solutions of Dirac's equation, and are represented relative to the Cartesian coordinate and spin frames given in the definition above. Then, transforming the spinor basis used in oblate spheroidal coordinates, we obtain expressions for solutions of the Dirac equation, in terms of series of spherical Bessel functions, that are eigenfunctions of \( Q \) and \( P \). The expressions for the components are given by

\[
\psi_1 = \sum_{L=0}^{\infty} \sum_{m=-L}^{L} \left[ (\beta - i \alpha) b L(r, m | L, K) \times C_{L+1/2}^{1/2} - \frac{1}{2} r m | L, 0) - m_\epsilon \epsilon \alpha C_{L+1/2}^{1/2} - \frac{1}{2} r m | L, 0) \right] \times C_{L+1/2}^{1/2} - \frac{1}{2} L, 0) \times i^L f_L(\beta r) u_m^{L-1/2} \cos \alpha \right] e^{i(m-1/2)\gamma}.
\]
\[ \psi_2 = \sum_{L=0}^{\infty} \sum_{m=-L}^{L} [i(\beta - i\sigma) b, C(\frac{1}{2}, \frac{1}{2}; r, m | L, K) \times C(\frac{1}{2}, -\frac{1}{2}; r, \frac{1}{2} | L, 0)] \\
+ im_c C(\frac{1}{2}, \frac{1}{2}; r, m | L, K) C(\frac{1}{2}, -\frac{1}{2}; r, -\frac{1}{2} | L, 0)] \]

\[ \times i^L \mathcal{E} (\beta r) u_m + \frac{1}{2} \cos \alpha \Theta (m + \frac{1}{2}) \gamma, \quad (2.22) \]

where \( C(j, m; l, n | L, M) \) are the Clebsch–Gordan coefficients of the rotation group. The expressions for \( \psi_3 \) and \( \psi_4 \) can be obtained by interchanging the expressions \( (\beta - i\sigma) \) and \( im_c \) in the expressions for \( \psi_1 \) and \( \psi_2 \) to obtain \( \psi_3 \) and \( \psi_4 \), respectively. The change in spin frame from Cartesian coordinates to the frame specified by the null tetrad given above can be readily computed. We standardize the choice of oblate coordinates as \( z^0 = t, z^1 = \sqrt{r^2 + a^2} \sin \alpha \cos \gamma, z^2 = \sqrt{r^2 + a^2} \sin \alpha \sin \gamma, \) and \( z^3 = r \cos \alpha \). The Lorentz transformation that maps the vector fields \( D^\mu = \partial/\partial z^\mu \) into the vector fields

\[ D^0 = \frac{1}{\sqrt{2}}(\mu + n^\mu) \frac{\partial}{\partial z^0}, \]
\[ D^1 = \frac{1}{\sqrt{2}}(\mu - n^\mu) \frac{\partial}{\partial z^1}, \quad (2.23) \]
\[ D^2 = \frac{1}{\sqrt{2}}(m^\mu + \bar{m}^\mu) \frac{\partial}{\partial z^2}, \]
\[ D^3 = \frac{1}{\sqrt{2}}(m^\mu - \bar{m}^\mu) \frac{\partial}{\partial z^3}, \]

according to \( D^\mu = L^\mu D^\nu \) can readily be calculated. The matrix elements \( L^\nu_{\mu} \) are given by

\[ L^0_0 = \frac{1}{\sqrt{2}} \left[ \frac{1 + (r^2 + a^2)}{\rho^2} \right], \]
\[ L^0_1 = \frac{1}{\sqrt{2}} \left[ r \sin \alpha \cos \gamma \left( \frac{1}{\sqrt{r^2 + a^2}} - \frac{r^2 + a^2}{\rho^2} \right) + a \sin \alpha \sin \gamma \left( -\frac{1}{\sqrt{r^2 + a^2}} - \frac{r^2 + a^2}{\rho^2} \right) \right], \]
\[ L^0_2 = \frac{1}{\sqrt{2}} \left[ r \sin \alpha \sin \gamma \left( \frac{1}{\sqrt{r^2 + a^2}} - \frac{r^2 + a^2}{\rho^2} \right) + a \sin \alpha \cos \gamma \left( \frac{1}{\sqrt{r^2 + a^2}} + \frac{r^2 + a^2}{\rho^2} \right) \right] \]

\[ (2.24) \]
The solutions of Dirac’s equation, relative to the frame (1.2), denoted by

\[ L^3 = \frac{1}{\sqrt{2}} \begin{bmatrix} \bar{r} \sin \alpha \sin \gamma & \frac{1}{\sqrt{r^2 + a^2}} + \frac{\sqrt{r^2 + a^2}}{\rho^2} \\ \bar{r} \sin \alpha \cos \gamma & \frac{1}{\sqrt{r^2 + a^2}} - \frac{\sqrt{r^2 + a^2}}{\rho^2} \end{bmatrix} + a \sin \alpha \cos \gamma \begin{bmatrix} 1 \\ \frac{1}{\sqrt{r^2 + a^2}} \end{bmatrix} \]

The solutions of Dirac’s equation, relative to the frame (1.2), denoted by

\[ 0\psi = \begin{bmatrix} \phi_x \\ \chi' y \end{bmatrix} \]

are obtained by applying \( \Lambda(r, \alpha, \gamma) \), the representative of \( L(r, \alpha, \gamma) \) acting on the Dirac spinors \( \Psi \). However, these are just the solutions of Dirac’s equation found in terms of Teukolsky functions above. What has been achieved here is expressions for these solutions in terms of sums of spherical Bessel functions and spherical harmonics. Clearly these expressions will, in general, be quite complex. We content ourselves with the derivation of an expansion formula for the functions \( R_{1/2} \). Indeed if we take \( \alpha = \theta = \pi/2, \omega = \sqrt{\rho^2 + a^2} \), then \( L \) can be factored in the form \( R_3(\varphi)N_1(A)R_3(\delta) \) where

\[ e^A = 1 + \frac{a^2}{2r} + \frac{a^2}{r} \left[ 1 + \frac{a^2}{4r^2} \right]^{1/2} \]

\[ e^B = \left[ (r + ia)(r + ia/2) \right]^{1/2} \left[ (r - ia)(r - ia/2) \right] \]

The corresponding transformation matrix acting on the Dirac spinor then has the form \( \Lambda(r, \pi/2, \gamma) = e^{ij} j^{\gamma} e^{ij} e^{ij} A^{ij} e^{ij} b^{ij} r^{j/2} \) and, consequently, \( 0\psi = \Lambda(r, \pi/2, \gamma) \psi \).

The small \( a \) expansion of \( P_{\pm} = V_{\pm} \pm V_{\mp} \) can be expressed in terms of Bessel functions. The first few terms in the expressions are

\[ P_+ = (m_\pm - \sigma)^{1/2} J_{j+1}(\beta r)^{1/2} - \frac{am}{2(j + 1)} \]

\[ \times (\sigma - m_\pm)^{1/2} J_{j+1}(\beta r)^{-1/2} + \frac{a\beta(\sigma + m_\pm)^{1/2}}{16\beta} \]

\[ \times [j(j + 1) + 2m^2] J_{j-1}(\beta r)^{-1/2} + \cdots, \quad (2.25) \]

An interesting application of these solutions would be the solution of the MIT bag model of confinement for which the boundary is the surface of an oblate spheroid. For this problem the appropriate boundary condition is

\[ i\gamma^\mu n_\mu \psi = \psi \quad \text{for} \quad r = r_0 \quad (2.26) \]

where \( n_\mu n_\mu = -1 \) and \( r = r_0 \) is the surface of the spheroidal bag. In the tetrad formalism the nonzero components of the unit normal spacelike vector \( n_\mu \) are \( n_{00} = \sqrt{\rho^2/a^2}, n_{11} = -\sqrt{\Delta/2\rho} \). If we naively apply the boundary conditions to a single solution of the Dirac equation as found in Sec. I this would require that \( R_{1/2} = \sqrt{-\rho/\rho^2} R_{-1/2} \) for \( r = r_0 \) which is clearly impossible. If, however, the boundary conditions are modified so as to be of the form

\[ i\gamma^\mu n_\mu \psi = (\cos \alpha + i\gamma^0 \sin \alpha) \psi, \quad (2.27) \]

where \( \gamma^0 = \sqrt{-\rho^2/\rho} \), then this boundary value problem can be more readily solved, as it reduces to the requirement that \( R_{1/2} = i\Delta^{-1/2} R_{-1/2} \) for \( r = r_0 \). The boundary conditions (2.27) do not adequately describe a quantum bag although they do imply that the probability density vanishes on the bag surface. In order to solve the bag model conditions a solution must be represented as a sum of eigenfunctions of the type developed above. There are then no problems in principle with the bag boundary conditions. We shall return to this problem subsequently.

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