# Complete sets of functions for perturbations of Robertson Walker cosmologies

E. G. Kalnins
*Department of Mathematics and Statistics, University of Waikato, Hamilton, New Zealand*
W. Miller Jr.
*School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455*

(Received 5 April 1990; accepted for publication 24 October 1990)

Crucial to a knowledge of the perturbations of Robertson Walker cosmological models is a knowledge of complete sets of functions with which to expand such perturbations. For the open Robertson Walker cosmology, this question will be completely answered. In addition, some observations will be made concerning explicit solution by separation of variables of wave equations for spin $s$ in a Riemannan space having an infinitesmal line element of which the Robertson Walker models are a special case.

## I. VECTOR AND TENSOR HARMONICS ON THREE-DIMENSIONAL SPACES OF CONSTANT RIEMANNIAN CURVATURE

The original investigations of Lifshitz[1] and Lifshitz and Khalatnikov[2] into the gravitational stability of the Robertson Walker (RW) isotropic cosmological models[3] demonstrated the utility of scalar, vector, and tensor harmonics in giving a complete description of small perturbations. In particular these authors[1,2] showed that in the synchronous gauge all perturbations involving pressure, density, velocity, and metric fluctuations can be obtained once a complete set of such functions is found for $S_3$ (three-dimensional sphere), $E_3$ (Euclidean three space), or $H_3$ (three-dimensional hyperbolic space). The choice of three-dimensional manifold is determined by whether the closed, flat or open RW model is used. In the book by Landau and Lifshitz[3] a complete set of basis functions is derived for the conformally flat RW model in which a general tensor field $h_{\alpha\beta}$ on $E_3$ can be expanded in terms of three families of functions related to three-dimensional plane waves.

(1) Using the scalar function $Q = e^{i\mathbf{n}\cdot\mathbf{r}}$ the tensor functions

$$Q_{\alpha\beta} = \frac{1}{3} g_{\alpha\beta} Q, \quad P_{\alpha\beta} = \left(\frac{1}{3} g_{\alpha\beta} - \frac{n^\alpha n^\beta}{(\mathbf{n}\cdot\mathbf{n})}\right) Q, \quad P^{\alpha}{}_{\alpha} = O \tag{1.1}$$

are formed. These plane waves in the conformally flat model correspond to perturbations in which the gravitational field, velocity, and density vary.

(2) With the transverse vector wave $\mathbf{S} = \mathbf{s}e^{i\mathbf{n}\cdot\mathbf{r}}$, $\mathbf{s}\cdot\mathbf{n} = 0$ the tensor $S_{\alpha\beta} = n_\alpha S_\beta + n_\beta S_\alpha$ satisfies $S^{\alpha}{}_{\alpha} = 0$. These waves correspond to perturbations in which the gravitational field and velocity vary but not the density.

(3) The transverse tensor waves $G_{\alpha\beta} = U_{\alpha\beta} e^{i\mathbf{n}\cdot\mathbf{r}}$ where the symmetric tensor $U_{\alpha\beta}$ satisfies $U_{\alpha}{}^{\beta} n_\beta = 0, U_{\alpha}{}^{\alpha} = 0$. These waves correspond to gravitational waves.

The expansion of a symmetric tensor $h_{\alpha\beta}$ can then be given in terms of the three families of functions. In fact the various families can be invariantly characterized on $E_3$ according to

$$\Delta W_{\alpha\beta} = (\nabla^\gamma \nabla_\gamma) W_{\alpha\beta} = -n^2 W_{\alpha\beta}, \tag{1.2}$$

where

$$W_{\alpha\beta} = Q_{\alpha\beta},\ P_{\alpha\beta},\ S_{\alpha\beta},\ G_{\alpha\beta},$$

$$\nabla^\alpha G_{\alpha\beta} = 0, \quad S^{\alpha}{}_{\alpha} = G^{\alpha}{}_{\alpha} = P^{\alpha}{}_{\alpha} = 0.$$

Accordingly, this set of functions is but one choice of many possible complete sets of functions which could be obtained from the above equations, e.g., we could have chosen spherical coordinates and expanded the components of the tensor $h_{\alpha\beta}$ in a suitable set of spherical waves. As the underlying space in this case is $E_3$ there is a six-dimensional isometry group $E_3$ consisting of translations and rotations. If we choose a basis of eigenvectors of the translation operators we recover the basis of plane waves discussed above. We note also that

$$\int W^{\alpha\beta} \overline{W}^{*}_{\alpha\beta}\, d\mathbf{r} = 0, \tag{1.3}$$

when $W_{\alpha\beta}$, $\overline{W}_{\alpha\beta}$ are not from the same type and that each contributing tensor harmonic satisfies

$$P_\alpha W_{\beta\gamma} = \partial_\alpha W_{\beta\gamma} = i n_\alpha W_{\beta\gamma}, \tag{1.4}$$

the $P_\alpha$ being the translation generators of the six-dimensional isometry group of $E_3$ (the others being rotations). The analogous problem for the closed RW universe has been solved by Gerlach and Sengupta.[4] A general tensor field on $S_3$ is expanded in terms of three families of functions in direct analogy with the flat space case.

(1) From scalar eigenfunctions of the Laplace operator $Q$ on $S_3$, viz.,

$$\Delta Q = (\nabla^\gamma \nabla_\gamma) Q = -(n^2 - 1) Q \tag{1.5}$$

and for $n$ an integer, the tensor fields

$$Q_{\alpha\beta} = \frac{1}{3} g_{\alpha\beta} Q,$$

$$P_{\alpha\beta} = \frac{1}{(n^2 - 1)} \nabla_\beta \nabla_\alpha Q + Q_{\alpha\beta}, \quad P^{\alpha}{}_{\alpha} = 0 \tag{1.6}$$

are constructed.

 0022-2488/91/051415-09$03.00  

(2) From vector eigenfunctions of the Laplace operator $S_\alpha$ which are divergenceless, a tensor $S_{\alpha\beta} = \nabla_\alpha S_\beta + \nabla_\beta S_\alpha$ can be constructed where

$$\Delta S_\alpha = -(n^2-2)S_\alpha, \quad \nabla^\alpha S_\alpha = 0. \tag{1.7}$$

(3) From tensor eigenfunctions of the Laplace operator $G_{\alpha\beta}$, one can construct solutions that are symmetric, divergenceless, traceless,

$$\Delta G_{\alpha\beta} = -(n^2-3)G_{\alpha\beta}, \quad \nabla^\alpha G_{\alpha\beta} = 0, \quad G^\alpha{}_\alpha = 0.$$

Gerlach and Sengupta[4] developed a complete set of solutions for tensors of these types in terms of an angular momentum basis. The results are correct but can be derived more neatly using a knowledge of the group representation theory of SO(4) acting on $S_3$. In the open RW model the problem of a complete set of basis functions has, as far as we know, yet to be fully elucidated. In this article we explicitly compute a basis with which to expand second-order tensors $h_{\alpha\beta}$ on $H_3$. We do this by using group theory and the inherent completeness results obtained by Naimark[5] and Gelfand *et al.*[6] The manifold $H_3$ is realized on the upper sheet of the two sheeted hyperboloid:

$$\nu_0^2 - \nu_1^2 - \nu_2^2 - \nu_3^2 = 1, \nu_0 > 1. \tag{1.8}$$

We choose spherical coordinates on the hyperboloid, viz.,

$$\begin{aligned}\nu &= (\nu_0,\nu_1,\nu_2,\nu_3)\\ &= (\cosh a, \sinh a \sin\theta\cos\phi,\\ &\quad \sinh a\sin\theta\sin\phi, \sinh a\cos\theta)\\ &\quad 0<a<\infty, \quad 0\leqslant\theta<\pi, \quad 0\leqslant\phi<2\pi,\end{aligned} \tag{1.9}$$

with line element

$$ds^2 = da^2 + \sinh^2 a(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{1.10}$$

In order to obtain a complete set of functions with which to expand second-order tensors we proceed as outlined above.

(1) Scalar functions $Q$ that satisfy

$$\Delta Q = -(1+\rho^2)Q \tag{1.11}$$

are readily obtained. A complete set of such functions in the coordinate basis given above is

$$\Phi^{\rho 0}_{00J}(a) D^J_{0M}(0,\theta,\phi), \quad 0<\rho<\infty;\ J=1,2,\ldots;\ |M|\leqslant J, \tag{1.12}$$

where $D^J_{MN}(\psi,\theta,\phi)$ is a matrix element of the rotation group in the Euler parametrization and $\Phi^{\rho m}_{l\lambda J}(a)$ the matrix elements of the group element $N_3(a)$ in an angular momentum basis for the unitary irreducible representation labeled by $[m,i\rho]$. These functions and their properties are discussed in the Appendix.

(2) Vector harmonics $S_\alpha$. The functions we require in this case must be eigenfunctions of $\Delta$ and divergenceless. Taking the choice of coordinates given in (1.9) we may write

$$\begin{aligned}\nu &= (\nu_0,\nu_1,\nu_2,\nu_3)\\ &= R_3(\phi)R_1(\theta)N_3(a)R_3(\alpha)R_1(\beta)R_3(\gamma)\bar{\nu}\\ &= R_3(\phi)R_1(\theta)N_3(a)\bar{\nu},\end{aligned} \tag{1.13}$$

where $\bar{\nu} = (1,0)$. Given a relativistic vector field $S_b, b=0,1,2,3$ the action on $S_b$ induced by the Lorentz group is

$$T_g S_b(x) = D^{\{0,2\}}{}_b{}^c(g) S_c(g^{-1}x), \tag{1.14}$$

where

$$x = r\nu, g\in SO(3,1), r>0.$$

This is just the normal transformation law for relativistic fields. We define new vector fields by

$$S'_b(g) = D^{\{0,2\}}{}_b{}^c(g) S_c(g^{-1}\bar{\nu}). \tag{1.15}$$

These new fields transform according to

$$\begin{aligned}T_g S'_b(g) &= D^{\{0,2\}}{}_b{}^c(gg') S_c(g'^{-1}g^{-1}\bar{\nu})\\ &= S'_b(gg'),\end{aligned} \tag{1.16}$$

i.e., the individual components of the new vector fields $S'_b$ transform independently. For the Euler parametrization of a Lorentz group element given in (1.13) we can write $S'_b(g)$ as

$$\begin{aligned}S'_b(g) &= D^{\{0,2\}}{}_b{}^c(R) S_c(a,\theta,\phi)\\ R &= R_3(-\gamma)R_1(-\beta)R_3(-\alpha).\end{aligned} \tag{1.17}$$

The functions $S'_b(g)$ transform under the Lorentz group according to the regular representation and are of the specific form given in (1.13). From the decomposition of the regular representation of the Lorentz group into its unitary irreducible components, a complete set of basis functions can be taken as

$$\begin{aligned}&D^l_{N\lambda}(\alpha,\beta,\gamma)\Phi^{\rho m}_{l\lambda J}(a) D^J_{\lambda m}(0,\theta,\phi)\\ &0<\rho<\infty; \quad m=0,\pm 1,\pm 2,\ldots;\\ &J,l = |m|,|m|+1,\ldots,\\ &|N|\leqslant l, \ |M|\leqslant J, \ |\lambda|\leqslant \min(l,J).\end{aligned} \tag{1.18}$$

For functions of the form (1.17) the expansion functions for $S_b(a,\theta,\phi)$ are

$$\begin{aligned}&\Phi^{\rho m}_{l\lambda J}(a) D^J_{\lambda M}(0,\theta,\phi)\\ &0<\rho<\infty;\ l,m=0,\pm 1;\ J=|m|,|m|+1,\ldots;\\ &|\lambda|\leqslant\min(l,J), |M|\leqslant J.\end{aligned} \tag{1.19}$$

If we choose a frame in space-time at each point we can, without loss of generality, choose the frame such that $\alpha=\beta=\gamma=0$ and identify $S_b(a,\theta,\phi)$ as our set of vector fields. The above expansion functions then form a complete set for a general vector field. Proca's equation (and hence Maxwell's equations) can be solved in these coordinates. Agamaliev, Atakashiev, and Verdiev[7] have indicated how this can be done in Minkowski space-time. Returning to the problem on the manifold $H_3$, we seek transverse fields corresponding to spin 1 as a result of the condition $\nabla^\alpha S_\alpha = 0$. These functions can be obtained from considerations in Minkowski space-time as follows. Consider a general point in Minkowski space-time as $x = r\nu$ and choose the frame of one-forms:

$$e_{(0)i}\,dx^i = dr, \quad e_{(1)i}dx^i = r\,da,$$

$$e_{(2)i}\,dx^i = \frac{1}{\sqrt{2}} r \sinh a (d\theta + i \sin\theta\, d\phi),$$

$$e_{(3)i}\,dx^i = \frac{1}{\sqrt{2}} r \sinh a (d\theta - i \sin\theta\, d\phi). \tag{1.20}$$

Then the components of the vector field $S_b$ referred to this frame, viz., $S_b$ can be expanded in terms of the functions

$$S_0 = f_1(r)\Phi^{\rho 0}_{00J}(a) D^J_{0M}(0,\theta,\phi),$$

$$S_1 = f_2(r)\Phi^{\rho m}_{10J}(a) D^J_{0M}(0,\theta,\phi), \quad m = 0, \pm 1,$$

$$S_2 = f_2(r)\Phi^{\rho m}_{11J}(a) D^J_{1M}(0,\theta,\phi), \quad m = 0, \pm 1,$$

$$S_3 = f_2(r)\Phi^{\rho m}_{1-1J}(a) D^J_{-1M}(0,\theta,\phi), \quad m = 0, \pm 1, \tag{1.21}$$

the $r$ dependence being chosen so as to obtain a complete set of functions on $H_3$. This is done by taking $f_1 = 0$ and choosing solutions of $\Delta' S_b = (\nabla^c \nabla_c) S_b = 0$ to have $f_2(r) = r^{i\rho}$. The vectors $S_\alpha$ are then solutions of

$$\Delta S_\beta = (\nabla^\alpha \nabla_\alpha) S_\beta = -(\rho^2 + 2) S_\beta, \quad \beta = 1,2,3, \tag{1.22}$$

and $\nabla^\beta S_\beta = 0$, i.e., a suitable basis for transverse vector functions relative to the frame $e_{(a)}, a = 1,2,3$ consists of the functions

$$S_1 = \Phi^{\rho \pm 1}_{10J}(a) D^J_{0M}(0,\theta,\phi),$$
$$S_2 = \Phi^{\rho \pm 1}_{11J}(a) D^J_{1M}(0,\theta,\phi),$$
$$S_3 = \Phi^{\rho \pm 1}_{1-1J}(a) D^J_{-1M}(0,\theta,\phi), \tag{1.23}$$

for

$$0 < \rho < \infty; \; J = 1,2, \ldots; \; |M| \leqslant J.$$

Even and odd parity states can be constructed by realizing that the parity operation corresponds to the replacement $a \to -a$ and the matrix element functions $\Phi^{\rho m}_{l\lambda J}(a)$ satisfy

$$\Phi^{\rho m}_{l\lambda J}(a) = (-1)^{l-J} \Phi^{\rho -m}_{l\lambda J}(-a). \tag{1.24}$$

(3) Tensor harmonics $G_{\alpha\beta}$. The functions we require in this case must be eigenfunctions of $\Delta$, traceless and divergenceless. As with the case of vector harmonics we consider the relativistic tensor fields that transform under the Lorentz group according to

$$T_g G_{bc}(x) = D^{[0,3]}{}_{bc}{}^{de}(g) G_{de}(g^{-1}x). \tag{1.25}$$

Defining new vector fields

$$G'_{bc}(g) = D^{[0,3]}{}_{bc}{}^{de}(g) G_{de}(g^{-1}\bar{v}), \tag{1.26}$$

then these fields transform according to

$$T_{g'} G'_{bc}(g) = D^{[0,3]}{}_{bc}{}^{de}(gg') G_{de}(g'^{-1} g^{-1} \bar{v}) = G'_{bc}(gg'). \tag{1.27}$$

Then writing

$$G'_{bc}(g) = D^{[0,3]}{}_{bc}{}^{de}(R) G_{ed}(a,\theta,\phi), \tag{1.28}$$

where $R = R_3(-\gamma) R_1(-\beta) R_3(-\alpha)$, we argue just as we did in the vector case that the suitable basis of expansion functions for functions $G_{cd}(a,\theta,\phi)$ are as in (1.18), but with

$$0 < \rho < \infty; \quad l,m = 0, \pm 1, \pm 2; \quad J = |m|, |m| + 1, \ldots;$$
$$|\lambda| \leqslant \min(l,J), |M| \leqslant J.$$

If we fix a frame as before by taking $\alpha = \beta = \gamma = 0$, we can identify $G_{cd}(a,\theta,\phi)$ as our set of tensor fields. In order to identify which components of $G_{cd}(a,\theta,\phi)$ enable the canonical action of the rotation group to be realized we use the tetrad defined by (1.20). A suitable choice of tensor harmonics is

$$G_{00} = f_3(r)\Phi^{\rho m}_{00J}(a) D^J_{0M}(0,\theta,\phi),$$

$$G_{11} = \left[\sqrt{\tfrac{2}{3}} f_1(r)\Phi^{\rho m}_{20J}(a) + \tfrac{1}{3} f_3(r)\Phi^{\rho m}_{00J}(a)\right] D^J_{0M}(0,\theta,\phi),$$

$$G_{01} = \left[(\tfrac{2}{3}) f_3(r)\Phi^{\rho m}_{00J}(a) - (1/\sqrt{2}) f_2(r)\Phi^{\rho m}_{10J}(a)\right] \times D^J_{0M}(0,\theta,\phi),$$

$$G_{02} = (i/\sqrt{2}) f_2(r)\Phi^{\rho m}_{1-1J}(a) D^J_{-1M}(0,\theta,\phi),$$

$$G_{03} = (i/\sqrt{2}) f_2(r)\Phi^{\rho m}_{11J}(a) D^J_{1M}(0,\theta,\phi),$$

$$G_{12} = (i/\sqrt{2}) f_1(r)\Phi^{\rho m}_{21J}(a) D^J_{1M}(0,\theta,\phi),$$

$$G_{13} = (i/\sqrt{2}) f_1(r)\Phi^{\rho m}_{2-1J}(a) D^J_{-1M}(0,\theta,\phi),$$

$$G_{33} = f_1(r)\Phi^{\rho m}_{2-2J}(a) D^J_{-2M}(0,\theta,\phi),$$

$$G_{22} = f_1(r)\Phi^{\rho m}_{22J}(a) D^J_{2M}(0,\theta,\phi),$$

$$G_{23} = (i/3) G_{00} - (1/\sqrt{6}) G_{11}. \tag{1.29}$$

Here, $m = 0, \pm 1, \pm 2$ where appropriate. The functions $f_i, i = 1,2,3$ are chosen in such a way as to make the orthogonality relations for the functions $G_{bc}$ coincide with those conditions given in the Appendix. If we now seek divergence-free solutions that satisfy $\nabla^b G_{bc} = 0$ we take $G_{0a} = 0$ for all $a$. Then we obtain the two independent solutions by taking $f_1 = r^{-1 + i\rho}$, which are solutions of

$$\Delta G_{\beta\gamma} = (\nabla^\alpha \nabla_\alpha) G_{\beta\gamma} = -(3 + \rho^2) G_{\beta\gamma} \tag{1.30}$$

and $\nabla^\alpha G_{\alpha\beta} = 0$. A suitable basis of functions is

$$G_{11} = \sqrt{2/3}\,\Phi^{\rho m}_{20J}(a) D^J_{0M}(0,\theta,\phi),$$
$$G_{12} = (i/\sqrt{2})\Phi^{\rho m}_{21J}(a) D^J_{1M}(0,\theta,\phi),$$
$$G_{13} = (i/\sqrt{2})\Phi^{\rho m}_{2-1J}(a) D^J_{-1M}(0,\theta,\phi),$$
$$G_{33} = \Phi^{\rho m}_{2-2J}(a) D^J_{-2M}(0,\theta,\phi),$$
$$G_{22} = \Phi^{\rho m}_{22J}(a) D^J_{2M}(0,\theta,\phi),$$
$$G_{23} = (-1/2) G_{11}, \quad m = \pm 2. \tag{1.31}$$

By using the forms of the transverse vector fields $S_\alpha$ and the scalar field $Q$, the traceless fields given previously and the recurrence formulas of the Appendix, all the traceless components in the expansion of the field $h_{\alpha\beta}$ are then given by allowing $m = 0, \pm 1, \pm 2$ in (1.31). The remaining component having trace is simply $G_{\alpha\beta} = g_{\alpha\beta}\Phi^{\rho 0}_{00J}(a) D^J_{0M}(0,\theta,\phi)$. This then gives the complete set of functions with which to expand a tensor on $H_3$.

## II. SEPARATION OF VARIABLES FOR GENERALIZATIONS OF ROBERTSON WALKER TYPE SPACE-TIMES

In addition to the problem of determining complete sets of functions for the expansion of vector and tensor fields on $H_3$ there has been considerable interest in the intrinsic characterization of solutions of the nonscalar equations of mathematical physics. Considerable attention has been paid to this topic and we mention, in particular, studies of the Dirac equation[7-9] and Maxwell's equations.[10] In this section we discuss some extensions of the results of Kamran and Fels.[11] These authors studied the metric given in local coordinates by the line element

$$ds^2 = dt^2 - a^2(t)(dx^2 + b^2(x)dy^2 + c^2(y)dz^2). \quad (2.1)$$

In the null frame specified by the one-forms

$$\begin{aligned}
e_{(0)i}\, dx^i &= (1/\sqrt{2})(dt - a\, dx),\\
e_{(1)i}\, dx^i &= (1/\sqrt{2})(dt + a\, dx),\\
e_{(2)i}\, dx^i &= (1/\sqrt{2})ab(dy + ic\, dz),\\
e_{(3)i}\, dx^i &= (1/\sqrt{2})ab(dy - ic\, dz).
\end{aligned} \quad (2.2)$$

Kamran and Fels[12] demonstrated that the Dirac equation could be solved by a separation of variables procedure that is described by second-order symmetries. We demonstrate that Maxwell's equations in their spinor and vector potential forms also admit separable solutions in direct analogy with what happens for the RW metrics, but that for spin $s \geqslant 2$ the solution mechanism breaks down. The null frame can be intrinsically characterized by using the observation that the Riemannian space with line element (2.1) admits a valence two Killing–Yano tensor having nonzero component $K^{yz} = 1/(abc)$. If we look for simultaneous eigenvectors of $K^{bc}$ and its dual $K^*_{bc} = \varepsilon_{bcde} K^{de}$ the corresponding eigenvectors are

$$\begin{aligned}
l^i_{(0)} &= (1,a,0,0), & l^i_{(1)} &= (1,-a,0,0),\\
l^i_{(2)} &= (0,0,1,i\sin\theta), & l^i_{(3)} &= (0,0,1,i\sin\theta),
\end{aligned} \quad (2.3)$$

with eigenvalues given according to Table I.

The null frame specified by the forms (2.2) is the natural one for the spinorial form of Maxwell's equations. However, for the vector potential form the quasidiagonal tetrad is more suitable. This can be characterized intrinsically by realizing that there is also a Killing–Yano tensor of valence 3 for the Riemannian space with line element (2.1) with components $K_{bcd} = \varepsilon_{bcde} K^e$ where the only nonzero element of $K_e$ is $K_t = a$. If we now look for simultaneous eigenvectors of $\hat{K}_{bc} = K_b K_c - (\nabla^d K_d) g_{bc}$ and $K_{bd}$ we recover eigenvectors in the quasidiagonal tetrad. In the case of the form of Maxwell's equations written in terms of the vector potential we solve the more general problem of the massive spin 1 equation, viz.,

$$\Delta' A_b - R^{\,c}_{\,b} A_c = m^2 A_b, \quad \nabla^d A_d = 0. \quad (2.4)$$

If instead of the frame $e^i_{(a)}$ we choose the quasidiagonal frame specified by

$$E^i_{(0)} = e^i_{(0)} + e^i_{(1)}, E^i_{(1)} = e^i_{(0)} - e^i_{(1)}, E^i_{(K)} = e^i_{(K)}\, i = 2,3$$

then Maxwell's equations have the form

$$\begin{aligned}
&\left[\Delta_{KG} + 3\frac{a_{tt}}{a} - 3\left(\frac{a_t}{a}\right)^2\right]A_0 + 2\frac{a_t}{a}\left(\partial_x + \frac{2b_x}{b}\right)A_1\\
&\quad - \frac{\sqrt{2}a_t}{a^2 b}\left[\left(\partial_y - \frac{i}{c}\partial_z + \frac{c_y}{c}\right)A_2\right.\\
&\quad \left. + \left(\partial_y + \frac{i}{c}\partial_z + \frac{c_y}{c}\right)A_3\right] = m^2 A_0,\\
&\left[\Delta_{KG} + \frac{a_{tt}}{a} + \left(\frac{a_t}{a}\right)^2 + \frac{2}{a^2}\right.\\
&\quad \left.\times\left[\left(\frac{b_x}{b}\right)^2 - \frac{b_{xx}}{b}\right]\right] A_1 + 2\frac{a_t}{a}\partial_x A_0\\
&\quad - \frac{\sqrt{2}b_x}{a^2 b^2}\left[\left(\partial_y - \frac{i}{c}\partial_z + \frac{c_y}{c}\right)A_2\right.\\
&\quad \left. - \left(\partial_y + \frac{i}{c}\partial_z + \frac{c_y}{c}\right)A_3\right] = m^2 A_1,\\
&\left[\Delta_{KG} - \frac{2c_y}{a^2 b^2 c} i\partial_z + \frac{a_{tt}}{a} + \left(\frac{a_t}{a}\right)^2 + \left(\frac{c_y}{abc}\right)^2\right.\\
&\quad \left. + \frac{1}{a^2}\left[\left(\frac{b_x}{b}\right)^2 - \frac{b_{xx}}{b}\right]\right] A_2 - \frac{\sqrt{2}a_t}{a^2 b}\left(\partial_y + \frac{i}{c}\partial_z\right)A_0\\
&\quad + \frac{\sqrt{2}b_x}{a^2 b^2}\left(\partial_y + \frac{i}{c}\partial_z\right)A_1 = m^2 A_2, \qquad (2.5)\\
&\left[\Delta_{KG} + \frac{2c_y}{a^2 b^2 c} i\partial_z + \frac{a_{tt}}{a} + \left(\frac{a_t}{a}\right)^2 + \left(\frac{c_y}{abc}\right)^2\right.\\
&\quad \left. + \frac{1}{a^2}\left[\left(\frac{b_x}{b}\right)^2 - \frac{b_{xx}}{b}\right]\right] A_3 - \frac{\sqrt{2}a_t}{a^2 b}\left(\partial_y - \frac{i}{c}\partial_z\right)A_0\\
&\quad + \frac{\sqrt{2}b_x}{a^2 b^2}\left(\partial_y - \frac{i}{c}\partial_z\right)A_1 = m^2 A_3,\\
&\left(\partial_t + \frac{3a_t}{a}\right)A_0 - \frac{1}{a}\left(\partial_x + \frac{2b_x}{b}\right)A_1\\
&\quad + \frac{1}{\sqrt{2}ab}\left[\left(\partial_y + \frac{-i}{c}\partial_z + \frac{c_y}{c}\right)A_2\right.
\end{aligned}$$

TABLE I. Eigenvalues for the corresponding eigenvectors given in Eq. (2.3).

| | Eigenvalues of $K_{bc}$ | Eigenvalues of $K^*_{bc}$ |
|---|---|---|
| $l_{(0)}$ | 0 | $1/bc$ |
| $l_{(1)}$ | 0 | $-1/bc$ |
| $l_{(2)}$ | $i$ | 0 |
| $l_{(3)}$ | $-i$ | 0 |

$$+\left(\partial_y+\frac{i}{c}\partial_z+\frac{c_y}{c}\right)A_3=0\Bigg],$$

where

$$\Delta_{KG}=g^{\alpha\beta}\partial_\alpha\partial_\beta.$$

There are two families of solutions for these equations.

(1) We write

$$A_0=a_0g_1(y)e^{-i\lambda z},\qquad A_1=a_1g_1(y)e^{-i\lambda z},$$
$$A_2=(1/\sqrt{2})a_2g_0(y)e^{-i\lambda z},\quad A_3=(1/\sqrt{2})a_3g_2(y)e^{-i\lambda z},\tag{2.6}$$

where the functions $g_i, i=0,1,2$ satisfy the first-order system

$$(\partial_y-(\lambda/c)+(c_y/c))g_0(y)=\lambda_4g_1(y),$$
$$(\partial_y-(\lambda/c))g_1(y)=\lambda_3g_2(y),$$
$$(\partial_y+(\lambda/c))g_1(y)=\lambda_2g_0(y),$$
$$(\partial_y+(\lambda/c)+(c_y/c))g_2(y)=\lambda_1g_1(y),\tag{2.7}$$

which is consistent if $\lambda_1\lambda_3=\lambda_2\lambda_4$. Then for the $x$ dependence of solutions of first type choose

$$a_1=ah_1,\quad a_2=\tfrac{1}{2}ah_0,\quad a_3=\tfrac{1}{2}ah_2,\quad a_0=0,\tag{2.8}$$

where

$$\frac{\lambda_4}{b}h_0+\left(u+\partial_x-\frac{2b_x}{b}\right)h_1=0,$$
$$\frac{\lambda_3}{b}h_1+\left(u+\partial_x-\frac{b_x}{b}\right)h_2=0,$$
$$\left(u-\partial_x-\frac{b_x}{b}\right)h_0+\frac{\lambda_2}{b}h_1=0,$$
$$\left(u-\partial_x-\frac{2b_x}{b}\right)h_1+\frac{\lambda_1}{b}h_2=0.\tag{2.9}$$

Then the function $a$ satisfies the differential equation

$$\left[\partial_t^2+\frac{3a_t}{a}\partial_t+\frac{a_{tt}}{a}+\left(\frac{a_t}{a}\right)^2+\frac{u}{a^2}\right]\hat{a}=m^2\hat{a}.\tag{2.10}$$

(2) For the second type of solution choose the components of the vector field as

$$a_0=\hat{a}_0b_0g_1e^{-i\lambda z},\qquad a_1=\hat{a}_1b_1g_1e^{-i\lambda z},$$
$$a_2=(1/\sqrt{2})\hat{a}_1b_2g_0e^{-i\lambda z},\quad a_3=(1/\sqrt{2})\hat{a}_1b_2g_2e^{-i\lambda z},\tag{2.11}$$

and require that the functions $b_i, i=0,1,2$ satisfy the consistent system of equations:

$$\partial_xb_0=-\varepsilon b_1,$$
$$(\partial_x+(2b_x/b))b_1=3\varepsilon b_0+(u/b)b_2,$$
$$\varepsilon b_2+(\lambda b_0/2b)=0,$$
$$\lambda=-\tfrac{1}{2}\lambda_2=-\tfrac{1}{2}\lambda_3,\quad \lambda_3=\lambda_4=\tfrac{1}{2}u.\tag{2.12}$$

Then the $\hat{a}_i$ functions satisfy

$$(\partial_t+(3a_t/a))\hat{a}_0-(3\varepsilon/a)\hat{a}_1=0,$$
$$\left[\partial_t^2+\frac{3a_t}{a}\partial_t+\frac{3\varepsilon^2}{a^2}+\frac{3a_{tt}}{a}-3\left(\frac{a_t}{a}\right)^2\right]\hat{a}_0+\frac{6a_t\varepsilon}{a^2}\hat{a}_1=m^2\hat{a}_0,$$
$$\left[\partial_t^2+\frac{3a_t}{a}\partial_t+\frac{3\varepsilon^2}{a^2}+\frac{3a_{tt}}{a}+\left(\frac{a_t}{a}\right)^2\right]\hat{a}_1-\frac{2a_t\varepsilon}{a^2}\hat{a}_0=m^2\hat{a}_1.\tag{2.13}$$

In particular, if the metric is chosen in local coordinates to correspond to the open RW cosmological model, then

$$a=\sinh^2(\psi/2),\quad t=\tfrac{1}{2}(\sinh\psi-\psi),$$
$$b=\sinh x,\quad c=\sin y.\tag{2.14}$$

Identifying

$$A_0=a_0\Phi^{\rho m}_{00J}(x)D^J_{0M}(0,y,z),$$
$$A_1=a_1\Phi^{\rho m}_{10J}(x)D^J_{0M}(0,y,z),$$
$$A_2=a_1\Phi^{\rho m}_{11J}(x)D^J_{1M}(0,y,z),$$
$$A_3=a_1\Phi^{\rho m}_{1-1J}(x)D^J_{-1M}(0,y,z),\quad m=0,\pm1,\tag{2.15}$$

we find that the solutions of Maxwell's equations (mass $=0$) are given by

$$a_0=0,a_1=(\cosh\psi/2)^{-3/2}P^{\pm1/2}_{-1/2+2i\rho}(\cosh\psi/2)\tag{2.16}$$

and

$$a_0=(\cosh\psi/2)^{-3/2}P^{\pm5/2}_{-1/2+2i\rho}(\cosh\psi/2),$$
$$a_1=(1+\rho^2)^{-1/2}(\partial_\psi+3\cosh\psi/2)a_0,$$

where $P^{\nu}_{\mu}(z)$ is a solution of Legendre's equation. The second solution does not represent electromagnetic waves and can be removed by a gauge-fixing transformation. This solution represents the solutions of Maxwell's equations in which the vector $A=(A_0,A_1,A_2,A_3)$ is simultaneously in the synchronous and de Donder gauges.

The systems of first-order differential equations (2.7), (2.9), (2.12) mimic the recurrence relations for the matrix elements $\Phi^{\rho m}_{l\lambda J}(a)$, $m=0,1$ and $D^J_{\lambda\lambda'}(0,\theta,\phi)$.

In fact if one examines the spinor equivalent of Maxwell's equations, which are a special case of massive equations due to Wünsch,[13] viz.,

$$\nabla^{AA'}\phi_{AB}=m\psi_B{}^{A'},$$
$$\nabla_{(AA'}\Psi^{A'}{}_{B)}=-m\phi_{AB},\tag{2.17}$$

then relative to the null frame $e^i_{(a)}$ solutions can be chosen such that

$$\phi_{00}=a_1h_0g_0e^{-i\lambda z},\quad \phi_{01}=a_1h_1g_1e^{-i\lambda z},$$
$$\phi_{11}=a_1h_2g_2e^{-i\lambda z},$$
$$\psi_{00'}=A_1h_1g_1e^{-i\lambda z},\quad \psi_{11'}=-A_1h_0g_0e^{-i\lambda z},$$
$$\psi_{01'}=A_1h_2g_2e^{-i\lambda z},\quad \psi_{10'}=-Ah_1g_1e^{-i\lambda z},\tag{2.18}$$

where the functions $a_1,A_1$ satisfy the coupled equations

$$\frac{-1}{\sqrt{2}}\left(\partial_t + \frac{2a_t}{a} - \frac{u}{a}\right)a_1 = mA_1,$$
$$\frac{1}{\sqrt{2}}\left(\partial_t + \frac{a_t}{a} + \frac{u}{a}\right)A_1 = ma_1. \qquad (2.19)$$

This separation of variables procedure does not work if an attempt is made to use the tetrad and to mimic the recurrence formulas relating the various components of $h_{ab}$. In fact this procedure will only work if the underlying infinitesmal distance $dx^2 + b^2(x)(dy^2 + c^2(y)dz^2)$ corresponds to a three-dimensional Riemannian space of constant Riemannian curvature, i.e., the case which includes the RW metrics. Rather than write out the equations in detail, we mention that the solution to the equation for gravitational waves in the simultaneous synchronous and de Donder gauges has the form (1.29) with $f_2 = f_3 = 0$, $m = \pm 2$ and $f_1$ given by

$$f_1 = (\cosh\psi/2)^{-3/2} P^{\pm 5i/2}_{-1/2+2i\rho}(\cosh\psi/2), \qquad (2.20)$$

where $P^{\nu}_{\mu}(z)$ is a Legendre function.

This is a solution of the equations

$$G_{ab} + 2R_{acbd}G^{cd} - 2R_{c(a}G_{b)}{}^{c} = 0,$$
$$\nabla^a G_{ac} = 0, \quad G^a{}_a = 0.$$

Any theory that explains exactly when a separation of variables procedure works would need to show exactly why it is that spin 1 equations in the case of infinitesmal distance (2.1) admit separable solutions whereas higher spin equations do not. This problem does not occur in the case of RW cosmological models, as group theory guarantees the results.

## ACKNOWLEDGMENT

W. M. was supported in part by the National Science Foundation under grant DMS 88-23054.

## APPENDIX: THE LORENTZ GROUP SO(3,1) AND COMPLETE SETS OF MATRIX ELEMENTS

We give here in summarized form, the relevant properties of the Lorentz group. We refer the reader to Gelfand, Minlos, and Shapiro.[6]

If $R_i(t)$ is the rotation about the $i$th spatial axis and $N_i(t)$ the hyperbolic rotation in the $0i$ plane $i = 1,2,3$ then the generators of these one parameter subgroups denoted by $M_i, N_i, i = 1,2,3$ satisfy the commutation relations

$$[M_i,M_j] = \varepsilon_{ijk}M_K, \quad [M_i,M_j] = \varepsilon_{ijk}N_K,$$
$$[N_i,N_j] = -\varepsilon_{ijk}M_K. \qquad (A1)$$

Each irreducible representation (IR) of SO(3,1) is labeled by a pair of numbers $[m,c]$ where $c$ is complex and $|m|$ a positive integer. There are two invariant operators

$$K_1 = \mathbf{M}^2 - \mathbf{N}^2, \quad K_2 = \mathbf{M}\cdot\mathbf{N} \qquad (A2)$$

such that in a given IR

$$K_1 = 1 - c^2 - m^2, \quad K_2 = icm. \qquad (A3)$$

The IRs of SO(3,1) are of two types.

### 1. Infinite-dimensional class

In this class $c^2 \neq (|m| + n)^2$ for any positive integer $n$. The action of the generators of the Lie algebra on a canonical SO(3) basis $f_{l\lambda}$ is

$$M_+ f_{l\lambda} = \alpha^l_{\lambda+1} f_{l\lambda+1},$$
$$M_- f_{l\lambda} = \alpha^l_{\lambda} f_{l\lambda-1},$$
$$iM_3 f_{l\lambda} = \lambda f_{l\lambda},$$
$$N_+ f_{l\lambda} = \alpha^l_{\lambda,\lambda+1} c_l f_{l-1,\lambda+1} - \alpha^l_{\lambda,-\lambda-1} A_l f_{l,\lambda+1} + \alpha^{l+1}_{\lambda,-\lambda-1} c_{l+1} f_{l+1,\lambda+1},$$
$$N_- f_{l\lambda} = \alpha^l_{\lambda,-\lambda+1} c_l f_{l-1,\lambda-1} - \alpha^l_{-\lambda,\lambda-1} A_l f_{l,\lambda-1} - \alpha^{l+1}_{\lambda,\lambda-1} c_{l+1} f_{l+1,\lambda-1},$$
$$iN_3 f_{l\lambda} = \alpha^l_{\lambda,-\lambda} c_l f_{l-1,\lambda} - \lambda A_l f_{l,\lambda} - \alpha^{l+1}_{-\lambda,\lambda} c_{l+1} f_{l+1,\lambda},$$
$$M_\pm = M_1 \pm iM_2, \quad N_\pm = N_1 \pm iN_2, \qquad (A4)$$

where

$$A_l = \frac{imc}{l(l+1)}, \quad c_l = \frac{i}{l}\sqrt{\left[\frac{(l^2-m^2)(l^2-c^2)}{4l^2-1}\right]},$$
$$\alpha^l_{\lambda u} = \sqrt{(l-\lambda)(l-u)}.$$

The $l,\lambda$ spectrum for the IR $[m,c]$ is

$|\lambda| \leqslant l, \quad l = |m|, |m|+1, \ldots .$

The representations are unitary if

(1) $c = i\rho, \quad 0 \leqslant \rho < \infty, \quad m = 0, \quad \pm\frac{1}{2}, \quad \pm 1, \quad \pm\frac{3}{2} \ldots$

(this is the principal series);

(2) $\operatorname{Im} c = 0, \quad 0 < c < 1, \quad m = 0$

(this is the complementary series).

### 2. Finite-dimensional class

In this class $c^2 = (|m| + n)^2$ for some positive integer $n$. The action of the generators on a canonical SO(3) basis is as in (A4). The $l,\lambda$ spectrum for the IR $[m,c]$ is

$$|\lambda| \leqslant l, \quad l = |m|, \quad |m|+1, \ldots, \quad |m|+n-1.$$

The unitary IR $[m,i\rho]$ can realized on the space of functions on the two-dimensional sphere via the orthonormal angular momentum basis functions:

$$f_{l\lambda} = \left(\frac{2l+1}{4\pi}\right)^{1/2} D^l_{\lambda m}(\phi,\theta,0), \quad |\lambda| \leqslant l,$$
$$l = |m|, |m|+1, \ldots . \qquad (A5)$$

The action of the Lorentz group can be induced from the action

$$T^{[m,i\rho]}(g)\Phi(z) = (\beta z + \gamma)^{m+i\rho-1}(\beta^* z^* + \gamma^*)^{-m+i\rho-1} \times \Phi\left(\frac{\alpha z + \delta}{\beta z + \gamma}\right), \qquad (A6)$$

via the identification

$$|z|^{-1} = \tan \tfrac{1}{2}\theta, \quad \arg z = \phi,$$

and with

$$f(\theta,\phi) = e^{-im\phi} (\sin^2 \theta/2)^{i\rho-1}\phi(z), \tag{A7}$$

the matrix element of $N_3(a)$ in the angular momentum basis has the integral representation

$$\Phi^{\rho m}_{l\lambda J}(a) = \frac{1}{2}\sqrt{(2l+1)(2J+1)} \times \int_{-1}^{1} dx\, (\cosh a + x \sinh a)^{i\rho-1}, \times d^{\,l}_{\lambda m}(x) d^{\,J}_{\lambda m}(x') \tag{A8}$$

where

$$x' = (x + \tanh a)/(1 + x \tanh a).$$

An explicit expression for these functions has been obtained by Duc and Van Hieu.[14] These functions satisfy the orthogonality relations

$$\sum_\lambda \int_0^\infty \Phi^{\rho m}_{l\lambda J}(a)\Phi^{\rho' m' *}_{l\lambda J}(a) \sinh^2 a\, da = N^{\rho m}_{lJ}\delta_{mm'}\delta(\rho-\rho')$$

$$\sum_{m=-j}^{j}\int_0^\infty \Phi^{\rho m}_{l\lambda J}(a)\Phi^{\rho m *}_{l\lambda J}(a')d\rho = N^{\rho m}_{lJ}\frac{\delta(a-a')}{\sinh^2 a}, j = \min(j,J). \tag{A9}$$

The normalization factor is

$$N^{\rho m}_{lJ} = 2\pi \frac{(L-j)![2(l+1)!]^2(j+|m|)!(j-|m|)!}{(L+j)!(L+m+l)!(L-m)!(L+l-m)!} \times \prod_{k=|m|+1}^{j}(\rho^2+k^2)\left|\frac{\Gamma(i\rho+|m|)}{\Gamma(i\rho+L+1)}\right|^2,$$

where

$$L = \max(l,J).$$

These functions obey the symmetry relations

$$\Phi^{\rho m}_{l-\lambda J}(a) = (-1)^{l-J}\Phi^{\rho-m}_{l\lambda J}(-a) = (-1)^{l-J}\Phi^{-\rho-m}_{l\lambda J}(a) = \Phi^{\rho-m}_{l-\lambda J}(a). \tag{A10}$$

We know from the group theory arguments that each component of a Lorentz invariant equation must be expandable in an appropriate choice of matrix elements. Recurrence formulas for the functions $\Phi^{\rho m}_{l\lambda J}(a)$ can be deduced by realizing the matrix element $D^{[m,c]}_{l\lambda J\lambda'}(g)$ in the generalized Euler parametrization in the form

$$D^{[m,i\rho]}_{l\lambda J\lambda'}(g) = \sum_\mu D^{\,l}_{\lambda\mu}(\phi,\theta,0)\Phi^{\rho m}_{l\mu J}(a)D^{\,J}_{\mu\lambda'}(\alpha,\beta,\gamma). \tag{A11}$$

For fixed $J,\lambda'$ these matrix elements provide a realization of the unitary IR $[m,i\rho]$ by the left regular representation:

$$T_{g'}D^{[m,i\rho]}{}_{l\lambda J\lambda'}(g) = D^{[m,i\rho]}{}_{l\lambda j\mu}(g')D^{[m,i\rho]j\mu}{}_{J\lambda'}(g). \tag{A12}$$

Consequently invoking the canonical action of the infinitesimal operators as in (A4) we deduce the recurrence relations that follow. These results are due to Ström:[15]

$$\sqrt{[(l+1)^2-\lambda^2]}(\partial_a - l\coth a)\Phi^{\rho m}_{l\lambda J}(a) + (1/(2\sinh a))\Big[\sqrt{(l-\lambda)(l-\lambda+1)(J-\lambda)(J+\lambda+1)}\Phi^{\rho m}_{l\lambda+1J}(a) + \sqrt{(l+\lambda)(l+\lambda+1)(J+\lambda)(J-\lambda+1)}\Phi^{\rho m}_{l\lambda+J}(a)\Big] = -\left[((l+1)^2-m^2)((l+1)^2+\rho^2)\left(\frac{2l+1}{2l+3}\right)\right]^{1/2}\Phi^{\rho m}_{l+1\lambda J}(a),$$

$$\sqrt{[l^2-\lambda^2]}(\partial_a + (l+1)\coth a)\Phi^{\rho m}_{l\lambda J}(a) + (-1/(2\sinh a))\Big[\sqrt{(l+\lambda)(l+\lambda+1)(J+\lambda)(J+\lambda+1)}\Phi^{\rho m}_{l\lambda+1J}(a) + \sqrt{(l-\lambda)(l-\lambda+1)(J+\lambda)(J-\lambda+1)}\Phi^{\rho m}_{l\lambda+1J}(a)\Big] = \left[(l^2-m^2)(l^2+\rho^2)\left(\frac{2l+1}{2l-1}\right)\right]^{1/2}\Phi^{\rho m}_{l-1\lambda J}(a),$$

$$(\lambda\partial_a + \lambda\coth a + im\rho)\Phi^{\rho m}_{l\lambda J}(a) = \frac{1}{2\sinh a}\Big[\sqrt{(l+\lambda)(l-\lambda+1)(J+\lambda)(J-\lambda+1)}\Phi^{\rho m}_{l\lambda-1J}(a) - \sqrt{(l-\lambda)(l+\lambda+1)(J+\lambda+1)(J-\lambda)}\Phi^{\rho m}_{l\lambda+1J}(a)\Big],$$

$$\left(\partial_a^2 + 2\coth a\partial_a - \frac{[l(l+1)+J(J+1)]}{\sinh^2 a} + (1+\coth^2 a)\lambda^2 + 1 + \rho^2 - m^2\right)\Phi^{\rho m}_{l\lambda J}(a) = -\frac{\coth a}{\sinh a}\Big[\sqrt{(l+\lambda)(l-\lambda)(J+\lambda)(J-\lambda+1)}\Phi^{\rho m}_{l\lambda-1J}(a) + \sqrt{(l+\lambda+1)(l-\lambda)(J+\lambda+1)(J-\lambda)}\Phi^{\rho m}_{l\lambda+1J}(a)\Big]. \tag{A13}$$

These relations then enable the uncoupling of the variable of $a_1$, in relativistically invariant equations. The matrix elements arising from the Euler parametrization have given one complete set of functions with which to expand relativistically invariant equations. There are however other systems of basis functions possible, corresponding to a different choice of group parametrization and coordinates on the hyperboloid. These functions are the analogs on $H_3$ of vector and tensor expansion functions corresponding to spherical, or cylindrical waves in Euclidean three-space. We list below a brief summary of other important sets of basis functions that are possible, together with the corresponding group parametrizations and coordinates on $H_3$. In each case the new basis functions are eigenfunctions of a definite subgroup chain of SO(3,1). In the case of spherical coordinates (1.9) the basis consists of sets of eigenfunctions of the operators $M^2$ (angular momentum) and $M_3$ (its third component).

Two other coordinate systems on the hyperboloid are the following.

(1) Hyperbolic coordinates

$$v = (\cosh a \cosh b, \cosh a \sinh b \cos \phi, \cosh a \sinh b \sin \phi, \sinh a),$$
$$-\infty < a < \infty, 0 \leqslant b < \infty, 0 \leqslant \phi < 2\pi. \quad \text{(A14)}$$

The corresponding group parametrization is

$$g = R_3(\phi) N_1(b) N_3(a) R_3(\alpha) R_1(\beta) R_3(\gamma). \quad \text{(A15)}$$

The appropriate basis functions are denoted by

$$H^{\rho m \epsilon}_{l\lambda j}(a) D^{j\epsilon}_{\lambda N}(0,b,\phi), \epsilon = \pm,$$

where $D^{j\epsilon}_{\lambda N}(\varphi,b,\phi)$ are the matrix elements of a general element of the SO(2,1) group given in terms of the Euler parametrization

$$g = R_3(\varphi) N_1(b) R_3(\phi)$$

and in the corresponding unitary irreducible representations labeled by $j,\epsilon = \pm$ where

$$j = -\tfrac{1}{2} + iq, 0 < q < \infty; j = \eta, \eta + 1, \ldots, |m| - 1;$$
$$m = j + 1, j + 2, \ldots; \epsilon = +,$$
$$m = j + 1, j + 2, \ldots; \epsilon = +,$$
$$m = -j - 1, -j - 2, \ldots; \epsilon = -,$$
$$\eta, |m| = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots; |m| - \eta$$

is an integer.

The functions $H^{\rho m \epsilon}_{l\lambda j}(a)$ have the integral representation[16]

$$H^{\rho m +}_{l\lambda j}(a) = \frac{1}{2}\sqrt{\left(l + \frac{1}{2}\right)\left(j + \frac{1}{2}\right)} \times \int_0^\infty (\cosh a \cosh b + \sinh a)^{i\rho - 1} i^{M - \lambda} \times d^{\,j}_{m\lambda}(\cosh b) d^{\,l}_{m\lambda}(\cos \theta_g) \sinh b \, db, \quad \text{(A16)}$$

where

$$\cos \theta_g = (\cosh b \sinh a + \cosh a)/(\cosh b \cosh a + \sinh a)$$

and

$$H^{\rho m -}_{l\lambda j}(a) = (-1)^{l - \lambda} H^{\rho - m +}_{l\lambda j}(-a). \quad \text{(A17)}$$

As expected, the recurrence formulas for these functions enable the complete decoupling of relativistically invariant equations from the dependence on $a,b,\phi$ in a frame corresponding to the one-forms:

$$e_{(1)i}\, dx^i = da, e_{(2)i}\, dx^i = (1/\sqrt{2}) \sinh a (db + i \sinh b \, d\phi),$$
$$e_{(3)i}\, dx^i = (1/\sqrt{2}) \sinh a (db - i \sinh b \, d\phi). \quad \text{(A18)}$$

Bearing in mind that if we consider spinor equations, the use of null tetrads is appropriate, the basis functions are eigenfunctions of $N_1^2 + N_2^2 - M_3^2$ and $M_3$ with eigenvalues $-j(j+1)$ and $M$, respectively.

(2) Horospherical coordinates

$$v = (\tfrac{1}{2} r^2 e^a + \cosh a, r e^a \cos \phi, r e^a \sin \phi, \tfrac{1}{2} r^2 e^a - \sinh a)$$
$$-\infty < a < \infty, 0 \leqslant r < \infty, 0 \leqslant \phi < 2\pi. \quad \text{(A19)}$$

The corresponding group parametrization is

$$g = R_3(\phi) T_1(r) N_3(a) R_3(\alpha) R_1(\beta) R_3(\gamma), \quad \text{(A20)}$$

where $T_1(r) = e^{(N_1 + M_2)r}$.

The appropriate basis functions are denoted by

$$E^{\rho m}_{l\lambda x}(a) J_{\lambda - M}(Xr) e^{iM\phi},$$

where $J_\nu(z)$ is a Bessel function. The functions $E^{\rho m}_{l\lambda x}(a)$ have the integral representation[16]

$$E^{\rho m}_{l\lambda x}(a) = \sqrt{l + \frac{1}{2}} \int_0^\pi \left(e^a \cos^2 \frac{1}{2}\theta\right)^{i\rho - 1} \times J_{m - \lambda}\left(e^{-a} x \tan \frac{1}{2}\theta\right) \times d^{\,l}_{m\lambda}(\cos \theta) \sin \theta \, d\theta. \quad \text{(A21)}$$

The corresponding frame of one-forms in which complete decoupling of relativistically invariant equations occurs from the variables $a,r,\phi$ is

$$e_{(1)i}\, dx^i = da, \quad e_{(2)i}\, dx^i = (1/\sqrt{2}) e^{-a}(dr + ir \, d\phi),$$
$$e_{(3)i}\, dx^i = (1/\sqrt{2}) e^{-a}(dr - ir \, d\phi), \quad \text{(A22)}$$

with suitable modification to include the use of null tetrads if spinor equations are included. The basis functions are eigenfunctions of $(N_1 + M_2)^2 + (N_2 - M_1)^2$ and $M_3$ with eigenvalues $-X^2$ and $M$, respectively. In fact, all possible subgroup chains for the Lorentz group are known and appropriate basis functions on $H_3$ for symmetric tensors can be computed in a suitable frame. For further details see Kalnins.[17]

Specifically for the case of perturbations of the RW cosmological models, we give the explicit expressions for the expansion functions in the coordinates. The functions $\Phi^{\rho m}_{l\lambda x}(a)$ satisfy the differential equation

$$[\partial_a^2 + 2(l+1) \coth a \partial_a + ([l(l+1) - J(J+1)]/\sinh^2 a) - 2im\rho \coth a + (l+1)^2 + \rho^2 - m^2] \Phi^{\rho m}_{lIJ}(a) = 0 \quad \text{(A23)}$$

and have the solution

$$\Phi^{\rho m}_{llJ}(a) = i^{J-l}(1-e^{-2a})^{J-l}\exp[-(l+1-m-i\rho)a] \times {}_2F_1(J+1-i\rho, J+1-m; 2J+2, 1-e^{-2a}). \tag{A24}$$

The other external matrix element can be obtained from the symmetry condition

$$\Phi^{\rho m}_{l-lJ}(a) = \Phi^{\rho-m}_{llJ}(a). \tag{A25}$$

The remaining functions can be obtained from the recurrence formulas as follows:

$m=0$

$$\Phi^{\rho 0}_{2-1J}(a) = \Phi^{\rho 0}_{21J}(a) = [2/\sqrt{J(J+1)-2}]\,(\sinh a\partial_a + \cosh a)\Phi^{\rho 0}_{22J}(a),$$

$$\Phi^{\rho 0}_{20J}(a) = [2/\sqrt{3J(J+1)}]\,(\sinh a\partial_a + \cosh a) \times \Phi^{\rho 0}_{21J}(a) + \sqrt{J(J+1)-2}\,\Phi^{\rho 0}_{22J}(a); \tag{A26}$$

$m=1$

$$\Phi^{\rho 1}_{2\pm 1}(a) = [(2/\sqrt{J(J+1)}) - 2] \times [\sinh a\partial_a + \cosh a \pm i\rho]\Phi^{\rho 1}_{2\pm 2J}(a),$$

$$\Phi^{\rho 0}_{20J}(a) = [2/\sqrt{3J(J+1)}] \times [(\sinh a\partial_a + \cosh a) \pm i\rho]\Phi^{\rho 1}_{2\pm 1J}(a) + \sqrt{J(J+1)-2}\,\Phi^{\rho 1}_{2\pm 2J}(a); \tag{A27}$$

$m=2$

$$\Phi^{\rho 2}_{2\pm 1J}(a) = [(2/\sqrt{J(J+1)}) - 2] \times [\sinh a\partial_a + \cosh a \pm i\rho]\Phi^{\rho 2}_{2\pm 2J}(a),$$

$$\Phi^{\rho 2}_{20J}(a) = [2/\sqrt{J(J+1)}] \times [\sqrt{3}\,(\sinh a\partial_a + \cosh a)\Phi^{\rho 2}_{2\pm 1J}(a) - \sqrt{3[J(J+1)-2]}\,\Phi^{\rho 2}_{2\pm 2J}(a). \tag{A28}$$

These expressions are deduced from the simplest recurrence relations that enable all other matrix elements $\Phi^{\rho m}_{l\lambda J}(a)$ to be deduced from the expressions for the extremal components.


[1] E. M. Lifshitz, Zh. Eksp. Teoret. Fiz. **16**, 587 (1946).

[2] E. M. Lifshitz and I. M. Khalatnikov, Adv. Phys. **12**, 185 (1963).

[3] E. M. Litfshitz and L. D. Landau, *The Classical Theory of Fields* (Pergamon, New York, 1975).

[4] V. H. Gerlach and U. K. Sengupta, Phys. Rev. D **18**, 1773 (1978).

[5] M. A. Naimark, *Linear Representations of the Lorentz Group* (Pergamon, New York, 1964).

[6] I. M. Gelfand, R. A. Minlos, and Z. Ya Shapiro, *Representations of the Rotation and Lorentz Groups and their Applications* (Pergamon, New York, 1963).

[7] A. K. Agamaliev, N. M. Atakashiev, and I. A. Verdiev, Sov. J. Nucl. Phys. **9**, 120 (1969).

[8] B. Carter and R. G. Mclenaghan, Phys. Rev. D **19**, 1093 (1979).

[9] R. G. Mclenaghan and Ph. Sphindel, Phys. Rev. D **20**, 409 (1979).

[10] N. Kamran and R. G. Mclenaghan, Phys. Rev. D **30**, 357 (1984).

[11] G. C. Williams, "Killing spinors, Teukolsky equations and the intrinsic characterisation of spin wave equations," Ph.D. thesis, University of Waikato (1989).

[12] M. Fels and N. Kamran, Proc. R. Soc. London Ser. A **428**, 229 (1990).

[13] V. Wünsch, Gen. Relat. and Gravit. **17**, 15 (1985).

[14] D. W. Duc and N. Van Hieu, Sov. Phys. Dokl. **12**, 312 (1967).

[15] S. Ström, Arkiv Fysik. **29**, 467 (1965).

[16] R. Delbourgo, K. Koller, and P. Mahanta, Nuovo Cimento A **52**, 1254 (1967).

[17] E. G. Kalnins, "Subgroup reductions of the Lorentz group," Ph.D. thesis, University of Western Ontario (1972).