Symmetry operators and separation of variables for spin-wave equations in oblate spheroidal coordinates

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A family of second-order differential operators that characterize the solution of the massless spin s field equations, obtained via separation of variables in oblate spheroidal coordinates and using a null tetrad is found. The first two members of the family also characterize the separable solutions in the Kerr space-time. It is also shown that these operators are symmetry operators of the field equations in empty space-times whenever the space-time admits a second-order Killing–Yano tensor.

I. INTRODUCTION

Interest in the separation and solution of the nonscalar equations of mathematical physics in Kerr space-time began when Teukolsky[1] found that separable solutions were possible for some of the Maxwell and Weyl scalars. Chandrasekhar[2] was later able to obtain a separable solution to the Dirac equation. Separable solutions to massless spin s equations were studied by Dudley and Finlay[3] while Carter and McLenaghan[4] were able to understand Chandrasekhar's separation of Dirac's equation in terms of a differential operator that characterized the separation constant appearing in the solution. That is, the separable solutions to Dirac's equation were found to be eigensolutions of the differential operator, the eigenvalue being the separation constant appearing in the solution. Similarly, the separation constant appearing in the solution to Maxwell's equations in Kerr geometry has been characterized by Kalnins *et al.*[5] in terms of a second-order differential operator. These differential operators characterizing the separation constants are also symmetry operators of the various field equations in question. That is, they map solutions of the field equations into solutions. The essential property that allows the construction of such operators is the existence of a Killing–Yano tensor in the Kerr space-time.

The other constants associated with the separable solutions of various field equations in the Kerr space-time are the Starobinsky constants. Torres del Castillo[6–8] has shown, for various fields in type D space-times, that one can construct differential operators of order $2s$, $s=0,\frac{1}{2},1$ that characterize these constants. Physically, Killing–Yano tensors and operators constructed from them have been associated with angular momentum by Carter and McLenaghan[4] and by Dietz and Rudiger.[9,10]

In this paper we take the Kerr metric and a Kinnersley null tetrad and subsequently place $M=0$. We then find that the solution to the massless spin s field equations obtained via separation of variables (and with the aid of a generalized Hertz potential) are characterized by a second-order differential operator. We also show that this differential operator is a symmetry operator of the field equations.

II. PRELIMINARIES

In this paper we will use the abstract index and spinor formalisms of Penrose and Rindler.[11] For the purpose of this paper we shall also refer to those components of a symmetric spinor that are of extreme helicity as the extremal components.

The Kerr metric describes the space-time in the region exterior to a rotating black hole, its line element being

$$ds^2=\left(1-\frac{2Mr}{\tilde\rho\tilde\rho^*}\right)dt^2-\frac{\tilde\rho\tilde\rho^*}{\Delta}dr^2 - \tilde\rho\tilde\rho^*\,d\theta^2-\left(r^2+a^2+\frac{2a^2Mr\sin^2\theta}{\tilde\rho\tilde\rho^*}\right) \times\sin^2\theta\,d\phi^2+\frac{4aMr\sin^2\theta}{\tilde\rho\tilde\rho^*}dt\,d\phi, \tag{1}$$

where

$$\tilde\rho=r+ia\cos\theta \text{ and } \Delta=r^2-2Mr+a^2. \tag{2}$$

We shall use the null tetrad

$$\begin{aligned}
l^a&=(1/\sqrt{2}\Delta)(r^2+a^2,\Delta,0,a),\\
n^a&=(1/\sqrt{2}\tilde\rho\tilde\rho^*)(r^2+a^2,-\Delta,0,a),\\
m^a&=(1/\sqrt{2}\tilde\rho)(ia\sin\theta,0,1,i\csc\theta),\\
\overline{m}^a&=(1/\sqrt{2}\tilde\rho^*)(-ia\sin\theta,0,1,-i\csc\theta).
\end{aligned}\tag{3}$$

In this tetrad the spin coefficients are

$$\begin{aligned}
&\epsilon=0,\quad \beta=\cot\theta/2\sqrt{2}\tilde\rho,\quad \alpha=\pi-\beta^*\\
&\gamma=\mu+\frac{(r-M)}{\sqrt{2}\tilde\rho\tilde\rho^*},\\
&\rho=-\frac{1}{\sqrt{2}\tilde\rho^*},\quad \tau=-\frac{ia\sin\theta}{\sqrt{2}\tilde\rho\tilde\rho^*},\quad \pi=\frac{ia\sin\theta}{\sqrt{2}\tilde\rho^{*2}}\\
&\mu=-\frac{\Delta}{\sqrt{2}\tilde\rho\tilde\rho^{*2}},\quad \kappa=0,\quad \sigma=0,\quad \lambda=0,\quad \nu=0,
\end{aligned}\tag{4}$$

while the only nonzero component of the Weyl spinor is

$$\Psi_2=-M/\tilde\rho^{*3}. \tag{5}$$

Following Chandrasekhar[12] we define the differential operators

$$\begin{aligned}
\mathscr{D}_s&=\partial_r+iK/\Delta+2s(r-M)/\Delta,\\
\mathscr{D}_s^\dagger&=\partial_r-iK/\Delta+2s(r-M)/\Delta,\\
\mathscr{L}_s&=\partial_\theta+Q+s\cot\theta,\\
\mathscr{L}_s^\dagger&=\partial_\theta-Q+s\cot\theta,
\end{aligned}\tag{6}$$

where

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$$K = \sigma(r^2 + a^2) + ma \text{ and } Q = \sigma a \sin\theta + m \csc\theta. \tag{7}$$

A second-order Killing–Yano tensor is an antisymmetric tensor K_{ab} that satisfies

$$\nabla_{(a}K_{b)c} = 0. \tag{8}$$

Being antisymmetric, K_{ab} can be written in terms of symmetric spinors as

$$K_{ab} = K_{AA'BB'} = \tfrac{1}{2}(\epsilon_{A'B'}K_{AB} + \epsilon_{AB}\widetilde{K}_{A'B'}). \tag{9}$$

The Killing spinors K_{AB} and $\widetilde{K}_{A'B'}$ as a consequence of (8) satisfy

$$\begin{aligned} &\nabla_{(AA'}K_{BC)} = 0, \\ &\nabla_{A(A'}\widetilde{K}_{B'C')} = 0, \\ &\nabla_{BA'}K_A{}^B + \nabla_{AB'}\widetilde{K}_{A'}{}^{B'} = 0. \end{aligned} \tag{10}$$

Defining the quantity $M_{AA'}$ by

$$M_{AA'} = \nabla_{BA'}K_A{}^B, \tag{11}$$

we can write the derivatives of the Killing spinors as

$$\begin{aligned} &\nabla_{AA'}K_{BC} = \tfrac{2}{3}\epsilon_{A(B}M_{C)A'}, \\ &\nabla_{AA'}\widetilde{K}_{B'C'} = -\tfrac{2}{3}\epsilon_{A'(B'}M_{AC')}. \end{aligned} \tag{12}$$

The derivative of $M_{AA'}$ is given by

$$\nabla_{AA'}M_{BB'} = \tfrac{1}{2}\epsilon_{A'B'}W_{AB} - \tfrac{1}{2}\epsilon_{AB}\widetilde{W}_{A'B'}, \tag{13}$$

where the symmetric spinors W_{AB} and $\widetilde{W}_{A'B'}$ are defined by

$$\begin{aligned} &W_{AB} = \tfrac{3}{2}\Psi_{ABCD}K^{CD} \\ &\widetilde{W}_{A'B'} = \tfrac{3}{2}\overline{\Psi}_{A'B'C'D'}\widetilde{K}^{C'D'}. \end{aligned} \tag{14}$$

Note also from (13) that $\nabla_{AA'}M_{BB'}$ is an antisymmetric tensor, that is, we have

$$\nabla_{(a}M_{b)} = 0, \tag{15}$$

which is the condition that M_a be a Killing vector. Other relations satisfied by the above quantities are

$$\begin{aligned} \Psi^E{}_{ABC}K_{DE} &= \tfrac{3}{4}\epsilon_{D(A}\Psi_{BC)EF}K^{EF} \\ &= \tfrac{1}{2}\epsilon_{D(A}W_{BC)} \\ \overline{\Psi}^{E'}{}_{A'B'C'}\widetilde{K}_{D'E'} &= \tfrac{3}{4}\epsilon_{D'(A'}\overline{\Psi}_{B'C')E'F'}\widetilde{K}^{E'F'} \\ &= \tfrac{1}{2}\epsilon_{D'(A'}\widetilde{W}_{B'C')}, \end{aligned} \tag{16}$$

and

$$\begin{aligned} &W_{AC}K_B{}^C = +\tfrac{1}{2}\epsilon_{AB}W_{CD}K^{CD}, \\ &\widetilde{W}_{A'C'}\widetilde{K}_{B'}{}^{C'} = +\tfrac{1}{2}\epsilon_{A'B'}\widetilde{W}_{C'D'}\widetilde{K}^{C'D'}. \end{aligned} \tag{17}$$

The antisymmetry in the free indices of the above two quantities being particularly useful. The derivatives of W_{AB} and $\widetilde{W}_{A'B'}$ are

$$\begin{aligned} &\nabla_{AA'}W_{BC} = 2\Psi_{ABC}{}^DM_{DA'}, \\ &\nabla_{AA'}\widetilde{W}_{B'C'} = -2\overline{\Psi}_{A'B'C'}{}^{D'}M_{AD'}, \end{aligned} \tag{18}$$

expressions which can be obtained by examining the consistency condition on $M_{AA'}$, that is, from

$$\begin{aligned} [\nabla_{AA'},\nabla_{BB'}]M_{CC'} = &-\epsilon_{A'B'}\Psi_{ABC}{}^DM_{DC'} \\ &-\epsilon_{AB}\overline{\Psi}_{A'B'C'}{}^{D'}M_{CD'}. \end{aligned} \tag{19}$$

We define the differential operator ${}_\eta J_{AA'}$ by

$$\begin{aligned} {}_\eta J_{AA'} &= 2K_{AA'}{}^{CC'}\nabla_{CC'} + (\eta/3)M_{AA'} \\ &= K_A{}^C\nabla_{CA'} + \widetilde{K}_{A'}{}^{C'}\nabla_{AC'} + (\eta/3)M_{AA'}. \end{aligned} \tag{20}$$

This operator will be the essential building block for the symmetry operators we shall encounter later. The commutator of $\nabla_{BB'}$ with ${}_\eta J_{AA'}$ is

$$\begin{aligned} [\nabla_{BB'},{}_\eta J_{AA'}] = &K_A{}^C[\nabla_{BB'},\nabla_{CA'}] + \widetilde{K}_{A'}{}^{C'}[\nabla_{BB'},\nabla_{AC'}] \\ &+ \tfrac{2}{3}(M_{AB'}\nabla_{BA'} - M_{BA'}\nabla_{AB'}) \\ &+ (\eta/6)(\epsilon_{B'A'}W_{BA} - \epsilon_{BA}\widetilde{W}_{B'A'}). \end{aligned} \tag{21}$$

We also define the vectors $U_{AA'}$ and $\widetilde{U}_{AA'}$ by

$$\begin{aligned} &K_A{}^BU_{BA'} = -\tfrac{1}{3}M_{AA'}, \\ &\widetilde{K}_{A'}{}^{B'}\widetilde{U}_{AB'} = \tfrac{1}{3}M_{AA'}. \end{aligned} \tag{22}$$

These two vectors $U_{AA'}$ and $\widetilde{U}_{AA'}$ will later be useful in choosing gauge fields that will in turn enable the separability of a decoupled equation for the extremal components of a generalized Hertz potential representing a massless spin s field. The derivative of the Weyl spinor is also related to the vector $U_{AA'}$ by

$$\nabla_{AA'}\Psi_{BCDE} = 5U_{(AA'}\Psi_{BCDE)}. \tag{23}$$

In the Kerr space-time the only solution of (10) to within a common multiplicative constant, is

$$\begin{aligned} &K_{01} = -\tilde{\rho}^*, \qquad K_{00} = K_{11} = 0, \\ &\widetilde{K}_{0'1'} = \tilde{\rho}, \qquad \widetilde{K}_{0'0'} = \widetilde{K}_{1'1'} = 0, \end{aligned} \tag{24}$$

whence the only nonzero components of $K_{AA'BB'}$ are

$$\begin{aligned} &K_{01'10'} = -K_{10'01'} = r, \\ &K_{00'11'} = -K_{11'00'} = ia\cos\theta. \end{aligned} \tag{25}$$

The components of $M_{AA'}$ are

$$\begin{aligned} &M_{00'} = -\frac{3}{\sqrt{2}}, \qquad M_{01'} = -\frac{3}{\sqrt{2}}\frac{ia\sin\theta}{\tilde{\rho}}, \\ &M_{10'} = \frac{3}{\sqrt{2}}\frac{ia\sin\theta}{\tilde{\rho}^*}, \qquad M_{11'} = -\frac{3}{\sqrt{2}}\frac{\Delta}{\tilde{\rho}\tilde{\rho}^*}, \end{aligned} \tag{26}$$

and the components of W_{AB} and $\widetilde{W}_{A'B'}$ are

$$\begin{aligned} &W_{01} = 3\tilde{\rho}^*\Psi_2, \qquad W_{00} = W_{11} = 0, \\ &\widetilde{W}_{0'1'} = -3\tilde{\rho}\Psi_2^*, \qquad \widetilde{W}_{0'0'} = \widetilde{W}_{1'1'} = 0, \end{aligned} \tag{27}$$

while those of the vectors $U_{AA'}$ and $\widetilde{U}_{AA'}$ are

$$\begin{aligned} &U_{00'} = \rho,\ U_{01'} = \tau,\ U_{10'} = -\pi,\ U_{11'} = -\mu, \\ &\widetilde{U}_{00'} = \tilde{\rho}^*,\ \widetilde{U}_{01'} = -\pi^*,\ \widetilde{U}_{10'} = \tau^*,\ \widetilde{U}_{11'} = -\mu^*. \end{aligned} \tag{28}$$

The minimally coupled first-order equation for a massless spin s field is

$$\nabla^A{}_{A'}\phi_{AA_2\cdots A_{2s}} = 0. \tag{29}$$

It is well known that in a space-time that is not conformally flat this equation is inconsistent for $s > 1$. In particular when $s > 1$ and for the case of an empty space-time $\phi_{A_1\ldots A_{2s}}$ must satisfy the consistency condition

$$\Psi^{BCD}{}_{(A_3}\phi_{A_4\ldots A_{2s})BCD} = 0. \tag{30}$$

In the Kerr space-time and using the null tetrad (3) and defining a new set of functions Φ_k for $k = 0, \ldots, 2s$ by $\Phi_k = \tilde{\rho}^{*k}\phi_k$ Eqs. (29) become

$$[\mathscr{L}_{s-p} - (2s-2p-1)(ia\sin\theta/\tilde{\rho}^*)]\,\Phi_p - [\mathscr{D}_0 + (2s-2p-1)(1/\tilde{\rho}^*)]\,\Phi_{p+1} = 0,$$
$$\Delta[\mathscr{D}^\dagger_{s-p} - (2s-2p-1)(1/\tilde{\rho}^*)]\,\Phi_p + [\mathscr{L}^\dagger_{p-s+1} + (2s-2p-1) \times (ia\sin\theta/\tilde{\rho}^*)]\,\Phi_{p+1} = 0, \quad (31)$$

where $p = 0,...,2s-1$.

The following method of obtaining a solution to the massless spin s field Eqs. (29) is due to Cohen and Kegeles.[13] If the potential $\overline{P}^{A'_1\cdots A'_{2s}}$ and an associated arbitrary gauge field $G_B{}^{A'_2\cdots A'_{2s}}$ both of which are symmetric in their primed indices satisfy

$$\nabla^{B(A'_1}\nabla_{BB'}\overline{P}^{A'_2\cdots A'_{2s})B'} - \nabla^{B(A'_1}G_B{}^{A'_2\cdots A'_{2s})} - (2s-1)(s-1)\overline{\Psi}_{B'C'}{}^{(A'_1A'_2}\overline{P}^{A'_3\cdots A'_{2s})B'C'} = 0, \quad (32)$$

then a spin s field constructed from the potential and gauge fields as follows:

$$\phi_{A_1\ldots A_{2s}} = \nabla_{(A_1A'_1}\nabla_{A_2A'_2}\cdots\nabla_{A_{2s-1}A'_{2s-1}} \times\Big[\nabla_{A_{2s})A'_{2s}}\overline{P}^{A'_1\ldots A'_{2s}} - G_{A_{2s})}{}^{A'_1\ldots A'_{2s-1}}\Big], \quad (33)$$

will satisfy the spin s field Eqs. (29) provided those equations are consistent. When the space-time admits a second-order Killing–Yano tensor and the quantity $\widetilde{U}_{AA'}$ as defined by (22) exists, we can make the following rather special choice of the gauge field:

$$G_B{}^{A'_2\ldots A'_{2s}} = -2s\widetilde{U}_{BA'}\overline{P}^{A'A'_2\ldots A'_{2s}}. \quad (34)$$

This choice of gauge field was made by Cohen and Kegeles though not in this covariant form. With this choice of gauge and in a type D space-time Eqs. (32) decouple. In addition, in the Kerr space-time the extremal components of the potential will now satisfy separable equations. That is, if we look for solutions of the form $f(r,\theta)e^{i\sigma t + im\varphi}$ for $\overline{P}^{0'\cdots 0'}$ and $\overline{P}^{1'\cdots 1'}$ we find that

$$[\Delta\mathscr{D}^\dagger_{1-s}\mathscr{D}_0 + \mathscr{L}^\dagger_{1-s}\mathscr{L}_s + 2(2s-1)i\sigma\tilde{\rho}^*]\overline{P}^{0'\cdots 0'} = 0, \quad (35)$$
$$[\Delta\mathscr{D}^\dagger_{1+s}\mathscr{D}_0 + \mathscr{L}^\dagger_{1+s}\mathscr{L}_{-s} - 2(2s+1)i\sigma\tilde{\rho}^*]\tilde{\rho}^{-2s}\overline{P}^{1'\cdots 1'} = 0,$$

which are separable and have solutions

$$\overline{P}^{0'\cdots 0'} = R_{-s}S_{+s}e^{i\sigma t + im\varphi},$$
$$\overline{P}^{1'\cdots 1'} = \tilde{\rho}^{2s}R_{+s}S_{-s}e^{i\sigma t + im\varphi}, \quad (36)$$

where the functions $R_{\pm s}$ and $S_{\pm s}$ satisfy Teukolsky's equations, namely,

$$[\Delta\mathscr{D}^\dagger_{1-s}\mathscr{D}_0 + 2(2s-1)i\sigma r]R_{-s} = \lambda R_{-s},$$
$$[\Delta\mathscr{D}_1\mathscr{D}^\dagger_s - 2(2s-1)i\sigma r]R_{+s} = \lambda R_{+s},$$
$$[\mathscr{L}^\dagger_{1-s}\mathscr{L}_s + 2(2s-1)\sigma a\cos\theta\,]S_{+s} = -\lambda S_{+s},$$
$$[\mathscr{L}_{1-s}\mathscr{L}^\dagger_s - 2(2s-1)\sigma a\cos\theta\,]S_{-s} = -\lambda S_{-s}. \quad (37)$$

If we form $\phi_{A_1\ldots A_{2s}}$ from a potential having $\overline{P}^{0'\cdots 0'}$ as its only nonzero component, then the extremal components of the field $\phi_{A_1\ldots A_{2s}}$ are

$$\phi_0 = [1/(\sqrt{2})^{2s}]\mathscr{D}_0^{2s}R_{-s}S_{+s}e^{i\sigma t + im\varphi},$$
$$\phi_{2s} = [1/(\sqrt{2})^{2s}\tilde{\rho}^{*2s}]\mathscr{L}_{1-s} \times\mathscr{L}_{2-s}\cdots\mathscr{L}_{s-1}\mathscr{L}_sR_{-s}S_{+s}e^{i\sigma t + im\varphi}. \quad (38)$$

Using the Teukolsky–Starobinsky identities[14] we can write these two components, up to some constant of proportionality, in the following form:

$$\phi_0 = R_{+s}S_{+s}e^{i\sigma t + im\varphi},$$
$$\phi_{2s} = (1/\tilde{\rho}^{*2s})R_{-s}S_{-s}e^{i\sigma t + im\varphi}. \quad (39)$$

III. INTRINSIC CHARACTERIZATION OF THE TEUKOLSKY SEPARATION CONSTANT

Suppose we form a solution $\phi_{A_1\ldots A_{2s}}$ for the massless spin s field Eqs. (29) by generating it from the extremal component of a generalized Hertz potential as in (33). We will also suppose that the space-time is the Kerr space-time if $s \leqslant 1$ while if $s > 1$ we will restrict ourselves to the oblate spheroidal coordinate system and null tetrad obtained by placing $M = 0$. The extremal components of the solution may then be written in the form given by (39). The other components of the field take on more complicated forms. The separation constant λ appearing in the solution is characterized by the following operator:

$$[K_{(A_1|}{}^{B'CC'}\nabla_{CC'} - \tfrac{1}{3}M_{(A_1|}{}^{B'}]\,[K^B{}_{B'}{}^{DD'}\nabla_{DD'} + (2s/3)M^B{}_{B'}]\phi_{B|A_2\ldots A_{2s})}$$
$$= \tfrac{1}{4} - 2J_{(A_1|}{}^{B'}{}_{4s}J^B{}_{B'}\phi_{B|A_2\ldots A_{2s})}$$
$$= \tfrac{1}{2}\lambda\phi_{A_1\cdots A_{2s}}. \quad (40)$$

For brevity we will also sometimes write the above as

$$\mathscr{I}\phi = \tfrac{1}{2}\lambda\phi. \quad (41)$$

The extremal components of this identity are relatively easy to verify using the form for the extremal components of the field given in (39). Since it is not possible to verify directly that the remaining components of this identity hold for arbitrary values of the spin parameter s we are forced to proceed by a different method. Firstly we will show that the operator and its action on the spin s field as given by (40) is a symmetry operator of the spin s field Eqs. (29) whenever those equations are consistent. That is the operator maps solutions into solutions. The following identity holds for any empty space-time which admits a second-order Killing–Yano tensor and for any spinor $\phi_{A_1\ldots A_{2s}}$. In particular we do not assume that $\phi_{A_1\ldots A_{2s}}$ satisfies a field equation of any sort. For $s > 1$ we have

$$
\begin{aligned}
&\nabla^{CC'}{}_\xi J_{(C|}{}^{A'}{}_\eta J^A{}_{A'}\phi_{A|B_2\cdots B_{2s})}\\
&= \big[{}_{\xi+2}J_{(B_2|}{}^{C'}{}_{\eta-2}J^A{}_{A'} - {}_\xi J_{(B_2|A'\eta}J^{AC'} - \tfrac{4}{9}M_{(B_2|}{}^{C'}M^A{}_{A'}\big] + (1/3s)\nabla_{(B_2|}{}^{C'}\big[K^{AD}M_{DA'} + \tilde{K}_{A'}{}^{D'}M^A{}_{D'}\big]\big]\nabla^{CA'}\phi_{AC|B_3\cdots B_{2s})}\\
&\quad + \big[(1/6s)((\eta-4s) - (\xi+2))\nabla_{(B_2|}{}^{C'}M^{CA'}{}_\eta J^A{}_{A'} + \tfrac{1}{6}(\xi+2)\big[W_{(B_2|}{}^C{}_\eta J^{AC'} + \epsilon_{(B_2|}{}^C\tilde{W}^{A'C'}{}_\eta J^A{}_{A'}\big]\\
&\quad - \tfrac{1}{6}((\eta-4s) + (2s-2)){}_{\xi+4}J_{(B_2|}{}^{C'}W^{AC} + \tfrac{1}{3}(2s-2)W_{(B_2|}{}^C{}_\eta J^{AC'}\big]\phi_{AC|B_3\cdots B_{2s})} + (2s-2)\big[-(1/12s)\nabla_{(B_2|}{}^{C'}\\
&\quad \big[K^{AC}W_{|B_3|}{}^M + 3\tilde{K}_{A'D'}\tilde{K}^{A'D'}\Psi^{ACM}{}_{|B_3|}\big] + {}_\xi J_{(B_2|}{}^{A'}\tilde{K}_{A'}{}^{C'}\Psi^{ACM}{}_{|B_3|} + \tilde{K}^{A'C'}\Psi^{CM}{}_{(B_2B_3|\eta}J^A{}_{A'}\big]\phi_{ACM|B_4\cdots B_{2s})}. \qquad (42)
\end{aligned}
$$

To prove this, first we split the left hand side of (42) into two parts. Removing the index C from the symmetrization we find that

$$
\begin{aligned}
&\nabla^{CC'}{}_\xi J_{(C|}{}^{A'}{}_\eta J^A{}_{A'}\phi_{A|B_2\cdots B_{2s})}\\
&= \nabla^{CC'}{}_\xi J_{(B_2|}{}^{A'}{}_\eta J^A{}_{A'}\phi_{AC|B_3\cdots B_{2s})} + (1/4s)\nabla_{(B_2|}{}^{C'}\big[{}_\xi J^C{}_{A'\eta}J^{AA'} - {}_\xi J^{AA'}{}_\eta J^C{}_{A'}\big]\phi_{AC|B_3\cdots B_{2s})}. \qquad (43)
\end{aligned}
$$

Taking the first term on the right hand side of (43) and applying (21) twice to pass $\nabla^{CC'}$ through ${}_\xi J_{(B_2|}{}^{A'}{}_\eta J^A{}_{A'}$ we obtain

$$
\begin{aligned}
&\nabla^{CC'}{}_\xi J_{(B_2|}{}^{A'}{}_\eta J^A{}_{A'}\phi_{AC|B_3\cdots B_{2s})}\\
&= {}_\xi J_{(B_2|}{}^{A'}\big[{}_\eta J^A{}_{A'}\nabla^{CC'} + \tfrac{2}{3}\big[M^{AC'}\nabla^C{}_{A'} - M^C{}_{A'}\nabla^{AC'}\big] + (\eta/6)\big[\epsilon^{C'}{}_{A'}W^{CA} - \epsilon^{CA}\tilde{W}^{C'}{}_{A'}\big] + K^{AD}\big[\nabla^{CC'},\nabla_{DA'}\big]\\
&\quad + \tilde{K}_{A'}{}^{D'}\big[\nabla^{CC'},\nabla^A{}_{D'}\big]\big]\phi_{AC|B_3\cdots B_{2s})} + \big[\tfrac{2}{3}\big[M_{(B_2|}{}^{C'}\nabla^{CA'} - M^{CA'}\nabla_{(B_2|}{}^{C'}\big] + (\xi/6)\big[\epsilon^{C'A'}W^C{}_{(B_2|} - \epsilon^C{}_{(B_2|}\tilde{W}^{C'A'}\big]\\
&\quad + K_{(B_2|}{}^D\big[\nabla^{CC'},\nabla_D{}^{A'}\big] + \tilde{K}^{A'D'}\big[\nabla^{CC'},\nabla_{(B_2|D'}\big]\big]{}_\eta J^A{}_{A'}\phi_{AC|B_3\cdots B_{2s})}. \qquad (44)
\end{aligned}
$$

By cycling the two contracted indices A' and the index C' we can write

$$
{}_\xi J_{(B_2|}{}^{A'}{}_\eta J^A{}_{A'}\nabla^{CC'} = \big[{}_\xi J_{(B_2|}{}^{C'}{}_\eta J^A{}_{A'} - {}_\xi J_{(B_2|A'\eta}J^{AC'}\big]\nabla^{CA'}. \qquad (45)
$$

Using (21) to pass $\nabla^{CA'}$ through ${}_\eta J^A{}_{A'}$ and applying some of the identities (10)–(18) we can also write

$$
\begin{aligned}
&\tfrac{2}{3}M_{(B_2|}{}^{C'}\nabla^{CA'}{}_\eta J^A{}_{A'}\phi_{AC|B_3\cdots B_{2s})}\\
&= \big[\tfrac{2}{3}M_{(B_2|}{}^{C'}{}_\eta J^A{}_{A'}\nabla^{CA'} - \tfrac{8}{9}M_{(B_2|}{}^{C'}M^A{}_{A'}\nabla^{CA'} - \tfrac{2}{9}(\eta-4)M_{(B_2|}{}^{C'}W^{AC} + \tfrac{2}{9}(2s-2)M_{(B_2|}{}^{C'}W^{AC}\big]\phi_{AC|B_3\cdots B_{2s})}. \qquad (46)
\end{aligned}
$$

Again applying the identities (10)–(18) we obtain

$$
\begin{aligned}
&\nabla^{CC'}{}_\xi J_{(B_2|}{}^{A'}{}_\eta J^A{}_{A'}\phi_{AC|B_3\cdots B_{2s})}\\
&= \big[\big[{}_{\xi+2}J_{(B_2|}{}^{C'}{}_{\eta-2}J^A{}_{A'}\nabla^{CA'} - {}_\xi J_{(B_2|A'\eta}J^{AC'}\nabla^{CA'} - \tfrac{4}{9}M_{(B_2|}{}^{C'}M^A{}_{A'}\big]\nabla^{CA'} - \tfrac{1}{6}(\eta-4){}_\xi J_{(B_2|}{}^{C'}W^{AC} - \tfrac{2}{9}(\eta-4)\\
&\quad \times M_{(B_2|}{}^{C'}W^{AC} - \tfrac{2}{3}M^{CA'}\nabla_{(B_2|}{}^{C'}{}_\eta J^A{}_{A'} + (\xi/6)W^C{}_{(B_2|\eta}J^{AC'} + \tfrac{1}{6}(\xi+4)\epsilon_{(B_2|}{}^C\tilde{W}^{A'C'}{}_\eta J^A{}_{A'}\\
&\quad + \tfrac{1}{6}(2s-2){}_{\xi+4}J_{(B_2|}{}^{C'}W^{AC} + \tfrac{1}{3}(2s-2)W_{(B_2|}{}^C{}_\eta J^{AC'}\big]\phi_{AC|B_3\cdots B_{2s})}\\
&\quad + (2s-2)\big[{}_\xi J_{(B_2|}{}^{A'}\tilde{K}_{A'}{}^{C'}\Psi^{ACM}{}_{|B_3|} + \tilde{K}^{A'C'}\Psi^{CM}{}_{(B_2B_3|\eta}J^A{}_{A'}\big]\phi_{ACM|B_4\cdots B_{2s})}. \qquad (47)
\end{aligned}
$$

Looking at the second term of the right hand side of (43) and using the definition (20) of the differential operator ${}_\xi J_{AA'}$, and making use of the symmetry in the indices A and C we have

$$
\begin{aligned}
&\nabla_{(B_2|}{}^{C'}\big[{}_\xi J^C{}_{A'\eta}J^{AA'} - {}_\xi J^{AA'}{}_\eta J^C{}_{A'}\big]\phi_{AC|B_3\cdots B_{2s})}\\
&= \nabla_{(B_2|}{}^{C'}\big[K^{CD}K^{AE}\big[\nabla_{DA'},\nabla_E{}^{A'}\big] + 2K^{CD}\tilde{K}^{A'E'}\big[\nabla_{DA'},\nabla^{A'}{}_{E'}\big] + \tilde{K}_{A'}{}^{D'}\tilde{K}^{A'E'}\big[\nabla^C{}_{D'},\nabla^A{}_{E'}\big] + 2K^{CD}(\nabla_{DA'}K^{AE})\nabla_E{}^{A'}\\
&\quad + 2K^{CD}(\nabla_{DA'}\tilde{K}^{A'E'})\nabla^A{}_{E'} + 2\tilde{K}_{A'}{}^{D'}(\nabla^C{}_{D'}K^{AE})\nabla_E{}^{A'} + 2\tilde{K}_{A'}{}^{D'}(\nabla^C{}_{D'}\tilde{K}^{A'E'})\nabla^A{}_{E'} + (2\eta/3){}_\xi J^C{}_{A'}M^{AA'}\\
&\quad + (2\xi/3)M^C{}_{A'0}J^{AA'}\big]\phi_{AC|B_3\cdots B_{2s})}. \qquad (48)
\end{aligned}
$$

Applying the identities (10)–(18) and noting that the field spinor and the Killing spinors are symmetric and that the quantities $K_{AC}W_B{}^C$ and $M^C{}_{A'}M^{AA'}$ are antisymmetric, and by cycling indices in some of the terms involving both the Killing vector $M_{AA'}$ and one or other of the Killing spinors, (48) becomes

$$
\begin{aligned}
&\nabla_{(B_2|}{}^{C'}\big[{}_\xi J^C{}_{A'\eta}J^{AA'} - {}_\xi J^{AA'}{}_\eta J^C{}_{A'}\big]\phi_{AC|B_3\cdots B_{2s})}\\
&= \tfrac{1}{3}\nabla_{(B_2|}{}^{C'}\big[4K^C{}_D\big[M^{AA'}\nabla^D{}_{A'} + M^{DA'}\nabla^A{}_{A'}\big]\phi_{AC|B_3\cdots B_{2s})}\\
&\quad + \big[2\eta{}_\xi J^C{}_{A'}M^{AA'} + 2\xi M^C{}_{A'\eta}J^{AA'}\big]\phi_{AC|B_3\cdots B_{2s})}\\
&\quad - (2s-2)\big[K^{AC}W^M{}_{|B_3|} + 3\tilde{K}_{A'D'}\tilde{K}^{A'D'}\Psi^{ACM}{}_{|B_3|}\big]\phi_{ACM|B_4\cdots B_{2s})}\big]. \qquad (49)
\end{aligned}
$$

We now note the following three relations: first,

$$
\begin{aligned}
&\tfrac{1}{3}M^{CA'}\nabla_{(B_2|}{}^{C'}{}_\eta J^A{}_{A'}\phi_{AC|B_3\cdots B_{2s})}\\
&= \big[\tfrac{1}{3}\nabla_{(B_2|}{}^{C'}M^{CA'}{}_\eta J^A{}_{A'} - \tfrac{1}{6}W_{(B_2|}{}^C{}_\eta J^{AC'}\\
&\quad + \tfrac{1}{6}\epsilon_{(B_2|}{}^C\tilde{W}^{A'C'}{}_\eta J^A{}_{A'}\big]\phi_{AC|B_3\cdots B_{2s})}. \qquad (50)
\end{aligned}
$$

Second, using the definition (20) of ${}_\eta J^A{}_{A'}$ we have

$$
\begin{aligned}
&(1/6s)\nabla_{(B_2|}{}^{C'}K^A{}_D M^{CA'}\nabla^D{}_{A'}\phi_{AC|B_3\cdots B_{2s})}\\
&= -(1/6s)\big[\nabla_{(B_2|}{}^{C'}M^{CA'}{}_\eta J^A{}_{A'}\\
&\quad + \nabla_{(B_2|}{}^{C'}\tilde{K}_{A'}{}^{D'}M^A{}_{D'}\nabla^{CA'}\big]\phi_{AC|B_3\cdots B_{2s})}, \qquad (51)
\end{aligned}
$$

and finally

$$(\eta/12s)\nabla_{(B_2|}{}^{C'}{}_{\xi}J^{C}{}_{A'}M^{AA'}\phi_{AC|B_3\cdots B_{2s})}$$
$$= (\eta/12s)\nabla_{(B_2|}{}^{C'}M^{CA'}{}_{\eta}J^{A}{}_{A'}\phi_{AC|B_3\cdots B_{2s})}, \quad (52)$$

since

$$_{\xi}J^{(A}{}_{A'}M^{C)A'} = 0. \quad (53)$$

Reforming (43) from its two pieces as given by (47) and (49) and making use of the three relations (50), (51), and (52) we obtain the desired result (42). One can also prove an analogous result for the cases $s = \frac{1}{2}s^1$. It therefore follows that if $\phi_{A_1\cdots A_{2s}}$ is a solution of

$$\nabla^{A}{}_{A'}\phi_{AA_2\cdots A_{2s}} = 0, \quad (54)$$

then the new field

$$\chi_{B_1\cdots B_{2s}} = {}_{\xi}J_{(B_1|}{}^{A'}{}_{\eta}J^{A}{}_{A'}\phi_{A|B_2\cdots B_{2s})} \quad (55)$$

is also a solution whenever

$$\begin{aligned}&\eta - 4s = \xi + 2 \text{ and } s \geqslant \tfrac{1}{2}, \quad \text{if } \Psi_{ABCD} = 0,\\ &\eta - 4s = 0 = \xi + 2 \text{ and } s \leqslant 1, \quad \text{if } \Psi_{ABCD} \neq 0.\end{aligned} \quad (56)$$

Thus under these conditions the differential operator (40) is a symmetry operator of the spin s field equations.

Having verified that the new field $\chi_{A_1\cdots A_{2s}}$ is a solution of the massless spin s field equations we now form the following field:

$$\zeta_{A_1\cdots A_{2s}} = \chi_{A_1\cdots A_{2s}} - \tfrac{1}{2}\lambda\phi_{A_1\cdots A_{2s}}, \quad (57)$$

where the field $\phi_{A_1\cdots A_{2s}}$ is obtained from a generalized Hertz potential which has $P^{0'\cdots 0'}$ as its only nonzero component and where λ is the separation constant appearing in the solution for this one nonzero component. Clearly the field $\zeta_{A_1\cdots A_{2s}}$ is a solution of the spin s field equations. It is our intention to show that this field is identically zero and hence that the operator and its action as given in (40) characterizes the separation constant λ. Carter and McLenaghan[4] and Kalnins *et al.*[5] have already shown for $s \leqslant 1$ that λ is characterized by the operator (40). We can therefore restrict ourselves to a flat space-time, i.e., place $M = 0$ and consider the cases where $s > 1$. One can verify by explicit computation that $\chi_0 = \frac{1}{2}\lambda\phi_0$ and $\chi_{2s} = \frac{1}{2}\lambda\phi_{2s}$ and so the extremal components of $\zeta_{A_1\cdots A_{2s}}$ must vanish. Further from the form of the operator (40) and since t and φ are ignorable coordinates we can also conclude that the other components of $\zeta_{A_1\cdots A_{2s}}$ must have the same t and φ dependence as $\phi_{A_1\cdots A_{2s}}$. Thus we can write

$$\zeta_0 = \zeta_{2s} = 0 \quad (58)$$

and

$$\zeta_j = f_j(r,\theta)e^{i\sigma t + im\varphi}. \quad (59)$$

We will now compare the behavior of the left-and right-hand sides of (57) as $\theta \to 0$ and also as $\theta \to \pi$. The argument is an inductive one in that we will show that if $\zeta_{j-1} = 0$ then $\zeta_j = 0$. Note that we already have $\zeta_0 = 0$. If we write $Z_k = \tilde{\rho}^{*k}\zeta_k$ and suppose that for some $j > 0$ we have $Z_{j-1} = 0$ then from (31) we find that Z_j must satisfy

$$[\mathscr{D}_0 + (2s - 2j + 1)(1/\tilde{\rho}^*)]Z_j = 0,$$
$$[\mathscr{L}^{\dagger}_{j-s} + (2s - 2j + 1)(ia\sin\theta/\tilde{\rho}^*)]Z_j = 0. \quad (60)$$

Intergrating these equations we have

$$\zeta_j = A\tilde{\rho}^{*-(2s-j+1)}e^{-i\sigma\tilde{\rho}}\left[\frac{a - ir}{\sqrt{r^2 + a^2}}\right]^m$$
$$\times (1 + \cos\theta)^m(\sin\theta)^{s-j-m}e^{i\sigma t + im\varphi}. \quad (61)$$

From the form of the solution it is clear that in the neighborhood of $\theta = 0$,

$$\zeta_j = \theta^{s-j-m}[b_0(r) + b_1(r)\theta + b_2(r)\theta^2 + \cdots]e^{i\sigma t + im\varphi}, \quad (62)$$

where we have dropped the subscript j from the functions $b_i(r)$. Letting $\tilde{\theta} = \pi - \theta$ we find that in the neighborhood of $\theta = \pi$,

$$\zeta_j = \tilde{\theta}^{s-j+m}[\tilde{b}_0(r) + \tilde{b}_1(r)\tilde{\theta} + \tilde{b}_2(r)\tilde{\theta}^2 + \cdots]e^{i\sigma t + im\varphi}. \quad (63)$$

We now consider the behavior of the right-hand side of (57) under the assumption that the field $\phi_{A_1\cdots A_{2s}}$ obtained from the generalized Hertz potential is regular at $\theta = 0$ and $\theta = \pi$. Firstly note that we can write $\phi_j = f_j(r,\theta)e^{i\sigma t + im\varphi}$ for each j and that from (31) we can generate a decoupled second order equation for ϕ_j. We find, when $M = 0$, that $\Phi_j = \tilde{\rho}^{*j}\phi_j$ satisfies the separable equation

$$[\mathscr{L}^{\dagger}_{j-s+1}\mathscr{L}_{s-j} + \Delta\mathscr{D}_1\mathscr{D}^{\dagger}_{s-j}$$
$$- 2(2s - 2j - 1)i\sigma\tilde{\rho}]\Phi_j = 0. \quad (64)$$

It therefore follows that we can write ϕ_j as a sum over λ of terms of the form

$$(1/\tilde{\rho}^{*j})R(r;\lambda)S(\theta;\lambda). \quad (65)$$

Now since to first order in θ the function $\tilde{\rho}^* = r - ia\cos\theta$ is independent of θ as $\theta \to 0$ and also as $\theta \to \pi$ it follows that to first order the θ dependence of ϕ_j in the neighborhood of $\theta = 0$ and $\theta = \pi$ will be determined by the behavior of the function S in these regions. From (64) we find that S satisfies

$$[\mathscr{L}^{\dagger}_{j-s+1}\mathscr{L}_{s-j} + 2(2s - 2j - 1)\sigma a\cos\theta]S(\theta;\lambda)$$
$$= -\lambda S(\theta;\lambda). \quad (66)$$

From an examination of this equation we find that the regular solution for S behaves in the neighborhood of $\theta = 0$ as

$$S = \theta^{|s-j+m|}(c_0 + c_1\theta + c_2\theta^2 + \cdots), \quad (67)$$

while in the neighborhood of $\theta = \pi$ we find

$$S = \tilde{\theta}^{|s-j-m|}(\tilde{c}_0 + \tilde{c}_1\tilde{\theta} + \tilde{c}_2\tilde{\theta}^2 + \cdots). \quad (68)$$

Given that ϕ is nonsingular we must have in the neighborhood of $\theta = 0$ that

$$\tfrac{1}{2}\lambda\phi_j = \theta^{|s-j+m|}[g_0(r) + g_1(r)\theta$$
$$+ g_2(r)\theta^2 + \cdots]e^{i\sigma t + im\varphi}. \quad (69)$$

Now $(\mathscr{T}\phi)_j$ will in general be formed from second-order derivatives of ϕ_j and first-order derivatives of both ϕ_{j-1} and ϕ_{j+1}. Thus in the neighborhood of $\theta = 0$ we will have

$$(\mathscr{T}\phi)_j = \theta^{|s-j+m|}[h_{-2}(r)\theta^{-2} + h_{-1}(r)\theta^{-1}$$
$$+ h_0(r) + h_1(r)\theta + \cdots]e^{i\sigma t + im\varphi}. \quad (70)$$

Subtracting (69) from (70) we find that we must have

$$\zeta_j = \theta^{|s-j+m|-2}[k_0(r) + k_1(r)\theta$$
$$+ k_2(r)\theta^2 + \cdots]e^{i\sigma t + im\varphi} \quad (71)$$

in the neighborhood of $\theta=0$. Similarly in the neighborhood of $\theta=\pi$ we have

$$\zeta_j=\tilde{\theta}^{|s-j-m|-2}[\tilde{k}_0(r)+\tilde{k}_1(r)\theta+\tilde{k}_2(r)\theta^2+\cdots]e^{i\sigma t+im\varphi}. \tag{72}$$

Now if $\zeta_{j-1}=0$ and we assume that $\zeta_j\neq 0$ then it is also required that ζ_j have the behavior specified in (62) and (63). Thus if the two differing behaviors of ζ_j are to be consistent we must have

$$\begin{aligned} s-j-m &\geqslant |s-j+m|-2, \\ s-j+m &\geqslant |s-j-m|-2. \end{aligned} \tag{73}$$

The above inequalities have a solution only when

$$-1\leqslant m\leqslant 1 \text{ and } j\leqslant s+1. \tag{74}$$

Thus for $|m|>1$ we must have $\zeta_j=0$ and so by induction all the components of $\zeta_{A_1\cdots A_{2s}}$ will vanish and so for $|m|>1$ we obtain

$$\mathscr{T}\phi=\tfrac{1}{2}\lambda\phi, \tag{75}$$

as desired.

To deal with the case where $|m|\leqslant 1$ we note that since we obtained $\phi_{A_1\cdots A_{2s}}$ from a generalized Hertz potential we can write ϕ_j as

$$\phi_j=\mathscr{H}_j R_{-s}S_{+s}, \tag{76}$$

where $\mathscr{H}_j$ is a differential operator of order $2s$. We can therefore write (75) as

$$\begin{aligned} &[\tilde{L}_{j-1}\mathscr{H}_{j-1}+L_j\mathscr{H}_j+\tilde{\tilde{L}}_{j+1}\mathscr{H}_{j+1}]R_{-s}S_{+s} \\ &\quad=\tfrac{1}{2}\lambda\mathscr{H}_j R_{-s}S_{+s}, \end{aligned} \tag{77}$$

where the differential operators L_j are of second order while the operators $\tilde{L}_j$ and $\tilde{\tilde{L}}_j$ are of first order. The only relations existing on the functions R_{-s} and S_{+s} by which this identity could hold are the Teukolsky equations for the functions R_{-s} and S_{+s}. Accordingly for some given j we must be able to write

$$\begin{aligned} &[\tilde{L}_{j-1}\mathscr{H}_{j-1}+L_j\mathscr{H}_j+\tilde{\tilde{L}}_{j+1}\mathscr{H}_{j+1}]-\tfrac{1}{2}\lambda\mathscr{H}_j \\ &\quad=\mathscr{G}_r\mathscr{T}_r+\mathscr{G}_\theta\mathscr{T}_\theta, \end{aligned} \tag{78}$$

where $\mathscr{G}_r$ and $\mathscr{G}_\theta$ are in general differential operators of order $2s$ and $\mathscr{T}_r$ and $\mathscr{T}_\theta$ are the "Teukolsky operators", that is

$$\begin{aligned} \mathscr{T}_r &\equiv \Delta\mathscr{D}^\dagger_{1-s}\mathscr{D}_0+2(2s-1)i\sigma r-\lambda, \\ \mathscr{T}_\theta &\equiv \mathscr{L}^\dagger_{1-s}\mathscr{L}_s+2(2s-1)\sigma a\cos\theta+\lambda, \end{aligned} \tag{79}$$

for which $\mathscr{T}_r R_{-s}=0$ and $\mathscr{T}_\theta S_{+s}=0$. Now we note that Eq. (78) may be split into two parts, namely those terms which are independent of λ and those terms which are linear in λ. We find that

$$\mathscr{G}_r-\mathscr{G}_\theta=\tfrac{1}{2}\mathscr{H}_j, \tag{80}$$

and hence that

$$\begin{aligned} &[\tilde{L}_{j-1}\mathscr{H}_{j-1}+L_j\mathscr{H}_j+\tilde{\tilde{L}}_{j+1}\mathscr{H}_{j+1}]-\tfrac{1}{2}\lambda\mathscr{H}_j \\ &-\tfrac{1}{2}\mathscr{H}_j\mathscr{T}_\theta=\mathscr{G}_\theta(\mathscr{T}_r+\mathscr{T}_\theta), \end{aligned} \tag{81}$$

thus $\mathscr{G}_r$ and $\mathscr{G}_\theta$ will be uniquely determined.

We have established that the above identities amongst the various differential operators must hold for $|m|>1$. Further, the m dependence of the various terms in any given identity is described by a polynomial in m. Since any given identity holds for an infinite number of values of m it must also hold when $m\leqslant 1$. We therefore have

$$\mathscr{T}\phi=\tfrac{1}{2}\lambda\phi, \tag{82}$$

for all values of m.

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