Matrix operator symmetries of the Dirac equation and separation of variables

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The set of all matrix-valued first-order differential operators that commute with the Dirac equation in n-dimensional complex Euclidean space is computed. In four dimensions it is shown that all matrix-valued second-order differential operators that commute with the Dirac operator in four dimensions are obtained as products of first-order operators that commute with the Dirac operator. Finally some additional coordinate systems for which the Dirac equation in Minkowski space can be solved by separation of variables are presented. These new systems are comparable to the separation in oblate spheroidal coordinates discussed by Chandrasekhar [S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford U.P., Oxford, 1983)].

I. INTRODUCTION

A complete theory of separation of variables for the nonscalar equations of mathematical physics has yet to be developed. Some partial results have been obtained for the Dirac equation, the Proca equation, and the Pauli–Fierz equation.[1–3] More recently there has been renewed interest in the separability properties of the equations for first-order perturbations of spin fields in a gravitational background. In particular Teukolsky[4] has shown that for massless fields of spin-0, -$\frac{1}{2}$, and -1, a form of separable solution does exist for perturbations in a Kerr metric gravitational background. Chandrasekhar[5] has shown that the Dirac equation also admits a separable solution in such a background. These results have been extended by several authors[6,7] and shown to hold for more general classes of type D vacuum metrics. More recently the constant of the motion associated with the separation of variables for the Dirac equation (the other two constants are associated with geometrical symmetries) has been characterized.[8–10] It is found that the additional constant of the motion is a matrix first-order differential operator that commutes with the Dirac Hamiltonian. This operator is associated with the generalized Killing tensors of Yano and Bochner.[11] Furthermore McLenaghan and Spindel[9] have established the general form of a matrix first-order operator that commutes with the Dirac Hamiltonian. An interesting feature of Chandrasekhar's work is that it also implies that the Dirac equation in Minkowski space admits a separable solution in oblate spheroidal coordinates. We should mention in this connection the work of Cook[12] on separation of variables for the Dirac equation. These results inject new life into the possibility of classifying all separable coordinate systems and solutions for the Dirac equation. To this end in Sec. II we compute the matrix operators which commute with the Dirac Hamiltonian in complex Euclidean n-space. We then study the first- and second-order matrix symmetries of the Dirac equation in four dimensions in Sec. III culminating in Theorem I. Finally, in Sec. IV we present several examples of separable solutions (of the Dirac equation) in four-dimensional Minkowski space and give their operator characterization.

II. FIRST-ORDER CONSTANTS OF THE MOTION FOR THE DIRAC HAMILTONIAN IN E_n

In complex Euclidean n-space E_n Cartesian coordinates will be denoted by z_i, $i = 1,...,n$, and the associated infinitesimal distance is

$$ds^2 = dz_i\, dz_i. \tag{2.1}$$

In this section repeated subscripts imply summation; we only use subscripts and work exclusively in Cartesian coordinates. Furthermore we will take the dimension to be $n = 2\nu$, i.e., even. Then, as is well known,[13] there is a unique representation of the Clifford algebra of dimension 2ν by $2\nu\times 2\nu$ matrices γ_i, which satisfy the anticommutation rules ({ , } is the anticommutator bracket)

$$\{\gamma_i,\gamma_j\} = \gamma_i\gamma_j + \gamma_j\gamma_i = 2I_n\delta_{ij}. \tag{2.2}$$

The associated Dirac Hamiltonian is

$$H = \gamma_i\partial_i + m \quad \left(\partial_i = \frac{\partial}{\partial z_i}\right). \tag{2.3}$$

Clearly the results of significance in this section are to be obtained by considering various real forms of complex Euclidean n-space. The resulting Ψ, which is a solution of $H\Psi = 0$, could then be interpreted as the solution of an appropriate wave equation in a physical theory realized in dimension n. The other advantage of working in complex Euclidean n-space is that a large number of different cases for operators of a given type correspond to a single class in this case. The classification problem is thus made considerably simpler.

We now search for operators $L = F_a\,\partial_a + G$, which commute with H:

$$[H,L\,] = 0. \tag{2.4}$$

Equating to zero the coefficients of the derivatives in this condition we obtain

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$$[\gamma_a, F_b] + [\gamma_b, F_a] = 0, \quad (2.5a)$$

$$[G,\gamma_a] = (\gamma_i\,\partial_i)F_a, \quad (2.5b)$$

$$(\gamma_i\,\partial_i)G = 0. \quad (2.5c)$$

In addition to the γ_i matrices we define $\gamma_{2\nu+1} = \omega = (1/(2\nu)!)\,\epsilon_{i_1\cdots i_{2\nu}}\gamma_{i_1}\cdots\gamma_{i_{2\nu}}$. (Here $\epsilon_{i_1\cdots i_{2\nu}}$ is the usual antisymmetric tensor.) This matrix satisfies

$$\{\omega,\gamma_j\} = 0 \quad j = 1,\ldots,2\nu. \quad (2.6)$$

A suitable basis[13] for the space of $2\nu\times 2\nu$ matrices is then

$$\gamma_{a_1}\cdots\gamma_{a_{2p}}, \quad p = 1,\ldots,\nu,$$
$$\omega\gamma_{a_1}\cdots\gamma_{a_{2p-1}}, \quad p = 1,\ldots,\nu. \quad (2.7)$$

where $a_i < a_j$ if $i<j$. We write

$$F_a = {}_1F_a I + {}_3F_{aa_1a_2}\gamma_{a_1}\gamma_{a_2} + \cdots + {}_{2\nu+1}F_{aa_1\cdots a_{2\nu}}\gamma_{a_1}\cdots\gamma_{a_{2\nu}} + {}_2F_{aa_1}\omega\gamma_{a_1} + \cdots + {}_{2\nu}F_{aa_1\cdots a_{2\nu-1}}\omega\gamma_{a_1}\cdots\gamma_{a_{2\nu-1}}, \quad (2.8)$$

where we take ${}_pF_{aa_1\cdots a_{p-1}} = {}_pF_{a[a_1\cdots a_{p-1}]}$. The square bracket denotes complete antisymmetrization. The conditions (2.5a) then imply

$${}_pF_{a_0\cdots a_{p-1}} = {}_pF_{[a_0\cdots a_{p-1}]}. \quad (2.9)$$

In particular

$${}_{2\nu+1}F_{a_0a_1\cdots a_{2\nu}} = 0 \text{ and } {}_{2\nu}F_{a_0a_1\cdots a_{2\nu-1}} = C\epsilon_{0,1,\ldots,2\nu-1}.$$

The conditions (2.5b) then imply

$$[1/p!]P_{a_0'\cdots a_{p-1}'}\left[\partial_{a_0'}\left({}_pF_{aa_1'\cdots a_{p-1}'}\right)\right] + \partial_c\left({}_{p+2}F_{aca_0\cdots a_{p-1}}\right) + 2\left({}_{p+2}G_{aa_0\cdots a_{p-1}}\right) = 0, \quad (2.10)$$

where round brackets denote symmetrization, the first summation is over all permutations $a_0'\cdots a_{p-1}'$ of the fixed set $a_0,\cdots,a_{p-1}$, and $P_{a_0'\cdots a_{p-1}'}$ is the sign of this permutation. From these equations we can deduce that

$$\partial_b\left({}_pF_{aa_1\cdots a_{p-1}}\right) + \partial_a\left({}_pF_{ba_1\cdots a_{p-1}}\right) = 0, \quad (2.11)$$

i.e., each ${}_pF_{a_1\cdots a_p}$ function is a generalized Killing–Yano tensor.[11] In particular, $\partial_a C = 0$ and $\partial_b({}_1F_a) + \partial_a({}_1F_b) = 0$. This last condition is just the statement that the ${}_1F_a$ are the components of a Killing vector. The remaining conditions are, in fact, redundant, since, for any general Killing–Yano tensor[11] ${}_pF_{a_1\cdots a_p}$, we have that

$$\partial_a\,\partial_b\left({}_pF_{a_1\cdots a_p}\right) = 0. \quad (2.12)$$

We thus see that the space of operators L is determined by the Killing–Yano tensors ${}_pF_{a_1\cdots a_p}$ and ${}_{p+1}G_{a_0\cdots a_{p+1}}$ via (2.10)

The general solution of the Killing–Yano equations (2.11) is known[11] to be

$${}_pF_{a_1\cdots a_p}(z_1,\ldots,z_n) = Az_i\epsilon_{ia_1\cdots a_p} + B\epsilon_{a_1\cdots a_p} \quad (2.13)$$

and the corresponding solutions for

$${}_{p+2}G_{aa_1\cdots a_p} = \tfrac{1}{2}p\,\partial_a\left({}_pF_{a_1\cdots a_p}\right) = A\,\tfrac{1}{2}p\,\epsilon_{aa_1\cdots a_p}. \quad (2.14)$$

A basis for the space $\{L\}$ consists of

$$L_{2q} = [1/(2q)!]P_{a'b'a_1'\cdots a_{2q}'}z_{a'}\,\partial_{b'}\,\gamma_{a_1'}\cdots\gamma_{a_{2q}'} + q\gamma_a\gamma_b\gamma_{a_1}\cdots\gamma_{a_{2q}}, \quad q = 0,1,\ldots,\nu, \quad (2.15)$$

$$L_{2q-1} = [1/(2q-1)!]P_{a'b'a_1'\cdots a_{2q-1}'}z_{a'}\,\partial_{b'}\,\omega\gamma_{a_1'}\cdots\gamma_{a_{2q-1}'} + (q-\tfrac{1}{2})\omega\gamma_a\gamma_b\gamma_{a_1}\cdots\gamma_{a_{2q-1}}, \quad q = 1,\ldots,\nu, \quad (2.16)$$

$$M_{2q} = [1/(2q)!]P_{b'a_1'\cdots a_{2q}'}\,\partial_{b'}\,\gamma_{a_1'}\cdots\gamma_{a_{2q}'}, \quad q = 0,1,\ldots,\nu, \quad (2.17)$$

$$M_{2q-1} = [1/(2q-1)!]P_{b'a_1'\cdots a_{2q-1}'}\,\partial_{b'}\,\omega\gamma_{a_1'}\cdots\gamma_{a_{2q-1}'}, \quad (2.18)$$

where the summations extend over a fixed set of indices (e.g., a, b, $a_1,\ldots,a_{2q}$ in the case of L_{2q}). This basis has a particular significance, which we can see as follows: consider operators of the type M_l. From (2.17) and (2.18) we have

$$M_l^2 = \partial_b^2 + \cdots + \partial_l^2, \quad (2.19)$$

i.e., M_l^2 is the second-order Casimir invariant for the subgroup E_{l+1}, whose Lie algebra has the basis

$$P_\lambda = \partial_\lambda,$$
$$M_{\lambda\mu} = z_\lambda\,\partial_\mu - z_\mu\,\partial_\lambda + \tfrac{1}{2}\gamma_\lambda\gamma_\mu,$$
$$\lambda,\mu = b,a_1,\ldots,a_l, \quad \lambda\neq\mu. \quad (2.20)$$

A similar result holds for the operators L_l; from (2.15) and (2.16) it follows that

$$L_l^2 = \sum_{\lambda>\mu}M_{\lambda\mu}^2 + \frac{1}{8}l(l+1)I_n, \quad (2.21)$$

i.e., to within a constant L_l^2 is the Casimir invariant for the subgroup $SO(l+1)$, whose Lie algebra has a basis

$$M_{\lambda\mu}, \quad \lambda,\mu = 1,b,a_1,\ldots,a_l, \quad \lambda>\mu.$$

These operators generalize the "square root of angular momentum" introduced by Dirac in this treatment of the electron.

It is in fact the study of orbits of commuting operators that should be of basic importance to a study of separation of variables. The particular example of Chandrasekhar[5] has highlighted this feature. From the point of view of separation of variables theory the operators that are associated with it could be second order. As a step in this direction we extend the studies of McLenaghan to second-order matrix differential operators that commute with H.

III. SECOND-ORDER CONSTANTS OF THE MOTION FOR THE DIRAC HAMILTONIAN IN E_4

In this section we study second-order operators of the type

$$\hat{\mathscr{L}} = K_{ab}\,\partial_a\,\partial_b + L_c\,\partial_c + M, \quad (3.1)$$

which commute with the Dirac Hamiltonian in complex Euclidean four-space. (To make the computations relatively straightforward we restrict ourselves to E_4.) The condition $[H,\hat{\mathscr{L}}] = 0$ is equivalent to the equations

$$[\gamma_{(a}, K_{bc)}] = 0, \quad (3.2a)$$

$$-2\gamma_d(\partial_d K_{ab}) + [\gamma_{(a}, L_{b)}] = 0, \quad (3.2b)$$

$$\gamma_d(\partial_d L_a) + [\gamma_a, M] = 0, \quad (3.2c)$$

$$\gamma_d(\partial_d M) = 0, \quad (3.2d)$$

where the ( ) subscripts denote complete symmetrization of the enclosed indices.

Our purpose is to show that all second-order constants of the motion of the type (2.1) can be constructed as products of the first-order ones, as calculated in Sec. II. We note that the set $\{\hat{\mathscr{L}}\}$ does not close under commutation; however, if L_1, L_2 are any two first-order matrix differential operators that commute with H, then $[L_1L_2, H] = 0$.

A suitable basis for $\{L\}$ in E_4 is

$$L_{abc} = \gamma_5[\gamma_{(a}z_b\,\partial_{c)} - \gamma_{(a}z_c\,\partial_{b)}] + \gamma_d,$$
$$a>b>c, \quad a,b,c,d \neq, \quad (3.3a)$$

$$Q_{ab} = \gamma_5\gamma_{[a}\,\partial_{b]}, \quad a>b, \quad (3.3b)$$

$$S_{abc} = \gamma_{(a}\gamma_b\,\partial_{c)}, \quad a>b>c, \quad (3.3c)$$

$$M = \epsilon_{abcd}\gamma^a\gamma^b z_c\,\partial_d + \tfrac{3}{2}\gamma_5, \quad (3.3d)$$

$$M_{ab} = z_{[a}\,\partial_{b]} + \tfrac{1}{2}\gamma_a\gamma_b, \quad a>b, \quad (3.3e)$$

$$P_i = \partial_i, \quad (3.3f)$$

$$H = \gamma_i\,\partial_i \quad (3.3g)$$

where all indices run from 1,...,4. From the first of conditions (3.2), if we write

$$K_{ab} = K_{ab}I + K_{abc}\gamma_c + K_{abcd}\gamma_c\gamma_d + \hat{K}_{abc}\gamma_5\gamma_c + \hat{K}_{ab}\gamma_5, \quad (3.4)$$

then the coefficients of K_{ab} must satisfy

$$K_{(abc)d} = 0, \quad (3.5)$$

$$K_{bcd}(\delta_{a\alpha}\delta_{d\beta} - \delta_{a\beta}\delta_{d\alpha}) + K_{acd}(\delta_{b\alpha}\delta_{d\beta} - \delta_{b\beta}\delta_{d\alpha})$$
$$+ K_{abd}(\delta_{c\alpha}\delta_{d\beta} - \delta_{c\beta}\delta_{d\alpha}) = 0, \quad (3.6)$$

$$\hat{K}_{(abc)} = 0, \quad (3.7)$$

$$\hat{K}_{ab} = 0, \quad (3.8)$$

in addition to the obvious symmetries

$$K_{[ab]} = 0, \quad (3.9)$$

$$K_{ab(cd)} = 0. \quad (3.10)$$

For the second set of conditions (3.2) we write

$$L_a = L_a I + L_{ab}\gamma_b + L_{abc}\gamma_b\gamma_c + \hat{L}_{ab}\gamma_5\gamma_b + \hat{L}_a\gamma_5, \quad (3.11)$$

with $L_{a(bc)} = 0$ and obtain

$$\partial_c K_{ab} + L_{(ab)c} + \partial_d K_{abdc} = 0, \quad (3.12)$$

$$\partial_{(c}K_{|ab|de)} + \epsilon_{fcde}\,\partial_f\,\hat{K}_{ab} + 2L_b\epsilon_{acde} + 2L_a\epsilon_{bcde} = 0, \quad (3.13)$$

$$\epsilon_{cdfe}\,\partial_c(K_{abd}) + \partial_{[f}\hat{K}_{|ab|e]} + \epsilon_{acfe}L_{bc} + \epsilon_{bcfe}L_{ac} = 0, \quad (3.14)$$

$$\partial_d(\hat{K}_{abd}) + 2(\hat{L}_{ab} + \hat{L}_{ba}) = 0, \quad (3.15)$$

$$\partial_c K_{abc} = 0. \quad (3.16)$$

For the third set of conditions, we write

$$M = MI + M_a\gamma_a + M_{ab}\gamma_a\gamma_b + M_c\gamma_5\gamma_c + M\gamma_5 \quad (3.17)$$

and obtain the conditions

$$\partial_c L_{ac} = 0, \quad (3.18)$$

$$\partial_b L_a + \partial_c L_{abc} - 2M_{ab} = 0, \quad (3.19)$$

$$\partial_{(b}L_{|a|cd)} + \epsilon_{ebcd}\,\partial_e\hat{L}_a + 2\epsilon_{abcd}\hat{M} = 0, \quad (3.20)$$

$$\partial_{[b}\,\hat{L}_{|a|c]} + \epsilon_{decb}\,\partial_d L_{ae} + 2\epsilon_{adcb}\,M_d + 2\epsilon_{bdcb}\,M_d = 0, \quad (3.21)$$

$$\partial_c\hat{L}_{ac} + 2\hat{M}_a = 0. \quad (3.22)$$

We will now show that the space of operators L in E_4 is spanned by all products of first-order matrix operators. From (3.12) and (3.5) we can see that

$$\partial_{(c}K_{ab)} = 0. \quad (3.23)$$

There are 50 independent operators $\tilde{L}$ of the form

$$\tilde{L} = K_{ab}\,\partial_a\,\partial_b + L_{abc}\gamma_b\gamma_c\,\partial_a + M \quad (3.24)$$

constructed from the symmetric products in the enveloping algebra, i.e., products of the form $\{P_i,P_j\}$, $\{P_i,M_{jk}\}$, $\{M_{ij},M_{kl}\}$. The conditions (3.23) are the equations for a symmetric Killing tensor to exist in four-dimensional flat space E_4. These conditions have been discussed by Katzin and Levin[14] and the above result is included in their work.

From (2.6) we deduce that

$$K_{abc} = 0, \quad a,b,c\neq, \quad (3.25)$$

$$-K_{aaa} + 2K_{abb} = 0, \quad a\neq b, \quad (3.26)$$

and consequently

$$K_{abc} = K_a\delta_{ac}. \quad (3.27)$$

From (2.16) we have that

$$\partial_a K_b + \partial_b K_a = 0;$$

these are just Killing's equations. There are therefore ten independent operators of the form

$$\hat{L} = K_{abc}\gamma_c\partial_a\,\partial_b + \hat{L}_{ab}\gamma_5\gamma_b\,\partial_a + L_{ab}\gamma_b\,\partial_a. \quad (3.28)$$

These operators are formed by taking symmetric products of the form $\{H, P_i\}$, $\{H,M_{ij}\}$.

From (3.7) and (3.14) we deduce that

$$\hat{K}_{(abc)} = 0, \quad (3.7)$$

$$\partial_{(a}\hat{K}_{bc)d} = 0. \quad (3.29)$$

The number of independent solutions can be calculated as follows. We note that for fixed d, $K^{(d)}_{bc} = \hat{K}_{bcd}$ satisfies the equations for a second-order Killing tensor in four-dimensional Euclidean space, i.e.,

$$\partial_{(a}K^{(d)}_{bc)} = 0. \quad (3.30)$$

It is known[14] that the vector space of second-order Killing tensors is in this case of dimension 50. Furthermore it is always possible to choose a basis of the form

$$\kappa^l_{bc} = A^{\,l}_{bc,ef}\,z_e z_f, \quad l = 1,...,20,$$

$$\mu^m_{bc} = B^{\,m}_{bc,e}z_e, \quad m = 1,...,20,$$

$$\nu^n_{bc} = C^{\,n}_{bc}, \quad n = 1,...,10, \quad (3.31)$$

where $A^{\,l}_{bc,ef}$, $B^{\,l}_{bc,e}$, and $C^{\,l}_{bc}$ are constants. Consequently because of (3.30) we may write

$$K^{(d)}_{bc} = c^{dl}\kappa^l_{bc} + d^{\,dm}\mu^m_{bc} + e^{dn}\nu^n_{bc}. \quad (3.32)$$

Our problem is to determine the number of independent coefficients c^{dl}, $d^{\,dm}$, e^{dn}, given that the $\hat{K}_{abc}$ are subject to the conditions (3.7) and

$$\partial_d\hat{K}_{(abc)} = 0, \quad (3.7)$$

$$\partial_d \partial_e \widehat{K}_{(abc)} = 0. \tag{3.33}$$

To determine the number of independent coefficients e^{dn} consider the $\widehat{K}_{abc}$ evaluated at $\mathbf{0}$, i.e., $\widehat{K}_{abc}(\mathbf{0})$. There are 40 such coefficients and they are subjected to 20 independent constraints $\widehat{K}_{(abc)}(\mathbf{0}) = 0$. There are therefore 20 independent constants e^{dn}. These are constructed from the 24 anticommutators $\{P_c, Q_{ab}\}$ subject to the four constraints

$$Q_{(ab}P_{c)} = 0. \tag{3.34}$$

To determine the number of independent coefficients d^{dm} we consider $\partial_a \widehat{K}_{bcd}(\mathbf{0})$. There are 80 unknown coefficients d^{dm} subject to the 51 constraints

$$\begin{aligned}
&\partial_a \widehat{K}_{daa}(\mathbf{0}) = 0, \quad a \neq d,\\
&\partial_d \widehat{K}_{aad}(\mathbf{0}) + 2\partial_d \widehat{K}_{add}(\mathbf{0}) = 0, \quad a \neq d,\\
&\partial_c \widehat{K}_{(aab)}(\mathbf{0}) = 0, \quad a,b,c \neq,\\
&\partial_a \widehat{K}_{(abc)}(\mathbf{0}) = 0, \quad a,b,c \neq,\\
&\partial_d \widehat{K}_{(abc)}(\mathbf{0}) = 0, \quad a,b,c,d \neq.
\end{aligned} \tag{3.35}$$

All indices are distinct in these conditions and conditions of the last type are subject to the restriction

$$\partial_a \widehat{K}_{(bcd)}(\mathbf{0}) + \partial_b \widehat{K}_{(cda)}(\mathbf{0}) + \partial_c K_{(dab)}(\mathbf{0}) + \partial_d \widehat{K}_{(abc)}(\mathbf{0}) = 0,$$

which follows from conditions (3.29).

A suitable basis for operators associated with these independent constants is obtained from the 52 anticommutators $\{P_a, L_{bcd}\}$, $\{M_{ab}, Q_{cd}\}$ subject to the 23 independent constraints

$$\begin{aligned}
&Q_{(ab}M_{c)d} = L_{abc}P_d + H/2,\\
&Q_{[a|b}M_{b|c]} = Q_{ca} - L_{abc}P_b,\\
&P_{[a}L_{bc|d]} = -M_{bc}Q_{ad} + M_{ad}Q_{bc},\\
&\sum_{\substack{a>b\\ c>d}} \epsilon_{abcd} M_{ab} Q_{cd} + \frac{H}{2} = 0.
\end{aligned} \tag{3.36}$$

To determine the number of independent coefficients c^{dl} we consider the second derivatives $\partial_a \partial_b \widehat{K}_{cde}(\mathbf{0})$. There are 80 unknown coefficients c^{dl} subject to 60 constraints:

$$\begin{aligned}
&\partial_a \widehat{K}_{daa}(\mathbf{0}) = 0, \quad \partial_d \partial_a \widehat{K}_{daa}(\mathbf{0}) = 0, \quad a \neq d,\\
&\partial_c \partial_a \widehat{K}_{daa}(\mathbf{0}) = 0, \quad a,c,d \neq,\\
&\partial_c^2 (\widehat{K}_{aab}(\mathbf{0}) + 2\widehat{K}_{baa}(\mathbf{0})) = 0, \quad ab,c \neq,\\
&\partial_c \partial_d (\widehat{K}_{aab}(\mathbf{0}) + 2\widehat{K}_{baa}(\mathbf{0})) = 0, \quad a,b,c \neq.
\end{aligned} \tag{3.37}$$

A suitable basis for operators associated with these independent constants is obtained from the 24 anticommutators $\{M_{ab}, L_{cde}\}$ subject to the four constraints

$$M_{a(b}L_{|a|cd)} = L_{bcd}. \tag{3.38}$$

From (3.5) and (3.13) we have the conditions

$$K_{(abc)d} = 0, \tag{3.39}$$

$$\partial_{(a}K_{bc)de} = 0. \tag{3.40}$$

We are, of course, also assuming that $K_{abcd} = K_{(ab)cd}$ and $K_{abcd} = K_{ab[cd]}$. Conditions (3.39) are then equivalent to the four types

$$K_{abcd} + K_{cabd} + K_{bcad} = 0, \quad a,b,c,d \neq, \tag{3.41a}$$

$$K_{acca} = 0, \quad a, \neq c, \tag{3.41b}$$

$$K_{aacd} + 2K_{caad} = 0, \quad a,c,d \neq, \tag{3.41c}$$

$$K_{aaad} = 0, \quad a, \neq d. \tag{3.41d}$$

Conditions (3.40) are equivalent to the five types

$$\partial_a K_{bcce} + \partial_c K_{abce} + \partial_b K_{acce} = 0, \quad a,b,c,e \neq, \tag{3.42a}$$

$$\partial_a K_{bccb} + \partial_c K_{bacb} + \partial_b K_{accb} = 0, \quad a,b,c \neq, \tag{3.42b}$$

$$2\partial_a K_{acde} + \partial_c K_{aade} = 0, \quad a,c,d,e \neq, \tag{3.42c}$$

$$2\partial_a K_{acce} + \partial_c K_{aace} = 0, \quad a,c,e \neq, \tag{3.42d}$$

$$\partial_a K_{aade} = 0, \quad a,d,e \neq. \tag{3.42e}$$

The number of independent solutions of these equations again can be determined by the constants $K_{abcd}(\mathbf{0})$ and all possible derivatives of K_{abcd} evaluated at $\mathbf{0}$. The number of independent components of $K_{abcd}(\mathbf{0})$ is 60 and the number of constraints of type (3.29) is 34. The number of derivatives $\partial_e K_{abcd}(\mathbf{0})$ is 240. The number of constraints on the first derivatives are obtained by counting the number of independent constraints from (3.40) and the derivatives of (3.39). There are 136 such conditions, as it can readily be verified that all the conditions so obtained are independent. If we now repeat these considerations for the derivatives $\partial_e \partial_f K_{abcd}(\mathbf{0})$ we obtain at first glance 600 such derivatives and 628 conditions on them obtained by differentiating conditions (3.39) twice and conditions (3.40) once. There are, in fact, only 600 independent conditions. This can be seen as follows. From the conditions

$$\begin{aligned}
&\partial_a (2\partial_a K_{acde} + \partial_c K_{aade}) = 0, \quad a,c,d,e \neq,\\
&\partial_e \partial_a K_{aade} = 0, \quad e,a,d \neq,
\end{aligned} \tag{3.43}$$

we deduce that

$$\partial_a^2 K_{acde} = 0, \quad a,c,d,e \neq. \tag{3.44}$$

Differentiating (3.41c) with respect to ∂_a and using (3.42e) we have that $\partial_a K_{caad} = 0$. Consequently the four conditions

$$\partial_c (\partial_a K_{bcce} + \partial_c K_{abce} + \partial_b K_{acce}) = 0, \quad a,b,c,e \neq, \tag{3.45}$$

which are obtained from (3.42a), are redundant.

Further, the conditions

$$\partial_a (2\partial_a K_{acce} + \partial_c K_{aace}) = 0, \quad a \neq c, \tag{3.46}$$

$$\partial_c \partial_a K_{aadc} = 0, \quad c,a,d \neq, \tag{3.47}$$

imply that

$$\partial_a^2 K_{acce} = 0, \quad a,c,e \neq. \tag{3.48}$$

The condition

$$\partial_c^2 (K_{aacd} + 2K_{caad}) = 0, \quad a,c,d \neq, \tag{3.49}$$

then implies

$$\partial_c^2 K_{aacd} = 0, \quad a,c,d \neq. \tag{3.50}$$

Then condition

$$\partial_b (\partial_c K_{bacd} + \partial_b K_{accb}) = 0, \quad a,b,c \neq, \tag{3.51}$$

implies that

$$\partial_b \partial_c K_{abbc} = 0, \quad a,b,c \neq. \tag{3.52}$$

Now conditions

$$\partial_a \partial_d (K_{aacd} + 2K_{caad}) = 0, \quad c,a,d \neq, \qquad (3.53)$$

also imply (3.52), so they are redundant. There are 24 of them. Thus we have succeeded in showing that there are only 600 independent conditions on the second derivatives $\partial_e \partial_f K_{abcd}(0)$. In fact, apart from the redundancies noted above, all these conditions are independent. These computations indicate that there must be 58 independent solutions to our original conditions. These solutions can be generated by anticommutators $\{H,Q_{ab}\}$, $\{H,L_{abc}\}$, $\{M,M_a\}$, $\{M,P_i\}$, $\{S_{abc},M_{ij}\}$, $\{S_{abc},P_i\}$. There are 60 of these combinations but there are two independent relations among them:

$$S_{(abc}P_{d)} = 0, \qquad (3.54)$$

$$\sum_{i>j>k} L_{ijk}(P_i + P_j + P_k) + \Big(\sum_{i>j} Q_{ij}\Big) H = 0. \qquad (3.55)$$

Theorem 1: Let $H = \gamma_a \partial_a + m$ be the Dirac operator in complex Euclidean four-space. Further let $\mathscr{L} = \{L\}$ be the space of all first-order differential operators $L = F_a \partial_a + G$ that commute with H. Then the space $\widehat{\mathscr{L}} = \{\widehat{L}\}$, consisting of all second-order operators of the type $\widehat{L} = K_{ab} \partial_a \partial_b + L_c \partial_c + M$, $K_{ab} \not\equiv 0$, that commute with H, is spanned by all products of element pairs of elements of $\mathscr{L}$.

Theorem 1 suggests that higher-order operators that commute with the Dirac operator also can be constructed as products of first-order symmetries, but we have not proved this.

IV. SEPARATION OF VARIABLES FOR THE DIRAC EQUATION IN MINKOWSKI SPACE

In this section we discuss how the first-order matrix operators L that commute with the Dirac Hamiltonian can be associated with separable solutions of Dirac's equation. This was implicitly shown by Chandrasekhar's analysis[5] of Dirac's equation in a Kerr background and explicitly by the detailed study[8] of Carter and McLenaghan. In the limiting case, where the Kerr metric degenerates to a flat space metric in oblate spheroidal coordinates, we have infinitesimal distance

$$ds^2 = dt^2 - \left[\frac{r^2 + a^2\cos^2\theta}{(r^2+a^2)} dr^2 + (r^2 + a^2\cos^2\theta) d\theta^2 + (r^2+a^2)\sin^2\theta\, d\phi^2 \right]. \qquad (4.1)$$

A more familiar version of this infinitesimal distance can be obtained by putting $r = a \sinh \eta$:

$$ds^2 = dt^2 - a^2[(\sinh^2\eta + \cos^2\theta)(d\eta^2 + d\theta^2) + \cosh^2\eta \sin^2\theta\, d\phi^2]. \qquad (4.2)$$

Dirac's equation in Newman–Penrose notation is

$$(D + \epsilon - p)F_1 + (\delta^* + \pi - \alpha)F_2 = imG_1,$$
$$(\Delta + \mu - \gamma)F_2 + (\delta + \beta - \tau)F_1 = imG_2,$$
$$(D + \epsilon^* - p^*)G_2 - (\delta + \pi^* - \alpha^*)G_1 = imF_2,$$
$$(\Delta + \mu^* - \gamma^*)G_1 - (\delta + \beta^* - \tau)G_2 = imF_1.$$

Here we have used Chandrasekhar's[5] notation for the spin coefficients and derivatives. A distinguishing feature of spinor equations is that the specification of coordinates does not determine uniquely the resulting form of the equation; one also needs to specify a (null) tetrad or moving reference frame in order to write the resulting equation. These ideas have their natural framework in the tetrad formalism.[5]

The first-order operators L, which commute with the Dirac operator H, are of crucial importance for the separable solutions of Dirac's equation computed by Chandrasekhar. We briefly review his procedure. Essentially Chandrasekhar has shown that Dirac's equation in a Kerr space-time background admits a solution which can be obtained from a separation of variables ansatz. Since oblate spheroidal coordinates in Minkowski space are a special case of the standard Kerr space-time metric, this implies that the Dirac equation in Minkowski space admits separable solutions in these coordinates.

What is interesting about Chandrasekhar's result is that the proper choice of null tetrad is unexpected. If we adopt coordinates t, r, θ, and ϕ, corresponding to the infinitesimal distance (4.1), the proper null tetrad has the components

$$l^i = (1,1,0, a/(r^2+a^2)), \qquad (4.3a)$$

$$n^i = [1/2(r^2 + a^2\cos^2\theta)](r^2+a^2, -r^2-a^2, 0, a), \qquad (4.3b)$$

$$m^i = [1/\sqrt{2}(r + ia\cos\theta)](ia\sin\theta, 0, 1, i/\sin\theta). \qquad (4.3c)$$

This is quite different from the appropriate choice in, say, the case of cylindrical coordinates with infinitesimal distance

$$ds^2 = dt^2 - dr^2 - r^2\, d\phi^2 - dz^2, \qquad (4.4)$$

$$l^i = (1/\sqrt{2})(1,0,0,1), \qquad (4.5a)$$

$$n^i = (1/\sqrt{2})(1,0,0,-1), \qquad (4.5b)$$

$$m^i = (1/\sqrt{2})(0,1,i/r,0). \qquad (4.5c)$$

This frame is simply related to the frame of orthogonal vectors

$$e^i_t = (1/\sqrt{2})(l^i + n^i), \quad e^i_z = (1/\sqrt{2})(l^i - n^i),$$
$$e^i_r = (1/\sqrt{2})(m^i + \overline{m}^i), \quad e^i_\phi = (1/\sqrt{2})(m^i - \overline{m}^i), \qquad (4.6)$$

where

$$e^i_\lambda e_{\mu i} = 0, \quad \text{if } \lambda \neq \mu,$$
$$e^i_\lambda e_{\lambda i} = \epsilon_\lambda,$$

$\epsilon_\lambda = +1$ if $\lambda = 1$ and -1 otherwise. For all coordinate systems that are characterized by the Casimir operators of some subgroup chain of the Poincaré group E(3,1), a choice of tetrad of this type will yield separable solutions and uncoupled equations.[15] There are sound group theoretical reasons for this, which we do not elaborate on here. Now the obvious choice for oblate spheroidal coordinates would be a null tetrad constructed via (4.6) from the orthogonal vectors

$$\tilde{e}^a_t = (1,0,0,0),$$
$$\tilde{e}^a_r = -\sqrt{(r^2+a^2)/(r^2+a^2\cos^2\theta)}\,(0,1,0,0),$$
$$\tilde{e}^a_\theta = (1/\sqrt{r^2+a^2\cos^2\theta})(0,0,1,0), \qquad (4.7)$$
$$\tilde{e}^a_\phi = (1/\sin\theta\sqrt{r^2+a^2})(0,0,0,1).$$

However, this choice does not lead to separable solutions. With the proper null tetrad (4.3) and

$$f_1 = (r - ia\cos\theta)F_1, \quad f_2 = F_2,$$
$$g_2 = (r + ia\cos\theta)G_2, \quad g_1 = G_1, \tag{4.8}$$

Dirac's equation becomes

$$\mathscr{D}_0 f_1 + 2^{-1/2}\mathscr{L}_{1/2} f_2 = m(ir + a\cos\theta)g_1,$$
$$\Delta\mathscr{D}^\dagger_{1/2} f_2 - 2^{1/2}\mathscr{L}_{1/2} f_1 = -2m(ir + a\cos\theta)g_2,$$
$$\mathscr{D}_0 g_2 - 2^{-1/2}\mathscr{L}^\dagger_{1/2} g_1 = m(ir - a\cos\theta)f_2, \tag{4.9}$$
$$\Delta\mathscr{D}^\dagger_{1/2} g_1 + 2^{1/2}\mathscr{L}_{1/2} g_2 = -2m(ir - a\cos\theta)f_1,$$

where

$$\mathscr{D}_0 = \partial_r + iK/\Delta, \quad \mathscr{D}^\dagger_{1/2} = \partial_r - iK/\Delta + r/\Delta,$$
$$\mathscr{L}_{1/2} = \partial_\theta + Q + \tfrac{1}{2}\cot\theta, \quad \mathscr{L}^\dagger_{1/2} = \partial_\theta - Q + \tfrac{1}{2}\cot\theta,$$

and

$$K = (r^2 + a^2)\sigma + am^*, \quad Q = a\sigma\sin\theta + m^*\csc\theta.$$

In these equations the t and ϕ dependence has been removed by assuming it to be of the form $e^{i(\sigma t + m^*\phi)}$ and factored out. These equations admit a separable solution if we make the substitution

$$f_1 = R_{-1/2}(r)S_{-1/2}(\theta), \quad f_2 = R_{1/2}(r)S_{1/2}(\theta),$$
$$g_1 = R_{1/2}(r)S_{-1/2}(\theta), \quad g_2 = R_{-1/2}(r)S_{1/2}(\theta). \tag{4.10}$$

The functions appearing in this substitution can be chosen to satisfy

$$\Delta^{1/2}\mathscr{D}_0 R_{-1/2} = (\lambda + imr)\Delta^{1/2}R_{1/2},$$
$$\Delta^{1/2}\mathscr{D}^\dagger_0 \Delta^{1/2}R_{1/2} = (\lambda - imr)R_{-1/2},$$
$$\mathscr{L}_{1/2}S_{1/2} = -(\lambda - am\cos\theta)S_{-1/2}, \tag{4.11}$$
$$\mathscr{L}^\dagger_{1/2}S_{-1/2} = (\lambda + am\cos\theta)S_{1/2},$$

where λ is a separation constant. If we write ψ as the column vector (f_1, f_2, g_1, g_2) the Dirac's equation has the form $H\psi = m\psi$.

The separation constant is also the eigenvalue of an operator L; i.e.,

$$L\psi = \lambda\psi, \tag{4.12}$$

where

$$L = \begin{bmatrix} 0 & 0 & (ia\cos\theta/2\rho)\Delta\mathscr{D}^\dagger_{1/2} & (-r/\sqrt{2}\rho)\mathscr{L}_{1/2} \\ 0 & 0 & (r/\sqrt{2}\rho)\mathscr{L}^\dagger_{1/2} & (ia\cos\theta/\rho)\mathscr{D}_0 \\ (-ia\cos\theta/\rho^*)\mathscr{D}_0 & (r/\sqrt{2}\rho^*)\mathscr{L}_{1/2} & 0 & 0 \\ (r/\sqrt{2}\rho^*)\mathscr{L}^\dagger_{1/2} & (-ia\cos\theta/2\rho^*)\Delta\mathscr{D}^\dagger_{1/2} & 0 & 0 \end{bmatrix}. \tag{4.13}$$

In addition we have that $[H,L] = 0$, so L is a first-order matrix symmetry operator. In fact,[8]

$$L = (1/\sqrt{2})(\widetilde{L}_{234} + a\widetilde{Q}_{14}), \tag{4.14}$$

where the operators $\widetilde{L}$ are those obtained from (3.3) for the corresponding realization in Minkowski space. In this particular case

$$\widetilde{L}_{234} = \gamma^5\gamma^2(x^4\partial_3 - x^3\partial_4) + \gamma^5\gamma^3(x^2\partial_4 - x^4\partial_2)$$
$$+ \gamma^5\gamma^4(x^3\partial_2 - x^2\partial_3) + \gamma^1, \tag{4.15}$$
$$\widetilde{Q}_{14} = \gamma^5\gamma^1\partial_4 + \gamma^5\gamma^4\partial_1,$$

where the γ^i matrices satisfy

$$\{\gamma^i, \gamma^j\} = g^{ij},$$

where $g^{ij} = \mathrm{diag}(1, -1, -1, -1)$ and contravariant coordinates x^i, $i = 1,2,3,4$.

If one is to construct a satisfactory theory of variable separation for equations of Dirac type, examples of this type need to be explained. In fact, from our knowledge of separable systems for the scalar wave equation we can construct additional such examples of separation. Consider, for instance, the coordinates

$$t = r\cosh\theta, \quad x = \sqrt{r^2 + a^2}\sinh\theta\cos\phi$$
$$y = \sqrt{r^2 + a^2}\sinh\theta\sin\phi, \quad z = z, \tag{4.16}$$
$$0 \leqslant r < \infty, \quad -\infty < \theta < \infty, \quad 0 \leqslant \theta < 2\pi, \quad -\infty < z < \infty.$$

This is clearly a slightly different variation of oblate spheroidal coordinates. The appropriate null tetrad has contravariant components

$$l^i = (1, 0, a/(r^2 + a^2), 1),$$
$$n^i = [1/2(r^2 + a^2\cosh^2\theta)](r^2 + a^2, 0, -a, -r^2 - a^2), \tag{4.17}$$
$$m^i = [1/\sqrt{2}(r + ia\cosh\theta)](0, 1, i/\sinh\theta, ia\sinh\theta).$$

These coordinates (4.16) and (4.17) enable Dirac's equation to be written as

$$\left(D_+ + \frac{1}{\bar{\eta}^*}\right)F_1 + \frac{1}{\sqrt{2}\bar{\eta}^*}\left(\mathscr{L}_- + \frac{1}{2}\coth\theta\right)F_2 = imG_1, \tag{4.18a}$$

$$\frac{(r^2 + a^2)}{2(r^2 + a^2\cosh^2\theta)}\left(D_- + \frac{2}{(r^2 + a^2)}\right)F_2 + \frac{1}{\sqrt{2}\bar{\eta}}\left(\mathscr{L}_+ + \frac{1}{2}\coth\theta - \frac{ia\sinh\theta}{\bar{\eta}^*}\right)F_1 = imG_2, \tag{4.18b}$$

$$\left(D_+ + \frac{1}{\bar{\eta}}\right)G_2 - \frac{1}{\sqrt{2}\bar{\eta}}\left(\mathscr{L}_- + \frac{1}{2}\coth\theta\right)G_1 = imF_2, \tag{4.18c}$$

$$\frac{(r^2 + a^2)}{2(r^2 + a^2\cosh^2\theta)}\left(D_- + \frac{r}{(r^2 + a^2)}\right)G_1 - \frac{1}{\sqrt{2}\bar{\eta}^*}\left(\mathscr{L}_- + \frac{1}{2}\coth\theta + \frac{ia\sinh\theta}{\bar{\eta}}\right)G_2 = imF_1, \tag{4.18d}$$

where $D_{\pm} = \partial_r \pm iam^*/(r^2+a^2) \pm i\tau$,

$\mathscr{L}_{\pm} = \partial_\theta \mp (m^*/\sinh\theta)$

$\pm a\tau \sinh\theta, \quad \bar\eta = r + ia\cosh\theta,$

and we have assumed ϕ, z dependence to be $e^{i(m^*\phi + \tau z)}$.

Putting $f_1 = \bar\eta^* F_1$, $g_2 = \bar\eta G_2$, $f_2 = F_2$, and $g_1 = G_1$, we can verify that separation of variables can be achieved via the substitution (4.10) and the coupled first-order equations

$$D_+ R_{-1/2} = (\lambda + mr) R_{1/2} \tag{4.19a}$$

$$D_- R_{-1/2} = (-\lambda + mr) R_{-1/2} \tag{4.19b}$$

$$(-1/2)\mathscr{L}_+ S_{-1/2} = (-\lambda + ima\cosh\theta) S_{+1/2}, \tag{4.19c}$$

$$(-1/2)\mathscr{L}_- S_{1/2} = (\lambda + ima\cosh\theta) S_{-1/2}. \tag{4.19d}$$

The operator whose eigenvalue is the separation parameter is $\widetilde{L}_{123} + a\widetilde{Q}_{14}$. For a genuinely new separable system differing from spheroidal coordinates consider the infinitesimal distance associated with the coordinate system (4.16),

$$ds^2 = (r^2 + a^2\cosh^2\theta)\left[\frac{dr^2}{(r^2+a^2)} - d\theta^2\right] - (r^2+a^2)\sinh^2\theta\, d\phi^2 - dz^2. \tag{4.20}$$

If we allow r and θ to be large and replace $a/2$ by b and ϕ by $2v$, we obtain the related infinitesimal distance

$$ds^2 = (r^2 + b^2 e^{2\theta})[dr^2/r^2 - d\theta^2] - r^2 e^{2\theta} dv^2 - dz^2. \tag{4.21}$$

We can then associate with this distance the null tetrad

$$l^i = (1,0,b/r^2,1), \tag{4.22a}$$

$$n^i = [r^2/2(r^2 + b^2 e^{2\theta})](1,0,-b/r^2,-1), \tag{4.22b}$$

$$m^i = [1/\sqrt{2}(r + ibe^\theta)](0,1,-ie^{-\theta},ibe^\theta). \tag{4.22c}$$

This coordinate system and frame clearly afford a separation of variables via the foregoing techniques. The separation equations can be obtained by the appropriate limits from the equations for the previous coordinate system. A suitable choice of space time coordinates is

$$t + x = \frac{1}{b} r e^\theta v^2 + \frac{r}{b} e^{-\theta} - \frac{b}{r} e^\theta, \qquad t - x = bre^\theta, \quad y = re^\theta v, \quad z = z. \tag{4.23}$$

The operator which describes the separation of the variables r and θ is

$$\widetilde{L}_{123} + b(\widetilde{Q}_{24} + \widetilde{Q}_{14}). \tag{4.24}$$

The examples of coordinate systems given here are based on the mechanism given in Chandrasekhar's original work. Clearly separation of variables for Dirac's equation depends on a simultaneous choice of coordinates and null tetrad. What is the connection between the null tetrad (4.3) and oblate spheroidal coordinates (4.1)? The operator L (4.14) has associated it with a Killing–Yano tensor $D^c{}_b$ with matrix elements

$$D^c{}_b = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & 0 & -x^4 & x^3 \\ 0 & x^4 & 0 & -x^2 \\ a & -x^3 & x^2 & 0 \end{bmatrix}. \tag{4.25}$$

If we compute the roots and eigenvectors of $D^a{}_b$,

$$(D^c{}_b - \lambda\delta^c{}_b)v^b = 0, \tag{4.26}$$

we obtain

$$\begin{aligned} &(i)\ \lambda = \pm ir, \quad v \sim (\pm ia\sin\theta, 0, 1, \pm i/\sin\theta), \\ &(ii)\ \lambda = \pm a\cos\theta, \quad v \sim (1, \pm 1, 0, a/(r^2+a^2)). \end{aligned} \tag{4.27}$$

Here we have written the eigenvectors relative to the t, r, θ, ϕ moving frame and have chosen Minkowski space-time coordinates as in (4.16). From (4.3) we see that these eigenvectors are, apart from normalization conventions, the basis vectors that define the null tetrad (4.3). We can go further than this. Consider the square of the matrix $D = (D^a{}_b)$. We know[11] that $\mathscr{D}^a{}_b = D^a{}_c D^c{}_b$ is a Killing tensor and, from what we have just observed, the above eigenvectors are also eigenvectors of $D^a{}_b$. In fact the Killing tensor $\mathscr{D}^{ab}$ is intimately related to the choice of coordinates t, r, θ, and ϕ. If we pass to the cotangent bundle (i.e., phase space in Minkowski space-time), then

$$\mathbb{D} = \mathscr{D}^{cb} p_c p_b = m_{23}^2 + m_{24}^2 + m_{34}^2 + ap_1 m_{23} + a^2(p_1^2 - p_4^2), \tag{4.28}$$

where we have used the notation

$$m_{ij} = x^i px^j - x^j px^i, \quad i,j = 2,3,4, \tag{4.29}$$

$$p_i = px^i, \quad i = 1,...,4. \tag{4.30}$$

These linear forms on the cotangent bundle, together with

$$n_{1j} = x^1 px^j + x^j px^1, \quad j = 2,3,4, \tag{4.31}$$

form the usual representation of the Poincaré algebra $\epsilon(3,1)$ with the Poisson bracket as commutator.

For additive separation of variables for the Hamilton–Jacobi equation

$$H = g^{cb} p_y c p_y b = p_{x^1}^2 - p_{x^2}^2 - p_{x^3}^2 - p_{x^4}^2 = E, \tag{4.32}$$

expressed in a given coordinate system $\{y^i\}$ (e.g., t, r, θ, ϕ, oblate spheroidal coordinates), there is a complete theory.[16,17]

Theorem: Necessary and sufficient conditions for the existence of an orthogonal separable coordinate system $\{y^i\}$ for the Hamilton–Jacobi equation

$$H = g^{ab}\partial_x a W \partial_x b W = E, \tag{4.33}$$

$g^{ij} = g^{ji}$, $1 \leqslant i,j \leqslant n$, are that there exist $n-1$ quadratic functions $A^\alpha = a^{(\alpha)ab} p_a p_b$ satisfying the following.

(1) The $\{A^\alpha\}$ are constants of the motion, i.e., $[H, A^\alpha] = 0$, $\alpha = 1,...,n-1$, where [,] is the Poisson bracket.

(2) The $\{A^\alpha\}$ are in involution: $[A^\alpha, A^\beta] = 0$, $1 \leqslant \alpha, \beta \leqslant n-1$.

(3) The set $\{H, A^1,...,A^{n-1}\}$ is linear independent (as n quadratic forms).

(4) At least one of the quadratic forms, say A^1, has simple roots.

(5) In a local coordinate system $\{z^j\}$ the quadratic forms satisfy the algebraic commutation property

$$a^{(\alpha)}{}_{ab} a^{(\beta)b}{}_c = a^{(\beta)}{}_{ab} a^{(\alpha)b}{}_c.$$

To obtain additive separable solutions we identify

$p_x i = \partial_x i W$ and look for solutions of the form $W = \Sigma_{i=1}^n W_i(y^i, \mathbf{c})$, which are a complete integral, i.e., $\det(\partial_y i \partial_{c_j} W) \neq 0$, $\mathbf{c} = (c_1, ..., c_n)$.

For oblate spheroidal coordinates a suitable choice of basis for the constants of the motion $\{A^{(\alpha)}\}$ is

$$A^1 = m_{23}^2 + m_{24}^2 + m_{34}^2 + a^2(p_2^2 + p_3^2), \quad A^1 = p_1^2, \quad A^3 = m_{23}^2. \tag{4.34}$$

Strictly speaking a set of constants of the motion in which A^1 is replaced by $\mathbb{D}$ will not satisfy the criteria of our theorem. In particular, condition (5) is not satisfied. However, it is obvious from (4.28) and (4.34) that in oblate spheroidal coordinates we could always choose separable solutions for which $D = c_1$, $p_1^2 = c_2$, $m_{23}^2 = c_3$, for fixed constants c_1, c_2, c_3. Thus there are involutive sets of operators that do not satisfy the criteria of our theorem but that admit additively separable solutions in which each element of this involutive set is a constant. Furthermore these involutive sets define different null tetrads from those defined for the involutive set satisfying the Theorem:

$$l^i = (1/\sqrt{2})(\tilde{e}_t^a + \tilde{e}_r^a),$$
$$n^i = (1/\sqrt{2})(\tilde{e}_t^a - \tilde{e}_r^a),$$
$$m^i = (1/\sqrt{2})(\tilde{e}_\theta^a + i\tilde{e}_\phi^a).$$

From the group-theoretic point of view it is not possible to use group motions under the adjoint action of the Lie algebra to transform the set $\{\mathbb{D}, p_1^2, m_{23}^2, H\}$ into $\{A^1, A^2, A^3, H\}$ as in (4.34). Thus if we classify orbits of triplets $\{L_1, L_2, L_2\}$ of second-order elements in the enveloping algebra (to within the addition of arbitrary multiples of H), the sets $\{\mathbb{D}, p_1^2, m_{23}^2\}$ and $\{A^1, A^2, A^3\}$ lie on different orbits. Although there is only one orbit that corresponds to the conditions of the theorem, there are, in general, several orbits of involutive sets of operators for which variable separation is possible in a fixed coordinate system. The analysis we have made of this specific case of separation of variables proceeds in an analogous way for the other two examples of variable separation we have given, viz. coordinates (4.16) and (4.23) and associated null tetrads (4.17) and (4.22), respectively. More recently Carter and McLenaghan[18] have given a master separation equation for all the separable perturbations of spin-0, -½, -1, and -2 in a Kerr space-time background. Kamran and McLenaghan[19] have also gone some way toward finding the conditions under which separation of variables occurs for the Dirac and neutral equations. It is our intention to pursue these matters further and develop an intrinsic theory of variable separation for equations of physical importance.

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[1] I. M. Gelfand, R. A. Minlos, and Z. Ya. Shaprio, *Representations of the Rotation and Lorentz Groups and their Applications* (Pergamon, New York, 1963).

[2] A. K. Agamaliev, N. M. Atakashiev, and I. A. Verdiev, "Invariant decomposition of the solution of the Dirac equation," Sov. J. Nucl. Phys. **10**, 106 (1970).

[3] N. M. Atakashiev and I. A. Verdiev, "Invariant expansion of solutions of relativistic Dirac type equations," Theor. Math. Phys. **4**, 833 (1970).

[4] S. A. Teukolsky, "Perturbations of a rotating black hole. I. Fundamental equations for gravitational, electromagnetic and neutronic field perturbations," Astrophys. J. **185**, 635 (1973).

[5] S. Chadrasekhar, *The Mathematical Theory of Black Holes* (Oxford U.P., Oxford, 1983).

[6] A. L. Dudley and J. D. Finlay, III, "Covariant perturbed wave equations in arbitrary type-D backgrounds," J. Math. Phys. **20**, 311 (1979).

[7] R. Guven, "The solution of Dirac's equation in a class of type D vacuum space times," Proc. R. Soc. London Ser. A **356**, 465 (1977).

[8] B. Carter and R. G. McLenaghan, "Generalized total angular momentum operator for the Dirac equation in curved space-time," Phys. Rev. D **19**, 1093 (1979).

[9] R. G. McLenaghan and Ph. Spindel, "Quantum numbers for Dirac spinor fields on a curved space time," Phys. Rev. D **20**, 409 (1979).

[10] R. G. McLenaghan and Ph. Spindel, "Integrales premieres des equations de dirac en espace courbe," Soc. Math. Belg. Bull. **31**, 65 (1979).

[11] K. Yano and S. Bochner, "Curvature and Bethe numbers," Ann. Math. Stud. **32** (1953).

[12] A. H. Cook, "On separable solutions of Diracs equation for the electron," Proc. R. Soc. London Ser. A **383**, 247 (1982).

[13] R. Brauer and H. Weyl, "Spinors in *n* dimensions," Am. J. Math. **57**, 425 (1935).

[14] G. H. Katzin and J. Levine, "Quadratic first integrals of the geodesics in spaces of constant curvature," Tensor **16**, 97 (1965).

[15] E. G. Kalnins, Ph. D. thesis, University of Western Ontario, 1972.

[16] E. G. Kalnins and W. Miller, Jr., "Killing tensors and variable separation for Hamilton-Jacobi and Helmholtz equations," SIAM. J. Math. Anal. **11**, 1011 (1980).

[17] V. N. Shapovalov, "Separation of variables in a second order linear differential equation," Diff. Eqs. **16**, 1212 (1981).

[18] B. Carter and R. G. McLenaghan, "Generalized master equations for wave equation separation in a Kerr or Kerr Newman black hole background," in *Proceeedings of the 2nd Marcel Grossmann Meeting on General Relativity* (North-Holland, Amsterdam, 1982), pp. 575–585.

[19] N. Kamran and R. G. McLenaghan, "Separation of variables and symmetry operators for the neutrino and Dirac equations in the space-times admitting a two parameter Abelian orthogonally transitive isometry group and a pair of shearfree geodesic null congruences," J. Math. Phys. **25**, 1019 (1984).