Generalized Stäckel matrices

E. G. Kalnins
Mathematics Department, University of Waikato, Hamilton, New Zealand
W. Miller, Jr.
School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455

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Stäckel and differential-Stäckel matrices are generalized so that the matrix elements may be functions of the derivatives of the dependent variable as well as the independent variable. The inverses of these matrices are characterized and it is shown that for significant classes of linear and nonlinear partial differential equations, variable separation is accomplished via this generalized Stäckel mechanism.

I. INTRODUCTION

In Ref. 1 the authors introduced a general definition of additive variable separation for a partial differential equation

$$H(x_I,u,u_{I,i}) = E, \tag{1.1}$$

where E is a parameter, $x_1,...,x_N$ are the independent variables, u is the dependent variable, and $u_{I,i} = \partial^i_{x_I}u$, $i = 1,2,...$. A separable solution of (1.1) is a solution of the form $u = \Sigma_{J=1}^N S^{(J)}(x_J,E)$. Our definition is a straightforward extension of Levi-Civita's definition for first-order equations.[2] We let n_I be the largest number l such that $\partial_{u_{I,l}}H \equiv H_{u_{I,l}} \neq 0$. To avoid discussion of degenerate cases we require $n_I > 0$ for each I (but n_I is finite).

Let the truncated differentiation operator $\widetilde{D}_I$ be defined by

$$\widetilde{D}_I = \partial_{x_I} + u_{I,1}\,\partial_u + u_{I,2}\,\partial_{u_{I,1}} + \cdots + u_{I,n_I}\partial_{u_{I,n_I-1}}. \tag{1.2}$$

In Ref. 1 we showed that every separable solution u of (1.1) satisfies the integrability conditions

$$H_{u_{I,n_I}}H_{u_{J,n_J}}(\widetilde{D}_I\widetilde{D}_JH) + H_{u_{I,n_I}u_{J,n_J}}(\widetilde{D}_IH)(\widetilde{D}_JH) - H_{u_{J,n_J}}(\widetilde{D}_IH)(\widetilde{D}_JH_{u_{I,n_I}}) - H_{u_{I,n_I}}(\widetilde{D}_JH)(\widetilde{D}_IH_{u_{J,n_J}}) = 0,$$
$$1 \leqslant I < J \leqslant N. \tag{1.3}$$

If (1.3) is an identity in the dependent variables $u, u_{K,k}$, we say that $\{x_I\}$ is a *regular separable coordinate system* for $H = E$. In this case the separable solutions involve $\Sigma_{J=1}^N n_J + 1$ independent parameters: u and the derivations $u_{I,i}$, $1 \leqslant I \leqslant N$, $1 \leqslant i \leqslant n_I$ can be prescribed arbitrarily at a given point $\mathbf{x}^0$. If conditions (1.3) do not hold identically then the separation is *nonregular* and separable solutions, if they exist, will involve strictly fewer parameters than the regular case. (The standard examples of variable separation for the differential equations of mathematical physics all correspond to regular separation.) Multiplicative separation is handled in this framework by passing to a new dependent variable $v = \ln u$. There is a modified definition of variable separation for (1.1) when $E \equiv 0$, which we will not discuss here.[1,3]

It would be of great interest to know the general solution of (1.3) so that the mechanism of variable separation could be determined in all cases. However, the general solution is not even known for the Levi-Civita case $n_I = \cdots = n_N = 1$. (The solution has recently been worked out for Hamilton–Jacobi equations on pseudo-Riemannian manifolds.[4])

Historically, the fundamental mechanism for variable separation has been the Stäckel matrix.[4] However, the Stäckel mechanism is not sufficient to encompass all types of separation given by solutions of (1.3). In order to describe the solutions of the integrability conditions for additive separation of linear equations $Lu = E$ and $Lu = 0$ the authors introduced differential-Stäckel (D-Stäckel) matrices, a nontrivial extension of Stäckel matrices.[3] Here, we further extend D-Stäckel matrices by permitting the matrix elements to be functions of the derivatives $u_{I,i}$ as well as the independent variables x_I. (For ordinary Stäckel matrices this is a straightforward extension. For D-Stäckel matrices it is more difficult.)

In Sec. II we define generalized D-Stäckel matrices and characterize their inverse matrices via a system of partial differential equations. This section is modeled on Ref. 3 (in which ordinary D-Stäckel matrices are treated) but Theorem 1 leads to some complications.

In Sec. III we present several classes of linear and nonlinear partial differential equations for which we can characterize the possible mechanisms of variable separation, and we show that they all correspond to generalized D-Stäckel form. For all cases treated we have $H_{u_{I,n_I}u_{J,n_J}} = 0$ in (1.3) for $I \neq J$. (The cases where the mixed partial derivatives do not vanish are much more complicated.) Even with this restriction we do not yet know if generalized D-Stäckel form is sufficient to describe all variable separation or if additional mechanisms exist.

All functions appearing in this paper are assumed to be locally real analytic. Furthermore, functions $f(x_K,u_{K,k})$ are assumed to be analytic as functions of the $u_{K,k}$ in the neighborhood of $u_{K,k} = 0$. If we require that a nonzero f is invertible we mean that $f(x_K,0) \neq 0$ for the $\{x_k\}$ in some neighborhood on R^N so that $f^{-1}(x_K,u_{K,k})$ is also analytic.

II. GENERALIZED D-STÄCKEL MATRICES

Consider a coordinate set $x_1,...,x_N$ and let $n_1,...,n_N$ be positive integers with $n = \Sigma_{I=1}^N n_I$. Let $S = (S_{(I,i),l}(x_I))$ be an $n \times n$ nonsingular matrix with the properties

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$$(1)\quad S_{(I,i),l}(x_I)=\frac{d^{\,i-1}}{dx_I^{i-1}}S_{(I,1),l}(x_I),\quad i=1,2,\ldots,n_I; \tag{2.1}$$

$$(2)\quad T^{1,(J,j)}\neq 0,\quad J=1,\ldots,N,\quad j=1,\ldots,n_J,$$

where $T=S^{-1}$, i.e.,

$$\sum_{l=1}^{u}S_{(I,i),l}(x_I)T^{l,(J,j)}=\delta_{(I,i)}^{(J,j)}. \tag{2.2}$$

We call a matrix S with these properties a *differential-Stäckel matrix* (D-Stäckel matrix). [Here the rows of S are designated by the index (I,i), where $I=1,\ldots,N$, $i=1,\ldots,n_I$, whereas the columns of S are designated by the index $l=1,2,\ldots,n$. Row (I,i) depends only on x_I and is the $i-1$ derivative of row $(I,1)$. The index notation for T is defined similarly but with rows and columns interchanged.] If $n_I=1$ for all I so $n=N$, then S is an ordinary Stäckel matrix.[5]

Set $H_{(J,j)}=T^{1,(J,j)}$. In Ref. 3 it is shown that for S a D-Stäckel matrix the system of equations

$$\partial_I\rho_{(J,j)}=(\rho_{(I,n_I)}-\rho_{(J,j)})\partial_I\ln H_{(J,j)}+(\rho_{(I,n_I)}-\rho_{(J,j-1)})\frac{H_{(J,j-1)}}{H_{(J,j)}}\delta_I^J, \tag{2.3}$$

$$I,J=1,\ldots,N,\quad h=1,\ldots,n_J,$$

admits a full linearly independent set of n vector valued solutions $\{\rho_{(J,j)}^l\}$, $l=1,\ldots,n$. Conversely if the n nonzero functions $\{H_{(J,j)}\}$ are such that (2.3) admits a linearly independent set of vector valued solutions then there is an $n\times n$ D-Stäckel matrix S such that $H_{(J,j)}=T^{1,(J,j)}$. See Refs. 6 and 7 for similar treatments of ordinary Stäckel matrices.

The integrability conditions for (2.3) are

$$\partial_{IJ}H_{(P,p)}-\partial_I H_{(P,p)}\partial_J\ln H_I-\partial_J H_{(P,p)}\,\partial_I H_J=0,$$
$$P\neq I,J,\quad p=1,\ldots,n_P, \tag{2.4a}$$

$$\partial_{IJ}H_{(J,j)}-\partial_I H_{(J,j)}\partial_J\ln H_I-\partial_J H_{(J,j)}\,\partial_I\ln H_J$$
$$=H_{(J,j-1)}\,\partial_I\ln H_J-\partial_I H_{(J,j-1)},\quad j=1,\ldots,n_J, \tag{2.4b}$$

where $I\neq J$, $H_I\equiv H_{(I,n_I)}$, and $H_{(J,0)}\equiv 0$. By Theorem 1 of Ref. 3, conditions (2.4) are necessary and sufficient that the n nonzero functions $\{H_{(J,j)}\}$ can be expressed in the form $H_{(J,j)}=T^{1,(J,j)}$ for T the inverse of a D-Stäckel matrix S.

When Eqs. (2.4) hold the partial differential equation

$$\sum_{I=1}^{N}\sum_{i=1}^{n_I}(D_I^{i-1}Q_I)H_{(I,i)}=\mathrm{E} \tag{2.5}$$

admits regular additive separation in the coordinates $x_1,\ldots,x_N$, where E is a parameter (which could be zero), $Q_I(x_I,u_{I,i})$ is a function of x_I, and a finite number of derivatives $u_{I,1},u_{I,2},\ldots,u_{I,q_I}$ with $m_I\geqslant 1$, $\partial_{u_{I,q_I}}Q_I\neq 0$, and

$$D_I=\partial_{x_i}+\sum_{i=1}^{\infty}u_{I,i+1}\,\partial_{u_{I,i}} \tag{2.6}$$

is the Ith total derivative. Indeed the separation equations are

$$D_I^{i-1}Q_I+\sum_{l=1}^{n}S_{(I,i),l}(x_I)\lambda_l=0, \tag{2.7}$$

$$1\leqslant I\leqslant N,\quad 1\leqslant i\leqslant n_I,\quad \lambda_1=-E,$$

where the λ_l are the separation parameters. The separable solutions $u=\Sigma_{I=1}^N u^{(I)}(x_I,\lambda_l)$ are obtained by integrating N ordinary differential equations (of order m_I)

$$Q_I(x_I,u_{I,i})+\sum_{l=1}^{n}S_{(I,1),l}(x_I)\lambda_l=0. \tag{2.8}$$

The remaining $n-N$ equations are redundant. The number of parameters in the solution u is $\Sigma_I q_I+n-N+1$.

Stäckel and D-Stäckel matrices can clearly be generalized to include dependent variables, thus incorporating a wider class of separable partial differential equations than (2.5). For this we consider coordinates $x_1,\ldots,x_N$, let $n=\Sigma_{I=1}^N n_I$, where the n_I are positive integers, and let $\Omega_1,\ldots,\Omega_N$ be non-negative integers. Then a nonsingular $n\times n$ matrix $S=(S_{(I,i),l}(x_I,u_{I,j}))$, with the properties

(1) $S_{(I,i),l}(x_I,u_{I,j})=D_I^{i-1}S_{(I,1),l}(x_I,u_{I,j})$, $i=1,2,\ldots,n_I$,

(2) $T^{1,(J,j)}\neq 0$, $J=1,\ldots,N$, $j=1,\ldots,n_J$, where $T=S^{-1}$,

(3) $S_{(I,1),l}=S_{(I,1),l}(x_I,u_{I,1},\ldots,u_{I,\Omega_I})$, with $\partial_{u_{I,\Omega_I}}S_{(I,1),l}\neq 0$ for some l if $\Omega_I>0$,

is a *(generalized) D-Stäckel matrix.*

A generalized D-Stäckel matrix S can be used to construct partial differential equations that permit regular separation in the coordinates x_I. Set $H_{(J,j)}=T^{1,(J,j)}$. It is then easy to show that equations of the form (2.5) permit regular separation.

Characterization of generalized D-Stäckel form in terms of differential equations satisfied by the $H_{(J,j)}$ is not particularly difficult. In analogy with the derivation of Eq. (1.3) in Ref. 3, we can easily show that

$$D_I\rho_{(J,j)}=(\rho_{(I,n_I)}-\rho_{(J,j)})D_I\ln H_{(J,j)}+(\rho_{(I,n_I)}-\rho_{(J,j-1)})(H_{(J,j-1)}/H_{(J,j)})\delta_I^J, \tag{2.9}$$

$$I,J=1,\ldots,N,\quad j=1,\ldots,n_J,$$

where $\rho_{(J,j)}$ (as well as $H_{(J,j)}$) depends only on the variables $x_I,u_{I,1}u_{I,2},\ldots,u_{I,s_I}$, $I=1,\ldots,N$, and $s_I=n_I+\Omega_I-1$ if $\Omega_I>0$, $s_I=0$ if $\Omega_I=0$, admit a full linearly independent set of n vector valued solutions $\{\rho_{(J,j)}^l\}$, $l=1,\ldots,n$ if and only if the nonzero functions $H_{(J,j)}$ are obtainable from a generalized D-Stäckel matrix S.

The total differential equation (2.9) is equivalent to a sequence of partial differential equations in which the left-hand side assumes the form $\partial_{u_{I,i}}\rho_{(J,j)}$, $i=0,\ldots,s_I$. (We make the convention that $x_I\equiv u_{I,0}$.) Indeed, equating coefficients of u_{I,s_I+1} on both sides of (2.9) we have

$$\partial_{u_{I,s_I}}\rho_{(J,j)}=(\rho_{(I,n_I)}-\rho_{(J,j)})\partial_{u_{I,s_I}}\ln H_{(J,j)}. \tag{2.10}$$

We can obtain the derivatives $\partial_{u_{I,i}}\rho_{(J,j)}$, $i=1,\ldots,s_I-1$ recursively from (2.9) and (2.10) through the relation $\partial_{u_{I,i-1}}=[\partial_{u_{I,i}},D_I]=\partial_{u_{I,i}}D_I-D_I\,\partial_{u_{I,i}}$. Finally, $\partial_{x_I}=D_I-\Sigma_{i=1}^{s_I}u_{I,i+1}\,\partial_{u_{I,i}}$, when applied to $\rho_{(J,j)}$.

We can obtain integrability conditions for Eq. (2.9) by computing $D_J(D_I\,\rho_{(K,k)})=D_I(D_J\,\rho_{(K,k)})$, $I\neq J$, and equating coefficients of $\rho_{(L,l)}$ on each side of the resulting expression:

$$D_I D_J H_{(P,p)} - D_I H_{(P,p)} D_J \ln H_I - D_J H_{(P,p)} D_I \ln H_J = 0,$$
$$P \neq I, J, \quad p = 1,...,n_P, \tag{2.11a}$$
$$D_I D_J H_{(J,j)} - D_I H_{(J,j)} D_J \ln H_I - D_J H_{(J,j)} D_I \ln H_J - H_{(J,j-1)} D_I \ln H_J + D_J H_{(J,j-1)} = 0,$$
$$j = 1,...,n_J. \tag{2.11b}$$

Here $H_J \equiv H_{(J,n_J)}$, $H_{(J,0)} \equiv 0$. It is not entirely clear, however, that Eqs. (2.11) are the complete set of integrability conditions. For these we need to compute $\partial_{u_{J,j}}(\partial_{u_{I,i}} \rho_{(K,k)}) = \partial_{u_{I,i}}(\partial_{u_{J,j}} \rho_{(K,k)})$, $i = 0,...,s_I$, $j = 0,...,s_J$.

Theorem 1: Conditions (2.11a) and (2.11b) are necessary and sufficient for complete integrability of Eqs. (2.9), hence they are the necessary and sufficient conditions for the existence of a generalized D-Stäckel matrix S such that $H_{(J,j)} = T^{1,(J,j)} \neq 0$.

Proof: It is already evident that conditions (2.11) are necessary for the existence of a generalized D-Stäckel matrix. To prove they are sufficient we consider the integrability conditions ($I \neq J$)

$$D_J(D_I \rho_{(K,k)}) - D_I(D_J \rho_{(K,k)}) = -\rho_{(J,n_J)} \frac{\overleftarrow{D}_I \overrightarrow{D}_J H_{(K,k)}}{H_{(K,k)}} + \rho_{(I,n_I)} \frac{\overleftarrow{D}_I \overrightarrow{D}_J H_{(K,k)}}{H_{(K,k)}}, \tag{2.12}$$

where $\overleftarrow{D}_I \overrightarrow{D}_J H_{(K,k)}$ is the left-hand side of expression (2.11a) if $K = P \neq I, J$ or expression (2.11b) if $K = J$. The left-hand side of (2.12) is computed directly from (2.9). Clearly (2.12) vanishes for a complete set of solutions ρ if and only if $\overleftarrow{D}_I \overrightarrow{D}_J H_{(K,k)} = 0$. The integrability condition

$$\partial_{u_{J,s_J}}(\partial_{u_{I,s_I}} \rho_{(K,k)}) - \partial_{u_{I,s_I}}(\partial_{u_{J,s_J}} \rho_{(K,k)}) \tag{2.13}$$

can be obtained from (2.12) by equating coefficients of $u_{J,s_J+1} u_{I,s_I+1}$:

$$\overleftarrow{\partial}_{u_{I,s_I}} \overrightarrow{\partial}_{u_{J,s_J}} H_{(K,k)} = 0. \tag{2.14}$$

Furthermore, equating coefficients of u_{J,s_J+1} and u_{I,s_I+1}, respectively, we find the conditions corresponding to $\partial_{u_{J,s_J}}(D_I \rho) - D_I(\partial_{u_{J,s_J}} \rho)$ and $D_J(\partial_{u_{I,s_I}} \rho) - \partial_{u_{I,s_I}}(D_J \rho)$:

$$\overleftarrow{D}_I \overrightarrow{\partial}_{u_{J,s_J}} H_{(K,k)} = 0, \quad \overleftarrow{\partial}_{u_{I,s_I}} \overrightarrow{D}_J H_{(K,k)} = 0. \tag{2.15}$$

Note that conditions (2.14) and (2.15) can be obtained directly from (2.11) by equating coefficients of $u_{J,s_J+1} u_{I,s_I+1}$, and u_{J,s_J+1} and u_{I,s_I+1}, respectively.

We can now derive the conditions corresponding to $\partial_{u_{J,j}}(D_I \rho) - D_I(\partial_{u_{J,j}} \rho)$ recursively from the above expression through repeated application of the identity

$$\partial_{u_{J,j}} = [\partial_{u_{J,j+1}}, D_J], \quad j = 1,2,...,s_J - 1. \tag{2.16}$$

From this result and (2.13) we can obtain the conditions corresponding to

$$\partial_{u_{J,j}}(\partial_{u_{I,i}} \rho) - \partial_{u_{I,i}}(\partial_{u_{J,j}} \rho) \tag{2.17}$$

recursively through application of the identity $\partial_{u_{I,i}} = [\partial_{u_{I,i+1}}, D_I]$, $i = 1,2,...,s_I - 1$. At each stage of this process the integrability conditions are linear combinations of (2.11), (2.14), (2.15), and their derivatives, hence they are implied by (2.11). Finally the conditions $\partial_{x_J}(\partial_{x_I} \rho) - \partial_{x_I}(\partial_{x_J} \rho)$ can be obtained from

$$[\partial_{x_I}, \partial_{x_J}] = \left[D_{x_I} - \sum_{i=1}^{s_I} u_{I,i+1} \partial_{u_{I,i}}, \quad D_J - \sum_{j=1}^{s_J} u_{J,j+1} \partial_{u_{J,j}} \right].$$

Again the integrability conditions are implied by (2.11a) and (2.11b). Q.E.D.

Now that we have succeeded in characterizing generalized D-Stäckel matrices in terms of the first column of their inverse matrices, we can extend the notion of a Stäckel multiplier to this situation. Suppose n nonzero functions $H_{(J,j)}(x_I, u_{I,i})$ satisfy conditions (2.11), and hence determine a generalized D-Stäckel matrix S. A nonzero function $f(x_I, u_{I,i})$ such that the functions $\widetilde{H}_{(J,j)} = H_{(J,j)}/f$ also satisfy conditions (2.11) is a *(generalized) D-Stäckel multiplier* for the system $\{H_{(J,j)}\}$.

Theorem 2: The following are equivalent characterizations of D-Stäckel multipliers f: (1) f satisfies the equations

$$D_I D_J f - D_I \ln H_J D_J f - D_J \ln H_I D_I f = 0,$$
$$I \neq J, \quad H_I = H_{(I,n_I)}. \tag{2.18a}$$

(2) there exist N functions $\varphi^J(x_J, u_{J,j})$, $D_I \varphi^J = 0$ for $I \neq J$, such that

$$f(x_K, u_{K,k}) = \sum_{J=1}^{N} \sum_{j=1}^{n_J} (D_J^{j-1} \varphi^J) H_{(J,j)}. \tag{2.18b}$$

Proof: It is obvious from Theorem 1 that (2.18a) is equivalent to the definition of a D-Stäckel multiplier. Now suppose f is a D-Stäckel multiplier. Then there is an $n \times n$ D-Stäckel matrix $\widetilde{S}$ (for $\widetilde{H}_{(J,j)} = H_{(J,j)}/f$) such that

$$\widetilde{T}^{1,(J,j)} = H_{(J,j)}/f. \tag{2.19}$$

The elements in the first column of the D-Stäckel matrix S can be denoted

$$\widetilde{S}_{(J,j),1} = \frac{d^{j-1}}{dx_J^{j-1}} \varphi^J(x_J, u_{J,l})$$

for N functions φ^J. Multiplying both sides of (2.19) by $\widetilde{S}_{(J,j),1}$ and summing over the index (J,j) we obtain (2.18b).

Conversely, suppose f is defined by (2.18b) for some functions $\varphi^J(x_J, u_{J,l})$. From this expression and conditions (2.10) it is straightforward to verify that f satisfies Eq. (2.18a). Hence f is a D-Stäckel multiplier. Q.E.D.

Although Theorem 1 is valid only when $H_{(J,j)} \neq 0$ for all (J,j), Eqs. (2.11) make sense as long as $H_{(J,n_J)} \equiv H_J \neq 0$, even if some of the remaining $H_{(J,j)}$ vanish. We need to determine the significance of those solutions of (2.11) for which it is only required that $H_J \neq 0$. Furthermore it will be useful to determine the effect on the solutions of replacing each H_J by $g_J(x_J, u_{J,j}) H_J$, where g_J is invertible in a neighborhood of the point $(x_J^0, 0)$, so that g_J^{-1} will also be analytic in the $u_{(J,j)}$ in a neighborhood of the point.

To answer these questions it is useful to write Eqs. (2.11) in the form

$$A_{IJ} H_{(P,p)} = 0, \quad P \neq I, J,$$
$$A_{IJ} H_{(J,j)} = B_{IJ} H_{(J,j-1)}, \quad H_{(J,0)} = 0, \quad I \neq J, \tag{2.20}$$

where

$$A_{IJ} = D_I D_J - D_J \ln H_I D_I - D_I \ln H_J D_J,$$
$$B_{IJ} = -D_I + D_I \ln H_J, \quad H_J = H_{(J,n_J)} \neq 0, \qquad (2.21)$$
$$1 \leqslant J \leqslant N, \quad 1 \leqslant j \leqslant n_J, \quad \sum_J n_J = n.$$

We require that the $H_{(J,j)}$ and the other functions appearing in the lemmas depend only on the variables x_I and $u_{I,i}$, $1 \leqslant I \leqslant N$, $1 \leqslant i \leqslant s_I$.

Suppose we are given N nonzero functions H_J satisfying $A_{IJ}H_P = 0$ for $P \neq I, J$, $I \neq J$, and N nonzero functions g_J satisfying $D_I g_J = 0$ for $I \neq J$. Our task will be to construct a finite set of functions $\mathscr{H}_{(J,j)}$ with $H_{(J,n_J)} = g_J H_J$ such that Eqs. (2.12) are satisfied. Initially the value of n_J is unknown.

The construction process is based on the second equation (2.20) which we rewrite as follows:

$$D_I\left(\frac{H_{(K,k-1)}}{H_K}\right) = \frac{-A_{IK}H_{(K,k)}}{H_K}, \quad I \neq K. \qquad (2.22)$$

If $H_{(K,k)}$ is known we can construct $H_{(K,k-1)}$ from (2.22) by quadrature.

Lemma 1: Suppose the N nonzero functions H_P satisfy $A_{IJ}H_P = 0$ for $P \neq I,J$ and the function $H_{K,k}$ (fixed K,k) satisfies $A_{IJ}H_{K,k} = 0, K \neq I, J, I \neq J$. Then the $N-1$ equations (2.20) are compatible and have the general solution

$$H_{(K,k-1)} = \widetilde{H}_{(K,k-1)} + f^{(k-1)}(x_K,u_{K,i})H_K, \qquad (2.23)$$

where $\widetilde{H}_{(K,k-1)}$ is a particular solution and $f^{(k-1)}$ is an arbitrary function of $x_K, u_{K,i}$. The solution satisfies

$$A_{IJ}H_{(K,k-1)} = 0, \quad K \neq I, J, \quad I \neq J. \qquad (2.24)$$

It follows that for each K we can always construct functions $H_{(K,k-1)}$ through a recursive procedure using (2.22) such that the first equation (2.20) is automatically satisfied. At each step the solution $H_{(K,k-1)}$ is arbitrary up to the additive term $f^{(k-1)}(x_K,u_{K,i})H_K$ and we choose one of these solutions. Thus we generate an infinite sequence $\{H_{(K,k)} = H_K^{(l)}\}$, $l = 0,1,2,\ldots$, where $n_K - l = k$ (but n_K is unknown) and

$$A_{IK}H_K^{(l)} = B_{IK}\mathrm{H}_K^{(l+1)}, \quad I \neq K, \quad H_K = H_K^{(0)}. \qquad (2.25)$$

Suppose there is a smallest finite positive integer n_K for which functions $f_{(i)}(x_K,u_{K,j})$ exist such that

$$H_K^{(m_K)} = \sum_{i=0}^{m_K-1} f_{(i)}(x_K,u_{K,j})H_K^{(i)}. \qquad (2.26)$$

The following lemmas can be verified by straightforward induction using the properties

$$(1) \quad B_{IK}F(x_j,u_{J,j}) = 0, \quad \text{for all } I \neq K,$$
$$\Leftrightarrow F = f(x_K,u_{K,j})H_K \qquad (2.27)$$
$$(2) \quad A_{IK}(f(x_K,u_{K,j})H_K^{(l)}) = B_{IK}(fH_K^{(l+1)} - D_K fH_K^{(l)}). \qquad (2.28)$$

Lemma 2: Each $H_K^{(m_K+s)}$, $s = 0,1,2,\ldots$, is a linear combination of the finite set $\{H_K^{(l)}: l = 0,\ldots,m_K - 1\}$ with coefficients that are functions of $x_K, u_{K,j}$.

Lemma 3: Let $\{\mathscr{H}_K^{(l)}\}$, $\{h_K^{(l)}\}$, $l = 0,1,2,\ldots$, be two sequences constructed by the procedure (2.22), (2.23) such that $\mathscr{H}_K^{(0)} \equiv \mathscr{H}_K = g_0 h_K$, where g_0 is invertible and $D_I g_0 = 0$ for $I \neq K$. Then there is a sequence $g_1, g_2, \ldots$ with $D_I g_i = 0$ for $I \neq K$ and expressions $L_{i,j}(g_0, g_1, \ldots, g_{i-j-1})$ with $L_{l,0} = 0$ for $l \geqslant 1$, $L_{i,i-1} = D_K g_0$, and $L_{i+1,j} = L_{i,j-1} + D_K g_{i-j} - D_K L_{i,j}$ such that

$$\mathscr{H}_K^{(i)} = g_0 h_K^{(i)} + \sum_{j=0}^{i-1}(g_{i-j} - L_{i,j})h_K^{(j)}, \quad i = 0,1,2,\ldots. \qquad (2.29)$$

Any such sequence $\{g_l\}$ together with $\{h_K^{(l)}\}$ determines a new sequence of solutions $\{\mathscr{H}_K^{(l)}\}$.

Let $\{H_K^{(l)}\}$ be the solution sequence with property (2.25). Then setting $h_K^{(i)} = H_K^{(i)}$ in (2.28) choosing $g_0 = 1, g_1, \ldots, g_{m_K-1}$ recursively such that

$$-f_{(j)} = g_{m_K-j} - L_{m_K,j}, \quad j = 0,1,\ldots,m_K - 1,$$

we have $\mathscr{H}_K^{(m_K)} = 0$. Thus there is a solution sequence $\{\mathscr{H}_K^{(l)}\}$ with $\mathscr{H}_K^{(0)},\ldots,\mathscr{H}_K^{(m_K-1)}$ nonzero and all further terms zero. By Lemma 3, all other solution sequences are linear combinations of these m_K nonzero terms.

Lemma 4: The integer m_K, if it exists, is unique.

In particular, modifying H_K, to $g_K H_K$ with g_K invertible and $D_I g_K = 0$ for $I \neq K$ does not change m_K.

Based on the preceding results, given any solution $\{H_{(K,k)}\}$ of Eq. (2.11) we can determine the integers m_K such that $1 \leqslant m_K \leqslant n_K$. Then there is another solution $\{\mathscr{H}_{(K,k)}\}$ with $m = \Sigma_{K=1}^N m_K$ nonzero terms such that each $H_{(K,k)}$ is a linear combination of the $\mathscr{H}_{(K,k)}$. Thus the original solution is associated with an $m \times m$ generalized D-Stäckel matrix.

III. SEPARABILITY CONDITIONS

Suppose we are given a partial differential equation

$$H(x_I,u,u_{I,i}) = E, \qquad (3.1)$$

which admits regular additive separability in the coordinates x_I. (Unless otherwise specified we will adhere to the notation and conventions for separation listed in the Introduction.) That is, suppose the integrability equations (1.3) are satisfied identically in u, $u_{I,i}$. What is the form of the separation and how can the separation equations be determined from (1.3)? In this section we will identify some classes of linear and nonlinear differential equations where the separation is achieved via generalized D-Stäckel matrices.

Our method of approach is exemplified by the following observation concerning (3.1).

Lemma 5: Suppose $\partial_u H = \partial_{u_{I,i}}\partial_{u_{J,j}} H = 0$ for all $I \neq J$ and $\partial_{u_{J,n_J}} H \equiv H_J(x_K,u_{K,k})$ is invertible for $1 \leqslant K \leqslant N$, $1 \leqslant k \leqslant n_K - 1$. Further suppose the $\{H_J\}$ generate a D-Stäckel matrix via the process (2.22). Then the differential equation $H = E$ is regular separable if and only if H is a generalized D-Stäckel multiplier.

Proof: The integrability conditions (1.3) for H are, in this case, equivalent to

$$(D_I D_J - D_I \ln H_J D_J - D_J \ln H_I D_I)H = 0, \quad I \neq J,$$

the condition that H be a D-Stäckel multipler.

Theorem 3: Suppose H takes the form

$$H = \sum_{J=1}^N H_J(x_K,u_{K,k})\mathscr{P}_J(x_J,u_{J,n_J},u_{J,j}) + V(x_K,u_{K,k}),$$
$$1 \leqslant K \leqslant N, \quad 1 \leqslant k \leqslant n_K - 1, \qquad (3.2)$$

where H_J is invertible, $D_I \mathscr{P}_J = 0$ for $I \neq J$, and

$\partial^2_{u_{J,n_J}} \mathscr{P}_J \neq 0$. Then the equation $H=E$ is regular separable if and only if the $\{H_J\}$ are in generalized Stäckel form $(m_J=1)$ and H is a generalized Stäckel multiplier with respect to this form.

Proof: In this case the integrability conditions (1.3) are equivalent to

$$A_{IJ}H_P=0, \quad I\neq J, \quad 1\leqslant I,J,P\leqslant N, \quad A_{IJ}H=0,$$

where A_{IJ} is defined by (2.21). Q.E.D.

Theorem 4: Suppose H takes the form

$$H=\sum_{J=1}^{N} H_J(x_K)\mathscr{P}_J(x_J,u_{J,j})u_{J,n_J}+V(x_K,u_{K,k}), \qquad 1\leqslant K\leqslant N, \quad 1\leqslant k\leqslant n_K-1, \quad 1\leqslant j\leqslant n_J-1, \tag{3.3}$$

where $H_J\mathscr{P}_J$ is invertible. Then the quasilinear equation $H=E$ is regular separable if and only if the functions H_J determine an (ordinary) D-Stäckel matrix via the process (2.22) and H is a D-Stäckel multiplier with respect to this form.

Proof: The integrability conditions (1.3) for $H=E$ are

$$\begin{aligned}
&(1) \quad \hat{A}_{IJ}H_P=0, \quad P\neq I,J,\\
&(2) \quad \hat{A}_{IJ}(H_I\mathscr{P}_I)=\hat{B}_{IJ}(\partial_{u_{I,n_I-1}}V),\\
&(3) \quad \hat{A}_{IJ}V=0,\\
&(4) \quad \partial_{u_{I,n_I-1}}\partial_{u_{J,n_J-1}}V=0,
\end{aligned}\tag{3.4}$$

for all $I\neq J$, where

$$\begin{aligned}
\hat{A}_{IJ}&=\hat{D}_I\hat{D}_J-\hat{D}_J\ln H_I\hat{D}_I-\hat{D}_I\ln H_J\hat{D}_J,\\
\hat{B}_{IJ}&=-\hat{D}_I+\hat{D}_I\ln H_J,\\
\hat{D}_I&=\partial_{x_I}+\sum_{i=1}^{n_I-2}u_{I,i+1}\partial_{u_{I,i}}.
\end{aligned}\tag{3.5}$$

Note that

$$D_I\ln(H_J\mathscr{P}_J)=D_I\ln H_J=\partial_{x_I}\ln H_J, \tag{3.6}$$

for $I\neq J$. Using (4) and differentiating (2) with respect to $u_{J,n_J-1}, \quad u_{J,n_J-2},\dots,u_{J,1}$, recursively, we obtain $\partial_{u_{I,n_I-1}}\partial_{u_{J,j}}V=0$. Then differentiating (3) with respect to $u_{I,i}$ and $u_{J,j}$ recursively, we obtain $\partial_{u_{I,i}}\partial_{u_{J,j}}V=0, I\neq J$. Thus we can write V uniquely in the form

$$V=V_0(x_K)+\sum_{J=1}^{N}V_J(x_K,u_{J,j}), \quad 1\leqslant K\leqslant N, \quad 1\leqslant j\leqslant n_J-1, \tag{3.7}$$

where $V_J(x_K,0)=0$, and (2) becomes

$$\hat{A}_{IJ}(H_I\mathscr{P}_I)=\hat{B}_{IJ}(\partial_{u_{I,n_I-1}}V_I). \tag{3.8}$$

Differentiating (3) with respect to $u_{I,i}$ we find

$$\hat{A}_{IJ}(\partial_{u_{I,i}}V_I)=\hat{B}_{IJ}(\partial_{u_{I,i-1}})V_I, \tag{3.9}$$

where the right-hand side of (3.9) vanishes for $i=1$. Then, using (2) we can verify the formulas

$$\begin{aligned}
&A_{IJ}(H_P\mathscr{P}_P)=0, \quad P\neq I,J,\\
&A_{IJ}(H_I\mathscr{P}_I)=B_{IJ}(\partial_{u_{I,n_I-1}}V_I+H_I\partial_{u_{I,n_I-1}}\mathscr{P}_I)\\
&A_{IJ}(\partial_{u_{I,i}}V_I+H_I\partial_{u_{I,i}}\mathscr{P}_I)\\
&\quad=B_{IJ}(\partial_{u_{I,i-1}}V_I+H_I\partial_{u_{I,i-1}}\mathscr{P}_I),\\
&\quad 1\leqslant i\leqslant n_I-1,
\end{aligned}$$

where $\partial_{u_{I,0}}V_I-H_I\partial_{u_{I,0}}\mathscr{P}_I\equiv 0$. Here the truncated derivatives $\hat{D}_J$ have been replaced by total derivatives D_J. These formulas agree with (2.11a) and (2.11b) for $\mathscr{H}_I=H_I\mathscr{P}_I$, $\mathscr{H}_{(I,i)}=\partial_{u_{I,i}}V_I+H_I\partial_{u_{I,i}}\mathscr{P}_I$, $1\leqslant i\leqslant n_I-1$. It follows from (2.6) and Lemma 3 that the H_I also generate solutions of (2.11) that are independent of the $u_{K,k}$, since $\mathscr{P}_I$ is invertible. Hence the integrability conditions (1.3) imply that $H_{(I,i)}$ determine a D-Stäckel matrix and that H is a D-Stäckel multiplier with respect to this form. Q.E.D.

Due to the property (3.6) it is not necessary to assume in Theorems 3 and 4 that the functions $\mathscr{P}_I(x_I,u_{I,i})$ are invertile in the strong sense of the Introduction; they may be permitted to vanish for $u_{I,i}=0$.

Corollary 1: If in (3.3) we have $\mathscr{P}_I\neq 0$ for all I then the functions $H_I(x_K)$ determine an (ordinary) D-Stäckel matrix and H is an (ordinary) D-Stäckel multiplier with respect to this form.

This result follows from Lemma 3.

Corollary 2: Consider the differential equation $L\psi=E\psi$, where L is the linear nth-order partial differential operator $(n>1)$

$$L=\sum_{J=1}^{N}H_J(x_K)\partial^n_{x_J}+\sum_{\substack{a_1+\cdots+a_N<n\\ a_I\geqslant 0}}H_{a_1,\dots,a_N}(x_K)\cdot\partial^{a_1}_{x_1},\dots,\partial^{a_N}_{x_N}, \tag{3.10}$$

with $H_J\neq 0$ for each J. This equation admits regular *multiplicative* separation in the coordinates x_K if and only if

$$L=\sum_{J=1}^{N}H_J(x_K)\left(\partial^n_{x_J}+\sum_{a=0}^{N-1}f^a_J(x_J)\partial^a_{x_J}\right), \tag{3.11}$$

where $\partial_{x_I}f^a_J=0$ for $I\neq J$ and the $\{H_p\}$ are in (ordinary) Stäckel form.

Proof: The equation $L\psi=E\psi$ admits regular multiplicative separation (by definition) provided the equation $H=E$, obtained by setting $\psi=e^u$ in $L\psi/\psi=E$, admits regular additive separation:

$$H=\sum_{J=1}^{N}H_J(x_K)u_{J,n}+V(x_K,u_{K,k}), \quad 1\leqslant k<n. \tag{3.12}$$

Here V is an nth-order polynomial in the derivatives $u_{K,k}$ whose nth-order terms take the form

$$\sum_{J=1}^{N}H_J(x_K)u^n_{J,1}.$$

Equating coefficients of $u^n_{P,1}$ on both sides of the integrability conditions $A_{IJ}H=0$, we find $A_{IJ}H_P=0$ for all P. Thus, $\{H_P\}$ is in (ordinary) Stäckel form. Furthermore, from (2.7) we see that there can be no cross terms in the potential V. This means that in (3.10) we can require $H_{a_1,\dots,a_N}=0$ if more than one a_I is nonzero. Since V must be a Stäckel multiplier with respect to the Stäckel form $\{H_P\}$ we obtain (3.11). Q.E.D.

Having brought up multiplicative separation of linear eigenvalue equations we might as well mention the additive separation case.

Proposition: The equation $Lu = Eu$, where

$$Lu = \sum_{J=1}^{N} \sum_{j=1}^{n_J} H_{(J,j)}(x_K) u_{J,j} + H_{(0)} u$$

and $H_J \equiv H_{(J, n_J)} \neq 0$ admits regular additive separation in the coordinates x_K if and only if $\partial_{x_I} H_{(J,j)} = 0$ for $I \neq J$ and $\partial_{x_I} H_{(0)} = 0$ for all I.

The proof is a straightforward application of the integrability conditions to $H = Lu/u$. Although additive separation is not very interesting for $Lu = Eu$, in the case of homogeneous equations $Lu = 0$ nontrivial D-Stäckel additive separation occurs even in coordinate systems for which there is no multiplicative separation.[3]

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