Completely integrable relativistic Hamiltonian systems and separation of variables in Hermitian hyperbolic spaces

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The Hamilton–Jacobi and Laplace–Beltrami equations on the Hermitian hyperbolic space HH(2) are shown to allow the separation of variables in precisely 12 classes of coordinate systems. The isometry group of this two-complex-dimensional Riemannian space, SU(2, 1), has four mutually nonconjugate maximal abelian subgroups. These subgroups are used to construct the separable coordinates explicitly. All of these subgroups are two-dimensional, and this leads to the fact that in each separable coordinate system two of the four variables are ignorable ones. The symmetry reduction of the free HH(2) Hamiltonian by a maximal abelian subgroup of SU(2, 1) reduces this Hamiltonian to one defined on an O(2, 1) hyperboloid and involving a nontrivial singular potential. Separation of variables on HH(2) and more generally on HH(n) thus provides a new method of generating nontrivial completely integrable relativistic Hamiltonian systems.

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I. INTRODUCTION

The purpose of this article is to discuss the separation of variables in the four (real)-dimensional Hermitian hyperbolic space HH(2) for the following two equations:

(i) The Hamilton–Jacobi equation (HJ)

\[ \sum_{ij} g_{ij} \frac{\partial S}{\partial x^i} \frac{\partial S}{\partial x^j} = E; \tag{1.1} \]

(ii) the Laplace–Beltrami equation (LB)

\[ \Delta \psi = \sum_{ij} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \sqrt{g} g^{ij} \frac{\partial}{\partial x^j} \psi = \lambda \psi. \tag{1.2} \]

In a previous paper (to be referred to as I) we have considered the separation of variables in complex projective spaces CP(n). The isometry group of CP(n) is the compact group SU(n + 1), and its Cartan subgroup was used to generate n ignorable variables and to reduce the problem of variable separation on CP(n) to the separation of variables on the real sphere Sn. We refer to this paper for a discussion of the motivation and for some historical background.

Here let us just mention the relation between separation of variables in the HJ equation and complete integrability of the corresponding Hamiltonian system. Indeed, separability for the HJ equation is defined to mean that a solution S of (1.1) exists satisfying

\[ S = \sum_i S_i(x_1, \ldots, x_n), \quad \det \frac{\partial^2 S}{\partial x^i \partial x^j} \neq 0, \tag{1.3} \]

where \( \lambda_i \) are n constants: the separation constants. We associate n second-order operators in involution with each separable coordinate system in an n-dimensional space (one of them is the Hamiltonian); the constants \( \lambda_i \) are the eigenvalues of these operators. The existence of these operators assures that the system is integrable.

For studies of the separation of variables in Hamilton–Jacobi equations on Riemannian and pseudo-Riemannian manifolds, see also Refs. 2–5.

The additive separation of variables (1.3) in the HJ equation corresponds to multiplicative separation in the LB equation (1.2):

\[ \psi = \prod_i \psi_i(x_1, \ldots, x_n). \tag{1.4} \]

Indeed, for Einstein spaces every coordinate system that separates the HJ equation will also separate the LB equation (the converse is always true). Separation of variables in LB equations makes it possible to use powerful methods of group theory to study broad classes of special functions.

II. THE SPACE HH(n) AND ITS ISOTROPY GROUP SU(n, 1)

We introduce the Hermitian hyperbolic (or complex hyperbolic) space HH(n) following Kobayashi and Nomizu and Helgason.\(^{11}\) Let \( (e_0, e_1, \ldots, e_n) \) be a standard basis in \( \mathbb{C}^{n+1} \) and consider the Hermitian form

\[ F(x, y) = -x_0 y_0 + \sum_{k=1}^{n} x_k y_k, \tag{2.1} \]

where the overbar denotes complex conjugation. This form is invariant under the action of the group U(n, 1):

\[ g \in U(n, 1), \quad F(gx, gy) = F(x, y), \quad x, y \in \mathbb{C}^{n+1}, \tag{2.2} \]

which acts transitively on the real hypersurface \( M \) in \( \mathbb{C}^{n+1} \) defined by

\[ F(x, y) = -1. \tag{2.3} \]
The group $U(1) = \{ e^{i\theta} \}$ acts freely on this manifold by $y \mapsto e^{i\theta} y$. The space of orbits with suitable complex manifold structure and Kaehler metric is identified as $HH(n)$. The corresponding natural projection $\pi: M \to HH(n)$ defines a principal bundle with $U(1)$ as structure group. The $U(n,1)$ action commutes with that of $U(1)$, and it hence projects to an action on the base $HH(n)$. The isotropy subgroup of $U(n,1)$ at the point $p_0 = \pi(e_0)$ is $U(1) \times U(n)$, and we obtain the diffeomorphism

$$U(n,1)/[U(1) \times U(1)] \sim HH(n).$$  \hspace{1cm} (2.4)

The group $SU(n,1)$ acts almost effectively on this space. In addition to the homogeneous coordinates 

$$\{ y_{0\alpha}, y_{1\alpha}, \ldots, y_{n\alpha} \},$$

let us introduce affine coordinates on $HH(n)$:

$$\pi(\{ y_{0\alpha}, y_{1\alpha}, \ldots, y_{n\alpha} \}) = (z_1, \ldots, z_n), \quad z_k = \frac{y_k}{y_{0\alpha}}, \quad k = 1, \ldots, n.$$   \hspace{1cm} (2.5)

The space $HH(n)$ can then be identified with an open unit ball in $\mathbb{C}^n$

$$z \in \mathbb{C}^n, \quad \sum_k |z_k|^2 < 1.$$   \hspace{1cm} (2.6)

The real part of the Hermitian form (2.1) determines in a natural manner a metric on $HH(n)$, which is the noncompact version of the well-known Fubini-Study metric:

$$ds^2 = -\frac{4}{c} \left(1 - \sum_k \bar{z}_k z_k\right) \left(\sum_k d\bar{z}_k dz_k + \sum_k \bar{z}_k dz_k \right),$$

$$\left(1 - \sum_k \bar{z}_k z_k\right)^2,$$   \hspace{1cm} (2.7)

where $c > 0$ is the (constant) holomorphic sectional curvature.

We now limit ourselves to the case under consideration, namely $n = 2$.

The Hamiltonian associated with the metric (2.7) for $n = 2$ ($c = -4$) is

$$H = 4(1 - |z_1|^2 - |z_2|^2)(|z_1|^2 - 1)p_1 \bar{p}_1 + (|z_2|^2 - 1)$$

$$+ p_2 \bar{p}_2 + z_1 \bar{z}_2 p_1 \bar{p}_1 + \bar{z}_1 z_2 p_1 \bar{p}_1.$$   \hspace{1cm} (2.8)

The Lie algebra $su(2,1)$ in the representation acting on the homogeneous coordinates $\{ y_{0\alpha}, y_{1\alpha}, y_{2\alpha} \}$ is realized by $3 \times 3$ complex matrices $X$ satisfying

$$X + J + JX = 0, \quad J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$   \hspace{1cm} (2.9)

(the superscript $+$ denotes Hermitian conjugation).

Two convenient bases are given by the matrices $X_i$, or alternatively $Y_i$, $i = 0, 1, \ldots, 8$:

$$X_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$X_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$X_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$X_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$X_9 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$X_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$X_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$X_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$X_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_7 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$X_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_9 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

The $Y_i$ basis is particularly appropriate for considering solvable subalgebras of $su(2,1)$.

With these conventions the second order Casimir operator of $su(2,1)$ can be written as

$$C_2 = X_1^2 + X_2^2 + X_3^2 - X_4^2 - X_5^2 - X_6^2 - X_7^2 + X_8^2$$

$$= \frac{1}{3} Y_1^2 - Y_2^2 - Y_3^2 - Y_4^2 - Y_5^2$$

$$+ [Y_2, Y_7] + [Y_3, Y_6] + [Y_4, Y_5],$$   \hspace{1cm} (2.11)

where $[\cdot, \cdot]$ denotes the anticommutator.

A Killing vector $L$ on the cotangent bundle with local coordinates $(z, \bar{z}, p_z, \bar{p}_z, i = 1, 2)$ is a linear polynomial in $p_z, \bar{p}_z$:

$$L = \sum_i c_i (x_i z \bar{z} + \bar{x}_i z \bar{z}) p_z + c.c.$$   \hspace{1cm} (2.12)

(where c.c. indicates the complex conjugate quantity), such that

$$[H, L]_p = 0.$$   \hspace{1cm} (2.13)

i.e., the Poisson bracket of $H$ with $L$ is zero. The Killing vectors for $HH(2)$ provide a realization of the algebra $su(2,1)$. Using the basis $X_i$ ($i = 1, \ldots, 8$) of (2.10) for the infinitesimal operators, we calculate the corresponding Killing vectors in affine and homogeneous coordinates to be, respectively,
\[ X_7 = i \left( z_1 z_2 p_z + (x_1^2 + 1) p_x \right) + \text{c.c.} \]
\[ = (i - y_2 p_x - y_0 p_y) + \text{c.c.}, \]
\[ (2.14) \]
\[ X_8 = i \sqrt{3}(z_1 p_z + z_2 p_y) + \text{c.c.} \]
\[ = (i/\sqrt{3}) (y_0 p_x - y_1 p_y - y_2 p_y) + \text{c.c.} \]
Throughout we shall make use of the moment map; whenever convenient we use the operators \( \partial / \partial z_1 \) instead of the functions \( P_z \) or \( P_z \), and commutator brackets instead of Poisson brackets.

**III. SUBGROUPS OF SU(2,1) AND COMPLETE SETS OF COMMUTING SECOND-ORDER OPERATORS**

According to the operator approach to the separation of variables,\(^7\) each separable system on HH(2) will be characterized by four second-order operators \( \{ H, T_1, T_2, T_3 \} \) that are in involution with respect to the appropriate Lie bracket (one of them being the Hamiltonian \( H \), or correspondingly the Laplace operator \( \Delta \)). The first task is to classify the triplets of operators \( \{ T_1, T_2, T_3 \} \) into equivalence classes under the action of the group SU(2,1), leaving \( H \) invariant.

The task in the present case of HH(2) is greatly simplified by two circumstances:

1. It has recently been shown\(^2\) that for HH(2) all second-order Killing tensors, i.e., operators

\[ T = \sum_{a,b=1}^8 \left( c_{ab} (z_1 \bar{z}_2, z_2 \bar{z}_1) p_1 p_k \right. \]
\[ + \left. d_{ab} (x_1 \bar{x}_2, x_2 \bar{x}_1) p_1 p_k + \text{c.c.} \right), \]

(3.1)

satisfying

\[ [T,H] = 0 \]

(3.2)

lie in the enveloping algebra of su(2,1). Each of the operators \( T_1 \) can hence be written in the form

\[ T_1 = \sum_{a,b=1}^8 A_{1ab} X_a X_b, \quad A_{1ab} = A_{1ba} \in \mathbb{R}. \]

(3.3)

2. We have shown in I, Theorem 4, that every separable coordinate system in CP(2) and HH(2) has precisely two ignorable variables. We recall that an ignorable variable in a certain coordinate system is a variable that does not figure in the metric tensor \( g_{ik} \) expressed in this system.\(^2\) An ignorable variable \( \phi \) is obtained by setting a Killing vector, say \( L_\phi \), equal to the momentum \( p_\phi \) canonically conjugate to \( \phi \). The square of this Killing vector is then a second-order Killing tensor

\[ T_1 = L_\phi^2 = p_\phi^2. \]

(3.4)

This can be done\(^3\) if the corresponding Killing tensor \( T_1 \) is the square of a Killing vector, i.e., in our case the square of an element of su(2,1). Since two variables must be ignorable in each separable coordinate system, it follows that two of the operators \( T_1 \), say \( T_1 \) and \( T_2 \), must be squares of elements of su(2,1):

\[ T_1 = L_\phi^2 = \left( \sum_{a=1}^8 a_a X_a \right)^2, \]
\[ T_2 = L_\psi^2 = \left( \sum_{a=1}^8 b_a X_a \right)^2. \]

(3.5)

Since \( T_1 \) and \( T_2 \) commute, the operators \( L_1 \) and \( L_2 \) must generate an abelian subalgebra of su(2,1). All subalgebras of su(2,1) are known,\(^4\) and work is in progress on the classification of the maximal abelian subalgebras (MASA's) of all classical Lie algebras.\(^5\) In particular, su(2,1) has four different MASA's [each representing a conjugacy class of MASA's under the action of SU(2,1)]. Each of them is two-dimensional.

The procedure of finding all triplets of operators \( \{ T_1, T_2, T_3 \} \) related to separable coordinates on HH(2) thus reduces to the following:

1. Take \( T_1 \) and \( T_2 \), as in (3.5), where \( L_1 \) and \( L_2 \) run through the four different MASA's of su(2,1).

2. For each MASA \( L_1, L_2 \), find the most general operator \( Q = T_3 \in \mathfrak{s}(\mathfrak{su}(2,1)) \) [second-order symmetric tensor in the enveloping algebra of su(2,1)] commuting with \( L_1 \) and \( L_2 \). The operator \( T_3 \) has the form (3.3).

3. Simplify each \( T_3 \) by linear combinations, with \( L_1^2 \), \( L_1^2 \), \( L_1 L_2 \), and \( C \) (2.11) and classify the operators \( T_3 \) into conjugacy classes under the action of the normalizer of \( \{ L_1, L_2 \} \) in SU(2,1) (the normalizer is the group of transformations leaving the algebra \( \{ L_1, L_2 \} \) invariant).

A particularly important and simple class of coordinates are called "subgroup type coordinates,"\(^5,7,8,13\) and they occur when \( T_3 \) is the Casimir operator of a subgroup of SU(2,1).

In Fig. 1 we show all subalgebras of su(2,1) that are relevant for our purposes (for a complete classification see Ref. 14). The basis elements \( \{ X_a \} \) and \( \{ Y_a \} \) are defined in (2.10), we use the two bases interchangeably. The lowest row in Fig. 1 is occupied by the four MASA's: \( \{ X_a, Y_a \} \) and \( \{ Y_1, Y_4, Y_5 - Y_4 \} \) are the compact and noncompact Cartan subalgebras, respectively. \( \{ Y_1, Y_4 \} \) contains a nilpotent element.

![Diagram](image-url)

**FIG. 1.** Maximal abelian subalgebras of su(2,1) and some subalgebras containing them. The basis elements \( X_a \) and \( Y_a \) are defined in (2.10). The four MASA's constitute the lowest row, and double boxes indicate their normalizers. \( A_{4,10}, A_{4,9}, A_{4,4}, \) and \( A_{4,1} \) are solvable algebras, and \( T \) denotes a translation type subalgebra.
\( Y_4 \) (or \( Y_4 \) is represented by a nilpotent matrix in any finite-dimensional representation). All elements of \( \{ Y_2, Y_4 \} \) are nilpotent, i.e., this is a maximal abelian nilpotent subalgebra (MANS).\(^{15,14}\) The letter \( T \) in the boxes denotes the presence of such nilpotent elements ("translations" on a light cone). The double boxes indicate normalizers of the MASA's. By definition, Cartan subalgebras are self-normalizing. A classification of all real Lie algebras of dimension \( d \leq 5 \) exists\(^{17}\); the notation \( A_{4,10}, A_{4,9}, A_{4,2} \) refers to that article. The algebras \( A_{4,10}, U(1,1) \), and \( u(2) \) are the only subalgebras of \( su(2,1) \) [up to conjugacy under \( SU(2,1) \)] containing at least one MASA and having a second-order Casimir operator.

These algebras and their Casimir operators play an important role below; so let us discuss them in more detail.

1. The \( su(2) \) subalgebras of \( u(2) \) is \( \{ X_1, X_2, X_3 \} \) and its Casimir operator is

\[
I_{su(2)} = X_1^2 + X_2^2 + X_3^2. \tag{3.6}
\]

2. Two mutually conjugate \( su(1, 1) \) subalgebras and their Casimir operators are

\[
\begin{align*}
&\{ X_4, X_5, X_6 \} \text{ with } X_4 = X_5 = X_6 = 0, \\
&\{ X_7, X_8, X_9 \} \text{ with } X_7 = X_8 = X_9 = 0.
\end{align*}
\]

3. The solvable algebra \( A_{4,10} \):

\[
\begin{align*}
&\{ Y_1, Y_2, Y_3, Y_4 \} \text{ with } Y_1 = Y_2 = Y_3 = 0, \\
&\{ X_5, X_6, X_7, X_8, X_9, X_{10} \} \text{ with } X_5 = X_6 = X_7 = X_8 = X_9 = X_{10} = 0.
\end{align*}
\]

Its invariant is\(^{14}\)

\[
I_{A_{4,10}} = 4Y_1Y_4 + 3(Y_2^2 + Y_3^2) \tag{3.9}
\]

Notice that one realization of \( A_{4,10} \) is related to the one-dimensional harmonic oscillator. If we put

\[
Y_1 = \frac{3}{2} \left( \frac{\partial^2}{\partial x^2} + x^2 \right), \quad Y_2 = x, \quad Y_3 = \frac{\partial}{\partial x}, \quad Y_4 = \frac{1}{2},
\]

then the commutation relations for \( Y \) are satisfied, and we have \( I_{A_{4,10}} = 4Y_1 \).

Let us now return to the classification of triplets of operators outlined above.

### A. The compact Cartan subalgebra

We have

\[
T_1 = X_1^2, \quad T_2 = X_2^2,
\]

and \( \{ T_1, T_2 \} = 0, \{ T_2, T_3 \} = 0 \) implies

\[
Q_1 = I_{su(2)} = I_{su(1,1)} + \frac{1}{2} I_{su(1,1)}. \tag{3.10}
\]

The Cartan subalgebras are self-normalizing; hence the only freedom left is to subtract some multiple of \( C_2 \). The following possibilities occur:

1. \( b = c = a = 0 \): \( Q_1 = I_{su(2)} \),
2. \( b = c = a \neq 0 \): \( Q_2 = I_{su(1,1)} \),
3. \( b = c = a \neq 0 \): \( Q_3 = I_{su(1,1)} + \mu I_{su(1,1)}, \mu > 0 \),
4. \( b = c = 0 \): \( Q_4 = I_{su(1,1)} + \mu I_{su(1,1)}, \mu > 0 \),
5. \( b = c = 0 \): \( Q_5 = I_{su(1,1)} + \mu I_{su(1,1)}, \mu < 0 \),
6. \( b = c = 0 \): \( Q_6 = I_{su(1,1)} + \mu I_{su(1,1)}, \mu < 0 \).

The case \( |\mu| > 1 \) can be rotated into one of the cases with \( |\mu| < 1 \).

### B. The noncompact Cartan subalgebra

\[
T_1 = \frac{1}{2} \{ X_3 + (1/\sqrt{3})X_9 \}^2, \quad T_2 = X_2^2,
\]

\[
Q_1 = I_{su(1,1)} + \frac{1}{2} I_{su(1,1)} + \frac{1}{2} I_{su(1,1)} + \frac{1}{2} I_{su(1,1)}, \tag{3.11}
\]

Two possibilities should be distinguished:

\[
Q_5: \quad b = 0, \quad a = 1,
\]

\[
Q_6: \quad b = a = 0.
\]

(The relative sign of \( b \) and \( a \) can be changed by a rotation through the angle \( \pi \), hence the restriction \( a \geq 0 \) in \( Q_6 \).

### C. The MASA \( \{ Y_5, Y_4 \} \)

\[
T_1 = Y_1^2 = \frac{1}{2} \{ X_3 + (1/\sqrt{3})X_9 \}^2,
\]

\[
T_2 = Y_2^2 = (X_9 - \frac{1}{2} X_5 - \frac{1}{2} \sqrt{3} X_9)^2,
\]

\[
Q_1 = I_{su(1,1)} + \frac{1}{2} I_{su(1,1)} + \frac{1}{2} I_{su(1,1)} + \frac{1}{2} I_{su(1,1)}.
\]

Four possibilities occur:

\[
Q_7: \quad a = 0, \quad b = 1,
\]

\[
Q_8: \quad a = 1, \quad b = 0,
\]

\[
Q_9: \quad a = b = 1,
\]

\[
Q_{10}: \quad a = b = 0.
\]

Indeed, if \( ab \neq 0 \), we make use of the external part of the normalizer of \( \{ Y_5, Y_4 \} \), namely the operator \( Y_7 \) to scale \( a \) with respect to \( b \): For \( ab > 0 \) we can scale so that we get \( a = b \), for \( ab < 0 \) so that we get \( a = -b \).

### D. The maximal abelian nilpotent subalgebra

\[
T_1 = Y_3^2 = (X_3 - X_9)^2,
\]

\[
T_2 = Y_2^2 = (X_9 - \frac{1}{2} X_5 - \frac{1}{2} \sqrt{3} X_9)^2,
\]

\[
Q_1 = I_{su(1,1)} + \frac{1}{2} I_{su(1,1)} + \frac{1}{2} I_{su(1,1)} + \frac{1}{2} I_{su(1,1)}
\]

\[
- \frac{1}{2} (Y_5 Y_3 - Y_3 Y_5) = 6(Y_3 Y_8 - Y_8 Y_3).
\]

Two cases should be distinguished:

\[
Q_{11}: \quad a = 1, \quad b = 0,
\]

\[
Q_{12}: \quad a = 0, \quad b = 1.
\]

Indeed, if \( a \neq 0 \), we set \( a = 1 \) and use the external part of the normalizer of \( \{ Y_5, Y_4 \} \) to transform \( b \to 0 \) [this is achieved by a transformation of the type \( Q' = \exp(\alpha Y_3)Q \exp(-\alpha Y_3) \)].

We have thus obtained 12 orbits of operators \( \{ T_1, T_2, T_3 \} \). Among them six are of the subgroup type, i.e., such that \( Q \) is the Casimir operator of some subgroup of \( SU(2,1) \). These are the sets involving \( Q_1, Q_2, Q_3, Q_7, Q_8, \) and \( Q_{11} \).

In the following section we shall establish a one-to-one correspondence between the above-classified triplets of operators in involution and 12 types of separable coordinates on HH(2).
IV. SEPARABLE COORDINATES ON HH (2)

A. Introduction of ignorable coordinates and reduction to separation on an O(2,1) hyperboloid

Our purpose now is to find all separable coordinates in HH(2), i.e., to transform from the affine coordinates \( \{ z_1, z_2, z_3 \} \) to four real variables \( \{ A, B, x, y \} \) such that \( x \) and \( y \) are ignorable and that Eqs. (1.1) and (1.2) separate in the new variables. This transformation can be performed in two different manners, starting with the affine coordinates \( z_i \), \( (i = 1, 2, 3) \) or the homogeneous coordinates \( y_{\mu} \), \( (\mu = 0, 1, 2) \), respectively. In each case the procedure is repeated four times, separately for each MASA of su(2,1).

Using affine coordinates, we proceed as follows:

1. Choose a basis \( \{ L_1, L_2 \} \) for the considered MASA, express \( L_1 \) and \( L_2 \) in terms of \( z_i \) as in (2.14) and put

\[
L_1 = P_x, \quad L_2 = P_y. \tag{4.1}
\]

Solve equations \((4.1)\). This provides the explicit dependence of \( z_1 \) and \( z_2 \) on the ignorable variables. The dependence on the essential variables \( A, B \) is as yet unknown and is contained in the integration "constants" of \((4.1)\).

2. To obtain the dependence on \( A, B \) make use of the procedure outlined in Ref. 4, for arbitrary four-dimensional Riemannian spaces. Since HH(2) is a positive-definite metric space and since each separable system must involve precisely two ignorable variables, only case "C" of Ref. 4 occurs. Hence a pseudogroup \( P \) of coordinate transformations (described in I and Ref. 4) must exist, transforming the Fubini–Study metric \((2.7)\) into a form in which the metric tensor satisfies:

\[
g_{AA} = g_{BB} = \frac{1}{k_1(A) + k_2(B)}, \quad g_{AB} = 0,
\]

\[
g_{xx} = \frac{c_1(A) + c_2(B)}{k_1(A) + k_2(B)} g_{yy} = \frac{f_1(A) + f_2(B)}{k_1(A) + k_2(B)},
\]

\[
g_{xy} = h_1(A) + h_2(B), \quad g_{yy} = g_{xx} = g_{PP} = 0, \tag{4.2}
\]

where \( k_i, c_i, f_i, \) and \( h_i \) are functions of the indicated variables satisfying

\[
\frac{\partial^2}{\partial A \partial B} \ln \left[ \frac{(k_1 + k_2)^2}{(c_1 + c_2)(f_1 + f_2) - (h_1 + h_2)^2} \right] = 0. \tag{4.3}
\]

[i.e., \( R_{AB} = 0 \), where \( R_{ij} \) is the Ricci tensor]. Solve Eqs. (4.2) and (4.3) to obtain the dependence of \( \{ z_1, z_2 \} \) on \( A \) and \( B \).

Following this procedure, we find that the MASA \( \{ X_3, X_4 \} \) leads to four different types of coordinates, \( \{ Y_1, Y_3 - Y_2 \} \) to two types, \( \{ Y_1, Y_4 \} \) to four types, and finally \( \{ Y_3, Y_4 \} \) to two types. The computations are quite long and involved, but the results are relatively simple and coincide with those obtained using a different, more geometrical and group-theoretical method, described below.

The second procedure is an adaptation of the general method of the reduction of phase space in classical mechanics by ignorable variables. The procedure is related to that used by Marsden and Weinstein and Kazhdan, Kostant, and Sternberg to obtain completely integrable Hamiltonian systems. In I we applied this procedure to reduce by the maximal torus, i.e., the Cartan subgroup of SU(n+1). We thus reduced the problem of separating variables on CP(n) to that of separating on the sphere \( S_{n+1} \). The free Hamiltonian on CP(n) was reduced to a singular Hamiltonian on \( S_{n+1} \) with a specific inverse square type potential. We shall see that the situation is very similar for HH(2) and that the reduction can be performed by any of the maximal abelian subgroups (not just the maximal torus).

Instead of MASA's of su(2,1), we shall use MASA's of \( u(2,1) \), i.e., to the basis \( L_1, L_2 \) of each MASA we add a further operator

\[
X_0 = y_0 p_\rho + y_1 p_x + y_2 p_y + c.c. \tag{4.4}
\]

When acting on functions \( f(y_0, y_1, y_2) \) that project properly onto HH(2), i.e., homogeneous functions satisfying

\[
f(y_0, y_1, y_2) = f(y_0/y_0, y_2/y_0) \tag{4.5}
\]

we have

\[
\left( \frac{y_0}{y_0} \frac{\partial}{\partial y_0} + y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2} \right) f = 0 \tag{4.6}
\]

and for the corresponding constant of motion on HH(2) we have

\[
X_0 = y_0 p_\rho + y_1 p_x + y_2 p_y = 0. \tag{4.7}
\]

The procedure is:

1. Choose a basis \( \{ L_1, L_2 \} \) for the considered MASA, express \( L_1, L_2 \), and \( X_0 \) in terms of \( y \) as in (2.14) and (4.4) and put

\[
L_1 = p_x, \quad L_2 = p_y, \quad X_0 = p_\rho. \tag{4.8}
\]

Solve equations \((4.8)\) to obtain the explicit dependence of \( y \) on the ignorable variables \( x, y \), and \( \rho \) [up on projection from \( C(3) \) to HH(2) \( \rho \) will cancel out].

The variables \( y_{\mu} \) depend on three more real variables, say \( s_0, s_1, \) and \( s_2 \), which are contained in the integration constants of Eqs. \((4.8)\). These must be introduced in such a manner that \( s_\mu \times x, y, \) and \( \rho \) parametrize all of \( C(3) \), that \( x \) and \( y \) project into ignorable variables on HH(2), and that the variables \( s_\mu \) are compatible with the projection, i.e.,

\[
| y_1^2 - 2y_0y_2 - |y_1|^2 - |y_2|^2 = s_0^2 - s_1^2 - s_2^2 = s^2 = \text{const}. \tag{4.9}
\]

In order to obtain the space HH(2), we put \( s^2 = 1 \); other homogeneous spaces with SU(2,1) actions are obtained by putting \( s^2 = -1 \) or \( s^2 = 0 \).

Express the su(2,1) infinitesimal operators \( X_i \) \( (i = 1, \ldots, 8) \) the Hamiltonian \( H \) and the Killing tensor \( T_i = \mathcal{Q} \) in terms of the variables \( x, y, s_0, s_1, s_2 \) \( \{ \text{setting } p_\rho = 0, \text{ or correspondingly dropping a term containing } \partial/\partial \rho \} \). The essential variables \( s_\mu \) are constrained by the condition \((4.9)\). The corresponding momenta \( p_{s_\mu} \) figure in the infinitesimal operators \( X_i \) only via the expressions

\[
I_{12} = s_1 p_{s_1} - s_2 p_{s_2}, \quad I_{01} = s_0 p_\rho + s_1 p_{s_1},
\]

\[
I_{02} = s_0 p_{s_0} + s_2 p_{s_2}. \tag{4.10}
\]

The quantities \( I_{uv} \) \( \{ u, v = 0, 1, 2 \} \) generate an o(2,1) algebra under the corresponding Lie bracket. This o(2,1) is in general not a subalgebra of su(2,1); however, if we restrict ourselves to the manifold \((4.9)\) by setting the ignorable variables equal
to zero, then we obtain
\begin{equation}
X_1 = I_{12}, \quad X_2 = I_{01}, \quad X_3 = I_{02},
\end{equation}
(4.11)
i.e., the O(2,1) group acting on the variables \( s \) coincides with the real O(2,1) subgroup of SU(2,1). In the new variables the Hamiltonian \( H \) and the Killing tensor \( Q \) are expressed as
\begin{equation}
H = I_{12} + I_{01}^2 - I_{02}^2 + f_1(s_s) p_s^2 + f_2(s_s) p_s P_s + f_3(s_s) p_s P_p,
\end{equation}
(4.12)
\begin{equation}
Q = \sum_{\mu \nu} A_{\mu \nu} \mu \nu \mu \nu + h_1(s_s) p_s^2 + h_2(s_s) p_s P_s + h_3(s_s) p_s P_p,
\end{equation}
(4.13)
where \( f_i \) and \( h_i \) are functions of the essential variables \( s_s \), and \( A_{\mu \nu \mu \nu} = A_{\mu \nu} \mu \nu \) is a symmetric constant matrix. The problem of separating variables for the free Hamiltonian on HH(2) has thus been reduced to that of separating variables in the Hamiltonian (4.12). This is an O(2,1) Hamiltonian, which is, however, not a free one: It includes a "potential" term depending on the O(2,1) variables \( s_s \). We recall that the momenta \( P_s \) and \( P_p \) corresponding to the ignorable variables should be set equal to constants
\begin{equation}
P_s = c_1, \quad P_p = c_2.
\end{equation}
(4.14)
Notice that we have
\begin{equation}
I_{12}^2 - I_{01}^2 - I_{02}^2 = (p_s^2 - p_s^2 - p_s^2),
\end{equation}
(4.15)
where we have used the fact that
\begin{equation}
\sum_{\mu = 0} s \nu p_{\mu} = 0.
\end{equation}
(4.16)
(3) Introduce separable coordinates on the hyperboloid (4.9), compatible with the form of the operator \( Q \) and the potential in (4.12).

Let us now implement the first two steps of this procedure for each of the four MASA’s of su(2,1).

1. The compact Cartan subalgebra \([X_3, (1/\sqrt{3})X_8, X_0]\)

We first introduce the ignorable variables \((\rho, \alpha, \mu)\), putting
\begin{align*}
[X_3 - (1/\sqrt{3})X_8] &= \rho, \\
-\frac{1}{2} [X_3 + (1/\sqrt{3})X_8] &= \rho, \\
X_0 &= \rho. 
\end{align*}
(4.17)
Using (2.14), we obtain a system of equations that is easily solved to express the homogeneous coordinates as
\begin{align*}
y_0 &= s \rho e^{i \rho \mu} - \alpha \mu, \\
y_1 &= s \rho e^{i \rho \mu} + 2 \alpha \mu, \\
y_2 &= s \rho e^{i \rho \mu} + 2 \alpha \mu. 
\end{align*}
(4.18)
The infinitesimal operators are expressed in these coordinates in the Appendix. Putting \( \alpha = \alpha = 0 \), we obtain (4.11); \( X_0X_3X_2X_1X_8 \) then involve only the essential variables and the momenta conjugate to the ignorable ones. Expressions (4.12) and (4.13) for the Hamiltonian \( H \) and Killing tensor \( Q \) (3.10) reduce to
\begin{equation}
H = - I_{12}^2 + I_{01}^2 + I_{02}^2 + \left[ \frac{1}{s_1} p_{\alpha}^2 + \frac{1}{s_2} p_{\alpha}^2 - \frac{1}{s_0} (p_{\alpha} + p_{\alpha})^2 \right],
\end{equation}
(4.19)
\begin{align*}
Q &= a \left[ I_{12}^2 + \left( 1 + \frac{s_1^2}{s_1^2} \right) p_{\alpha}^2 + \left( 1 + \frac{s_2^2}{s_2^2} \right) p_{\alpha}^2 \right] \\
&\quad + b \left[ I_{01}^2 + \left( -1 + \frac{s_1^2}{s_1^2} \right) (p_{\alpha} + p_{\alpha})^2 \right] \\
&\quad + c \left[ I_{02}^2 + \left( -1 + \frac{s_2^2}{s_2^2} \right) (p_{\alpha} + p_{\alpha})^2 \right]. 
\end{align*}
(4.20)
Setting \( p_{\alpha} = 0 \) we obtain a free O(2,1) Hamiltonian and a Killing tensor of a specific type: it involves the squares \( I_{\mu}^2 \) only. Separation of variables on an O(2,1) hyperboloid \( H_2 \) is discussed below.\(^5\)\(^7\)\(^9\)\(^11\) Nine distinct separable coordinate systems exist on \( H_2 \) but only four of them have Killing tensors of the type \( Q \). Precisely these four occur in our HH(2) problem.

Setting \( p_{\alpha} = c_1 \neq 0 \), we reduce (4.19) to an O(2,1) Hamiltonian with an inverse square type singular potential, and \( Q \) reduces to the corresponding integral of motion. We have thus generated a nontrivial relativistic completely integrable Hamiltonian system. Similar systems with singular inverse square potentials have been studied in a nonrelativistic context.\(^2\)\(^3\)\(^4\)\(^5\)

2. The noncompact Cartan subalgebra \([X_3 + (1/\sqrt{3})X_8, X_0]\)

Introduction the ignorable variables \((\rho, \alpha, \mu)\) by putting
\begin{equation}
-\frac{1}{2} [X_3 + (1/\sqrt{3})X_8] = \rho, \quad X_3 = \rho, \quad X_0 = \rho,
\end{equation}
(4.21)
Expressing \( X_i \) in terms of the homogeneous coordinates \( y_i \), we obtain a system of partial differential equations that can be solved to yield
\begin{align*}
y_0 &= e^{i \rho \mu} \rho \mu (i s_\rho \rho \mu + s_\mu \rho \mu), \quad 0 < \rho < 2 \pi, \quad 0 < \alpha < 2 \pi, \\
y_1 &= e^{i \rho \mu} \rho \mu (i s_\rho \rho \mu - s_\mu \rho \mu), \quad 0 < \mu < \infty, \\
y_2 &= e^{i \rho \mu} \rho \mu (2 s_\rho \rho \mu) i s_\mu \rho \mu.
\end{align*}
(4.22)
The infinitesimal operators are given in the Appendix. Putting \( \alpha = u = 0 \), we again obtain (4.11). The Hamiltonian and Killing tensor \( Q \) (3.10) in this case are
\begin{align*}
H &= - I_{12}^2 + I_{01}^2 + I_{02}^2 + \left[ -s_0^2 - s_1^2 \right] p_{\alpha}^2 \\
&\quad + \frac{s_0^2 - s_1^2}{(s_0^2 + s_1^2)^2} p_{\alpha}^2 + \frac{4 s_0 s_1}{(s_0^2 + s_1^2)^2} p_{\alpha} p_{\alpha}, \\
Q &= a \left[ I_{01}^2 + \left( s_0^2 - s_1^2 \right)^2 p_{\alpha}^2 - \left( s_0^2 + s_1^2 \right) p_{\alpha} p_{\alpha} \right] \\
&\quad + b \left[ I_{12}^2 I_{02}^2 \right] + 2 \frac{s_0 s_1}{s_0^2 + s_1^2} p_{\alpha}^2 \\
&\quad + 2 \frac{s_0 s_1}{s_0^2 + s_1^2} \left( s_0^2 - s_1^2 \right) p_{\alpha} p_{\alpha}.
\end{align*}
(4.23)
Setting \( p_s = p_u = 0 \), we again obtain a free O(2, 1) Hamiltonian and a specific O(2, 1) Killing tensor (leading to only two of the nine separable systems on \( H_3 \)). For \( p_u = c_1 \) and \( p_s = c_2 \), we obtain a new nontrivial completely integrable Hamiltonian system with a singular potential.

3. The orthogonally decomposable MASA \( \{ Y, Y_a \} \)

To introduce the ignorable variables \( \{ p, \alpha, t \} \), we put
\[
- \frac{1}{2} Y_1 = p_\alpha, \quad Y_4 = p_t, \quad X_0 = p_\rho \tag{4.25}
\]
and obtain
\[
y_0 = e^{\frac{4}{3} p_\rho - \alpha^3} \left[ s_0 + (s_0 - s_1) t \right], \quad -\infty < t < \infty,
\]
\[
y_1 = e^{\frac{4}{3} p_\rho - \alpha^3} \left[ s_1 + (s_0 - s_1) t \right], \quad 0 < \rho < 2\pi, \quad 0 < \alpha < 2\pi,
\]
\[
y_2 = e^{\frac{4}{3} p_\rho + 2\alpha^3} s_2.
\]

The infinitesimal operators are given in the Appendix. The Hamiltonian and Killing tensor (3.12) are
\[
H = - I_{12}^2 + I_{01}^2 + I_{02}^2
+ \left( \frac{1}{s_2^2} p_\alpha^2 + \frac{s_0 + s_1}{(s_0 - s_1)^2} p_\rho^2 + \frac{2}{(s_0 - s_1)^2} p_\alpha p_t \right).
\tag{4.27}
\]
\[
Q_{III} = 3a \left[ (I_{02} - I_{12})^2 + \frac{(s_0 - s_1)^2}{s_2^2} p_\alpha^2 + \frac{s_1^2}{(s_0 - s_1)^2} p_\rho^2 + 2p_\alpha p_t \right]
+ \left[ I_{01}^2 + \frac{(s_0 + s_1)^2}{(s_0 - s_1)^2} p_\rho^2 + \frac{2(5s_0 + s_1)}{s_0 - s_1} p_\alpha p_t \right].
\tag{4.28}
\]

For \( \alpha = t = 0 \) we again have pure O(2, 1) quantities. The specific form of \( Q_{III} \) leads to four of the nine separable O(2,1) systems. For \( p_u = c_1 \) and \( p_s = c_2 \), we obtain yet another O(2,1) Hamiltonian with a new nontrivial singular interaction.

4. The maximal abelian nilpotent subalgebra \( \{ Y_p, Y_u \} \)

To introduce the ignorable variables \( \{ p, t, u \} \), we put
\[
Y_3 = p_t, \quad Y_4 = - p_u, \quad X_0 = p_\rho \tag{4.29}
\]
and obtain
\[
y_0 = e^{\rho} \left[ (s_0 - s_1) (u - \frac{1}{2} it^2) + s_2 t - is_0 \right], \quad -\infty < u < \infty,
\]
\[
y_1 = e^{\rho} \left[ (s_0 - s_1) (u - \frac{1}{2}it^2) + s_2 t - is_1 \right], \quad -\infty < t < \infty,
\]
\[
y_2 = e^{\rho} \left[ -is_2 - (s_0 - s_1) t \right], \quad 0 < \rho < 2\pi.
\tag{4.30}
\]

The infinitesimal operators are in the Appendix; the Hamiltonian and Killing tensor (3.13) are
\[
H = - I_{12}^2 + I_{01}^2 + I_{02}^2 + \left( \frac{1}{(s_0 - s_1)^2} p_\rho^2 + \frac{3s_0^2 - 4s_0 s_1 + s_1^2}{(s_0 - s_1)^4} p_\alpha^2 \right).
\tag{4.31}
\]
\[
Q_{IV} = 3a \left[ (I_{12} - I_{02})^2 + \left( \frac{p_\rho - \frac{2s_0}{s_0 - s_1} p_\alpha}{s_0 - s_1} \right)^2 \right]
+ 3b \left[ (I_{02} - I_{12})^2 + \frac{2s_0}{s_0 - s_1} p_\rho^2 + \frac{s_0 - s_1}{(s_0 - s_1)^3} p_\alpha^2 \right]
+ 2\left( \frac{5s_0 + s_1}{s_0 - s_1} p_\alpha p_t \right).
\tag{4.32}
\]

For \( p_t = p_\rho = 0 \) the operator \( Q_{IV} \) reduces to an O(2,1) operator related to variable separation in two of the nine separable systems on \( H_3 \). For \( p_t = c_1 \) and \( p_\rho = c_2 \) we again obtain a nontrivial interaction term in (4.31).

B. Separation of variables on an O(2,1) hyperboloid

Let us now consider the separation of variables in the free Hamilton–Jacobi equation of free Laplace–Beltrami equation on the O(2,1) homogeneous space
\[
s^2 = s_0^2 - s_1^2 - s_2^2 = K^2 \quad (K^2 = \pm 1 \text{ or } 0).
\tag{4.33}
\]

Nine separable coordinate systems have been shown to exist\(^5\) and to be in one-to-one correspondence with orbits of second-order operators in the enveloping algebra of O(2,1).\(^7\) Since the results are not readily available and were not presented in a convenient form for our purposes, we summarize them here.

Let \( I_\mu \), be the O(2,1) operators (4.10), satisfying
\[
[I_{01}, I_{02}] = - I_{12}, \quad [I_{13}, I_{01}] = I_{02}, \quad [I_{13}, I_{02}] = - I_{01}.
\tag{4.34}
\]

A general second-order operator in the O(2,1) enveloping algebra can be written as
\[
R = (I_{12}I_{01}I_{02}) X \begin{pmatrix} I_{01} \\ I_{02} \end{pmatrix}, \quad X = X^T \in \mathbb{R}^{3 \times 3}.
\tag{4.35}
\]

Under an O(2,1) transformation, \( R \) is transformed into \( R' \) given by (4.35) with \( X \) replaced by \( X' \):
\[
X' = G^{-1} X G, \quad (GJG)^T = J,
\tag{4.36}
\]

where \( J \) is a nonsingular \( 3 \times 3 \) real symmetric matrix with signature \((- + +)\). We rewrite (4.36) as
\[
X' = G^{-1} X G, \quad \tilde{X} = X J, \quad \tilde{X}^T J = J \tilde{X}.
\tag{4.37}
\]

Thus, \( X \) is symmetric under the involution that defines o(2,1). Such symmetric matrices have recently been classified for all classical Lie algebras.\(^5\) For O(2,1) the results are quite simple, namely any pair of matrices \( (X, J) \) satisfying (4.37) is SL(3,\( \mathbb{R} \)) conjugate to one of the following:

(I) \( \tilde{X} \) orthogonally decomposable with three real eigenvalues:
\[
\tilde{X} = \begin{pmatrix} -c \\ a \\ b \end{pmatrix}, \quad J = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \quad a, b, c \in \mathbb{R}.
\tag{4.38}
\]

(II) \( \tilde{X} \) orthogonally decomposable with one real eigenvalue and one pair of complex conjugate eigenvalues:
3. Elliptic I: \( R_1 \) with \(-\lambda_3 \neq \lambda_1 \neq \lambda_2 \neq -\lambda_3\), 
\((\lambda_1 + \lambda_3)/(\lambda_2 + \lambda_3) > 0\)

\[
\begin{align*}
\bar{X}_{\text{III}} &= \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & c \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad a,b,c \in \mathbb{R}, \quad b > 0.
\end{align*}
\] (4.39)

(III) \( \bar{X}_{\text{III}} \) orthogonally decomposable with two real eigenvalues:

\[
\bar{X}_{\text{III}} = \begin{pmatrix} a & 0 & 0 \\ 1 & a & 0 \\ 0 & 0 & b \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad a,b \in \mathbb{R}.
\] (4.40)

(IV) \( \bar{X}_{\text{IV}} \) indecomposable (one real eigenvalue):

\[
\bar{X}_{\text{IV}} = \begin{pmatrix} a & 0 & 0 \\ 1 & a & 0 \\ 0 & 0 & a \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad a \in \mathbb{R}.
\] (4.41)

Returning to a basis in which \( J \) is as (4.38) and simplifying by linear combinations with the O(2,1) Casimir operator

\[
\Delta = I_{\bar{0}1}^2 + I_{02}^2 - I_{12}^2,
\] (4.42)

we obtain four classes of quadratic operators \( R_1 \):

\[
\begin{align*}
R_1 &= \lambda_3 I_{12}^2 + \lambda_1 I_{\bar{0}1}^2 + \lambda_2 I_{02}^2, \quad \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}, \\
R_{\bar{1}} &= \lambda I_{\bar{0}1}^2 + [I_{\bar{0}1} I_{02} - I_{12} I_{\bar{0}1}], \\
R_{\text{III}} &= \lambda I_{\bar{0}1}^2 + \mu(I_{02} - I_{12})^2, \quad \mu \neq 0, \quad \lambda, \mu \in \mathbb{R}, \\
R_{\text{IV}} &= [I_{02} - I_{12} I_{\bar{0}1}]
\end{align*}
\] (4.43)

(the brackets \{ , \} denote an anticommutator). The operator \( R_1 \) can be further simplified by combinations with \( \Delta \); in \( R_{\bar{1}} \) we can assume \( \lambda \geq 0 \); in \( R_{\text{III}} \) we can scale \( \mu \) with respect to \( \lambda \) by means of the O(2,1) transformation exp \( I_{\bar{0}1} \) and hence only distinguish three cases: \( \lambda = 0, \mu = 1; \lambda = \mu = 1; \lambda = -\mu = 1 \).

Finally we obtain nine classes of operators \( R_1 \), \( a = 1, \ldots, 9 \) and the corresponding coordinate systems for which the O(2,1) Hamilton-Jacobi and Laplace-Beltrami equations separate. The separable coordinates, Hamiltonians \( H \) and integrals of motion \( R_1 \), for the two-sheeted hyperboloid, i.e., \( K^2 = 1 \) are as follows.

1. Spherical: \( R_1 \) with \( \lambda_1 = \lambda_2 \neq \lambda_3 \)

\[
\begin{align*}
s_0 &= \cosh A, \quad s_1 = \sinh A \cos B, \quad s_2 = \sinh A \sin B, \\
0 < A < \infty, \quad 0 < B < 2\pi,
\end{align*}
\] (4.44)

\[
H = p_A^2 + \frac{1}{\sinh A^2} p_B^2, \quad R_1 = I_{12}^2 = p_B^2.
\]

2. Hyperbolic: \( R_1 \) with \( -\lambda_3 = \lambda_2 \neq \lambda_1 \)

\[
\begin{align*}
s_0 &= \cosh A \cosh B, \quad s_1 = \cosh A \sinh B, \\
s_2 &= \sinh A, \quad A, B \in \mathbb{R}, \\
\end{align*}
\] (4.45)

\[
H = p_A^2 + \frac{1}{\cosh A^2} p_B^2, \quad R_1 = I_{\bar{0}1}^2 = p_B^2.
\]

3. Elliptic I: \( R_1 \) with \( -\lambda_3 \neq \lambda_1 \neq \lambda_2 \neq -\lambda_3 \), 
\((\lambda_1 + \lambda_3)/(\lambda_2 + \lambda_3) > 0\)

\[
\begin{align*}
s_0^2 &= v p/a, \quad s_1^2 = (v - 1)/(\rho - 1)/a, \\
s_2^2 &= (v - a)/(\rho - 1)/a, \quad 1 < \rho < a < 0, \quad 1 < a,
\end{align*}
\] (4.46)

\[
H = [4/(v - \rho)] \left[ \nu(v - 1)/(v - a) p_v^2 + \rho/(\rho - 1)(a - p_v^2) \right],
\]

\[
R_3 = a I_{\bar{0}1}^2 + I_{02}^2
\]

\[
= [4v p/(v - \rho)] \left[ (v - 1)/(v - a) p_v^2 + (\rho - 1)(a - p_v^2) \right].
\]

4. Elliptic II: \( R_1 \) with \( -\lambda_3 \neq \lambda_1 \neq \lambda_2 \neq -\lambda_3 \), 
\((\lambda_1 + \lambda_3)/(\lambda_2 + \lambda_3) < 0\)

\[
\begin{align*}
s_0^2 &= (v - 1)/(\rho - 1)/a, \quad s_1^2 = -v p/a, \\
s_2^2 &= (v - a)/(\rho - 1)/a, \quad \rho < 0, \quad 1 < a < 0, \quad 0 < a < 1,
\end{align*}
\] (4.47)

\[
H = [4/(v - \rho)] \left[ \nu(v - 1)/(v - a) p_v^2 + \rho/(\rho - 1)(a - p_v^2) \right],
\]

\[
R_4 = (a - 1) I_{\bar{0}1}^2 - I_{02}^2
\]

\[
= -[4(1 - \rho)(v - 1)/(v - \rho)] \left[ (v - a) p_v^2 + (a - \rho) p_\rho^2 \right]
\].

5. Complex elliptic: \( R_1 \)

\[
\begin{align*}
\{s_0 + i s_1\}^2 &= (v - a)(\rho - a)/(a - a^*), \quad s_1^2 = -v p/|a|^2, \\
v < 0 < \rho, \quad a = \alpha + i \beta, \quad \beta > 0, \quad \alpha, \beta \in \mathbb{R},
\end{align*}
\] (4.48)

\[
H = [4/(\rho - v)] \left[ \nu(\rho - a)/(\rho - a^*) p_v^2 - \nu(v - a)(v - a^*) p_\rho^2 \right],
\]

\[
R_5 = a I_{\bar{0}1}^2 - \beta [I_{\bar{0}1} I_{02}]
\]

\[
= [4v p/(\rho - v)] \left[ (\rho - a)/(\rho - a^*) p_v^2 - (v - a)(v - a^*) p_\rho^2 \right].
\]

6. Horospheric: \( R_{\text{wi}} \) with \( \lambda = 0 \)

\[
\begin{align*}
s_0 &= \cosh A + \frac{1}{2} r^2 e^{-A}, \quad s_1 = \sinh A + \frac{1}{2} r^2 e^{-A}, \\
s_2 &= r e^{-A}, \quad -\infty < A < \infty, \quad -\infty < r < \infty,
\end{align*}
\] (4.49)

\[
H = p_A^2 + e^{2A} p_r^2, \quad R_0 = (I_{02} - I_{12})^2 = p_r^2.
\]

7. Elliptic parabolic: \( R_{\text{wi}} \) with \( \lambda \mu > 0 \)

\[
\begin{align*}
s_0^2 &= \frac{1}{4}(v + \rho)^2/vp, \quad s_1^2 = \frac{1}{4}(v + \rho - 2vp)^2/vp, \\
s_2^2 &= (1 - v)(\rho - 1), \quad 0 < \rho < 1 < \rho,
\end{align*}
\] (4.50)

\[
H = [4/(\rho - v)] \left[ \nu^2(\rho - 1) p_v^2 + \nu^2(1 - \nu p_v^2) \right],
\]

\[
R_7 = I_{\bar{0}1}^2 + (I_{02} - I_{12})^2
\]

\[
= [4vp/(\rho - v)] \left[ (\rho - 1) p_v^2 + \nu(1 - \nu p_v^2) \right].
\]
8. Hyperbolic parabolic: $R_{\mu
u}$ with $\lambda \mu < 0$

\[ s_0^3 = (v + \rho - 2v\rho)^2/(1 - 4v\rho), \quad s_1^3 = (v + \rho)^2/(1 - 4v\rho), \]
\[ s_2^3 = (1 - v\rho)(\rho - 1), \quad v < 0 < 1 < \rho, \] (4.51)
\[ H = [4/(v - \rho)] \left[ \rho^2(\rho - 1) p_\rho^2 + v^2(1 - v) p_v^2 \right], \]
\[ R_\alpha = I_{11} - (I_{02} - I_{12})^2 \]
\[ = [4\rho v/(v - \rho)] \left[ \rho(\rho - 1) p_\rho^2 + v(1 - v) p_v^2 \right]. \]

9. Semicircular parabolic: $R_{\nu
u}$

\[ s_0^3 = 1/(1 - 16(\rho v)^2) \left[ (\rho - v)^2 + \rho^2 v^2 \right]^2, \]
\[ s_1^3 = 1/(1 - 16(\rho v)^2) \left[ (\rho - v)^2 - \rho^2 v^2 \right]^2, \]
\[ s_2^3 = (\rho + v)^2/(1 - 4\rho v), \quad v < 0 < \rho, \] (4.52)
\[ H = [4/(\rho - v)] \left[ \rho^2 p_\rho^2 - v^2 p_v^2 \right], \quad R_\alpha = |I_{02} - I_{12}I_{01}| \]
\[ = 2\left[ \rho v/(v - \rho) \right] \left[ \rho^2 p_\rho^2 - v^2 p_v^2 \right]. \]

Three of these coordinate systems are of the “subgroup type,” namely spherical, hyperbolic, and horospheric, corresponding to the group reductions

\[ O(2,1) \supset O(2), \quad O(2,1) \supset O(1,1), \quad \text{and} \quad O(2,1) \supset T, \]

respectively [$T$ being the group of translations generated by $I_{02} - I_{12}I_{01}$].

All coordinate systems are written so as to parametrize the upper sheet of a one-sheeted hyperboloid. It is not difficult to modify the coordinates so as to parametrize the one-sheeted hyperboloid ($s^3 = -1$).

C. Separable coordinates on HH(2) and the Hamiltonian systems

In Sec. III we have classified triplets of operators $\{T_1,T_2,T_3\}$ into 12 orbits under SU(2,1). In Sec. IVA we have introduced ignorable variables on HH(2). Each different MASA of SU(2,1) leads to specific coordinates in which the Hamiltonian $H$ and integral of motion $Q = T_3$, reduce to an $O(2,1)$ form corresponding to an $O(2,1)$ Hamiltonian system with a nontrivial interaction. In Sec. IVB we reviewed separation on the O(2,1) hyperboloid $s^3 = 1$. Combining all these results together, we obtain the following theorem.

**Theorem 1:** (1) There exist precisely 12 systems of coordinates on HH(2) in which the Hamiltonian–Jacobi and Laplace–Beltrami equations separate.

(2) Each separable system has two ignorable and two nonignorable variables. The nonignorable variables are introduced so as to separate variables on the O(2,1) hyperboloid $s^3 = s_0^3 - s_1^3 - s_2^3 = 1$.

(3) The separable coordinate systems in HH(2) are in one-to-one correspondence with orbits of triplets of second-order operators $\{T_1,T_2,T_3\}$ in the enveloping algebra of su(2,1). The operators $T_i$ are in involution, two of them, $T_1 = L_1$ and $T_2 = L_2$, are squares of the generators $L_1,L_2$ of a MASA of su(2,1), the third $T_3$ is a general operator of the form (3.3). The operator $Q$ takes one of the forms $Q_1,...,Q_{12}$ listed in Sec. III.

(4) The compact Cartan subalgebra $\{X_3,X_\alpha\}$ for which $Q$ has the form $Q_3$ of (4.20) leads to four types of coordinate systems, namely, (4.18) with $(s_\mu,s_\nu,s_\lambda)$ expressed in spherical $Q_1$, hyperbolic $Q_2$, elliptic $Q_3$, or elliptic II $Q_4$ coordinates on the 0(2,1) hyperboloid $H_2$.

(5) The noncompact Cartan subalgebra $\{X_3 + (1/\sqrt{3})X_\alpha,X_\beta\}$ for which $Q$ has the form $Q_{11}$ of (4.24) leads to two types of coordinate systems, namely, (4.22) with $(s_\mu,s_\nu,s_\lambda)$ expressed in hyperbolic $Q_2$, or complex elliptic $Q_3$ coordinates on $H_2$.

(6) The decomposable non-Cartan subalgebra $\{X_3,X_\alpha\}$ for which $Q$ has the form $Q_{11}$ of (4.28) leads to four separable coordinate systems, namely, (4.26) with $(s_\mu,s_\nu,s_\lambda)$ expressed in hyperbolic $Q_2$, or elliptic II $Q_4$ coordinates on $H_2$.

(7) The MANS $\{Y_3,Y_\alpha\}$ for which $Q$ has the form $Q_{11}$ of (4.32) leads to two separable systems, namely, (4.30) with $(s_\mu,s_\nu,s_\lambda)$ expressed in horospheric $Q_3$ or semicircular $Q_4$ coordinates on $H_2$.

Finally, let us list the 12 separable coordinate systems and in the process show that the “potentials” in the O(2,1) Hamiltonians are indeed compatible with separation in each of the 12 cases. We shall use the affine coordinates (2.5).

1. The compact Cartan subalgebra $\{X_3,X_\alpha\}$

\[ [X_3 - (1/\sqrt{3})X_\alpha] = p_{\alpha}, \quad c_{\alpha}, \]
\[ - [X_3 + (1/\sqrt{3})X_\alpha] = p_{\alpha}, \quad c_{2.} \]

a. Spherical coordinates:

\[ z_1 = \tanh A \cos Be^{\alpha}, \quad z_2 = \tanh A \cos Be^{\alpha}, \]
\[ Q_1 = \rho^2 + (1/\cos^2 B) p_{\alpha}^2, \quad Q_2 = (1/\sinh^2 A) Q_1 - (1/\cosh^2 A) (p_{\alpha} + p_{\alpha})^2 = E. \] (4.53)

b. Hyperbolic coordinates:

\[ z_1 = \tanh Be^{\alpha}, \quad z_2 = (\tanh A/\cos B) e^{\alpha}, \]
\[ Q_1 = \rho^2 + (1/\sinh^2 B) p_{\alpha}^2, \quad Q_2 = (1/\sinh^2 A) Q_1 - (1/\cosh^2 B) (p_{\alpha} + p_{\alpha})^2 = c_{2}, \]
\[ H = p_{\alpha}^2 \left[ (1/2\sinh^2 A) + (1/2\cosh^2 B) \right] + \frac{1}{2} E. \] (4.54)
c. Elliptic I coordinates:

\[ z_1^+ = \frac{[\alpha(v-1)\beta/1-(v-1)\beta]}{1-(v-1)\beta} e^{i\alpha}, \]
\[ z_2^+ = \left[\frac{v-a-(\beta-(a-1)\beta)}{1-(v-1)\beta}\right] e^{i\alpha}, \]
\[ Q_1 = \left[\frac{1-(v-1)\beta}{1-(v-1)\beta}\right] 4\rho v(v-1)(v-a) p_0^+ + 4\rho((v-1)\beta) p_0^- \\
+ [(v-a)\beta/1-(v-1)\beta] p_0^+ + a[(1-(v-1)\beta)(v-a) + (v-1)\beta/1-(v-1)\beta] p_0^+ \\
- a(v-a)\beta/1-(v-1)\beta p_{\alpha} + p_{\alpha} + p_{\alpha}^2 = c_3, \]
\[ H = \left[\frac{1-(v-1)\beta}{1-(v-1)\beta}\right] 4\rho v(v-1)(v-a) p_0^- + 4\rho((v-1)\beta) p_0^- \\
+ [(v-a)\beta/1-(v-1)\beta] p_0^- + a[(1-(v-1)\beta)(v-a) + 1/(v-1)\beta] p_0^- \\
- a(1-(v-1)\beta)(v-a) + p_{\alpha} + p_{\alpha}^2 = E. \]

d. Elliptic II coordinates:

\[ z_1^- = -(v-a)\beta/1-(v-1)\beta, \quad z_2^- = (v-a)(v-a)/(v-1)(1-v), \]
\[ \rho < 0, \quad 0 < \alpha < 1, \quad H = \left[\frac{1-(v-1)\beta}{1-(v-1)\beta}\right] 4\rho v(v-1)(v-a) p_0^- + 4\rho((v-1)\beta) p_0^- + a[-1/1-(v-1)\beta] p_{\alpha} \\
+ a(1-(v-1)\beta)(v-a) + p_{\alpha} + p_{\alpha}^2 = E, \]
\[ Q_2 = \left[\frac{1-(v-1)\beta}{1-(v-1)\beta}\right] (v-a) p_0^+ + a(1-(v-1)\beta)(v-a) + p_{\alpha} + p_{\alpha}^2 = c_3. \]

2. The noncompact Cartan subalgebra \( \{X_3 + (1/\sqrt{3})X_0,X_0\} \)

\[ -\frac{1}{2}(X_3 + (1/\sqrt{3})X_0) = p_{\alpha} = c_1, \quad X_3 = p_{\alpha} = c_2. \]

e. Hyperbolic coordinates:

\[ z_1 = \frac{i \sin B \cosh u - \cos B \sin u}{i \cosh B \sinh u}, \quad z_2 = \frac{e^{i\alpha}}{\cosh B \sinh u}, \]
\[ Q_3 = p_0^+ + (1/\cosh^2 B)(p_0^2 - p_0^-) - (2 \sinh 2B/cosh^2 2B) p_0^+ p_0^- \]
\[ H = p_0^+ + (1/\cosh A) Q_3 + (1/\sinh^2 A) p_0^-. \]

f. Complex elliptic coordinates: The coordinates are

\[ z_1 = \frac{is_0 \cosh u - s_0 \sinh u}{is_0 \cosh u + s_0 \sinh u}, \quad z_2 = \frac{s_2}{is_0 \cosh u + s_0 \sinh u} \]
with \( s_0, s_1, \) and \( s_2 \) as in (4.48)
\[ Q_5 = \left[\frac{1}{1-(v-1)\beta}\right] 4\rho[(v-a)(\rho-a)](v-a)(v-a) p_0^+ \\
+ \frac{1}{2} [a-(a^*)(v-a)] p_0^+ - 4\rho(v-a)(v-a) p_0^- \\
+ \frac{1}{2} [a-(a^*)] p_0^+ - (v-1)(v-a)(v-a)](v-a) - \frac{1}{2} a^2 p_0^- \]
\[ + \frac{1}{2} i[a-(a^*)] \left[ 2[a^2] p_0^- + \frac{1}{2} (v-a)(v-a) - \frac{1}{2} a^2 (v-a) \rho + a^2 \right] p_0^- \]
\[ H = \left[\frac{1}{1-(v-1)\beta}\right] 4\rho[(v-a)(\rho-a)](v-a)(v-a) p_0^- \\
+ \frac{1}{2} [a-(a^*)] p_0^- - 4\rho(v-a)(v-a) p_0^+ \\
+ \frac{1}{2} [a-(a^*)] p_0^- - (v-1)(v-a)(v-a)](v-a) - \frac{1}{2} a^2 p_0^+ \]
\[ + \frac{1}{2} i[a-(a^*)] \left[ 2[a^2] p_0^+ + \frac{1}{2} (v-a)(v-a) - \frac{1}{2} a^2 (v-a) \rho + a^2 \right] p_0^+ \]
\[ \]

3. The orthogonally decomposable MASA \( \{Y_1, Y_2\} \)

\[ z_1 = (\sinh B + i e^{-B})/(\cosh B + i e^{-B}), \]
\[ z_2 = \tanh A e^{i\alpha}/(\cosh B + i e^{-B}), \]
\[ Q_7 = p_0^+ + (1/cosh^2 A) Q_3 + (1/\sinh^2 A) p_{\alpha} = E. \]
h. Horospheric coordinates \([b = 0 \text{ in (4.28)}]\):
\[
z_1 = (\frac{1}{\rho} - 1 + e^{2\theta} + B^2 + 2i)\sqrt{1 + e^{2\theta} + B^2 + 2i},
\]
\[
z_2 = B e^{\alpha} [1 + e^{2\theta} + B^2 + 2i],
\]
\[
Q_b = p^b + [(1/B) p_a + B p_a],
\]
\[
H = p^b + e^{2\theta} Q_b + e^{4\theta} p^2.
\]

i. Elliptic parabolic coordinates \([3a = b \text{ in (4.28)}]\):
\[
z_1 = (\frac{1}{\rho} - 1 - 2i\rho + 2iv)\sqrt{1 + \rho + 2iv},
\]
\[
z_2^2 = 4v(1 - v)/(\rho - 1)^2 + 1/(\rho + 2iv),
\]
\[
Q_9 = [1/(\rho - v)] [4v^2 \rho(p - 1) p^2 + 1/v(1 - v) p^2 - 1/(\rho - 1)] p^2,
\]
\[
H = [1/(\rho - v)] [4v^2 \rho(p - 1) p^2 + 1/(1 - v) + 1/(\rho - 1)] p^2,
\]
\[
+ \sqrt{1 - v^2} (\rho - 1) p^2 + (1 - v)/(\rho - 1) p^2.
\]

j. Hyperbolic parabolic coordinates \([3a = -b \text{ in (4.28)}]\):
\[
z_1 = (\frac{1}{\rho} - 1 - 2i\rho + 2iv)\sqrt{1 + \rho + 2iv},
\]
\[
z_2^2 = [4v^2 (1 - v) [\rho(p - 1) p^2 + 1/v(1 - v) p^2] + [\rho(1 - v)/\sqrt{1 - v^2} + \rho(1 - 1)/\rho^2] p^2,
\]
\[
Q_{12} = p^2 + [1/(\rho - v) + 1/(\rho + 1)] p^2,
\]
\[
H = [1/(\rho - v)] [4v^2 \rho(p - 1) p^2 + 1/(1 - v) + 1/(\rho - 1)] p^2,
\]
\[
+ \sqrt{1 - v^2} (\rho - 1) p^2 + (1 - v)/(\rho - 1) p^2.
\]

4. The maximal abelian nilpotent subalgebra \([Y_3, Y_4]\):
\[
Y_3 = X_3 - X_7 = p_5 = c_3,
\]
\[
Y_4 = X_5 - X_9 = \frac{1}{2} X_5 + \frac{1}{2} X_9 = p_5 = c_2.
\]

k. Horospheric coordinates \([b = 0 \text{ in (4.32)}]\):
\[
z_1 = [2(u + Bt) - i(e^{2\theta} + B^2 + t^2 - 1)]/[2(1 + Bt) - i(e^{2\theta} + B^2 + t^2 + 1)],
\]
\[
z_2 = -2it + iBv/[2(u + Bt) - i(e^{2\theta} + B^2 + t^2 + 1)],
\]
\[
Q_{11} = p^2 + (\rho - 2Bp_a)^2 = c_3,
\]
\[
H = p^2 + e^{2\theta} Q_{11} + e^{4\theta} p^2 = E.
\]

l. Semicircular parabolic coordinates \([a = 0 \text{ in (4.32)}]\):
\[
z_1 = \frac{2p^2 v^2 u - 2pv(\rho + \rho v + i[(\rho - v)^2 + \rho^2 v^2(t^2 - 1)])}{2p^2 v^2 u - 2pv(\rho + \rho v + i[(\rho - v)^2 + \rho^2 v^2(t^2 + 1)])},
\]
\[
z_2 = \frac{2p^2 v^2 u - 2pv(\rho + \rho v + i[(\rho - v)^2 + \rho^2 v^2(t^2 + 1)])}{2p^2 v^2 u - 2pv(\rho + \rho v + i[(\rho - v)^2 + \rho^2 v^2(t^2 - 1)])},
\]
\[
Q_{12} = [2/(\rho - v)] [4v^2 p^2 + 4v^2 p^2 + (v^2 - \rho/v) p^2 + 4v(\rho^3 - \rho/v) p^2
\]
\[
+ (4v^2 - 4p^2 v^2 + \rho - v) p_a p_i],
\]
\[
H = [4/(\rho - v)] [\rho^3 p^2 + (4v - 1/v) p^2 + 4(\rho^3 - 1/v) p^2 + 4(\rho^3 - 1/v^2) p_a p_i].
\]

To summarize: The nonsubgroup type coordinates on \(H_3\), namely elliptic I and II, complex elliptic, elliptic parabolic, hyperbolic parabolic, and semicircular parabolic each occur precisely once. The subgroup type coordinates on \(H_2\) occur as follows: spherical coordinates once (since the compact subalgebra \(u(2)\) contains only one \(\text{MASA}\)), hyperbolic coordinates three times (\(u(1,1)\) contains three \(\text{MASA}\)’s) and horospheric coordinates twice (\(A_{4,10}\) contains two \(\text{MASA}\)’s (see Fig. 1)).

V. Conclusion

The results of this article should be viewed in the context of three different but related research programs. One is a systematic study of the group theoretical, algebraic, and geometrical aspects of the separation of variables in linear and nonlinear partial differential equations. From this point of view we should stress that the hermitian hyperbolic space \(\text{HH}(2)\) is a noncompact manifold of nonconstant curvature (it does, however, have constant holomorphic sectional curvature). The fact that it has a large isometry group, namely \(SU(2,1)\), made it possible to apply essentially the same techniques as for spaces with constant curvature. We have shown that all 12 separable coordinate systems on \(\text{HH}(2)\) have their origin in the properties of the algebra \(su(2,1)\), its subalgebras, and its enveloping algebra.

The second context is that of the classification of subgroups of Lie groups, in particular, maximal Abelian sub-
groups of classical Lie groups, and its application to the study of differential equations. Indeed, the classification of all MASA’s of su(2, 1) into four conjugacy classes was the basis of our calculations providing the explicit forms of the 12 separable coordinate systems. In passing, we comment that other applications of this classification are being pursued. In addition to the separation of variables, these include the derivation of superposition principles for certain systems of nonlinear differential equations and the symmetry reduction of certain nonlinear partial differential equations to ordinary ones.

Finally, the reduction of the problem of separating variables for the free Hamiltonian on HH(2) to that of a Hamiltonian with a nontrivial interaction, defined on a lower-dimensional manifold, namely the O(2, 1) hyperboloid H₂, is an example of a more general method of introducing interactions, in particular completely integrable Hamiltonian systems, by symmetry reduction on group manifolds or homogeneous spaces.

All three above aspects are being actively pursued. In particular, we are currently generalizing the results of this paper to the space HH(n) making use of the MASA’s of SU(n,1). The completely integrable Hamiltonian systems obtained in this article are being investigated (explicit solutions, properties of trajectories, special functions occurring as solutions of the Laplace–Beltrami equations, etc.). The related problem of separating variables in Hamiltonians on HH(2) with specific potentials that reduce by symmetry to more general completely integrable relativistic Hamiltonian systems than the ones treated in this article is also under consideration.

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APPENDIX: THE su(2,1) INFINITESIMAL OPERATORS IN TERMS OF IGNORABLE VARIABLES AND O(2,1) VARIABLES

1. Compact Cartan subalgebra variables

\[ X_1 = \cos(\alpha - \alpha_2)I_{12} + \sin(\alpha - \alpha_2)[(s_2/s_1)p_{a_s} + (s_1/s_2)p_{a_r}], \]

\[ X_2 = -\sin(\alpha - \alpha_2)I_{12} + \cos(\alpha - \alpha_2)[(s_2/s_1)p_{a_s} + (s_1/s_2)p_{a_r}], \]

\[ X_3 = p_{a_r} - p_{a_s}, \]

\[ X_4 = \cos(\alpha_1 - \alpha_2)I_{12} - \sin(\alpha_1 - \alpha_2)[(s_2/s_1)p_{a_s} + (s_1/s_2)p_{a_r}], \]

\[ X_5 = \sin(\alpha_1 - \alpha_2)I_{12} + \cos(\alpha_1 - \alpha_2)[(s_2/s_1)p_{a_s} + (s_1/s_2)p_{a_r}], \]

\[ X_6 = \cos(\alpha_1 - \alpha_2)I_{12} - \sin(\alpha_1 - \alpha_2)[(s_2/s_1)p_{a_s} + (s_1/s_2)p_{a_r}], \]

\[ X_7 = \sin(\alpha_1 - \alpha_2)I_{12} + \cos(\alpha_1 - \alpha_2)[(s_2/s_1)p_{a_s} + (s_1/s_2)p_{a_r}], \]

\[ X_8 = -\sqrt{3}(p_{a_r} + p_{a_s}). \]

2. Noncompact Cartan subalgebra variables

\[ X_1 = \cosh \varphi_1 \cosh (s_1/s_2)p_{a_s} + \sinh \varphi_2 \sinh (s_1/s_2)p_{a_r}, \]

\[ X_2 = \cosh \varphi_1 \cosh (s_1/s_2)p_{a_s} + \sinh \varphi_2 \sinh (s_1/s_2)p_{a_r}, \]

\[ X_3 = \cosh \varphi_1 \cosh (s_1/s_2)p_{a_s} + \sinh \varphi_2 \sinh (s_1/s_2)p_{a_r}, \]

\[ X_4 = \cosh \varphi_1 \cosh (s_1/s_2)p_{a_s} + \sinh \varphi_2 \sinh (s_1/s_2)p_{a_r}, \]

\[ X_5 = \cosh \varphi_1 \cosh (s_1/s_2)p_{a_s} + \sinh \varphi_2 \sinh (s_1/s_2)p_{a_r}, \]

\[ X_6 = \cosh \varphi_1 \cosh (s_1/s_2)p_{a_s} + \sinh \varphi_2 \sinh (s_1/s_2)p_{a_r}, \]

\[ X_7 = \cosh \varphi_1 \cosh (s_1/s_2)p_{a_s} + \sinh \varphi_2 \sinh (s_1/s_2)p_{a_r}, \]

\[ X_8 = -\sqrt{3}(p_{a_r} + p_{a_s}). \]

3. Variables corresponding to orthogonally decomposable non-Cartan subalgebra

\[ Y_1 = -\varphi_1, \]

\[ Y_2 = -\cos(\alpha_1 - \alpha_2)I_{12} + \sin(\alpha_1 - \alpha_2)[(s_1/s_2)p_{a_s} + (s_2/s_1)p_{a_r}], \]

\[ Y_3 = -\sin(\alpha_1 - \alpha_2)I_{12} + \cos(\alpha_1 - \alpha_2)[(s_1/s_2)p_{a_s} + (s_2/s_1)p_{a_r}], \]

\[ Y_4 = -p_{a_s}, \]

\[ Y_5 = \sqrt{3}(s_1/s_2)p_{a_s} + (s_2/s_1)p_{a_r}, \]

\[ Y_6 = -2\varphi_1, \]

\[ Y_7 = -\varphi_2, \]

\[ Y_8 = -\sqrt{3}(p_{a_r} + p_{a_s}). \]

4. Variables corresponding to the maximal abelian nilpotent subalgebra

\[ Y_1 = 3[t_1(t_2 - t_0) + s_1/(s_0 - s_1)]p_{a_s} + (t_1^2 - s_2^2)/(s_0 - s_1)^2 p_{a_r}, \]
\( Y_2 = I_{12} - I_{02} + 2tp_u, \quad Y_3 = p_t, \)
\( Y_4 = -p_u, \quad Y_5 = I_{01} + tp_t + 2tp_u, \)
\( Y_6 = 2uI_{01} + t^3(I_{02} - I_{12}) + t(I_{02} + I_{12}) \)
\[- \frac{3s_2t^2}{(s_0 - s_1)^3} \frac{u}{(s_0 - s_1)} p_t \]
\[+ \left( t^4 + \frac{3t^2s_2^2}{(s_0 - s_1)^2} - 2u^2 \right) \frac{s_2(s_0 + s_1)}{(s_0 - s_1)^3} p_u, \]
\( Y_7 = I_{12} + 12t^2(I_{02} - I_{12}) \)
\[- t \left( t^2 - 1 - \frac{3s_2^2}{(s_0 - s_1)^3} \right) p_u, \]
\( Y_8 = tI_{01} + u(I_{12} - I_{02}) + \left( t^2 + \frac{s_1^2 - s_2^2}{2} \frac{s_0}{(s_0 - s_1)^3} \right) p_t \)
\[+ \left( 2tu + \frac{s_1(s_0^2 - s_2^2 + s_3^2)}{(s_0 - s_1)^3} \right) p_u. \]


