The general theory of $R$-separation for Helmholtz equations

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We develop the theory of $R$-separation for the Helmholtz equation on a pseudo-Riemannian manifold (including the possibility of null coordinates) and show that it, and not ordinary variable separation, is the natural analogy of additive separation for the Hamilton–Jacobi equation. We provide a coordinate-free characterization of variable separation in terms of commuting symmetry operators.

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1. INTRODUCTION

Let $V_\alpha$ be a (local) pseudo-Riemannian manifold. The Helmholtz equation for $V_\alpha$ is expressed in local coordinates \{y\} by

$$\Delta \psi(y) = E \psi(y), \quad \text{(1.1)}$$

where $E$ is a nonzero constant and $\Delta$ is the Hamiltonian or Laplace–Beltrami operator\textsuperscript{1}

$$\Delta = \frac{1}{g^{1/2}} \sum_{i,j=1}^n \partial_i (g^{1/2} g^{ij} \partial_j). \quad \text{(1.2)}$$

Here, $\partial_i = \partial_{x_i}$, the metric on $V_\alpha$ is $ds^2 = g_{ij} dy^i dy^j$, $g = \text{det}(g_{ij}) \neq 0$, and $\sum_k g^{ik} g_{kj} = \delta^{ij}$. The Helmholtz equation is closely associated with the Hamilton–Jacobi equation\textsuperscript{2}

$$H[\partial_i W] = \sum_{i,j=1}^n g^{ij} \partial_i W \partial_j W = E, \quad \text{(1.3)}$$

where $H$ is the Hamiltonian function

$$H(p_i) = \sum_{i,j=1}^n g^{ij} p_i p_j. \quad \text{(1.4)}$$

Both $\Delta$ and $H$ are defined independent of local coordinates.

In Ref. 3 the authors presented a theory of orthogonal $R$-separation for (1.1). [By $R$-separation we mean separation up to a fixed factor:

$$\psi(y) = R(y) \prod_{j=1}^n \psi_j(y_j). \quad \text{(1.5)}$$

Ordinary separation corresponds to $R \equiv 1$ and trivial $R$-separation to $\partial_j$ in $R = 0$ for $i \neq j$.] We found necessary and sufficient conditions that an additively separable orthogonal coordinate system for the Hamilton–Jacobi equation will also $R$-separate the Helmholtz equation. [An $R$-separable system for (1.1) always separates (1.3).] Further, we found a coordinate-free characterization of orthogonal $R$-separable coordinate systems in terms of families of commuting symmetry operators for $\Delta$.

In this paper we extend the ideas of Ref. 3 to provide a general theory of $R$-separation for the Helmholtz equation, encompassing both orthogonal and nonorthogonal coordinate systems. A major new complication is the possibility of type 2 (null) coordinates. Our principal result is Theorem 3, which provides an intrinsic characterization of an $R$-separable coordinate system in terms of a family of commuting symmetry operators. (In particular, given the operators, expressed in an arbitrary coordinate system, one can compute the $R$-separable coordinates.)

Although $R$-separation has long been a useful tool in the study of the Laplace equation [$E = 0$ in (1.1)], its relevance to the Helmholtz equation was, until recently, virtually ignored. Our results show clearly that $R$-separation, rather than ordinary separation, for the Helmholtz equation is the proper analog to additive separation of the Hamilton–Jacobi equation. In fact, the problem of extending a separable system for (1.3) to an $R$-separable system for (1.1) reduces to an exercise in quantization theory.

In Sec. 2 we give a precise operational definition of $R$-separation for the Helmholtz equation. (We expect, though we have not tried to verify, that any coordinate system which $R$-separates in accordance with some more intuitive definition of separability can be shown to be equivalent to one of our canonical systems.) In Theorem 1 we derive necessary and sufficient conditions that a Hamilton–Jacobi separable system be $R$-separable for the Helmholtz equation, and we look at the special case of ordinary separation ($R = 1$), obtaining a new generalization of the Robertson condition for orthogonal separability. In Sec. 3 we develop the symmetry operator approach to $R$-separation and review the corresponding Hamilton–Jacobi theory. Section 4 contains our main result, Theorem 3, which gives the intrinsic symmetry operator characterization of $R$-separation. Finally, in Sec. 5 we provide some examples of $R$-separation and briefly discuss the significance of our results.

The theory presented here is local rather than global. All functions are assumed to be locally analytic.

2. TECHNICAL CONSIDERATIONS

Let \{x\} be a local coordinate system on the pseudo-Riemannian manifold. We present here an operational definition of $R$-separation for the Helmholtz equation

$$\Delta \psi = \frac{1}{g^{1/2}} \partial_i (g^{1/2} g^{ij} \partial_j) \psi = E \psi \quad \text{(2.1)}$$

in the coordinates \{x\} and derive necessary and sufficient conditions for the existence of this phenomenon. Let $\{S_\beta(x)\}$

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be a Stäckel matrix, i.e., an \( N \times N \) nonsingular matrix whose \( \alpha \)th row depends only on the variable \( x^\alpha \) and set \( S = \det(S_\alpha) \). We divide the coordinates \( x^1 \) into three disjoint classes: essential of type 1, essential of type 2, and ignorable. We further order the indices so that \( n_1 \) coordinates \( x^\alpha, 1 < \alpha < n_1 \), are essential of type 1, the \( n_2 \) coordinates \( x^\alpha, n_1 + 1 < \alpha < n_1 + n_2 \), are essential of type 2, and the \( n_3 \) coordinates \( x^\alpha, n_1 + n_2 + 1 < \alpha < n_1 + n_2 + n_3 = n \), are ignorable. (In the following, unless otherwise stated, indices \( a, b, c \) range from 1 to \( n_1 \), indices \( r, s, t \) range from \( n_1 + 1 \) to \( n_1 + n_2 \), indices \( \alpha, \beta, \gamma \) range from \( n_1 + n_2 + 1 \) to \( n \), and indices \( i, j, k \) range from 1 to \( n_1 \).) The ignorable coordinates are defined to be all \( x^\beta \) such that \( \partial_x g^{\beta \alpha}(x) = 0 \) for all \( j, k, \beta \). Finally, set \( N = n_1 + n_2 \), let \( \lambda_1 \ldots \lambda_N \) be complex parameters, and define differential operators \( K_a K_b \) by

\[
K_a = \partial_{x^a} + l_a(x^\alpha)\partial_{x^\alpha} + m_a(x^\alpha) + \sum_{\alpha \neq \beta} A_{a \beta}(x^\alpha)\partial_{x^\beta} + \sum_{\alpha = 1}^{n_1} \lambda_a S_\alpha(x^\alpha),
\]

for \( a = 1, \ldots, n_1 \) and

\[
K_r = 2\sum_{\alpha = 0} B_{a \beta}(x^\alpha)\partial_{x^\beta} + m_r(x^\alpha) + \sum_{\alpha \neq \beta} A_{a \beta}(x^\alpha)\partial_{x^\beta} + \sum_{\alpha = 1}^{n_1} \lambda_a S_\alpha(x^\alpha)
\]

for \( r = n_1 + 1, \ldots, N \).

We say that the coordinates \( \{x^\alpha\} \) are \( R \)-separable for the Helmholtz equation (2.1) provided there exist functions \( g_a(x) \) and \( R(x^\alpha, x^\beta) \) \( (R \neq 0) \) such that

\[
R^{-1} \Delta R - E = \sum_{\alpha = 1}^{N} g_a(x)K_a.
\]

(2.4)

Here

\[
R^{-1} \Delta R = \Delta + g^{\alpha \beta}(\partial_{x^\alpha} \ln R \partial_{x^\beta}) + R^{-1}(\Delta R)
\]

as an operator, where

\[
\Delta = g^{\alpha \beta}\partial_{x^\alpha} \partial_{x^\beta} + \frac{1}{g^{1/2}} \partial_{x^\alpha} g^{1/2} \partial_{x^\alpha}.
\]

(2.6)

If the coordinates are \( R \)-separable then the function

\[
\psi(x) = R(x^\alpha, x^\beta) \prod_{\alpha} \psi^{\alpha}(x^\alpha) \prod_{\beta} \psi^{\beta}(x^\beta) \exp \left( \sum_{\alpha} \lambda_\alpha x^\alpha \right)
\]

is a solution of \( \Delta \psi = E \psi \) whenever the \( \psi^{\beta} \) satisfy separation equations

\[
K_a \left[ \psi^{\alpha} \exp(\lambda_\alpha x^\alpha) \right] = 0, \quad a = 1, \ldots, n_1,
\]

\[
K_r \left[ \psi^{\alpha} \exp(\lambda_\alpha x^\alpha) \right] = 0, \quad r = n_1 + 1, \ldots, N.
\]

(2.8)

Here the \( \lambda_\alpha \) are arbitrary complex constants and \( \lambda_1, \ldots, \lambda_n \) are the separation parameters. Note that the function exp\( (\lambda_\alpha x^\alpha) \) can be factored out of expressions (2.8), thus reducing these expressions to ordinary differential equations. The type 1 coordinates \( x^\alpha \) have the property that the corresponding separation equations are second order ODE's, whereas for type 2 coordinates \( x^\beta \) the separation equations are first order ODE's. The solutions \( \psi(x, \lambda) \) (2.7), depend on the separation parameters \( \lambda \), but \( R(x^\alpha, x^\beta) \) is independent of these parameters.

It follows from (2.2)–(2.4) that a necessary condition for \( R \)-separation is

\[
g_a(x) = S_a^{\alpha \beta} / S, \quad k = 1, \ldots, N
\]

(2.9)

where \( S_a^{\alpha \beta} \) is the \((k,1)\) minor of \( (S_\alpha) \).

Thus the metric must take the form

\[
g^{\alpha \beta} = a^{\alpha \beta} S_a^{\alpha \beta} / S, \quad g^{\alpha \beta} = g^{\alpha \alpha} = 0, \quad g^{\alpha \beta} = 0,
\]

(2.10)

Note that

\[
\begin{pmatrix}
\delta_{\alpha \beta} g^{\alpha \alpha} & 0 & 0 \\
0 & 0 & g^{\alpha \beta} \\
0 & g^{\alpha \beta} & 0
\end{pmatrix}
\]

(2.11)

Conditions (2.10) are necessary but not sufficient for \( R \)-separation. Before determining the remaining conditions, however, it is worthwhile to point out the significance of these restrictions on the metric. Consider the Hamilton–Jacobi equation associated with the Helmholtz equation (2.1):

\[
g^{\alpha \beta} \partial_{x^\alpha} \partial_{x^\beta} W = E.
\]

(2.12)

It has recently been established that conditions (2.10) are necessary and sufficient for (additive) separation of the Hamilton–Jacobi equation in the coordinates \( \{x^\alpha\} \).

\[
W(x) = \sum a W^{(a)}(x^\alpha, \lambda) + \sum \psi^{(a)}(x^\alpha, \lambda) + \sum \lambda_\alpha x^\alpha
\]

(2.13)

Indeed, Benenti has shown that every system which separates (2.12), according to the intuitive definition of Levi-Civita, is equivalent to a system in the canonical form (2.10).

Proposition 1: A coordinate system that is \( R \)-separable for the Helmholtz equation is also separable for the Hamilton–Jacobi equation. Let

\[
H_i = \frac{S_i}{S}, \quad i = 1, \ldots, N.
\]

(2.14)

If conditions (2.10) hold then \( S^{(i)} \neq 0 \) since \( g \neq 0 \). We can associate with our coordinate system \( \{x^\alpha\} \) on \( V_a \), an orthogonal coordinate system \( \{y^i, x^\alpha\} \) on a space \( V_\alpha \) with metric

\[
dS^2 = \sum_{i=1}^{N} H_i \, (dx_i)^2.
\]

(2.15)

By (2.14), this metric is in Stäckel form. Recall that necessary and sufficient conditions that \( dS^2 \) be expressible in the form (2.14) for some Stäckel matrix are (Ref. 1, Appendix 13)

\[
\partial_{x^\alpha} \ln H_i - \partial_{y^i} \ln H_i - \partial_{x^k} \ln H_i - \partial_{y^j} \ln H_i = 0,
\]

(2.16)

We further recall some useful results from Ref. 6. Given a metric \( dS^2 = \sum H_i \, (dx_i)^2 \) in Stäckel form, we say that the
function \( Q(x) \) is a Stäckel multiplier for \((ds^2)\) if the metric \( ds^2 = Qdx^2 \) is also in Stäckel form with respect to the coordinates \( \{x^j\} \). It can be shown that \( Q \) is a Stäckel multiplier if and only if there exist functions \( \psi_j = \psi_j(x^i) \) such that
\[
Q(x) = \sum_{j=1}^{N} \psi_j(x^i) H_j^{-1}.
\]
(2.17)

Equivalent necessary and sufficient conditions are
\[
\partial_a Q = \partial_j Q \partial_i \ln H_j^{-1} - \partial_j Q \partial_i \ln H_j^{-2} = 0, \quad j \neq k.
\]
(2.18)

We can now reformulate conditions (2.10).

**Proposition 2:** A necessary requirement for R-separation of (2.1) in the coordinates \( \{x^{ij} = 1, ..., n\} \) is that
\[
g_{a\beta} = H^{-2}, \quad g_{a\beta} = B_{a\beta}(x^i) H^{-2},
\]
(2.19)

and that each \( g_{a\beta} \) be a Stäckel multiplier for the Stäckel form metric \( ds^2 = \Sigma_{k=1}^{N} H_k^{-2} (dx^k)^2 \). All other matrix elements \( g^{ij} \) must vanish.

To obtain sufficient conditions for R-separation we must also demand equality of the coefficients of \( \partial_j \) and the zeroth order terms on each side of (2.5):
\[
f_a + 2 \partial_a \ln R = l_a(x^i),
\]
(2.20)
\[
\sum_j g_{a\beta}(f_{a\beta} + 2 \partial_a \ln R) = \sum_k H_k^{-2} n_k^a(x^i),
\]
(2.21)
\[
R^{-1}(\Delta R) = \sum_k H_k^{-2} m_k(x^i).
\]
(2.22)

Here,
\[
f_a = \partial_a f, \quad f = \ln g^{1/2}/S,
\]
(2.23)
\[
f_{a\beta} = \partial_{a\beta} \ln g^{1/2}/S = f_a + \partial_a \ln B_{a\beta}(x^i).
\]
Solving for \( R \) from (2.19) we find
\[
R = \left( \frac{S}{g} \right)^{1/2} \exp\left[ \sum_a A_a(x^i) + Q(x^i) \right],
\]
(2.24)
and substituting (2.23) into (2.20) and (2.21) we ultimately obtain the following result.

**Theorem 1:** Necessary and sufficient conditions that the coordinates \( \{x^i\} \) be R-separable for the Helmholtz equation
\[
\frac{1}{g^{1/2}} \partial_{a}(g^{1/2} g^{ij} \partial_j \psi) = E \psi
\]
are

1. The requirements of Proposition 2 are satisfied, i.e., the coordinates \( \{x^i\} \) are separable for the Hamilton–Jacobi equation \( g^{ij} \partial_a \partial_j W = E \).
2. \( \sum_a g^{ij} \partial_a Q \) is a Stäckel multiplier for each \( a \).
3. \( \sum_k H_k^{-2} n_k^a(x^i) \) is a Stäckel multiplier, where \( f_a = \partial_a \ln g^{1/2}/S \) and \( S \) is the determinant of the Stäckel matrix.

If these conditions are satisfied then
\[
R(x) = \left( \frac{S}{g^{1/2}} \right)^{1/2} \left[ \sum_a A_a(x^i) + Q(x^i) \right]
\]
where the \( A_a = A_a(x^i) \) are arbitrary.

We say that the coordinates \( \{x^i\} \) are separable for the Helmholtz equation provided they are R-separable with \( R = 1 \). Furthermore, R-separable coordinates are trivially R-separable if \( R = \Pi_{j=1}^{n} R_j(x^i) \) and (since coordinates are trivially R-separable if and only if they are separable) we regard trivial R-separation as equivalent to ordinary separation.

Especially interesting is the case of ordinary separation. Then \( R = 1 \) and expression (2.23) becomes
\[
\ln \left( \frac{S}{g^{1/2}} \right) = \sum_a A_a(x^i) + Q(x^i).
\]
(2.25)

**Corollary 1** (Generalized Robertson Condition): Necessary and sufficient conditions that the coordinates \( \{x^i\} \) be separable for the Hamilton–Jacobi equation are

1. The coordinates are separable for the Hamilton–Jacobi equation,
2. \( f_{a_j} = 0 \) for \( j = 1, ..., N, j \neq a \),
3. \( \sum_a g^{ij} f_a \) is a Stäckel multiplier for each \( a \).

where \( f = \ln g^{1/2}/S \) and \( f_a = \partial_a f \).

The original Robertson condition was concerned with the case of orthogonal separation. By permitting a type 1 coordinate to be ignorable if necessary, we can identify this case with \( n_1 = n, n_2 = n_3 = 0 \). Robertson showed that an orthogonal separable system for the Hamilton–Jacobi equation separated the Helmholtz equation if and only if \( f_{a_j} = 0 \) for \( a \neq b \). Since \( n_2 = 0 \) this agrees with Corollary 1.

Eisenhart showed that the Robertson condition is equivalent to the requirement
\[
R_{ab} = 0, \quad a \neq b
\]
(2.26)
where \( R_{ab} \) is the Ricci tensor expressed in terms of the orthogonal coordinates \( \{x^i\} \). (For an explicit definition of the Ricci tensor \( R_{a\beta} \) in terms of the metric \( g^{ij} \) together with related computational formulas we refer the reader to Chap. 1 of Eisenhart's text.) Benenti studied nonorthogonal separation for the Helmholtz equation in which no nonignorable null coordinates were allowed (\( n_2 = 0 \) in our formalism). His requirement for Helmholtz separation agrees with our condition (2). Benenti further showed that his requirement was equivalent to (2.20) again and that \( R_{ab} = 0 \) automatically for Hamilton–Jacobi separable systems. By a tedious but straightforward computation we have established

**Lemma 1:** Condition (2) of Corollary 1, namely
\[
f_{a_j} = 0 \quad \text{for} \quad j = 1, ..., N, \quad j \neq a
\]
is equivalent to
\[
R_{ab} = 0, \quad a \neq b, \quad R_{ab} = 0,
\]
(2.27)
where \( R_{a\beta} \) is the Ricci tensor for \( V_a \) expressed in the coordinates \( \{x^i\} \). Furthermore, \( R_{a\beta} = 0 \) automatically if \( \{x^i\} \) separates the Hamilton–Jacobi equation.

It is perhaps somewhat surprising that requirements (2.25) continue to hold even with the presence of type 2 coordinates. Condition (3) of Corollary 1 appears not to be expressible in terms of the Riemann curvature tensor and its covariant derivatives. However, this condition is vacuous for \( n_2 < 1 \). Since \( g^{ij} = 0 \), type 2 coordinates are null and any such coordinates are orthogonal. Thus, for separation on a proper Riemannian space \( V_a \) we must have \( n_2 = 0 \) and for a pseudo-Riemannian \( V_a \) with signature \((-1,1^{n_3-1})\) we must have \( n_2 < 1 \).

**Corollary 2:** In order that Hamilton–Jacobi separable coordinates \( \{x^i\} \) separate the Helmholtz equation on a seu-
do-Riemannian manifold with signature \((1^r)(-1^{n-r})\) it is necessary and sufficient that
\[
R_{ab} = 0, \quad a \neq b, \quad R_{aa} = 0.
\]

3. CONSTANS OF THE MOTION

Let us suppose that the coordinates \(\{x^i\}\) \(R\)-separate the Helmholtz equation. Then expanding the corresponding Stäckel matrix in (2.2), (2.3) by the \(l\)th, rather than just the 1st, column we obtain operators \(A_l, l = 1, \ldots, N,\) such that
\[
A_l \psi = -\lambda_l \psi \text{ for an } R\text{-separated solution } \psi:
\]
\[
A_l = \sum_{\alpha} S_{\alpha l} \left( \partial_{\alpha} + \sum_{\alpha} A_{\alpha}^{\beta} \partial_{\alpha} + \sum_{\alpha} n_{\alpha} \partial_{\alpha} + m_{\alpha} + \lambda_{\alpha} \partial_{\alpha} \right)
\]
\[
+ \sum_{\alpha} \sum_{\beta} S_{\alpha l}^{\beta} \left( 2 A_{\alpha}^{\beta} \partial_{\alpha} + \sum_{\alpha} A_{\alpha}^{\beta} \partial_{\alpha} \right)
\]
\[
+ \sum_{\alpha} n_{\alpha}^2 - 2 B_{\alpha} \partial_{\alpha} \ln R \partial_{\alpha} + m_{l}.
\]
(3.1)

(Note that \(A_l = \Lambda_l\).) These expressions are not as complicated as they appear. It can be directly verified (and we will show this later) that
\[
\begin{align*}
[A_l, A_k] &= 0, \\
[A_l, L_{\alpha}] &= 0, \\
A_l A_{\alpha} &= A_{\alpha} A_l,
\end{align*}
\]
(3.2)

where
\[
L_{\alpha} = \partial_{\alpha}, \quad \alpha = N + 1, \ldots, n
\]
and \([A_l, \partial]\) = \(\partial - R A_l\). Thus the operators \(A_l, 2 < k < N,\) \(L_{\alpha}\) form a commuting family of symmetry operators for \(\Lambda_l\), i.e., they commute with \(\Lambda_l\) and with each other. Furthermore, the \(R\)-separated solutions of (2.2) are simultaneous eigenfunctions of the symmetry operators:
\[
A_l \psi = -\lambda_l \psi, \quad L_{\alpha} \psi = \lambda_{\alpha} \psi.
\]
(3.4)

Our construction has started with an \(R\)-separable coordinate system \(\{x^i\}\) and produced a commuting family of symmetry operators \(\{A_l, L_{\alpha}\}\). It is our principal task in this paper to characterize those families of commuting symmetry operators that correspond to \(R\)-separation.

In Ref. 1 we studied the corresponding problem for the Hamilton–Jacobi equation (2.12). In that case we utilized the natural symplectic structure on the cotangent bundle \(\tilde{V}_n\) of \(V_n\). Corresponding to local coordinates \(\{x^i\}\) on \(V_n\), we have coordinates \(\{x^i, p_i\}\) on the 2n-dimensional space \(\tilde{V}_n\). The Poisson bracket of two functions \(F(x^i, p_i), G(x^i, p_i)\) on \(\tilde{V}_n\) is defined by
\[
\{F, G\} = \sum_{i=1}^n \left( \partial_{p_i} F \partial_{x^i} G - \partial_{x^i} F \partial_{p_i} G \right).
\]
(3.5)

Let \(\{x^i\}\) be a separable coordinate system for the Hamilton–Jacobi equation (2.12) with coordinates of type 1, \(x^s,\) of type 2, \(x^t,\) and ignorable, \(x^u,\) as usual. Then the metric \(g^i\) in these coordinates takes the standard form (2.10).

It is convenient at this point to introduce the functions \(\rho_{jk}^i = \rho_{jk}^i(x^s, x^u),\) where
\[
S_{jk}^i = \frac{\rho_{jk}^i H_j}{S} = \frac{H_{jk}}{S}, \quad 1 < j, k < N,
\]
and \(S_j\) is the Stäckel matrix corresponding to the separable \(\{x^i\}\). Then \(\rho_{jk}^i = 1\) and it can be shown that (Ref. 1, Appendix 13)
\[
\partial_j \rho_{jk}^i = (\rho_{kj}^i - \rho_{jk}^i) \partial_j \ln H_j - 2, \quad 1 < i, j, k, < N.
\]
(3.7)

Let \(H = \Sigma_n \rho_{jk}^i \rho_{ij}^k\) be the Hamiltonian corresponding to (2.12). In Ref. 6 we constructed quadratic forms \(A_i (A_i = H)\), given by
\[
A_i = \sum_{j} \rho_{jk}^i H_{jk} = \left( p_j^2 + \sum_{\alpha} A_{\alpha}^{\beta} p_{\alpha} p_{\beta} \right)
\]
\[
+ \sum_{j} \rho_{jk}^i H_{jk} = \left( \sum_{\alpha} B_{\alpha} p_{\alpha} + \sum_{\alpha} A_{\alpha}^{\beta} p_{\alpha} p_{\beta} \right)
\]
for \(l = 1, \ldots, N\) and \(n_{\alpha}\) linear forms \(L_{\alpha}\).

These polynomials in the \(p_i\)'s were shown to satisfy
\[
\begin{align*}
\{A_i, A_k\} &= 0, \\
\{L_{\alpha}, L_{\beta}\} &= 0, \\
\{A_i, L_{\alpha}\} &= 0, \quad \alpha = N + 1, \ldots, n
\end{align*}
\]
(3.10)

and when evaluated for \(p_{ij} = \partial_i W, p_{\alpha} = \partial_{\alpha} W\) with \(W\) a separable solution of (2.12), they satisfy
\[
A_i = -\lambda_i, \quad L_{\alpha} = \lambda_{\alpha},
\]
(3.11)

where \(\lambda_i = -\lambda_1, \ldots, \lambda_n\) are the separation parameters.

Let \(\rho^{i}(y)\) be a symmetric contravariant 2-tensor on \(V_n\), expressed in terms of local coordinates \(\{y^i\}\), and let \(g^{ij}(y)\) be the contravariant metric tensor. A root \(\rho(y)\) of \(\rho^{i}\) is a solution of the characteristic equation
\[
\det(\rho^{i} - \rho^{ij}(y)) = 0
\]
(3.12)

and an eigenform \(\omega = \Sigma_{\lambda} \lambda_{\lambda} dy^\lambda\) corresponding to \(\rho\) is a non-zero 1-form such that
\[
\sum_{j=1}^n (\rho^{i} - \rho^{ij}(y)) \lambda_{\lambda} = 0, \quad i = 1, \ldots, n.
\]
(3.13)

Roots and eigenforms are defined independent of local coordinates.

Note from (3.8) that for a separable \(\{x^i\}\) the \(\rho^{i}(y)\) are simple roots of the \(A_i\) with simultaneous eigenforms \(dx^i\), and the \(\rho^{i}\) are roots of multiplicity 2 but with a single eigenform \(dx^i\). Here \(dx^i, dx^j\) are also eigenforms for the products \(L_{\alpha} L_{\beta}\).

Let \(\{y^i\}\) be a local coordinate system on a pseudo-Riemannian manifold and let \(\omega_{1,h} = \lambda_{1,h} dy^h, 1 < h < n,\) be a local basis of 1-forms on \(V_n\). The dual basis of vector fields is \(\bar{X}^{(1)} = \lambda_{1,h} \partial_h, 1 < h < n,\) where \(\lambda^{(1,h)} \lambda_{1,h} = \delta_{1,h}\). The inner product of two 1-forms \(\omega_{1,h} \omega_{1,k} = G(j, k) = \lambda_{1,h} \lambda_{1,k}\).

In Ref. 6 we proved

Theorem 2: Let \(\theta\) be a vector subspace of quadratic forms on \(V_n\), such that \(H \theta \theta\) and
\[
\begin{align*}
(1) &\quad |A, B| = 0 \text{ for each } A, B \in \theta, \\
(2) &\quad \text{there is a basis of 1-forms } \omega_{1,h} = \lambda_{1,h} dy^h, 1 < h < n, \text{ such that} \\
&\quad \text{(i) the } n_1 \text{ forms } \omega_{1,h} \text{ are simultaneous eigenforms for each } A \in \theta \text{ with root } \rho^{i}_{s}, \\
&\quad \text{(ii) the } n_2 \text{ forms } \omega_{n_1} \text{ are simultaneous eigenforms for each } A \in \theta \text{ with root } \rho^{i}_{s}, \text{ the root } \rho^{i}_{s} \text{ has} \end{align*}
\]
(3) \(|L_\alpha, L_\beta| = 0\) and \(L_\alpha, L_\beta \notin \theta\), where \(L_\alpha = \Lambda^{\alpha\alpha}p_\alpha\), 
\(\alpha \beta = n_1 + n_2 + 1, \ldots, n\).
(4) \([A, L_\alpha] = 0\) for each \(A \in \theta\),
(5) \(\sum_{i} (\frac{\partial}{\partial L_\alpha}) (\frac{\partial}{\partial L_\beta}) = \rho - \sum_{i} (\frac{\partial}{\partial L_\alpha}) (\frac{\partial}{\partial L_\beta})\),
(6) \(\dim \theta = \frac{1}{2} (2n + n_2 - n_2), \) where \(n_2 = n - n_1 - n_2\),
(7) \(G(a, b) = 0\) if \(a \neq b\), and \(G(a, r) = G(a, \alpha) = G(r, s) = 0\).

Then there exist local coordinates \([x^i]\) for \(V_{\alpha}\) and functions \(f(x)\) such that \(\omega_{\alpha\beta} = f^{ij} dx^j\) (with a suitable modification of the \(\omega_{\alpha\beta}\) and the Hamilton–Jacobi equation is separable in these coordinates. Conversely, to every separable coordinate system \([x^i]\) for the Hamilton–Jacobi equation there corresponds a subspace \(\theta\) of quadratic forms on \(V_{\alpha}\) with properties (1)–(7).

In the following section we will show that, with suitable modifications, this result also characterizes R-separable systems for the Helmholtz equation.

4. THE BASIC RESULT

Let \(a\) be the Hamiltonian operator (1.2), expressed in terms of local coordinates \([x^i]\). Suppose \(\mathcal{A}\) is a second order symmetry operator for \(a\), i.e., a differential operator such that \([\mathcal{A}, a] = 0\) and which in local coordinates can be written
\[
\mathcal{A} = a^\alpha(y) \frac{\partial}{\partial y} + b^\alpha(y) \frac{\partial}{\partial \beta} + c(y), \quad \partial_i = \partial_{x_i} \tag{4.1}
\]
where \(a^\alpha = a^\beta\) and not all \(a^\alpha\) vanish. As shown in Ref. 3 we can decompose \(\mathcal{A}\) uniquely in the form
\[
\mathcal{A} = \mathcal{A}' + \mathcal{L}', \tag{4.2}
\]
where
\[
\mathcal{A}' = \frac{1}{2 (n+1)} \partial_i (g^{lj} a_{ij} \partial_j) + c,
\]
\[
\mathcal{L}' = b^\alpha \partial_i,
\]
\[
[A, a] = [\mathcal{A}', a] = 0. \tag{4.3}
\]
Furthermore, this decomposition is coordinate independent. Decomposing the operators \(\mathcal{A}\), (3.1), in this form we find
\[
\mathcal{A}' = \mathcal{A}' + \mathcal{L}', \tag{4.4}
\]
\[
\mathcal{A}' = \frac{1}{2 (n+1)} \partial_i (g^{lj} a_{ij} \partial_j)
+ \sum_{j} \rho_{ij}^j H_j - m_j
+ \frac{1}{2} a_{ij} [f_{ij} - l_{ij}]
+ \sum_{j} \rho_{ij}^j H_j - m_j, \tag{4.5}
\]
\[
\mathcal{L}' = \left[ \sum_{j} \rho_{ij}^j H_j - m_j \right] \partial_j + \frac{1}{2} a_{ij} \ln \partial_j \partial_j \partial_j
\]
is the quadratic form (3.8). Note that \(\mathcal{L}'\) is not only a symmetry operator for \(A\), but in addition is functionally dependent on the first order symmetries \(\mathcal{L}'\), (3.9). That is, there exist functions \(g^j(x)\) such that
\[
\mathcal{A}' = \sum_{j} g^j(x) \mathcal{L}'_j. \tag{4.7}
\]

Returning to the general symmetry operator \(\mathcal{A}\), (4.1)–(4.4), we can uniquely associate this operator with the quadratic form \(A\) on \(V_{\alpha}\), defined in local coordinates by
\[
A = \sum_{ij} a_{ij} p_i p_j. \tag{4.8}
\]
We can talk about the roots and eigenforms of \(\mathcal{A}\), meaning by this the roots and eigenforms of \(A\). The following analogy of Theorem 2 holds.

Theorem 3: Let \([\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_N]\) be a set of second order symmetry operators for \(A\) with \([A_i, A_j]\) linearly independent, and let \([\mathcal{L}_{\alpha,1}, \ldots, \mathcal{L}_{\alpha,N}]\) be a linearly independent set of first order symmetry operators such that
(1) \([\mathcal{A}_i, \mathcal{L}_{\alpha}'] = 0\), \([\mathcal{A}_i, \mathcal{L}_{\alpha}] = 0\), \([\mathcal{L}_{\alpha}, \mathcal{L}_{\alpha}'] = 0\),
(2) each \(\mathcal{L}_{\alpha}'\) is functionally dependent on the set \([\mathcal{L}_{\alpha}]\),
where \(\mathcal{A}_i = \mathcal{L}_i + \mathcal{L}_i'\) is the canonical decomposition (4.1)–(4.4) of \(\mathcal{A}_i\),
(3) no \(\mathcal{A}_i\) belongs to the associative algebra generated by \([\mathcal{L}_{\alpha}']\), i.e., \(\mathcal{A}_i\) cannot be expressed as \(c_{ij}^{\alpha\beta} \mathcal{L}_{\alpha} \mathcal{L}_{\beta}\) for constants \(c_{ij}^{\alpha\beta}\),
(4) there is a basis of forms \(\omega_{\alpha\beta} = \delta_{ij} \mathcal{L}_{\alpha} \mathcal{L}_{\beta}\), \(1 \leq j \leq n\), such that \(n_1 + n_2 = N\)
(i) the \(n_1\) forms \(\omega_{ij}\) are simultaneous eigenforms for each \(A_i\) with root \(\rho_{ij}\),
(ii) the \(n_2\) forms \(\omega_{ij}\) are simultaneous eigenforms for each \(A_i\) with double root \(\rho_{ij}\); the root corresponds to only one eigenform,
(iii) \(\mathcal{L}_\alpha = \Lambda^{\alpha\alpha} \partial_i\),
(iv) \(X^{\alpha\beta} (\frac{\partial}{\partial \alpha} - \frac{\partial}{\partial \beta}) = a_{ij} \partial_i \partial_j \),
(vi) \(G(a, b) = 0\) if \(a \neq b\), and \(G(a, a) = G(a, \alpha) = G(r, s) = 0\). Then there exist local coordinates \([x^i]\) for \(V_{\alpha}\) and functions \(f^{ij}(x)\) such that \(\omega_{\alpha\beta} = f^{ij} dx^j\) (with a suitable modification of the \(\omega_{\alpha\beta}\) and the Helmholtz equation (2.1) is R-separable in these coordinates. Conversely, to every R-separable coordinate system \([x^i]\) for the Helmholtz equation there corresponds operators \(\mathcal{A}_i, \mathcal{L}_\alpha\) on \(V_{\alpha}\) with properties (1)–(6).

Proof: Suppose conditions (1)–(6) are satisfied. Comparing coefficients of the highest order (nonvanishing) derivative terms in condition (1) we find
\[
[A_i, L_{\alpha,1}] = 0, \quad [A_i, L_{\alpha}] = 0, \quad [L_{\alpha}, L_{\beta}] = 0,
\]
where \(L_{\alpha} = \Lambda^{\alpha\alpha} p_\alpha\). It follows from this and conditions (3)–(6) that the hypotheses of Theorem 2 are satisfied. Indeed the subspace \(\theta\) is that with basis \([A_i, L_{\alpha}, L_{\beta}]\). Hence, there exists a local coordinate system \([x^i]\) such that the functions \(A_i, L_{\alpha}\) can be expressed in the form (3.8). If \(A_i = a_{ij} p_i p_j\) then by condition (2) and the fact that \(\det(\rho_{ij}) \neq 0\) we can write \(\mathcal{A}_i = \mathcal{A}'_i + \mathcal{L}'_i\), where
\[
\mathcal{A}'_i = \frac{1}{2 (n+1)} \partial_i (g^{lj} a_{ij} \partial_j) + \sum_{j} \rho_{ij}^j H_j - m_j, \tag{4.9}
\]
\[
\mathcal{L}'_i = \sum_{j} \rho_{ij}^j H_j - m_j \partial_j, \tag{4.10}
\]
and
\[
\sum_{k=1}^{N} H_k - \hat{m}_j = 0, \quad \sum_{k=1}^{N} H_k - \hat{m}_j a_{ij} \partial_j, \tag{4.10}
\]
for \(l = 1, \ldots, N\), where
\[
A_i = a_{ij} p_i p_j. \tag{4.6}
\]
a suitably small $x'$-coordinate neighborhood. Assuming this done and dropping the primes we set $\alpha = \beta$ in (4.14) to obtain
\[
\partial_{\xi} (T^{\alpha}_{\beta}/B^{\alpha}_{\beta}) = \partial_{\xi} (T^{\alpha}_{\beta}/B^{\alpha}_{\beta}), \quad r \neq s.
\]
(4.15)
Substituting this result back into (4.14) and simplifying we obtain
\[
\left( \frac{B^{\alpha}_{\beta}B^{\beta}_{\rho} - B^{\alpha}_{\beta}B^{\beta}_{\rho}}{B^{\alpha}_{\rho}B^{\beta}_{\rho}B^{\beta}_{\rho}} \right) \left( \partial_{\xi} (T^{\alpha}_{\rho}) - \partial_{\rho} (T^{\alpha}_{\beta}) \right) = 0.
\]
(4.16)
It follows from (4.16) that
\[
T^{\alpha}_{\beta} = B^{\alpha}_{\beta} (x') Z_{\alpha} + P^{\alpha}_{\beta} (x')
\]
(4.17)
and from (4.15) that $\partial_{\xi} Z_{\alpha} = \partial_{\xi} Z_{\beta}, \quad r \neq s$.

Thus there exists a function $Q (x')$ (depending on type 2 variables only) such that $Z_{\alpha} = -2\partial_{\xi} Q$.

We conclude that
\[
\xi^{\alpha} = -2B^{\alpha}_{\rho} \partial_{\xi} Q (x') + P^{\alpha}_{\beta} (x'), \quad \xi^{\alpha} = V_{\alpha} (x')
\]
(4.18)
Substituting this result into (4.10) we see that $\Sigma g^{\alpha \beta} \partial_{\xi} Q$ is a Stäckel multiplier. Thus all conditions of Theorem 1 are satisfied and the coordinates $\{x'\}$ (hence the coordinates $\{x\}$) $R$-separate the Helmholtz equation. [We note that the first derivative terms in (4.11d) yield no new restrictions.]

Conversely, if the coordinates $\{x'\}$ $R$-separate the Helmholtz equation we can reverse the order of the above argument and verify conditions (1)–(6).

Q.E.D.

5. DISCUSSION AND EXAMPLES

Theorem 2 states that a Hamilton–Jacobi separable system $\{x'\}$ is $R$-separable for the Helmholtz equation if and only if the involutive family of Killing tensors $A_{\alpha \beta \gamma}$ corresponds to a commutative family of symmetry operators $A_{\alpha \beta \gamma}$. The technical conditions (2) and (3) of Theorem 1 are necessary and sufficient that such a correspondence exists. In this sense our results have a close relationship with quantization theory.

Note that if the operators $A_{\alpha \beta \gamma}$ satisfy the hypotheses of Theorem 3, except for requirement (2), then the operators $A_{\alpha \beta \gamma}$ define an $R$-separation of the Helmholtz equation.

Our generalization of variable separation for the Helmholtz equation to $R$-separation and including null coordinates would be of little value unless nontrivial $R$-separation exists. In fact, all of the phenomena discussed in this paper do occur. For examples of ordinary separation involving type 2 (null) coordinates see Refs. 4, 5, and 11. For examples (and a theory) of nontrivial orthogonal $R$-separation see Refs. 3 and 12. Here, we merely recall one example of non-orthogonal $R$-separation from Ref. 12 to show how it relates to the general theory. The example is a $V_{4}$ with local coordinates $\{x', x_2, x_3, x_4\}$ and metric

\[
\begin{pmatrix}
0 & 0 & e^x & 1 \\
0 & 0 & e^x & 1 \\
e^x & e^x & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix} \quad (5.1)
\]

Thus, $n_2 = n_3 = 2, n = 4$. The coordinates are easily checked to be Hamilton–Jacobi separable and $f =
\[ \ln(g^{1/2}/S) = -\ln(e^\rho - e^\sigma). \]
Since \(n_1 = 0\), condition (3) of Theorem 1 is satisfied. We first check ordinary separability. Here \(H_x^2 = H_y^2 = 1\) and 
\[ g^{x\sigma}f_x + g^{y\sigma}f_y = -e^\rho - e^\sigma, \]
\[ g^{x\rho}f_x + g^{y\rho}f_y = -1 \] so \(g^{\sigma\rho}\) is always a Stäckel multiplier. It follows that the Helmholtz equation separates in the coordinates \(\{x^i\}\). We have shown that \(Q = f\) satisfies condition (2) in Theorem 1. However, once we have separation we can achieve further \(R\)-separation by choosing \(Q\) to be any other function satisfying condition (2). In particular choose \(Q = 0\). Then the Helmholtz equation \(R\)-separates in the coordinates \(\{x^i\}\) with \(R = (e^\rho - e^\sigma)^{1/2}\). (The phenomenon of multiple \(R\)-separation for a single coordinate system is possible only if type 2 coordinates are present.) In Ref. 12 we give the operator characterizations of these coordinates in accordance with Theorem 3.

Upon comparison of Theorem 2 and 3 it is clear that \(R\)-separation and not just ordinary separation is the appropriate Helmholtz analogy of separation for the Hamilton–Jacobi equation.

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10S. Benenti, "Integrabilita per separazione delle variabili delle equazioni alle derivate parziali lineari del secondo ordine interessanti la fisica-mate