The general theory of R-separation for Helmholtz equations

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We develop the theory of R-separation for the Helmholtz equation on a pseudo-Riemannian manifold (including the possibility of null coordinates) and show that it, and not ordinary variable separation, is the natural analogy of additive separation for the Hamilton–Jacobi equation. We provide a coordinate-free characterization of variable separation in terms of commuting symmetry operators.

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1. INTRODUCTION

Let V_n be a (local) pseudo-Riemannian manifold. The *Helmholtz equation* for V_n is expressed in local coordinates $\{y^j\}$ by

$$\Delta\psi(\mathbf{y}) = E\psi(\mathbf{y}), \tag{1.1}$$

where E is a nonzero constant and Δ is the Hamiltonian or Laplace–Beltrami operator[1]

$$\Delta = \frac{1}{g^{1/2}} \sum_{i,j=1}^{n} \partial_i (g^{1/2} g^{ij} \partial_j). \tag{1.2}$$

Here, $\partial_j = \partial_{y^j}$, the metric on V_n is $ds^2 = \Sigma_{i,j} g_{ij} dy^i dy^j$, $g = \det(g_{ij}) \neq 0$, and $\Sigma_k g^{ik} g_{kj} = \delta^i_j$. The Helmholtz equation is closely associated with the *Hamilton–Jacobi equation*[2]

$$H(\partial_i W) \equiv \sum_{i,j=1}^{n} g^{ij} \partial_i W \partial_j W = E, \tag{1.3}$$

where H is the Hamiltonian function

$$H(p_i) = \sum_{i,j=1}^{n} g^{ij} p_i p_j. \tag{1.4}$$

Both Δ and H are defined independent of local coordinates.

In Ref. 3 the authors presented a theory of orthogonal R-separation for (1.1). [By R-separation we mean separation up to a fixed factor:

$$\psi(\mathbf{y}) = R(\mathbf{y}) \prod_{j=1}^{n} \psi^{(j)}(y^j). \tag{1.5}$$

Ordinary separation corresponds to $R \equiv 1$ and trivial R-separation to $\partial_{ij} \ln R = 0$ for $i \neq j$.] We found necessary and sufficient conditions that an additively separable orthogonal coordinate system for the Hamilton–Jacobi equation will also R-separate the Helmholtz equation. [An R-separable system for (1.1) always separates (1.3).] Further, we found a coordinate-free characterization of orthogonal R-separable coordinate systems in terms of families of commuting symmetry operators for Δ.

In this paper we extend the ideas of Ref. 3 to provide a general theory of R-separation for the Helmholtz equation, encompassing both orthogonal and nonorthogonal coordinate systems. A major new complication is the possibility of type 2 (null) coordinates. Our principal result is Theorem 3, which provides an intrinsic characterization of an R-separable coordinate system in terms of a family of commuting symmetry operators. (In particular, given the operators, expressed in an arbitrary coordinate system, one can compute the R-separable coordinates.)

Although R-separation has long been a useful tool in the study of the Laplace equation [$E = 0$ in (1.1)], its relevance to the Helmholtz equation was, until recently, virtually ignored. Our results show clearly that R-separation, rather than ordinary separation, for the Helmholtz equation is the proper analog to additive separation of the Hamilton–Jacobi equation. In fact, the problem of extending a separable system for (1.3) to an R-separable system for (1.1) reduces to an exercise in quantization theory.

In Sec. 2 we give a precise operational definition of R-separation for the Helmholtz equation. (We expect, though we have not tried to verify, that any coordinate system which R-separates in accordance with some more intuitive definition of separability can be shown to be equivalent to one of our canonical systems.) In Theorem 1 we derive necessary and sufficient conditions that a Hamilton–Jacobi separable system be R-separable for the Helmholtz equation, and we look at the special case of ordinary separation ($R = 1$), obtaining a new generalization of the Robertson condition for orthogonal separability. In Sec. 3 we develop the symmetry operator approach to R-separation and review the corresponding Hamilton–Jacobi theory. Section 4 contains our main result, Theorem 3, which gives the intrinsic symmetry operator characterization of R-separation. Finally, in Sec. 5 we provide some examples of R-separation and briefly discuss the significance of our results.

The theory presented here is local rather than global. All functions are assumed to be locally analytic.

2. TECHNICAL CONSIDERATIONS

Let $\{x^j\}$ be a local coordinate system on the pseudo-Riemannian manifold. We present here an operational definition of R-separation for the Helmholtz equation

$$\Delta\psi \equiv \frac{1}{g^{1/2}} \partial_i (g^{1/2} g^{ij} \partial_j)\psi = E\psi \tag{2.1}$$

in the coordinates $\{x^j\}$ and derive necessary and sufficient conditions for the existence of this phenomenon. Let $(S_{ij}(x^i))$

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be a *Stäckel matrix*, i.e., an $N\times N$ nonsingular matrix whose ith row depends only on the variable x^i and set $S=\det(S_{ij})$. We divide the coordinates x^j into three disjoint classes: *essential of type 1*, *essential of type 2*, and *ignorable*. We further order the indices so that n_1 coordinates x^a, $1\leqslant a\leqslant n_1$, are essential of type 1, the n_2 coordinates x^r, $n_1+1\leqslant r\leqslant n_1+n_2$, are essential of type 2, and the n_3 coordinates x^α, $n_1+n_2+1\leqslant \alpha\leqslant n_1+n_2+n_3=n$, are ignorable. (In the following, unless otherwise stated, indices a, b, c range from 1 to n_1, indices r, s, t range from n_1+1 to n_1+n_2, indices α,β,γ range from n_1+n_2+1 to n, and indices i,j,k range from 1 to n.) The *ignorable* coordinates are defined to be all x^i such that $\partial_i g^{jk}(\mathbf{x})=0$ for all j, k. Finally, set $N=n_1+n_2$, let $\lambda_1=-E,\lambda_2,\ldots,\lambda_N$ be complex parameters, and define differential operators K_a, K_r by

$$K_a=\partial_{aa}+l_a(x^a)\partial_a+m_a(x^a)+\sum_{\alpha,\beta}A_a^{\alpha,\beta}(x^a)\partial_{\alpha\beta}+\sum_{\alpha}n_a^\alpha(x^a)\partial_\alpha+\sum_{i=1}^{N}\lambda_iS_{ai}(x^a), \qquad (2.2)$$

for $a=1,\ldots,n_1$ and

$$K_r=2\sum_{\alpha}B_r^\alpha(x^r)\partial_{r\alpha}+m_r(x^r)+\sum_{\alpha,\beta}A^{\alpha,\beta}(x^r)\partial_{\alpha\beta}+\sum_{\alpha}n_r^\alpha(x^r)\partial_\alpha+\sum_{i=1}^{N}\lambda_iS_{ri}(x^r) \qquad (2.3)$$

for $r=n_1+1,\ldots,N$.

We say that the coordinates $\{x^j\}$ are R-separable for the Helmholtz equation (2.1) provided there exist functions $g_k(\mathbf{x})$ and $R(x^a,x^r)$ $(R\neq 0)$ such that

$$R^{-1}\Delta R-E\equiv\sum_{k=1}^{N}g_k(\mathbf{x})K_k. \qquad (2.4)$$

Here

$$R^{-1}\Delta R=\Delta+g^{ij}\partial_i\ln R\partial_j+R^{-1}(\Delta R) \qquad (2.5)$$

as an operator, where

$$\Delta=g^{ij}\partial_{ij}+\frac{1}{g^{1/2}}\partial_i(g^{1/2}g^{ij})\partial_j. \qquad (2.6)$$

If the coordinates are R-separable then the function

$$\psi(\mathbf{x})=R(x^b,x^s)\prod_a\psi^{(a)}(x^a)\prod_r\psi^{(r)}(x^r)\exp\Big[\sum\lambda_\alpha x^\alpha\Big] \qquad (2.7)$$

is a solution of $\Delta\psi=E\psi$ whenever the $\psi^{(j)}$ satisfy *separation equations*

$$K_a[\psi^{(a)}\exp(\lambda_\alpha x^\alpha)]=0,\quad a=1,\ldots,n_1,$$
$$K_r[\psi^{(r)}\exp(\lambda_\alpha x^\alpha)]=0,\quad r=n_1+1,\ldots,N. \qquad (2.8)$$

Here the λ_α are arbitrary complex constants and $\lambda_1,\ldots,\lambda_n$ are the *separation parameters*. Note that the function $\exp(\lambda_\alpha x^\alpha)$ can be factored out of expressions (2.8), thus reducing these expressions to ordinary differential equations. The *type 1* coordinates x^a have the property that the corresponding separation equations are second order ODE's, whereas for *type 2* coordinates x^r the separation equations are first order ODE's. The solutions $\psi(\mathbf{x},\boldsymbol{\lambda})$ (2.7), depend on the separation parameters λ_i but $R(x^b,x^s)$ is independent of these parameters.

It follows from (2.2)–(2.4) that a necessary condition for R-separation is

$$g_k(\mathbf{x})=S^{k1}/S,\quad k=1,\ldots,N \qquad (2.9)$$

where S^{k1} is the $(k,1)$ minor of (S_{ij}).

Thus the metric must take the form

$$g^{ab}=\delta^{ab}\frac{S^{a1}}{S},\quad g^{ar}=g^{a\alpha}=0,\quad g^{rs}=0,$$
$$g^{r\alpha}=B_r^\alpha(x^r)\frac{S^{r1}}{S}, \qquad (2.10)$$
$$g^{\alpha\beta}=\frac{1}{2}\sum_{i=1}^{N}A_i^{\alpha,\beta}(x^i)\frac{S^{i1}}{S},\quad \alpha\neq\beta$$
$$g^{\alpha\alpha}=\sum_{i=1}^{N}A_i^{\alpha,\alpha}(x^i)\frac{S^{i1}}{S}.$$

Note that

$$\begin{array}{cc} & \begin{array}{ccc} n_1 & n_2 & n_3 \end{array} \\ (g^{ij})= & \left(\begin{array}{ccc} \delta^{ab}g^{aa} & 0 & 0 \\ 0 & 0 & g^{r\alpha} \\ 0 & g^{\alpha r} & g^{\alpha\beta} \end{array}\right)\begin{array}{c} n_1 \\ n_2. \\ n_3 \end{array} \end{array} \qquad (2.11)$$

Conditions (2.10) are necessary but not sufficient for R-separation. Before determining the remaining conditions, however, it is worthwhile to point out the significance of these restrictions on the metric. Consider the *Hamilton–Jacobi equation* associated with the Helmholtz equation (2.1):

$$g^{ij}\partial_iW\partial_jW=E. \qquad (2.12)$$

It has recently been established,[4–7] that conditions (2.10) are necessary and sufficient for (additive) separation of the Hamilton–Jacobi equation in the coordinates $\{x^j\}$

$$W(\mathbf{x})=\sum_a W^{(a)}(x^a,\boldsymbol{\lambda})+\sum_r W^{(r)}(x^r,\boldsymbol{\lambda})+\sum_\alpha\lambda_\alpha x^\alpha \qquad (2.13)$$

Indeed, Benenti[7] has shown that every system which separates (2.12), according to the intuitive definition of Levi-Civita,[8] is equivalent to a system in the canonical form (2.10).

Proposition 1: A coordinate system that is R-separable for the Helmholtz equation is also separable for the Hamilton–Jacobi equation. Let

$$H_i^{-2}=\frac{S^{i1}}{S},\quad i=1,\ldots,N. \qquad (2.14)$$

If conditions (2.10) hold then $S^{i1}\neq 0$ since $g\neq 0$. We can associate with our coordinate system $\{x^j\}$ on V_n an orthogonal coordinate system $\{x^1,\ldots,x^N\}$ on a space V_N with metric

$$ds^2=\sum_{i=1}^{N}H_i^2(dx^i)^2. \qquad (2.15)$$

By (2.14), this metric is in Stäckel form.[2] Recall that necessary and sufficient conditions that ds^2 be expressible in the form (2.14) for some Stäckel matrix are (Ref. 1, Appendix 13)

$$\partial_{jk}\ln H_i^{-2}+\partial_j\ln H_i^{-2}\partial_k\ln H_i^{-2}$$
$$-\partial_j\ln H_i^{-2}\partial_k\ln H_j^{-2}$$
$$-\partial_k\ln H_i^{-2}\partial_j\ln H_k^{-2}=0,$$
$$j\neq k;\quad i,j,k=1,\ldots,N. \qquad (2.16)$$

We further recall some useful results from Ref. 6. Given a metric $ds^2=\Sigma_i H_i^2(dx^i)^2$ in Stäckel form, we say that the function $Q(\mathbf{x})$ is a *Stäckel multiplier* for (ds^2) if the metric $d\hat{s}^2 = Qds^2$ is also in Stäckel form with respect to the coordinates $\{x^j\}$. It can be shown that Q is a Stäckel multiplier if and only if there exist functions $\psi_j = \psi_j(x^j)$ such that

$$Q(\mathbf{x}) = \sum_{j=1}^{N} \psi_j(x^j) H_j^{-2}. \tag{2.17}$$

Equivalent necessary and sufficient conditions are

$$\partial_{jk} Q - \partial_j Q \partial_k \ln H_j^{-2} - \partial_k Q \partial_j \ln H_k^{-2} = 0, \quad j \neq k. \tag{2.18}$$

We can now reformulate conditions (2.10).

Proposition 2: A necessary requirement for R-separation of (2.1) in the coordinates $\{x^i : i = 1,...,n\}$ is that

$$g^{aa} = H_a^{-2}, \quad g^{r\alpha} = B_r^\alpha(x^r) H_r^{-2}, \tag{2.19}$$

and that each $g^{\alpha\beta}$ be a Stäckel multiplier for the Stäckel form metric $ds^2 = \Sigma_{k=1}^N H_k^2 (dx^k)^2$. All other matrix elements g^{ij} must vanish.

To obtain sufficient conditions for R-separation we must also demand equality of the coefficients of ∂_j and the zeroth order terms on each side of (2.5):

$$f_a + 2\partial_a \ln R = l_a(x^a), \tag{2.20}$$

$$\sum_r g^{r\alpha}(f_{r\alpha} + 2\partial_r \ln R) = \sum_{k=1}^{N} H_k^{-2} n_k^\alpha(x^k), \tag{2.21}$$

$$R^{-1}(\Delta R) = \sum_{k=1}^{N} H_k^{-2} m_k(x^k). \tag{2.22}$$

Here,

$$f_a = \partial_a f, \quad f = \ln(g^{1/2}/S), \tag{2.23}$$

$$f_{r\alpha} = \partial_r \ln(g^{1/2} g^{r\alpha}) = f_r + \partial_r \ln B_r^\alpha(x^r).$$

Solving for R from (2.19) we find

$$R = \left(\frac{S}{g}\right)^{1/2} \exp\left[\sum_a A_a(x^a) + Q(x^s)\right], \tag{2.24}$$

and substituting (2.23) into (2.20) and (2.21) we ultimately obtain the following result.

Theorem 1: Necessary and sufficient conditions that the coordinates $\{x^j\}$ be R-separable for the Helmholtz equation

$$\frac{1}{g^{1/2}} \partial_i (g^{1/2} g^{ij} \partial_j \psi) = E\psi$$

are

(1) The requirements of Proposition 2 are satisfied, i.e., the coordinates $\{x^j\}$ are separable for the Hamilton–Jacobi equation $g^{ij}\partial_i W \partial_j W = E$,

(2) $\Sigma_r g^{r\alpha} \partial_r Q$ is a Stäckel multiplier for each α,

(3) $\Sigma_a H_a^{-2}(f_{aa} + \frac{1}{2} f_a^2)$ is a Stäckel multiplier, where $f_a = \partial_a \ln(g^{1/2}/S)$ and S is the determinant of the Stäckel matrix.

If these conditions are satisfied then

$$R(\mathbf{x}) = \left(\frac{S}{g^{1/2}}\right)^{1/2} \exp\left[\sum_a A_a(x^a) + Q(x^s)\right],$$

where the $A_a = A_a(x^a)$ are arbitrary.

We say that the coordinates $\{x^j\}$ are *separable* for the Helmholtz equation provided they are R-separable with $R \equiv 1$. Furthermore, R-separable coordinates are *trivially R-separable* if $R = \Pi_{i=1}^n R_i(x^i)$ and (since coordinates are trivially R-separable if and only if they are separable) we regard trivial R-separation as equivalent to ordinary separation.

Especially interesting is the case of ordinary separation. Then $R \equiv 1$ and expression (2.23) becomes

$$\tfrac{1}{2}\ln\left(\frac{g^{1/2}}{S}\right) = \sum_a A_a(x^a) + Q(x^s). \tag{2.25}$$

Corollary 1 (Generalized Robertson Condition): Necessary and sufficient conditions that the coordinates $\{x^j\}$ be separable for the Helmholtz equation are

(1) the coordinates are separable for the Hamilton–Jacobi equation,

(2) $f_{aj} = 0$ for $j = 1,...,N, j \neq a$,

(3) $\Sigma_r g^{r\alpha} f_r$ is a Stäckel multiplier for each α.

Here $f = \ln(g^{1/2}/S)$ and $f_i = \partial_i f$.

The original Robertson condition[9] was concerned with the case of orthogonal separation. (By permitting a type 1 coordinate to be ignorable if necessary, we can identify this case with $n_1 = n$, $n_2 = n_3 = 0$.) Robertson showed that an orthogonal separable system for the Hamilton–Jacobi equation separated the Helmholtz equation if and only if $f_{ab} = 0$ for $a \neq b$. (Since $n_2 = 0$ this agrees with Corollary 1.)

Eisenhart[2] showed that the Robertson condition is equivalent to the requirement

$$R_{ab} = 0, \quad a \neq b \tag{2.26}$$

where R_{ab} is the Ricci tensor expressed in terms of the orthogonal coordinates $\{x^a\}$. (For an explicit definition of the Ricci tensor R_{ij} in terms of the metric g^{ij} together with related computational formulas we refer the reader to Chap. 1 of Eisenhart's text.[1]) Benenti[10] studied nonorthogonal separation for the Helmholtz equation in which no nonignorable null coordinates were allowed ($n_2 = 0$ in our formalism). His requirement for Helmholtz separation agrees with our condition (2). Benenti further showed that his requirement was equivalent to (2.20) again and that $R_{\alpha\alpha} = 0$ automatically for Hamilton–Jacobi separable systems. By a tedious but straightforward computation we have established

Lemma 1: Condition (2) of Corollary 1, namely

$$f_{aj} = 0 \quad \text{for} \quad j = 1,...,N, \quad j \neq a$$

is equivalent to

$$R_{ab} = 0, \quad a \neq b, \quad R_{ar} = 0, \tag{2.27}$$

where R_{ij} is the Ricci tensor for V_n expressed in the coordinates $\{x^j\}$. Furthermore, $R_{\alpha\alpha} = 0$ automatically if $\{x^j\}$ separates the Hamilton–Jacobi equation.

It is perhaps somewhat surprising that requirements (2.25) continue to hold even with the presence of type 2 coordinates. Condition (3) of Corollary 1 appears not to be expressible in terms of the Riemann curvature tensor and its covariant derivatives. However, this condition is vacuous for $n_2 \leqslant 1$. Since $g^{rs} = 0$, type 2 coordinates are null and any two such coordinates are orthogonal. Thus, for separation on a proper Riemannian space V_n we must have $n_2 = 0$ and for a pseudo-Riemannian V_n with signature $(-1, 1^{n-1})$ we must have $n_2 \leqslant 1$.

Corollary 2: In order that Hamilton–Jacobi separable coordinates $\{x^j\}$ separate the Helmholtz equation on a pseudo-Riemannian manifold with signature (1^n) or $(-1,1^{n-1})$ it is necessary and sufficient that

$$R_{ab}=0, \quad a\neq b, \quad R_{ar}=0.$$

3. CONSTANTS OF THE MOTION

Let us suppose that the coordinates $\{x^j\}$ R-separate the Helmholtz equation. Then expanding the corresponding Stäckel matrix in (2.2), (2.3) by the lth, rather than just the 1st, column we obtain operators $\mathscr{A}_l$, $l=1,\ldots,N$, such that $\mathscr{A}_l\psi=-\lambda_l\psi$ for an R-separated solution ψ:

$$\begin{aligned}\mathscr{A}_l=\sum_a\frac{S^{al}}{S}\Big(&\partial_{aa}+f_a\partial_a+\sum_{\alpha,\beta}A_a^{\alpha,\beta}\partial_{\alpha\beta}+\sum_\alpha n_a^\alpha\partial_\alpha\\&+m_a+\tfrac{1}{2}\partial_a[f_a-l_a]+\tfrac{1}{2}[f_a^2-l_a^2]\Big)\\&+\sum_r\frac{S^{rl}}{S}\Big(2\sum_\alpha B_r^\alpha\partial_{r\alpha}+\sum_{\alpha,\beta}A_r^{\alpha,\beta}\partial_{\alpha\beta}\\&+\sum_\alpha(n_r^\alpha-2B_r^\alpha\partial_r\ln R)\partial_\alpha+m_r\Big).\end{aligned}\tag{3.1}$$

(Note that $\mathscr{A}_1=\Delta$.) These expressions are not as complicated as they appear. It can be directly verified (and we will show this later) that

$$\begin{aligned}&[\mathscr{A}_l,\mathscr{A}_k]=0, \quad [\mathscr{L}_\alpha,\mathscr{L}_\beta]=0,\\&[\mathscr{A}_l,\mathscr{L}_\alpha]=0, \quad 1\leqslant l,k\leqslant N\end{aligned}\tag{3.2}$$

where

$$\mathscr{L}_\alpha=\partial_\alpha, \quad \alpha=N+1,\ldots,n,\tag{3.3}$$

and $[\mathscr{A},\mathscr{B}]=\mathscr{A}\mathscr{B}-\mathscr{B}\mathscr{A}$. Thus the operators $\mathscr{A}_k$ $(2\leqslant k\leqslant N)$, $\mathscr{L}_\alpha$ form a commuting family of symmetry operators for Δ, i.e., they commute with Δ and with each other. Furthermore, the R-separated solutions of (2.2) are simultaneous eigenfunctions of the symmetry operators:

$$\mathscr{A}_l\psi=-\lambda_l\psi, \quad \mathscr{L}_\alpha\psi=\lambda_\alpha\psi.\tag{3.4}$$

Our construction has started with an R-separable coordinate system $\{x^l\}$ and produced a commuting family of symmetry operators $\{\mathscr{A}_l,\mathscr{L}_\alpha\}$. It is our principal task in this paper to characterize those families of commuting symmetry operators that correspond to R-separation.

In Ref. 6 the authors solved the corresponding problem for the Hamilton–Jacobi equation (2.12). In that case we utilized the natural symplectic structure on the cotangent bundle $\widetilde{V}_n$ of V_n. Corresponding to local coordinates $\{x^i\}$ on V_n we have coordinates $\{x^i,p_i\}$ on the $2n$-dimensional space $\widetilde{V}_n$. The *Poisson bracket* of two functions $F(x^j,p_j)$, $G(x^j,p_j)$ on $\widetilde{V}_n$ is defined by

$$\{F,G\}=\sum_{l=1}^n(\partial_{p_l}F\partial_{x^l}G-\partial_{x^l}F\partial_{p_l}G).\tag{3.5}$$

Let $\{x^i\}$ be a separable coordinate system for the Hamilton–Jacobi equation (2.12) with coordinates of type 1, x^a, of type 2, x^r, and ignorable, x^α, as usual. Then the metric g^{ij} in these coordinates takes the standard form (2.10).

It is convenient at this point to introduce the functions $\rho_j^{(k)}(x^1,\ldots,x^N)$, where

$$\frac{S^{jk}}{S}=\rho_j^{(k)}H_j^{-2}, \quad \frac{S^{j1}}{S}=H_j^{-2}, \quad 1\leqslant j,k\leqslant N,\tag{3.6}$$

and S_{ij} is the Stäckel matrix corresponding to the separable system $\{x^i\}$. Then $\rho_j^{(1)}=1$ and it can be shown that (Ref. 1, Appendix 13)

$$\partial_i\rho_j^{(k)}=(\rho_i^{(k)}-\rho_j^{(k)})\partial_i\ln H_j^{-2}, \quad 1\leqslant i,j,k,\leqslant N.\tag{3.7}$$

Let $H=\Sigma_{i,j}g^{ij}p_ip_j$ be the Hamiltonian corresponding to (2.12). In Ref. 6 we constructed quadratic forms A_l $(A_1=H)$, given by

$$\begin{aligned}A_l=\sum_a\rho_a^{(l)}H_a^{-2}&\Big(p_a^2+\sum_{\alpha,\beta}A_a^{\alpha,\beta}p_\alpha p_\beta\Big)\\&+\sum_r\rho_r^{(l)}H_r^{-2}\Big(\sum_\alpha B_r^\alpha p_rp_\alpha+\sum_{\alpha,\beta}A_r^{\alpha,\beta}p_\alpha p_\beta\Big)\end{aligned}\tag{3.8}$$

for $l=1,\ldots,N$ and n_3 linear forms L_α,

$$L_\alpha=p_\alpha, \quad \alpha=N+1,\ldots,n.\tag{3.9}$$

These polynomials in the p's were shown to satisfy

$$\begin{aligned}&\{A_l,A_k\}=0, \quad \{L_\alpha,L_\beta\}=0,\\&\{A_l,L_\alpha\}=0, \quad l,k=1,\ldots,N,\end{aligned}\tag{3.10}$$

and when evaluated for $p_a=\partial_aW, p_r=\partial_rW, p_\alpha=\partial_\alpha W$ with W a separable solution of (2.12), they satisfy

$$A_l=-\lambda_l, \quad L_\alpha=\lambda_\alpha,\tag{3.11}$$

where $\lambda_1=-E,\ldots,\lambda_n$ are the separation parameters.

Let $a^{ij}(y)$ be a symmetric contravariant 2-tensor on V_n, expressed in terms of local coordinates $\{y^k\}$, and let $g^{ij}(\mathrm{y})$ be the contravariant metric tensor. A *root* $\rho(y)$ of a^{ij} is a solution of the characteristic equation

$$\det(a^{ij}-\rho g^{ij})=0\tag{3.12}$$

and an *eigenform* $\omega=\Sigma\lambda_kdy^k$ corresponding to ρ is a nonzero 1-form such that

$$\sum_{j=1}^n(a^{ij}-\rho g^{ij})\lambda_j=0, \quad i=1,\ldots,n.\tag{3.13}$$

Roots and eigenforms are defined independent of local coordinates.

Note from (3.8) that for a separable system $\{y^j\}$ the $\rho_a^{(l)}$ are simple roots of the A_l with simultaneous eigenforms dx^a, and the $\rho_r^{(l)}$ are roots of multiplicity 2 but with a single eigenform dx^r. Here dx^a,dx^r are also eigenforms for the products $L_\alpha L_\beta$.

Let $\{y^j\}$ be a local coordinate system on a pseudo-Riemannian manifold and let $\omega_{(j)}=\lambda_{i(j)}dy^i$, $1\leqslant j\leqslant n$, be a local basis of 1-forms on V_n. The *dual basis* of vector fields is $\underline{\overline{X}}^{(h)}=\wedge^{i(h)}\partial_i$, $1\leqslant h\leqslant n$, where $\wedge^{i(h)}\lambda_{i(j)}=\delta_{(j)}^{(h)}$. The *inner product* of two 1-forms $\omega_{(j)},\omega_{(k)}$ is $G(j,k)=\lambda_{i(j)}g^{il}\lambda_{l(k)}$.
In Ref. 6 we proved

Theorem 2: Let θ be a vector subspace of quadratic forms on V_n such that $H\in\theta$ and

(1) $\{A,B\}=0$ for each $A,B\in\theta$,

(2) there is a basis of 1-forms $\omega_{(j)}=\lambda_{i(j)}dy^i$, $1\leqslant j\leqslant n$, such that

(i) the n_1 forms $\omega_{(a)}$ are simultaneous eigenforms for each $A\in\theta$ with root ρ_a^A,

(ii) the n_2 forms $\omega_{(r)}$ are simultaneous eigenforms for each $A\in\theta$ with root ρ_r^A; the root ρ_r^A has multiplicity 2 but corresponds to only one simultaneous eigenform,

(3) $\{L_\alpha,L_\beta\}=0$ and $L_\alpha L_\beta\in\theta$, where $L_\alpha = \wedge^{i(\alpha)}p_i$, $\alpha,\beta=n_1+n_2+1,\ldots,n$,
(4) $\{A,L_\alpha\}=0$ for each $A\in\theta$,
(5) $\underline{\overline{X}}^{(r)}(\lambda_{i(\alpha)}a^{ij}\lambda_{j(\beta)})=\rho_r{}^A\underline{\overline{X}}^{(r)}(\lambda_{i(\alpha)}g^{ij}\lambda_{j(\beta)})$,
(6) $\dim\theta=\frac{1}{2}(2n+n_3{}^2-n_3)$, where $n_3=n-n_1-n_2$,
(7) $G(a,b)=0$ if $a\neq b$, and $G(a,r)=G(a,\alpha)=G(r,s)=0$.

Then there exist local coordinates $\{x^j\}$ for V_n and functions $f^{(j)}(\mathbf{x})$ such that $\omega_{(j)}=f^{(j)}dx^j$ (with a suitable modification of the $\omega_{(\alpha)}$) and the Hamilton–Jacobi equation is separable in these coordinates. Conversely, to every separable coordinate system $\{x^j\}$ for the Hamilton–Jacobi equation there corresponds a subspace θ of quadratic forms on V_n with properties (1)–(7).

In the following section we will show that, with suitable modifications, this result also characterizes R-separable systems for the Helmholtz equation.

4. THE BASIC RESULT

Let Δ be the Hamiltonian operator (1.2), expressed in terms of local coordinates $\{x^j\}$. Suppose $\mathscr{A}$ is a *second order symmetry operator* for Δ, i.e., a differential operator such that $[\mathscr{A},\Delta]=0$ and which in local coordinates can be written

$$\mathscr{A}=a^{ij}(\mathbf{y})\partial_{ij}+\bar{b}^{i}(\mathbf{y})\partial_i+c(\mathbf{y}),\quad \partial_i=\partial_{y^i} \tag{4.1}$$

where $a^{ij}=a^{ji}$ and not all a^{ii} vanish. As shown in Ref. 3 we can decompose $\mathscr{A}$ uniquely in the form

$$\mathscr{A}=\mathscr{S}+\mathscr{L}, \tag{4.2}$$

where

$$\mathscr{S}=\frac{1}{g^{1/2}}\partial_i(g^{1/2}a^{ij}\partial_j)+c,$$

$$\mathscr{L}=b^{i}\partial_i, \tag{4.3}$$

$$[\mathscr{S},\Delta]=[\mathscr{L},\Delta]=0. \tag{4.4}$$

Furthermore, this decomposition is coordinate independent. Decomposing the operators $\mathscr{A}$, (3.1), in this form we find

$$\mathscr{A}_l=\mathscr{S}_l+\hat{\mathscr{L}}_l,$$

$$\begin{aligned}\mathscr{S}_l=&\frac{1}{g^{1/2}}\partial_i(g^{1/2}a^{ij}_{(l)}\partial_j)\\&+\sum_a\rho^{(l)}_aH_a^{-2}(m_a+\tfrac{1}{2}\partial_a[f_a-l_a]\\&+\tfrac{1}{4}[f_a^2-l_a^2])+\sum_r\rho^{(l)}_rH_r^{-2}m_r,\end{aligned} \tag{4.5}$$

$$\begin{aligned}\hat{\mathscr{L}}_l=\bigg[&\sum_{i=1}^{N}\rho^{(l)}_iH_i^{-2}n_i^\alpha\\&-\sum_r\rho^{(l)}_rH_r^{-2}B_r^\alpha(\partial_r\ln B_r^\alpha+\partial_rQ)\bigg]\partial_\alpha,\end{aligned}$$

for $l=1,\ldots,N$, where

$$A_l=a^{ij}_{(l)}p_ip_j \tag{4.6}$$

is the quadratic form (3.8). Note that $\hat{\mathscr{L}}_l$ is not only a symmetry operator for Δ, but it in addition is *functionally dependent* on the first order symmetries $\mathscr{L}_\alpha$, (3.3). That is, there exist functions $g_l^\alpha(\mathbf{x})$ such that

$$\hat{\mathscr{L}}_l=\sum_\alpha g_l^\alpha(\mathbf{x})\mathscr{L}_\alpha. \tag{4.7}$$

Returning to the general symmetry operator $\mathscr{A}$, (4.1)–(4.4), we can uniquely associate this operator with the quadratic form A on $\tilde{V}_n$, defined in local coordinates by

$$A=\sum_{i,j}a^{ij}p_ip_j. \tag{4.8}$$

We can talk about the *roots* and *eigenforms* of $\mathscr{A}$, meaning by this the roots and eigenforms of A. The following analogy of Theorem 2 holds.

Theorem 3: Let $\{\mathscr{A}_1=\Delta,\mathscr{A}_2,\ldots,\mathscr{A}_N\}$ be a set of second order symmetry operators for Δ with $\{A_l\}$ linearly independent, and let $\{\mathscr{L}_{N+1},\ldots,\mathscr{L}_n\}$ $(n-N=n_3)$ be a linearly independent set of first order symmetry operators such that
(1) $[\mathscr{A}_l,\mathscr{A}_k]=0$, $[\mathscr{A}_l,\mathscr{L}_\alpha]=0$, $[\mathscr{L}_\alpha,\mathscr{L}_\beta]=0$,
(2) each $\hat{\mathscr{L}}_l$ is functionally dependent on the set $\{\mathscr{L}_\alpha\}$, where $\mathscr{A}_l=\mathscr{S}_l+\hat{\mathscr{L}}_l$ is the canonical decomposition (4.1)–(4.4) of $\mathscr{A}_l$,
(3) no $\mathscr{A}_l$ belongs to the associative algebra generated by $\{\mathscr{L}_\alpha\}$, i.e., $\mathscr{A}_l$ cannot be expressed as $c_l^{\alpha\beta}\mathscr{L}_\alpha\mathscr{L}_\beta$ for constants $c_l^{\alpha\beta}$,
(4) there is a basis of 1-forms $\omega_{(j)}=\lambda_{i(j)}dy^i$, $1\leqslant j\leqslant n$, such that $(n_1+n_2=N)$
  (i) the n_1 forms $\omega_{(a)}$ are simultaneous eigenforms for each A_l with root $\rho^{(l)}_a$,
  (ii) the n_2 forms $\omega_{(r)}$ are simultaneous eigenforms for each A_l with double root $\rho^{(l)}_r$; the root corresponds to only one eigenform,
  (iii) $\mathscr{L}_\alpha=\wedge^{i(\alpha)}\partial_i$,
(5) $\underline{\overline{X}}^{(r)}(\lambda_{i(\alpha)}a^{ij}_{(l)}\lambda_{j(\beta)})=\rho^{(l)}_r\underline{\overline{X}}^{(r)}(\lambda_{i(\alpha)}g^{ij}\lambda_{j(\beta)})$,
(6) $G(a,b)=0$ if $a\neq b$, and $G(a,r)=G(a,\alpha)=G(r,s)=0$.
Then there exist local coordinates $\{x^j\}$ for V_n and functions $f^{(j)}(\mathbf{x})$ such that $\omega_{(j)}=f^{(j)}dx^j$ (with a suitable modification of the $\omega_{(\alpha)}$) and the Helmholtz equation (2.1) is R-separable in these coordinates. Conversely, to every R-separable coordinate system $\{x^j\}$ for the Helmholtz equation there correspond operators $\mathscr{A}_j,\mathscr{L}_\alpha$ on V_n with properties (1)–(6).

Proof: Suppose conditions (1)–(6) are satisfied. Comparing coefficients of the highest order (nonvanishing) derivative terms in condition (1) we find

$$\{A_l,A_k,\}=0,\ \{A_l,L_\alpha\}=0,\ \{L_\alpha,L_\beta\}=0,$$

where $L_\alpha=\wedge^{i(\alpha)}p_i$. It follows from this and conditions (3)–(6) that the hypotheses of Theorem 2 are satisfied. Indeed the subspace θ is that with basis $\{A_l,L_\alpha L_\beta\alpha\leqslant\beta\}$. Hence, there exists a local coordinate system $\{x^j\}$ such that the functions A_l,L_α can be expressed in the form (3.8). If $A_l=a^{ij}_{(l)}p_ip_j$ then by condition (2) and the fact that $\det(\rho^{(l)}_k)\neq0$ we can write $\mathscr{A}_l=\mathscr{S}_l+\hat{\mathscr{L}}_l$, where

$$\mathscr{S}_l=\frac{1}{g^{1/2}}\partial_i(g^{1/2}a^{ij}_{(l)}\partial_j)+\sum_{k=1}^{N}\rho^{(l)}_kH_k^{-2}\xi^k, \tag{4.9}$$

$$\hat{\mathscr{L}}_l=\sum_{k=1}^{N}\rho^{(l)}_kH_k^{-2}\xi^{k\alpha}\partial_\alpha,$$

and

$$\sum_{k=1}^{N}H_k^{-2}\xi^k=0,\quad \sum_{k=1}^{N}H_k^{-2}\xi^{k\alpha}=0, \tag{4.10}$$

since $\mathscr{A}_1 = \Delta$ and $\rho_k^{(1)} = 1$.

We have not yet fully utilized condition (1). Since $\mathscr{S}_l$ is self adjoint and $\hat{\mathscr{L}}_l, \mathscr{L}_\alpha$ are skew adjoint,[3] the first two equations in condition (1) yield

$$[\hat{\mathscr{L}}_l, \mathscr{L}_\alpha] = 0, \tag{4.11a}$$

$$[\hat{\mathscr{L}}_l, \mathscr{L}_k] = 0, \tag{4.11b}$$

$$[\mathscr{S}_l, \mathscr{S}_k] = 0, \tag{4.11c}$$

$$[\mathscr{S}_l, \hat{\mathscr{L}}_k] + [\hat{\mathscr{L}}_l, \mathscr{S}_k] = 0. \tag{4.11d}$$

Equation (4.11a) yields $\partial_\alpha \xi^{k\beta} = 0$ and (4.11b) is satisfied identically. Equating coefficients of ∂_{ij} on both sides of (4.11c) we find $\partial_a f_b = \partial_b f_a, \partial_r f_a = \partial_r f_{ra}$, a result already known. Equating coefficients of ∂_i on both sides of (4.11c) and using $\det(\rho_k^{(l)}) \neq 0$ we find

$$\partial_b \xi^r = 0, \quad \partial_b (2\xi^a - f_{aa} - \tfrac{1}{2} f_a^2) = 0, \quad a \neq b,$$

$$\partial_s (2\xi^a - f_{aa} - \tfrac{1}{2} f_a^2) = 0,$$

$$B_r^\alpha \partial_r \xi^s = B_s^\alpha \partial_s \xi^r, \quad r \neq s \quad \text{(no sum)}.$$

Since the last equality must hold for all α, we have $\partial_s \xi^r = 0$ for $r \neq s$. Thus

$$\xi^a = \tfrac{1}{2}[f_{aa} + \tfrac{1}{2} f_a^2 + 2P_a(x^a)],$$

$$\xi^r = P_r(x^r)$$

and from (4.10) we see that

$$\sum_a H_a^{-2}(f_{aa} + \tfrac{1}{2} f_a^2) \tag{4.12}$$

is a Stäckel multiplier. Thus condition (3) [and condition (1)] of Theorem 1 are satisfied. [The zeroth order terms in (4.11c) give no new requirements.]

The only constraints remaining to us are (4.11d). Equating coefficients of ∂_{ab} in this expression we find

$$\partial_b \xi^{r\alpha} = 0, \quad \partial_b \xi^{a\alpha} = 0, \quad b \neq a.$$

Equating coefficients of $\partial_{\alpha\beta}$ we find

$$B_r^\beta \partial_r \xi^{a\alpha} + B_r^\alpha \partial_r \xi^{a\beta} = 0,$$

$$B_r^\beta \partial_r \xi^{s\alpha} + B_r^\alpha \partial_r \xi^{s\beta} = B_s^\beta \partial_s \xi^{r\alpha} + B_s^\alpha \partial_s \xi^{r\beta}, \quad r \neq s.$$

Thus

$$\xi^{r\alpha} = T_r^\alpha(x^t), \quad \xi^{a\alpha} = V_a^\alpha(x^a), \tag{4.13}$$

where

$$B_r^\beta \partial_r T_s^\alpha + B_r^\alpha \partial_r T_s^\beta = B_s^\beta \partial_s T_r^\alpha + B_s^\alpha \partial_s T_r^\beta, \quad r \neq s, \quad \text{no sum.} \tag{4.14}$$

To solve relations (4.14) for T_s^α we use the fact that the $n_2 \times n_3$ matrix $(B_r^\beta(x^r))$ has rank n_2. The ignorable coordinates $\{x^\alpha\}$ are not unique. A new set of ignorable coordinates $\{x'^\beta\}$, where $x'^\beta = C_\alpha^\beta x^\alpha$ and (C_α^β) is a nonsingular constant matrix, will do as well. One effect of such a choice of new ignorable coordinates is to provide a new matrix $(B'^\beta_r(x^r))$ constructible from the original matrix by a sequence of elementary column transformations. Conversely, elementary column transformations of (B_r^β) induce transformations of ignorable coordinates. Assuming $n_2 \geqslant 2$ [since otherwise (4.14) is vacuous] we can always choose a new set of ignorable coordinates $\{x'^\beta\}$ such that every matrix element B'^α_r and every 2×2 minor in the new matrix are nonvanishing in a suitably small $\mathbf{x}'$-coordinate neighborhood. Assuming this done and dropping the primes we set $\alpha = \beta$ in (4.14) to obtain

$$\partial_r (T_s^\beta / B_s^\beta) = \partial_s (T_r^\beta / B_r^\beta), \quad r \neq s. \tag{4.15}$$

Substituting this result back into (4.14) and simplifying we obtain

$$\left(\frac{B_s^\alpha B_r^\beta - B_r^\alpha B_s^\beta}{B_s^\alpha B_r^\alpha B_s^\beta B_r^\beta}\right)\left(\partial_r\left(\frac{T_s^\alpha}{B_s^\alpha}\right) - \partial_r\left(\frac{T_s^\beta}{B_s^\beta}\right)\right) = 0. \tag{4.16}$$

It follows from (4.16) that

$$T_s^\alpha = B_s^\alpha(x^s) Z_s + P_s^\alpha(x^s) \tag{4.17}$$

and from (4.15) that $\partial_r Z_s = \partial_s Z_r$, $r \neq s$.
Thus there exists a function $Q(x^t)$ (depending on type 2 variables only) such that $Z_s = -2\partial_s Q$.
We conclude that

$$\xi^{r\alpha} = -2B_r^\alpha \partial_r Q(x^s) + P_r^\alpha(x^r), \quad \xi^{a\alpha} = V_a^\alpha(x^a). \tag{4.18}$$

Substituting this result into (4.10) we see that $\Sigma_r g^{r\alpha} \partial_r Q$ is a Stäckel multiplier. Thus all conditions of Theorem 1 are satisfied and the coordinates $\{\mathbf{x}'\}$ (hence the coordinates $\{\mathbf{x}\}$) R-separate the Helmholtz equation. [We note that the first derivative terms in (4.11d) yield no new restrictions.]

Conversely, if the coordinates $\{x^j\}$ R-separate the Helmholtz equation we can reverse the order of the above argument and verify conditions (1)–(6). Q.E.D.

5. DISCUSSION AND EXAMPLES

Theorem 2 states that a Hamilton–Jacobi separable system $\{x^j\}$ is R-separable for the Helmholtz equation if and only if the involutive family of Killing tensors A_l, L_α corresponds to a commutative family of symmetry operators $\mathscr{A}_l, \mathscr{L}_\alpha$. The technical conditions (2) and (3) of Theorem 1 are necessary and sufficient that such a correspondence exists. In this sense our results have a close relationship with quantization theory.

Note that if the operators $\mathscr{A}_l, \mathscr{L}_\alpha$ satisfy the hypotheses of Theorem 3, except for requirement (2), then the operators $\mathscr{S}_l, \mathscr{L}_\alpha$ define an R-separation of the Helmholtz equation.

Our generalization of variable separation for the Helmholtz equation to R-separation and including null coordinates would be of little value unless nontrivial R-separation exists. In fact, all of the phenomena discussed in this paper do occur. For examples of ordinary separation involving type 2 (null) coordinates see Refs. 4, 5, and 11. For examples (and a theory) of nontrivial orthogonal R-separation see Refs. 3 and 12. Here, we merely recall one example of nonorthogonal R-separation from Ref. 12 to show how it relates to the general theory. The example is a V_4 with local coordinates $(x^1,\ldots,x^4) \equiv (x,y,\alpha,\beta)$ and metric

$$(g^{ij}) = \begin{pmatrix} 0 & 0 & e^x & 1 \\ 0 & 0 & e^y & 1 \\ e^x & e^y & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}. \tag{5.1}$$

Thus, $n_2 = n_3 = 2$, $n = 4$. The coordinates are easily checked to be Hamilton–Jacobi separable and $f =$

$\ln(g^{1/2}/S) = -\ln(e^y - e^x)$. Since $n_1 = 0$, condition (3) of Theorem 1 is satisfied. We first check ordinary separability. Here $H_x^{-2} = H_y^{-2} = 1$ and $g^{x\alpha}f_x + g^{y\alpha}f_y = -e^x - e^y$, $g^{x\beta}f_x + g^{y\beta}f_y = -1$ so $\Sigma_r g^{r\gamma}f_r$ is always a Stäckel multiplier. It follows that the Helmholtz equation separates in the coordinates $\{x^j\}$. We have shown that $Q = f$ satisfies condition (2) in Theorem 1. However, once we have separation we can achieve further R-separation by choosing Q to be any other function satisfying condition (2). In particular choose $Q = 0$. Then the Helmholtz equation R-separates in the coordinates $\{x^j\}$ with $R = (e^y - e^x)^{1/2}$. (The phenomenon of multiple R-separation for a single coordinate system is possible only if type 2 coordinates are present.) In Ref. 12 we give the operator characterizations of these coordinates in accordance with Theorem 3.

Upon comparison of Theorem 2 and 3 it is clear that R-separation and not just ordinary separation is the appropriate Helmholtz analogy of separation for the Hamilton–Jacobi equation.

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