Nonorthogonal R-separable coordinates for four-dimensional complex Riemannian spaces

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We classify all R-separable coordinate systems for the equations $\Delta_4\Psi = \Sigma_{i,j=1}^4 g^{-1/2}\partial_j(g^{1/2}g^{ij}\partial_i\Psi) = 0$ and $\Sigma_{i,j=1}^4 g^{ij}\partial_i W\partial_j W = 0$ with special emphasis on nonorthogonal coordinates, and give a group theoretic interpretation of the results. For flat space we show that the two equations separate in exactly the same coordinate systems and present a detailed list of the possibilities. We demonstrate that every R-separable system for the Laplace equation $\Delta_4\Psi = 0$ on a conformally flat space corresponds to a separable system for the Helmholtz equations $\Delta_4\Phi = \lambda\Phi$ on one of the manifolds E_4, $S_1\times S_3$, $S_2\times S_2$, and S_4.

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1. INTRODUCTION

In this article we study the problem of R separation of variables for the Laplace and Hamilton–Jacobi equations

$$\text{(a)}\,\Delta_4\Psi = \sum_{i,j=1}^{4} g^{-1/2}\partial_i(g^{1/2}g^{ij}\partial_j\Psi) = 0\,,$$
$$\text{(b)}\,\sum_{i,j=1}^{4} g^{ij}\partial_i W\partial_j W = 0\,. \qquad (1.1)$$

ric, $g = \det(g_{ij})\neq 0$, $\Sigma_{i,j=1}^4 g^{ij}g_{jk} = \delta^i_k$, $g_{ij} = g_{ji}$, and $\partial_j\Psi = \partial_{x^j}\Psi$. Some aspects of R separation for these equations have been treated in an earlier paper.[1] In that paper we studied the *orthogonal* coordinate systems for which Eqs. (1.1) are R separable. For conformally flat spaces it was shown that each R-separable orthogonal coordinate system for Eq. (1.1a) corresponds to coordinates which permit pure separation for the Helmholtz equation $\Delta_4\Phi = \lambda\Phi$ on one of the manifolds E_4 (flat space), $S_1\times S_3$, $S_2\times S_2$, or S_4, where S_j is the j dimensional sphere. In this paper we show that the same basic results hold for *nonorthogonal* coordinate systems. However, our methods here differ considerably from those of Ref. 1. It is easy to show that if a coordinate system $\{x^j\}$ (orthogonal or not) is R separable for Eq. (1.1a) on a given Riemannian space, then it is also additively separable for Eq. (1.1b). For orthogonal coordinates on conformally flat spaces the condition that an additively separable system for Eq. (1.1b) also R separates Eq. (1.1a) could be completely solved by employing the Robertson condition in the geometrical form due to Eisenhart.[2,3] However, the Robertson condition no longer holds in general for nonorthogonal coordinates[4] and in this paper we find it necessary to employ detailed facts concerning the structure of the conformal symmetry group of Eq. (1.1b) in order to obtain our results. Indeed the use of Lie theory appears to be absolutely essential in this regard.

The paper is arranged as follows: In Sec. 2 we classify the possible types of separable systems for the Hamilton–Jacobi equation (1.1b) and in Sec. 3 we give the corresponding (crude) classification of R-separable systems for the Laplace equation (1.1a). Then in Sec. 4 we study in detail the nonorthogonal separable systems for conformally flat spaces and obtain an explicit list. Finally, in Sec. 5 we use our detailed results to show that, even allowing nonorthogonal coordinates, the flat space equations (1.1a) and (1.1b) separate in exactly the same systems and that on a conformally flat space every R-separable system for Eq. (1.1a) corresponds to a separable system for the Helmholtz equation on one of the manifolds E_4, $S_1\times S_3$, $S_2\times S_2$, and S_4. Nonorthogonal coordinates arise only from E_4 and S_4. The extreme importance of these constant curvature manifolds for variable separation on conformally flat spaces is now clear.

The authors have already given an exhaustive study of nonorthogonal separation for the Helmholtz equations on E_4, S_2, and S_3.[4,5,7] The remaining case S_4 will be treated in a forthcoming paper. This paper will then conclude our analysis of variable separation for the Hamilton–Jacobi, Helmholtz, and Laplace equations on three and four dimensional Riemannian spaces.[6–9]

2. SEPARABLE SYSTEMS FOR THE HAMILTON–JACOBI EQUATION

We now discuss the classification of separable systems for Eq. (1.1b). Recall that separation of variables for this equation means $W = \Sigma_{i=1}^4 W^{(i)}(x^i)$. The existence of separable systems for Eq. (1.1b) is closely related to the symmetries of this equation. To define symmetry operators we employ a phase space formalism. The coordinates of this space are (x^j, p_j), where $p_j = \partial_{x^j}W$, $j = 1,2,3,4$. The Poisson bracket of two functions F, G on phase space is the function

$$\{F,G\}(x,p) = \sum_{j=1}^{4}(\partial_{x^j}G\partial_{p_j}F - \partial_{x^j}F\partial_{p_j}G)\,. \qquad (2.1)$$

A *first order symmetry* of Eq. (1.1b) is a function

$$\mathscr{L} = \sum_{i=1}^{4}\xi^i(x)p_i \qquad (2.2)$$

such that $\{\mathscr{L},\Sigma_{i,j=1}^4 g^{ij}p_ip_j\} = \rho(x)\,(\Sigma_{i,j=1}^4 g^{ij}p_ip_j)$ for

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some analytic function ρ. The $\{\xi^i(x)\}$ are just the conformal Killing vector fields for the metric $\{g_{ij}\}$. The first order symmetries form a Lie algebra $\mathscr{H}$ under the Poisson bracket with $\dim\mathscr{H}\leqslant 15$ and the maximum dimension is achieved if and only if g_{ij} is conformally flat, in which case $\mathscr{H}\cong O(6,\mathbb{C})$. A (*strictly*) *second order symmetry* is a function

$$\mathscr{L}' = \sum_{i,j=1}^{4} \eta^{ij}(x) p_i p_j, \quad \eta^{ij} = \eta^{ji}, \tag{2.3}$$

such that

$$\left\{\mathscr{L}', \sum_{i,j=1}^{4} g^{ij} p_i p_j\right\} = \mu(s,p)\left(\sum_{i,j=1}^{4} g^{ij} p_i p_j\right),$$

where $\mu(x,p)$ is a linear function of the p_l. The vector space of second order symmetries can be decomposed into orbits under the adjoint action of $\mathscr{H}$. We will show explicitly that every class of separable solutions W of Eq. (1.1b) is characterized by a triplet of first or second order symmetries $\mathscr{L}_1$, $\mathscr{L}_2$, $\mathscr{L}_3$ which are in involution, i.e., $\{\mathscr{L}_i, \mathscr{L}_j\} = 0$ for $i\neq j$. The exact characterization is $\mathscr{L}_i = \lambda_i$ $(i = 1,2,3)$, where the λ_i are the separation constants.

Our classification of separable systems is based on the number of ignorable and essential variables. A variable x^i in a separable system is termed *ignorable* if $\mathscr{L} = p_i$ is a symmetry for Eq. (1.1b), where $p_i = \partial_{x^i} W$. Otherwise the variable x^i is *essential*. If the separated ordinary differential equation in the essential variable x^i is first degree, then x^i is of *type* 1; if second degree, then x^i is of *type* 2.

We consider a separable system for Eq. (1.1b) with two essential variables of type 2 (x^1,x^2), one essential variable of type 1 (x^3), and one ignorable variable (x^4). (This is called a type G equation.) With $W=\Sigma_{i,j=1}^{4} W^{(j)}(x^j)$, $W_j=\partial_j W$ we can write the separated ordinary differential equations in the form

$$\begin{aligned}
&W_1^2 + f_1 W_4^2 + \lambda_1 a_1 + \lambda_2 b_1 \equiv \Phi_1 = 0,\\
&W_2^2 + f_2 W_4^2 + \lambda_1 a_2 + \lambda_2 b_2 \equiv \Phi_2 = 0,\\
&W_3 W_4 + \lambda_1 a_3 + \lambda_2 b_3 \equiv \Phi_3 = 0,\\
&W_4 = \lambda_3,
\end{aligned} \tag{2.4}$$

where f_j, a_j, b_j are function of x^j and $\lambda_1, \lambda_2, \lambda_3$ are the separation constants. Making the trivial change of variable $x^j = X^j(\tilde{x}^j)$ if necessary, we can assume without loss of generality that $a_1 = b_2 = a_3 = 1$. To relate Eq. (1.1b) with Eqs. (2.4) we seek functions $\Theta_j(x^1,\ldots,x^4)$ such that

$$\sum_{j=1}^{3} \Theta_j \Phi_j \equiv \sum_{i,j}^{4} g^{ij} W_i W_j \tag{2.5}$$

identically in the separation constants, i.e., the coefficients of $\lambda_1, \lambda_2, \lambda_3$ should vanish in Eq. (2.5). As is easily verified, this condition determines the Θ_j up to an arbitrary multiple $Q(x^1,\ldots,x^4)$ and leads to the Hamilton–Jacobi equation

$$\text{(G)}\quad Q\,[(a_2 b_3 - 1)(W_1^2 + f_1 W_4^2) + (b_1 - b_3)\times(W_2^2 + f_2 W_4^2) + (1 - a_2 b_1) W_3 W_4] = 0, \tag{2.6}$$

with symmetry operators

$$\begin{aligned}
&\mathscr{L}_1 = (a_2 b_1 - 1)^{-1}\,(p_1^2 + f_1 p_4^2 - b_1(p_2^2 + f_2 p_4^2)),\\
&\mathscr{L}_2 = (a_2 b_1 - 1)^{-1}\,(p_2^2 + f_2 p_4^2 - a_2(p_1^2 + f_1 p_4^2)),\\
&\mathscr{L}_3 = p_4.
\end{aligned} \tag{2.7}$$

The most general metric tensor yielding separation of this type can be read off from Eq. (2.6) and the separation is characterized by $\mathscr{L}_j = \lambda_j, j = 1,2,3.$

In addition to the type G separable equations above, the following Hamilton–Jacobi equations admit separation:

(A) Four ignorable variables:

$$\text{(A)}\quad Q\sum_{i=1}^{4} p_i^2 = 0, \mathscr{L}_i = p_i^2, \quad i = 1,2,3; \tag{2.8}$$

(B) Three ignorable variables:

$$\text{(B)}\quad Q\sum_{i,j=1}^{4} G^{ij}(x^4) p_i p_j = 0, \mathscr{L}_i = p_i, \quad i = 1,2,3; \tag{2.9}$$

(C) Two ignorable variables with two essential variables of type 2:

$$\begin{aligned}
\text{(C)}\quad & Q\,[p_1^2 + p_2^2 + (e_1 + e_2) p_3^2 + 2(h_1 + h_2) p_3 p_4 + (f_1 + f_2) p_4^2] = 0,\\
& \mathscr{L}_1 = p_3, \quad \mathscr{L}_2 = p_4,\\
& \mathscr{L}_3 = p_1^2 + e_1 p_3^2 + 2h_1 p_3 p_4 + f_1 p_4^2;
\end{aligned} \tag{2.10}$$

(D) Two ignorable variables with one essential variable of each type: [It can be shown that (D2) is a special case of (D1).]

$$\begin{aligned}
\text{(D1)}\quad & Q\,[p_1^2 + 2a_2 p_2 p_3 + 2b_2 p_2 p_4 + d_1 p_3^2 + 2(f_1 + f_2) p_3 p_4 + e_1 p_4^2] = 0,\\
& \mathscr{L}_1 = p_3, \mathscr{L}_2 = p_4, \mathscr{L}_3 = 2a_2 p_2 p_3 + 2b_2 p_2 p_4 + 2 f_2 p_3 p_4,
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
\text{(D2)}\quad & Q\,[p_1^2 + 2 p_2 p_4 + (d_1 + d_2) p_3^2 + 2 f_1 p_3 p_4 + e_1 p_4^2] = 0,\\
& \mathscr{L}_1 = p_3, \mathscr{L}_2 = p_4, \mathscr{L}_3 = 2 p_2 p_4 + d_2 p_3^2;
\end{aligned} \tag{2.12}$$

(E) Two ignorable variables with two essential variables of type 1:

$$\begin{aligned}
\text{(E1)}\quad & Q\,(2a_1 p_1 p_3 + 2 p_1 p_4 + 2a_2 p_2 p_3 + 2 p_2 p_4 + (c_1 - c_2) p_3^2) = 0,\\
& \mathscr{L}_1 = p_3, \mathscr{L}_2 = p_4, \mathscr{L}_3 = 2a_2 p_2 p_3 + 2 p_2 p_4 + c_2\, p_3^2,
\end{aligned} \tag{2.13}$$

$$\begin{aligned}
\text{(E2)}\quad & Q\,(2 p_1 p_4 + 2 p_2 p_3 + 2b_2 p_2 p_4 + (d_1 + d_2) p_3^2) = 0, \quad b_2 \neq 0,\\
& \mathscr{L}_1 = p_3, \mathscr{L}_2 = p_4, \mathscr{L}_3 = 2 p_2 p_3 + 2b_2 p_2 p_4 + c_2 p_3^2,
\end{aligned} \tag{2.14}$$

$$\begin{aligned}
\text{(E3)}\quad & Q\,(2 p_1 p_4 + 2 p_2 p_3 + c_1 p_3^2 + d_2 p_4^2) = 0,\\
& \mathscr{L}_1 = p_3, \mathscr{L}_2 = p_4, \mathscr{L}_3 = 2 p_2 p_3 + d_2 p_4^2;
\end{aligned} \tag{2.15}$$

(F) One ignorable variable with three essential variables of type 2:

$$\begin{aligned}
\text{(F)}\quad & Q\,((q_2 - q_3) p_1^2 + (q_3 - q_1) + p_2^2 + (q_1 - q_2) p_3^2 + [r_1(q_2 - q_3) + r_2(q_3 - q_1) + r_3(q_1 - q_2)]\, p_4^2) = 0,\\
& \mathscr{L}_1 = p_4^2, \mathscr{L}_2 = \mathscr{Q}\,[(q_3^2 - q_2^2)\mathscr{P}_1^2 + (q_1^2 - q_3^2)\mathscr{P}_2^2 + (q_2^2 - q_1^2)\mathscr{P}_3^2],\\
& \mathscr{L}_3 = \mathscr{Q}\,[q_2 q_3 (q_2 - q_3)\mathscr{P}_1^2 + q_1 q_3 (q_3 - q_1)\mathscr{P}_2^2 + q_1 q_2 (q_1 - q_2)\mathscr{P}_3^2],
\end{aligned} \tag{2.16}$$

where

$$\mathscr{Q} = [(q_1 - q_2)(q_1 - q_3)(q_2 - q_3)]^{-1}, \mathscr{P}_i^2 = p_i^2 + r_i p_4^2,$$

$i = 1,2,3$;

(H) No ignorable variables

$$\text{(H)} \quad Q\Big(\sum_{j=1}^{4} M_{j,1} p_j^2\Big) = 0, \qquad (2.17)$$

$$\mathscr{L}_i = \sum_{j=1}^{4} M_{j,i+1} p_j^2, \quad i = 1,2,3,$$

where M_{jl} is the (j,l) minor of a 4×4 Stäckel matrix $[\Phi_{km}(x^k)]$.

Just as noted in the case of three dimensions, there are no strictly R-separable solutions of the Hamilton–Jacobi equation which are not equivalent to one of the separable types listed above.[6] [An R-separable solution would have the form $W = W^0(x^1, x^2, x^3, x^4) + \Sigma_{j=1}^4 W^{(j)}(x^j)$.]

3. *R*-SEPARABLE SYSTEMS FOR THE LAPLACE EQUATION $\Delta_4 \Psi = 0$

Here we classify the systems for which the Laplace equation (1.1a) admits R separation of variables. Again the separable systems can be characterized by a triplet of commuting symmetry operators. Recall that

$$L = \sum_{j=1}^{4} \xi^j(x)\partial_{x^j} + \xi(x) \qquad (3.1)$$

is a *first order symmetry operator* for Eq. (1.1a) if $[L, \Delta_4] = \rho(x)\Delta_4$ for some analytic function ρ. The set of all first order symmetries L forms a Lie algebra G under the commutator bracket $[A,B] = AB - BA$, called the *symmetry algebra* of Eq. (1.1a). The ξ^j satisfy the Killing equations for a conformal Killing vector relative to the metric g_{ij} and (factoring out the ideal generated by the trivial symmetry $L = 1$) G is a subalgebra of the infinitesimal conformal group of the metric. When g_{ij} corresponds to flat space then $\mathscr{G} \cong \mathrm{O}(6,\mathbb{C})$, a 15-dimensional complex Lie algebra.

Similarly,

$$L' = \sum_{i,k=1}^{4} \eta^{jk}(x)\partial_{x^j x^k} + \sum_{l=1}^{4} \eta^l(x)\partial_{x^l} + \eta(x) \qquad (3.2)$$

is a *second order symmetry operator* for Δ_4 if $[L', \Delta_4] = K\Delta_4$, where K is a first order differential operator of the form (3.1) (but K is not necessarily a symmetry). If every L' acting on the solution space of Eq. (1.1a) agrees with a linear combination of first and second order operators in the enveloping algebra of $\mathscr{G}$, then Eq. (1.1a) is said to be of *class* I; otherwise it is of *class* II.

We now proceed to classify all systems for which Eq. (1.1a) is R separable, i.e., for which Eq. (1.1a) admits solutions of the form $\Psi = e^R \Pi_{i=1}^4 \Psi^{(i)}(x^i)$, where each $\Psi^{(i)}(x^i)$ satisfies an ordinary differential equation and R is some specified function of the x^j. Substituting $\Psi = e^R\Phi$ into $\Delta_4\Psi = 0$ we obtain the equation

$$\sum_{i,j=1}^{4} b^{ij}\partial_{x^i x^j}\Phi + \sum_{i=1}^{4} b^i\partial_{x^i}\Phi + b_0\Phi = 0, \qquad (3.3)$$

where

$$b^{ij} = g^{ij}, \quad b^i = \sum_{j=1}^{4} g^{ij}\partial_{x^j}\ln[g^{1/2}g^{ij}M^2],$$

$$b_0 = M^{-1}(\Delta_4 M), \quad M = e^R.$$

Clearly, an R-separable solution of Eq. (1.1a) corresponds to a purely separable solution of Eq. (3.3). In proceeding to classify R-separable systems we do not distinguish between purely separable and strictly R-separable systems for Eq. (1.1a) because the conditions for pure separation can be obtained from those for R separation by setting $M = 1$.

The classification of R-separable types proceeds along the lines of the systems treated for the Hamilton–Jacobi equation. A variable x^i in a separable system is *ignorable* if for some analytic function ρ, $L = \partial_{x^i} + \rho(x)$ is a symmetry operator for Eq. (1.1a); otherwise x^i is *essential*. If the separated equation in the essential variable x^i is first order, then x^i is of *type* 1; if second order, then x^i is of *type* 2. It is readily seen that for a given metric the separation of Eq. (1.1b) is necessary for the R separation of Eq. (1.1a). Thus, the only possible systems permitting R separation of Eq. (1.1a) are those listed in Sec. 2. However, there are additional conditions that must be satisfied by the multiplier M in order for variables to R separate.

To explain our method we treat one example, the analogy of the type G equation for Sec. 2, in detail. Here there are two essential variables of type 2 (x^1,x^2), one essential variable of type 1 (x^3), and one ignorable variable (x^4). With $\Psi = M\,\Pi_{j=1}^4 \Psi^{(j)}(x^j)$ we can write the separated ordinary differential equations as

$$\begin{aligned}
&\Psi_{11}^{(1)} + h_1\Psi_1^{(1)} + (f_1\lambda_3^2 + \lambda_1 a_1 + \lambda_2 b_1 + K_1)\Psi^{(1)} \\
&\quad \equiv \Phi_1\Psi^{(1)} = 0, \\
&\Psi_{22}^{(2)} + h_2\Psi_2^{(2)} + (f_2\lambda_3^2 + \lambda_1 a_2 + \lambda_2 b_2 + K_2)\Psi^{(2)} \\
&\quad \equiv \Phi_2\Psi^{(2)} = 0, \qquad (3.4)\\
&\Psi_3^{(3)}\lambda_3 + (\lambda_1 a_3 + \lambda_2 b_3 + K_3)\Psi^{(3)} \equiv \Phi_3\Psi^{(3)} = 0, \\
&\Psi_4^{(4)} = \lambda_3\Psi^{(4)},
\end{aligned}$$

where $\Psi_{jj}^{(j)} = \partial_{jj}\Psi^{(j)}$. To relate Eqs. (3.3) with (3.4) one looks for functions $\Theta_j\ (x^1,...,x^4)$ such that

$$\Phi\sum_{j=1}^{3}\Theta_j\Phi_j \equiv \sum_{i,j=1}^{4} b^{ij}\partial_{ij}\Phi + \sum_{i=1}^{4} b^i\partial_i\Phi + b_0\Phi = 0, \qquad (3.5)$$

where $\Phi = \Pi_{j=1}^4\Psi^{(j)}(x^j)$. Comparison of the coefficients of the second derivative terms and the λ_i terms on both sides of Eq. (3.5) leads to the same solutions for Θ_j and g^{ij} as found in Eq. (2.6). Comparison of the coefficients of the first derivative and constant terms yields the R-separation conditions

$$\begin{aligned}
\frac{M^2}{Q} &= \Big[\prod_{i=1}^{3} A_i(x^i)\Big] \\
&\quad\times \exp(\alpha x^4)(1 - a_2 b_1)[(a_2 b_3 - 1)(b_1 - b_3)]^{1/2}, \\
\Delta_4 M &= MQ\,[K_1(a_2 b_3 - 1) + K_2(b_1 - b_3) + K_3(1 - a_2 b_1)], \\
\alpha &\in \mathbb{C}. \qquad (3.6)
\end{aligned}$$

The symmetry operators $\mathscr{L}'_j$ for Eq. (3.5) such that $\mathscr{L}'_j\Phi = \lambda_j\Phi, j = 1,2,3$ can easily be obtained by solving for λ_1, λ_2, and λ_3 in Eqs. (3.4). The simplest of these is $\mathscr{L}'_3 = \partial_4$; the other two operators while straightforward to compute have rather lengthy expressions which we will not bother to put down. Finally, the symmetry operators $\mathscr{L}_j$ for Eq. (1.1a) such that $\mathscr{L}_j\Psi = \lambda_j \Psi$ are given by $\mathscr{L}_j = M\mathscr{L}'_j M^{-1}$. The simplest of these is

$$\mathscr{L}_3 = \partial_4 - \frac{1}{2}\frac{Q_4}{Q} - \frac{1}{2}\alpha .$$

In the following we list the R-separable coordinates for Eq. (1.1a) in each of the cases (A)–(H), excluding (G) already listed. In each case we give the form of the metric ds^2 and the necessary and sufficient conditions to ensure R separation. The Hamilton–Jacobi equations and the defining triplet of commuting symmetry operators can be obtained in a straightforward manner from these results:

(A) All variables ignorable:

$$ds^2 = Q\left(\sum_{i=1}^{4}(dx^i)^2\right), \qquad M^2Q = \exp\left(\sum_{i=1}^{4}\alpha_i x^i\right), \Delta_4 M = \frac{M}{Q}\alpha_0, \quad \alpha_j \in \mathbb{C}; \tag{3.7}$$

(B) Three ignorable variables:

$$ds^2 = Q\left(\sum_{i,j=1}^{4} g_{ij}(x^1)dx^i dx^j\right), \qquad M^2Q = f(x^1)\exp\left(\sum_{i=1}^{4}\alpha_i x^i\right), \tag{3.8}$$

$$\Delta_4 M = \frac{M}{Q}h(x^1), \quad \alpha_j \in \mathbb{C};$$

(C) Two ignorable variables and two essential variables of type 2:

$$ds^2 = Q\Big((dx^1)^2 + (dx^2)^2 + \frac{1}{(ef-h^2)} \times \{f(dx^3)^2 + e(dx^4)^2 - 2h dx^3 dx^4\}\Big),$$

$$M^2Q = A_1(x^1)A_2(x^2)\exp(\alpha_3 x^3 + \alpha_4 x^4)[ef-h^2]^{1/2}, \tag{3.9}$$

$$\Delta_4 M = \frac{M}{Q}(K_1 + K_2),$$

$$e = e_1 + e_2, h = h_1 + h_2, f = f_1 + f_2;$$

(D) Two ignorable variables with one essential variable of each type:

(D1) $$dx^2 = Q\,[(dx^1)^2 + (2b_2 f - e_1 - b_2^2 d_1)^{-1}\{(e_1 d_1 - f^2)(dx^2)^2 - (b_2 dx^3 - dx^4)^2 + 2(b_2 f - e_1)dx^2 dx^3 + 2(f - b_2 d_1)dx^2 dx^4\}], f = f_1 + f_2, \tag{3.10}$$

$$M^2Q = A_1(x^1)A_2(x^2)\exp(\alpha_3 x^3 + \alpha_4 x^4) \times [2b_2 f - e_1 - b_2^2 d_1]^{1/2},$$

$$\Delta_4 M = \frac{M}{Q}(K_1 + K_2),$$

(D2) $$ds^2 = Q\Big[(dx^1)^2 + \frac{1}{d}\{(f_1^2 - e_1 d)(dx^2)^2 + (dx^3)^2 - 2f_1 dx^2 dx^3 + 2d\, dx^2 dx^4\}\Big],$$

$$M^2Q = A_1(x^1)A_2(x^2)\exp(\alpha_3 x^3 + \alpha_4 x^4)(d)^{1/2}, \tag{3.11}$$

$$\Delta_4 M = \frac{M}{Q}(K_1 + K_2), \quad d = d_1 + d_2;$$

(E) Two ignorable variables with two essential variables of type 1: For systems of this type we supply some of the details of the R-separation conditions:

(E1) $$ds^2 = Q\Big[-\Big(\frac{c_1 - c_2}{a_1 - a_2}\Big)(dx^1 - dx^2)^2 + 2\,dx^3(dx^1 - dx^2) + 2dx^4(a_1 dx^2 - a_2 dx^1)\Big]. \tag{3.12}$$

The (first derivative conditions for R separation of $\Delta_4\Psi = 0$ are equivalent to

$$(\partial_1 + \partial_2)\ln(M^2Q) = a_1 + a_2,$$
$$(a_1\partial_1 + a_2\partial_2)\ln(M^2Q) = b_1 + b_2, \tag{3.13}$$
$$\partial_j \ln(M^2Q) = \alpha_j \in \mathbb{C}, j = 3,4.$$

The solutions of these conditions fall into two classes:

Class (i): (a) $a_1 = \cosh x^1$, $a_2 = \cosh x^2$ or (b) $a_1 = e^{x^1}$, $a_2 = e^{x^2}$; (3.14)

then

$$M^2Q = \Big[\sinh\frac{1}{2}(x^1 - x^2)\Big]^{\alpha} A_1(x^1)A_2(x^2) \times \exp(\alpha_3 x^3 + \alpha_4 x^4), \tag{3.15}$$

$$\Delta_4 M = \frac{M}{Q}(K_1 + K_2), \ \alpha, \ \alpha_j \in \mathbb{C};$$

Class (ii): a_1, a_2 are not of the form (3.14). (3.16)

The the R-separation conditions are of the form (3.15) wih $\alpha = 0$:

(E2) $$ds^2 = Q\,[-(d_1 + d_2)(b_2\, dx^1 - dx^2)^2 + 2dx^3(dx^2 - b_2\, dx^1) + 2dx^1\, dx^4]. \tag{3.17}$$

The (first derivative) conditions for R separation of $\Delta_4\Psi = 0$ are equivalent to

$$(\partial_1 + b_2\partial_2)\ln(M^2Q) = a_1 + a_2,$$
$$\partial_2 \ln(M^2Q) = b_1 + b_2, \partial_j \ln(M^2Q) = \alpha_j \in \mathbb{C}, \quad j = 3,4. \tag{3.18}$$

There are two solutions to these conditions:

Class (i): $b_2 \neq x^2$, (3.19)

$$M^2Q = A_1(x^1)A_2(x^2)\exp(\alpha_3 x^3 + \alpha_4 x^4),$$

$$\Delta_4 M = \frac{M}{Q}(K_1 + K_2), \tag{3.20}$$

Class (ii): $b_2 = x^2$. (3.21)

Conditions (3.20) hold, except that now

$$M^2Q = A_1 A_2 \exp(x^2 e^{-x^1} + \alpha_3 x^3 + \alpha_4 x^4), \tag{3.22}$$

(E3) $$ds^2 = Q\,[-d_2(dx^1)^2 - c_1(dx^2)^2 + 2dx^1\, dx^4 + 2dx^2 dx^3]. \tag{3.23}$$

The first derivative R-separation conditions are

$$\partial_1 \ln(M^2Q) = a_1 + a_2, \quad \partial_2 \ln(M^2Q) = b_1 + b_2,$$
$$\partial_j \ln(M^2Q) = \alpha_j, j = 3,4. \tag{3.24}$$

These conditions have the general solution

$$M^2Q = A_1 A_2 \exp(\epsilon x^1 x^2 + \alpha_3 x^3 + \alpha_4 x^4), \quad \epsilon = 0,1,$$

$$\Delta_4 M = \frac{M}{Q}(K_1 + K_2). \quad (3.25)$$

(F) One ignorable variable with three essential variables of type 2:

$$ds^2 = Q\left[\frac{(dx^1)^2}{q_2 - q_3} + \frac{(dx^2)^2}{q_3 - q_1} + \frac{(dx^3)^2}{q_1 - q_2} + \frac{(dx^4)^2}{[r_1(q_2 - q_3) + r_2(q_3 - q_1) + r_3(q_1 - q_2)]}\right],$$
$$M^2 Q = A_1 A_2 A_3 S \exp(\alpha_4 x^4),$$
$$S = [(q_2 - q_3)(q_3 - q_1)(q_1 - q_2)\{r_1(q_2 - q_3) + r_2(q_3 - q_1) + r_3(q_1 - q_2)\}]^{1/2}, \quad (3.26)$$
$$\Delta_4 M = \frac{M}{Q}[K_1(q_2 - q_3) + K_2(q_3 - q_1) + K_3(q_1 - q_2)];$$

(H) No ignorable variables:

$$ds^2 = Q\left(\sum_{i=1}^{4}\frac{S}{M_{i1}}\right), \quad S = \det\Phi \neq 0,$$
$$\Phi = (\Phi_{ij}(x^i)), \quad QM^2 = \frac{A_1 A_2 A_3 A_4}{S}\left[\prod_{j=1}^{4} M_{j1}\right]^{1/2},$$
$$\Delta_4 M = \frac{M}{Q}\left(\sum_{j=1}^{4}\frac{B_j M_{j1}}{S} + \alpha\right). \quad (3.27)$$

Here M_{j1} is the $(j, 1)$ cofactor of the 4×4 Stäckel matrix Φ.

4. CONFORMALLY FLAT NONORTHOGONAL *R*-SEPARABLE SYSTEMS

Here we specialize the results of Sec. 2 and 3 to flat space, limiting ourselves to nonorthogonal coordinates. (The orthogonal case has already been treated in Ref. 1.) In principle, the classification is straightforward: One need only compute the Riemann curvature tensor for each of the separable nonorthogonal metrics (A)–(G), require that it vanish identically, and classify all possibilities. In practice, however, the computations are hopelessly complicated. The problem becomes tractible only if detailed use is made of the conformal symmetry algebra O(6,ℂ) of the flat space Laplace equation.

A basis for O(6,ℂ) is given by

$$P_j = \partial_{z^j}, \quad j = 1,2,3,4,$$
$$I_{kl} = z^k\partial_{z^l} - z^l\partial_{z^k} = -I_{lk}, \quad 1 \leqslant k < l \leqslant 4,$$
$$D = -\left(1 + \sum_{i=1}^{4} z^i\partial_{z^i}\right), \quad (4.1)$$
$$K_j = 2z^j + (2(z^j)^2 - z\cdot z)\partial_{z^j} + 2z^j z^l\partial_{z^l} + 2z^j z^m\partial_{z^m} + 2z^j z^n\partial_{z^n},$$

where $j,l,m,n = 1,2,3,4$ and no two are equal. Now every nonorthogonal R-separable system for the flat space Laplace equation

$$\sum_{j=1}^{4}\partial^2_{z^j}\Psi = 0, \quad (4.2)$$

or any other Laplace equation, contains at least one ignorable variable x^1. Clearly there must exist an analytic function ρ such that $\partial_{x^1} + \rho = L \in O(6,\mathbb{C})$. In general, a system with m ignorable variables is associated with an m dimensional Abelian subalgebra of O(6,ℂ). Since we identify two systems if one can be obtained from the other by an action of the conformal symmetry group, to classify all possibilities for ignorable variables associated with Eq. (4.2) it is necessary and sufficient to determine all equivalence classes of Abelian O(6,ℂ) subalgebras under the adjoint action of O(6,ℂ).

We first list the classes of one dimensional subalgebras of O(6,ℂ). To obtain most easily the results of Table I we have made use of the well known isomorphism $O(6,\mathbb{C}) \cong sl(4,\mathbb{C})$ and the Jordan canonical forms for 4×4 matrices. We have also identified in this and the higher dimensional cases those subalgebras which can be mapped into one another under the outer automorphisms of spatial reflection and inversion. For each equivalence class we exhibit a representative element.

Suppose that t is an ignorable variable belonging to the R-separable system $\{t, x^2, x^3, x^4\}$. Then we can assume the corresponding symmetry operator $L = \partial_t + \rho$ is identical with one of the five operators listed in Table I. From this relationship we can determine how the "standard" coordinates $z^1,...z^4$ are associated to t and the general form of the metric ds^2 in terms of dt. The ignorable variable t is *orthogonal* if the corresponding metric can be written

$$ds^2 = Q\left[dt^2 + \sum_{i,j=2}^{4} g_{ij}(t,x^2,x^3,x^4)dx^i dx^j\right]. \quad (4.3)$$

Otherwise, t is *nonorthogonal*. In Table II we list the metrics and coordinates corresponding to the operators in Table I.

It follows from Table II that the only operators associated with orthogonal ignorable variables are I_{14}, I_{23}, D, and P_3. Among nonorthogonal ignorable variables the only one for which the $(dt)^2$ term doesn't occur in the metric ds^2 is associated with the operator $P_3 + iP_4$. (Note that in each case t is not unique; it can be replaced by $t' = t + f$ for arbitrary f. For nonorthogonal variables the assertion is that, no matter what the choice of f, the metric contains cross terms of the form $dt'\,da$.) This last possibility is of great interest, for it leads to "heat type" variables (slightly renormalized):

$$z^1 = a, z^2 = b, z^3 - iz^4 = 2t, z^3 + iz^4 = c. \quad (4.4)$$

If in these coordinates we assume a solution of Eq. (1.1a) of the form $\Psi = \Phi(a,b,c)e^{\beta t}$, the resulting equation becomes

$$(\partial_a^2 + \partial_b^2)\Phi = \beta\partial_c\Phi. \quad (4.5)$$

Note that here the ignorable variable t is characterized by its nonorthogonality and the fact that there is no $(dt)^2$ term in the metric.

TABLE I. One-dimensional subalgebras of O(6,ℂ).

1. $\alpha I_{14} + \beta I_{23} + \gamma D$, $\alpha,\beta,\gamma \in \mathbb{C}$
2. $\alpha(-I_{14} + I_{23} - iD) + P_2 + iP_3 - I_{34} - iI_{24} + iI_{13} - I_{12}$, $i = \sqrt{-1}$
3. $\alpha I_{14} + P_3$
4. $\alpha I_{21} + \beta(I_{34} - iD) + P_3 + iP_4$
5. $\frac{1}{2}(P_2 + iP_3) - iI_{24} - I_{34}$

TABLE II. Metrics and coordinates associated with ignorable variables.

1. $z^1 = a\,e^{-\gamma t}\cos(\alpha t + c), \quad z^2 = b\,e^{-\gamma t}\cos\beta t$
$z^3 = b\,e^{-\gamma t}\sin\beta t, \quad z^4 = a\,e^{-\gamma t}\sin(\alpha t + c)$
$ds^2 = e^{-2\gamma t}[da^2 + db^2 - 2\gamma dt\,(a da + b db) + \{\gamma^2(a^2 + b^2) + b^2\beta^2\}dt^2 + a^2(\alpha dt + dc)^2]$

Nonorthogonal unless two of α, β, γ are zero.

2. $z^1 - iz^4 = a\,e^{2i\alpha t}, \quad z^1 + iz^4 = b + 2t^2$
$z^2 + iz^3 = (-2at + c)e^{2i\alpha t}, \quad z^2 - iz^3 = 2t$
$ds^2 = e^{2i\alpha t}[da\,db + 2dt\,(dc + i\alpha a\,db) + 4(i\alpha c - a)dt^2].$

Nonorthogonal.

3. $z^1 = a\cos\alpha t, \quad z^2 = b, \quad z^3 = t + c, \quad z^4 = a\sin\alpha t$
$ds^2 = da^2 + \alpha^2 a^2 dt^2 + db^2 + (dt + dc)^2$

Nonorthogonal unless $\alpha = 0$.

4. $z^2 + iz^1 = a\,e^{i(\beta+\alpha)t}, z^2 - iz^1 = b\,e^{i(\beta-\alpha)t}$
$z^3 - iz^4 = 2t + c, z^3 + iz^4 = f e^{2i\beta t}$
$ds^2 = e^{2i\beta t}[da\,db + i(\beta + \alpha)a\,dt\,db + i(\beta - \alpha)b\,dt\,da + df\,dc + 2dt\,(df + i\beta f\,dc) + \{ab\,(\alpha^2 - \beta^2) + 4i\beta f\}dt^2]$

Nonorthogonal. No dt^2 term only if $\alpha = \beta = 0$.

5. $z^1 = a, z^2 + iz^3 = i\sqrt{2}\,(b e^{\sqrt{2}t} - c\,e^{-\sqrt{2}t})$
$z^2 - iz^3 = t, z^4 = b\,e^{\sqrt{2}t} + c\,e^{-\sqrt{2}t}.$

Nonorthogonal.

A representative basis for each equivalence class of two-dimensional abelian subalgebras of O(6,ℂ) is listed in Table III.

The corresponding results for three dimensional abelian subalgebras are listed in Table IV.

Finally, there is only one equivalence class of four dimensional abelian subalgebras of O(6,ℂ). A representative has basis P_1, P_2, P_3, P_4.

The above results apply with only slight modification to the flat space Hamilton–Jacobi equation

$$\sum_{i=1}^{4} (\partial_{z^i} W)^2 = 0\,. \tag{4.6}$$

The symmetry algebra of this equation is again O(6,ℂ) with basis

$$P_j = p_j, \quad j = 1,2,3,4,$$
$$I_{kl} = z^k p_l - z^l p_k = -I_{lk}, \quad 1 \leqslant k < l \leqslant 4, \tag{4.7}$$
$$D = -\sum_{i=1}^{4} z^i p_i,$$
$$K_j = (2(z^j)^2 - z\cdot z)\,p_j + 2z^j z^l p_l + 2z^j z^m p_m + 2z^j z^n p_n\,,$$

where $j,l,m,n = 1,2,3,4$ and no two are equal. To find all nonorthogonal metrics for Eq. (4.6) it is clearly sufficient to examine each of the general nonorthogonal separable metrics from the list (A)–(H) of Sec. 2 and determine which of these is conformally flat. All orthogonal separable metrics for Eq. (4.6) were already computed in Ref. 1, so here we omit the systems of type (A), (F), and (H). Note that every R-separable system for the Laplace equation (4.2) must correspond to one of these conformally flat metrics. We will show later that this correspondence is one to one.

The necessary and sufficient condition that a metric $ds^2 = Q\,(\Sigma g_{ij} dx^i\,dx^j) = Q\,d\hat{s}^2$ be conformally flat is that the conformal tensor C_{ijkl} of the metric $d\hat{s}^2$ be identically zero.[10] Here,

$$C_{ijkl} = R_{ijkl} + \tfrac{1}{2}(g_{ik}R_{jl} - g_{il}R_{jk} + g_{jl}R_{ik} - g_{jk}R_{il}) + \tfrac{1}{6}R\,(g_{il}g_{jk} - g_{ik}g_{jl})\,, \tag{4.8}$$

where R_{ijkl} is the Riemann curvature tensor, R_{jl} is the Ricci tensor and R is the scalar curvature. We will use this condition to determine the number of nonorthogonal conformally flat metrics of each type.

TABLE III. Two-dimensional abelian subalgebras of O(6,ℂ).

1. $P_3 + iP_4 + \delta M_{23}, I_{34} - iD$
2. $P_3 + iP_4, P_1 + iP_2$
3. $P_3 + iP_4, I_{24} + iI_{32}$
4. $P_3 + iP_4, P_1 - iP_2 - iI_{14} + I_{31} + I_{24} + iI_{32}$
5. $P_3 + iP_4, P_1 - iP_2 - I_{34} + iD + I_{12}$
6. $P_3 + iP_4, P_3 - iP_4 - I_{24} - iI_{32} - iI_{14} + I_{31}$
7. $I_{41} + I_{23} - iD, P_2 + iP_3 - I_{34} - iI_{24} + iI_{13} - I_{12}$
8. $I_{43} + iI_{24} + I_{12} + iI_{13}, i(P_3 + K_3) + P_2 + K_2 + 2iI_{24} - 2I_{12}$
9. $I_{14} + P_3, I_{14} - iP_2$
10. $D, I_{24} + iI_{34}$
11. $P_3, P_2 + iP_3 + 2iI_{14}$
12. $iI_{24} + I_{43}, iI_{13} + I_{12}$
13. $iI_{24} + I_{43}, i(P_1 - K_1) - P_2 + K_2 + I_{43} - iI_{42} + I_{12} + iI_{13}$
14. $P_3 + I_{24} + iI_{12}, P_2 + I_{34} + iI_{13}$
15. $I_{23} - iD, P_2 + iP_3 + 2iI_{14}$
16. $I_{41} + I_{23} + \alpha(I_{23} - iD), I_{14} + iD + \beta(I_{23} - iD)$
17. $\alpha I_{14} + \beta(I_{23} - iD), P_2 + iP_3 + \delta(I_{23} - iD)$
18. $P_3 - iP_4 + \beta(D - iI_{34} - iI_{21}), D + P_2 + iP_1 - iI_{34} - iI_{21} - I_{14} + I_{32} + iI_{24} + iI_{31}$

TABLE IV. Three-dimensional abelian subalgebras of O(6,ℂ).

1. $P_3 + iP_4, I_{43} + iD - I_{12}, P_1 - iP_2 - iI_{14} + I_{31} + I_{24} + iI_{32}$
2. $P_3 + iP_4, I_{43} + iD - I_{12}, iI_{24} - I_{32} + I_{14} + iI_{31}$
3. $P_3 + iP_4, I_{43} + iD - I_{12}, P_1 + iP_2$
4. $P_3 + iP_4, P_1 + I_{24} + iI_{32}, P_2 + I_{14} + iI_{31}$
5. $P_3 + iP_4, P_1 - iP_2 + iI_{14} - I_{31} + I_{24} + iI_{32}, P_1 + iP_2$
6. $P_3 + iP_4, P_1 - iP_2 + I_{14} + iI_{31} + iI_{24} - I_{32}, I_{14} + iI_{31} - iI_{24} + I_{32}$
7. $P_3 + iP_4, I_{24} + iI_{32}, I_{14} + iI_{31}$
8. $P_3 + iP_4, P_1, P_2 + iI_{24} - I_{32}$
9. $P_3 + iP_4, P_1 - iP_2, I_{14} + iI_{31} - iI_{24} + I_{32}$
10. D, I_{14}, I_{23}
11. $D, I_{23} + I_{41}, I_{42} + iI_{21} + iI_{43} - I_{31}$
12. P_2, P_3, I_{41}
13. P_1, P_2, P_3

(B) Three ignorable variables: For forms of type (B) the conformal flatness conditions prove too complicated to solve explicitly. Fortunately, group theory comes to our rescue: The possible separable systems of this type correspond to the three dimensional Abelian subalgebras listed in Table IV. Subalgebras 10, 12, 13 correspond to orthogonal coordinates. The subalgebra 4 does not give a separable system because the three Lie derivatives are functionally dependent. (In order to define a separable coordinate system the three Lie derivatives must be functionally independent.) The remaining subalgebras yield nonorthogonal coordinates all of heat type except 11 which, once a radial variable is separated, corresponds to the single nonorthogonal separable system for the Helmholtz equation on the complex sphere S_3.[7] [However, this system also arises in E_4 where the diagonalization of D is accomplished by the diagonalization of the Casimir operator for the subalgebra O(4,$\mathbb{C}$) generated by the I_{jk}.]

The coordinates and their relationship to the standard coordinates z^j can be obtained from Tables I and II. For example, a suitable choice of coordinates for the operators of type 3 is

$$z^1 + iz^2 = e^{-2is}, \quad z^1 - iz^2 = t,$$
$$z^3 + iz^4 = w\, e^{-2is}, \quad z^3 - iz^4 = u, \tag{4.9}$$

where $\partial_s - i = I_{43} - I_{12} + iD$, $\partial_t = \frac{1}{2}(P_1 + iP_2)$, and $\partial_u = \frac{1}{2}(P_3 + iP_4)$. The corresponding differential form is

$$ds^2 = e^{-2is}[du\, dw - 2i\, ds(w\, du + dt)]. \tag{4.10}$$

We note that this metric also provides a separation of variables for the flat space Helmholtz equation $\Delta_4\Psi = E\Psi$. Indeed, if we set $x^1 = e^{-2is}$, $x^2 = t/2$, $x^3 = w/2$, $x^4 = u$ (these are equivalent coordinates), we obtain

$$ds^2 = 2\, dx^1 dx^2 + 2\, dx^4(x^3 dx^1 + x^1 dx^3) \tag{4.11}$$

and the Helmholtz equation is

$$2\left[\partial_{12} + \frac{1}{x^1}(-x^3\partial_{23} + \partial_{34})\right]\Psi = E\Psi \tag{4.12}$$

with separation equations

$$\partial_2\Psi_2 = l_1\Psi_2, \quad \partial_4\Psi_4 = l_2\Psi_4,$$
$$(-2l_1x^3\partial_3 + 2l_2\partial_3)\Psi_3 = l_3\Psi_3, \tag{4.13}$$
$$(2l_1\partial_1 + 2l_3/x^1)\Psi_1 = E\Psi_1,$$

where $\Psi = \Pi_{j=1}^4 \Psi_j(x^j)$. The operators $\mathscr{L}_j$ which describe this separation are $\mathscr{L}_1 = \frac{1}{2}(P_1 + iP_2)$, $\mathscr{L}_2 = \frac{1}{2}(P_3 + iP_4)$, $\mathscr{L}_3 = \frac{1}{4}\{P_3 - iP_4, I_{13} + iI_{23} + iI_{14} - I_{24}\}$. The operators which characterize separation in this case are not all first order and would also suffice to describe the separation in the case of the Laplace equation $E = 0$. (The significance of two separate operator characterizations of the same coordinate system will be the topic of a separate paper.) Similar comments hold for subalgebras 1 and 2 on Table IV. Subalgebras 5 to 9 clearly directly define separation of the flat space Helmholtz equation. Thus, nonorthogonal coordinates of type (B) all correspond to coordinates that separate the Helmholtz equation on E_4.

(C) Two ignorable variables and two essential variables of type 2: It would be possible but extremely complicated to derive these metrics by directly requiring the metric (3.9) to be conformally flat. An easier method follows from the observation that for a conformally flat space the two Lie symmetries corresponding to the ignorable variables x^3, x^4 are taken from the list of commuting pairs of symmetries in Table III. For each pair of symmetries from this list there are constraints on the form ds^2 and the way in which the differentials dx^3 and dx^4 appear in it. For subalgebras 1–9 the corresponding metric is such that $e = 0$, i.e., the ignorable variable x^4 is nonorthogonal and there is no $(dx^4)^2$ term appearing in ds^2. Thus, to compute all coordinates corresponding to subalgebras 1–9 we can simply require that the metric

$$d\hat{s}^2 = (l_1 - l_2)[(dx^1)^2 + (dx^2)^2] + 2dx^3\, dx^4 + \left(\frac{m_1 - m_2}{l_1 - l_2}\right)(dx^3)^2 \tag{4.14}$$

be conformally flat.

Subalgebras 10–13 each contain an orthogonal ignorable variable so they do not correspond to type (C) metrics. Subalgebras 14–18 are somewhat more awkward to treat but in each case one can show that the metrics associated with these subalgebras are not of the form (3.9). Thus, none of these subalgebras correspond to type (C) coordinates.

Now suppose the conformally flat metric is of the form (4.14). The conditions of conformal flatness are

$$C_{1221} = \tfrac{1}{3}R_{1221} = 0, C_{1442} = R_{1442} = 0,$$
$$C_{1332} = R_{1332} + \tfrac{1}{3}\left(\frac{m_1 - m_2}{l_1 - l_2}\right)^2 R_{1442} = 0,$$
$$C_{1331} = \tfrac{1}{2}(R_{1331} - R_{2332}) + \frac{1}{(l_1 - l_2)^2} \times(\tfrac{1}{3} - \tfrac{1}{2}(m_1 - m_2))R_{1221} = 0, \tag{4.15}$$
$$C_{2332} = \tfrac{1}{2}(R_{2332} - R_{1331}) + \frac{1}{(l_1 - l_2)^2} \times(\tfrac{1}{3} - \tfrac{1}{2}(m_1 - m_2))R_{1221} = 0.$$

These conditions imply $R_{ijkl} = 0$ so the metrics $d\hat{s}^2$ are flat. We then obtain the following distinct solutions:

$$d\hat{s}^2 = (x^1 - x^2)\left[\frac{(dx^1)^2}{x^1} - \frac{(dx^2)^2}{x^2}\right] + 2\, dx^3dx^4 + (x^1 + x^2)(dx^4)^2, \tag{4.16}$$

$$d\hat{s}^2 = (x^1 - x^2)[(dx^1)^2 - (dx^2)^2] + 2\, dx^3dx^4 + (x^1 + x^2)(dx^4)^2, \tag{4.17}$$

$$d\hat{s}^2 = (dx^1)^2 + (dx^2)^2 + 2\, dx^3dx^4 + (ax^1 + bx^2)(dx^4)^2. \tag{4.18}$$

The remaining conformally flat metrics of this type are of the form

$$d\hat{s}^2 = d\sigma^2 + 2\, dx^3dx^4, \tag{4.19}$$

where $d\sigma^2$ is a separable metric in Euclidean two-space (see Ref. 11). Thus, all conformally flat metrics of type (C) correspond to coordinates that separate the flat space Helmholtz equation.

(D) Two ignorable variables and one essential variable of each type: We look for conformally flat metrics of type (D1) for which the Lie symmetries corresponding to the ignorable variables x^3, x^4 are taken from the list of commuting pairs on Table III. Proceeding through the list we find that there can be no conformally flat metrics of this type, which are not already of type (D2).

For forms of type (D2) two of the conditions of conformal flatness are

$$C_{1331} = \tfrac{1}{3}R_{3113} = 0\,, C_{1223} = R_{1223} - \tfrac{1}{2}f_1 R_{1332} = 0\,. \tag{4.20}$$

These two conditions imply $d_1' = 0$ and $f_1 = 0$. The remaining conformal flatness condition is then $C_{1221} = \frac{1}{2} \times (R_{1221} - d\,R_{3223}) = 0$, which is equivalent to $d_2'' - \frac{3}{5} \times (d_2')^2/d_2 - d_2 e_1'' = 0$. This equation can be solved to give the forms

$$d\hat{s}^2 = (dx^3)^2 + [1 + (x^2)^2](dx^1)^2 + 2\,dx^2dx^4 + \frac{1}{1+(x^2)^2}(x^1dx^2)^2\,, \tag{4.21}$$

$$d\hat{s}^2 = (dx^3)^2 + x^2(dx^1)^2 + 2\,dx^2dx^4 + \frac{(x^1dx^2)^2}{x^2}\,, \tag{4.22}$$

$$d\hat{s}^2 = (x^2dx^1)^2 + ax^1\left(\frac{dx^2}{x^2}\right)^2 + (dx^3)^2 + 2\,dx^2dx^4\,. \tag{4.23}$$

(For these forms we have redefined x^2 and multiplied by a suitable function of x^2.) The forms (4.21)–(4.23) all define separation for the flat space Helmholtz equation.[8]

(E) Two ignorable variables and two essential variables of type 1: For metrics of type (E1) the conditions of conformal flatness imply that the metric $d\hat{s}^2$ is flat where $ds^2 = Q d\hat{s}^2$ is given by Eq. (3.12). Thus, from Ref. 8 we obtain the possibilities

$$d\hat{s}^2 = 2dx^3(dx^1 - dx^2) + 2\,dx^4\,[(x^1)^2dx^2 - (x^2)^2dx^1]\,, \tag{4.24}$$

$$d\hat{s}^2 = \left[A\,(x^1+x^2) + \frac{B\,(x^1+x^2)+C}{x^1-x^2}\right](dx^1-dx^2)^2 + 2dx^3(dx^1-dx^2) + 2\,dx^4(x^1dx^2 - x^2dx^1)\,, \tag{4.25}$$

$$d\hat{s}^2 = \left[\frac{A}{x^1} + \frac{B}{(x^1)^2} + \frac{C}{x^2} + \frac{D}{(x^2)^2}\right]\frac{(x^2dx^1 - x^1dx^2)^2}{x^1-x^2} + 2dx^3(x^2dx^1 - x^1dx^2) + 2\,dx^4(dx^2-dx^1)\,. \tag{4.26}$$

For metrics of type (E2), (i) the conformal flatness conditions are

$$C_{1223} = \frac{1}{b_2}C_{2113} = \tfrac{1}{2}R_{1223} = 0\,,$$
$$C_{1221} = R_{1221} + (d_1 + d_2)(R_{2113} - b_2R_{1223}) = 0\,. \tag{4.27}$$

Solving these equations we obtain the conformally flat metric

$$d\hat{s}^2 = \left(\frac{B}{e^{2x^1}} + \frac{B}{e^{x^1}} + \frac{C}{(x^2)^2} + \frac{D}{x^2}\right)(x^2dx^1 - dx^2)^2 + 2dx^3(dx^2 - x^2dx^1) + 2dx^1dx^4\,. \tag{4.28}$$

This form is conformal to the type (E2), (i) metric of Ref. 8 which defines separation of the flat space Helmholtz equation. A similar computation shows that there are no type (E2), (ii) conformally flat metrics.

For metrics of type (E3) the relevant conformal flatness conditions are $C_{1221} = R_{1221} = 0$ and we obtain the metrics

$$d\hat{s}^2 = \left(\frac{x^2dx^1}{x^1}\right)^2 - (dx^2)^2 + 2\,dx^1dx^4 + 2(x^1)^2dx^2dx^3\,, \tag{4.29}$$

$$d\hat{s}^2 = (x^2dx^1)^2 - (x^1dx^2)^2 + 2\,dx^1dx^4 + 2\,dx^2dx^3\,, \tag{4.30}$$

$$d\hat{s}^2 = x^2\left(\frac{dx^1}{x^1}\right)^2 + x^1(dx^2)^2 + 2\,dx^1dx^4 + 2(x^1)^2dx^2dx^3\,, \tag{4.31}$$

$$d\hat{s}^2 = x^2(dx^1)^2 + x^1(dx^2)^2 + 2\,dx^1dx^4 + 2\,dx^2dx^3\,, \tag{4.32}$$

$$d\hat{s}^2 = x^2\left(\frac{dx^1}{x^1}\right)^2 + 2\,dx^1dx^4 + 2(x^1)^2dx^2dx^3\,. \tag{4.33}$$

As shown in Ref. 8 these metrics define variable separation for the flat space Helmholtz equation.

(G) One ignorable variable, two essential variables of type 2 and one of type 1: Comparing the type (G) metric with the metrics on Table II we see that the symmetry operator $P_3 + iP_4$ must correspond to the ignorable variable x^4. With this restriction only the nontrivial conformal flatness conditions are $C_{1331} = C_{2332} = 0$ and we obtain two groups of metrics:

$$\text{I}: d\hat{s}^2 = (dx^1)^2 + \frac{e^{2ix^1}}{(x^3+a)}d\omega_k^2 - \frac{(dx^3)^2}{4(x^3+a)^2}\,, \tag{4.34}$$

$$\text{II}: d\hat{s}^2 = (dx^1)^2 + d\omega_k^2 + Ax^1(dx^3)^2\,, \tag{4.35}$$

where

$$\begin{aligned} d\omega_1^2 &= x^3(dx^2)^2 + 2\,dx^3dx^4 + \frac{(x^2dx^3)^2}{4x^3}\,,\\ d\omega_2^2 &= (1+(x^3)^2)(dx^2)^2 + 2\,dx^3dx^4 - \frac{(x^2dx^3)^2}{1+(x^3)^2}\,,\\ d\omega_3^2 &= (x^3dx^2)^2 + 2\,dx^3dx^4 - Bx^2\left(\frac{dx^3}{x^3}\right)^2\,,\\ d\omega_4^2 &= (dx^2)^2 + 2\,dx^3dx^4 + Ax^2(dx^3)^2\,. \end{aligned} \tag{4.36}$$

The metrics of type I determine separation for the Helmholtz equation on the four sphere S_4 and those of type II determine separation for the flat space Helmholtz equation.

This completes our classification of conformally flat nonorthogonal separable forms.

5. R-SEPARABLE COORDINATES FOR $\Delta_4\Psi = 0$

In our treatment of conformally flat metrics in Sec. 4 the original flat space metric was chosen in the form $ds^2 = Q\,(\Sigma g_{ij}dx^idx^j) = Q\,d\hat{s}^2$. In addition to the condition of conformal flatness for the metric $d\hat{s}^2$ the function $Q = e^{2\lambda}$ is determined by solving the equations

$$\lambda_{ij} = \tfrac{1}{2}(\tfrac{1}{6}g_{ij}R - R_{ij}) - \tfrac{1}{2}g_{ij}\left(\sum_{k,l=1}^{4} g^{kl}\lambda_{,k}\lambda_{,l}\right), \tag{5.1}$$

where $\lambda_{ij} = \lambda_{,ij} - \lambda_{,i}\lambda_{,j} = \partial_{x^i}\lambda$ and $\lambda_{,ij}$ is the second covariant derivative of λ with respect to g_{ij} (see Ref. 10).

As we have shown, the metrics $d\hat{s}^2$ correspond to only two manifolds: E_4 and S_4. The possible functions Q relating flat space and these two manifolds are independent of coordinates and were already computed in Ref. 1. Furthermore, it was shown in that reference that always

$$\hat{\Delta}_4 Q^{1/2} + \frac{R}{6}Q^{1/2} = 0\,, \tag{5.2}$$

where R is the (constant) scalar curvature, and $\hat{\Delta}_4$ is the Laplace–Beltrami operator on the manifold with metric $d\hat{s}^2$. When we studied orthogonal separation for $\Delta_4\Psi = 0$ in Ref.

1 we showed that we could always choose the multiplier $M = Q^{-1/2}$. Using this result as a guide we consider one of the nonorthogonal metrics $ds^2 = Qd\hat{s}^2$ listed in Sec. 4 and set $\Psi = Q^{-1/2}\Phi$. Substituting this expression into $\Delta_4\Psi = 0$ and making use of Eq. (5.2) we obtain

$$\hat{\Delta}_4\Phi + \frac{R}{6}\Phi = 0 \tag{5.3}$$

so that Φ satisfies a Helmholtz equation on the manifold corresponding to $d\hat{s}^2$. Since the Helmholtz equation separates in the coordinates x^j corresponding to $d\hat{s}^2$, we can find separable solutions $\Phi = (\Pi_{j=1}^4 A_j(x^j)$ for Eq. (5.3) and R-separable solutions $\Psi = Q^{-1/2}\Pi_{j=1}^4 A_j(x^j)$ for the flat space Laplace equation. This proves that *all* nonorthogonal coordinate systems which separate Eq. (4.6) also R-separate Eq. (4.2). Combining these results with those of Ref. 1 we obtain the following:

Theorem: Let $\{x^j\}$ be a coordinate system (orthogonal or not) for which the equation

$$\sum_{l=1}^{4} (\partial_{z^l} W)^2 = 0 \tag{5.4}$$

is separable. Then

$$dx^2 = \sum_{l=1}^{4} (dz^l)^2 = Q\Big(\sum_{i,j=1}^{4} g^{ij} dx^i dx^j\Big) = Q\, d\hat{s}^2 ,$$

where $d\hat{s}^2$ is a metric on one of the spaces $\mathscr{M} = E_4$, $S_3 \times S_1$, $S_2 \times S_2$, S_4 and the coordinates $\{x^j\}$ are separable for the Helmholtz equation on $\mathscr{M}$. If $\{x^j\}$ is nonorthogonal, then we can assume that $\mathscr{M}$ is one of E_4 or S_4. The function Q satisfies Eq. (5.3), where R is the (constant) scalar curvature of $\mathscr{M}$. Furthermore, the Laplace equation

$$\sum_{j=1}^{4} \partial^2_{z^j} \Psi = 0 \tag{5.5}$$

is R separable in the coordinates $\{x^j\}$:

$$\Psi = Q^{-1/2} A_1(x^1) A_2(x^2) A_3(x^3) A_4(x^4) .$$

All separable systems for the Helmholtz equation on $\mathscr{M}$ yield R-separable systems for the flat space Laplace equation.

Corollary: Equations (5.4) and (5.5) separate in exactly the same coordinate systems (orthogonal or not).

Corollary: If $\{x^j\}$ is a separable coordinate system for the Laplace equation on a conformally flat space, then these coordinates permit separation of the Helmholtz equation on one of the manifolds E_4, $S_3 \times S_1$, $S_2 \times S_2$, or S_4.

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