Lie theory and the wave equation in space–time. 4. The Klein–Gordon equation and the Poincaré group

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A detailed classification is made of all orthogonal coordinate systems for which the Klein–Gordon equation in space–time, \( \psi_{tt} - \Delta \psi = -\lambda \psi \), admits a separation of variables. We show that the Klein–Gordon equation is separable in 261 orthogonal coordinate systems. In each case the coordinate systems presented are characterized in terms of three symmetric second order commuting operators in the enveloping algebra of the Poincaré group. This paper also constitutes an important step in the study of separation of variables for the wave equation in space–time \( \psi_{tt} - \Delta \psi = 0 \), and its relation to the underlying conformal symmetry group \( O(4,2) \) of this equation.

INTRODUCTION

In this paper we continue\(^1\) an investigation of the connection between separation of variables for the wave equation in space–time

\[
\psi_{tt} - \Delta \psi = 0,
\]

and the \( O(4,2) \) symmetry group of this equation. Here we study all the orthogonal coordinate systems for which the Klein–Gordon equation

\[
\square \psi = \psi_{tt} - \Delta \psi = \lambda \psi, \quad \lambda \neq 0
\]

admits a separation of variables. (By simply setting \( \lambda = 0 \) in our results we will obtain orthogonal separable systems for (0.1).) The method used to obtain all such coordinate systems is an adaptation of that used by Eisenhart\(^4\) in the case of the Helmholtz equation in three-dimensional Euclidean space. The work of Eisenhart enables us to classify all distinct orthogonal differential forms

\[
ds^2 = \sum_{i=1}^{4} H_i^2 \, dx_i^2,
\]

and hence coordinate systems for which (0.2) admits a separation of variables. In (0.3) the \( H_i^2 \) are real functions of the new variables \( x_i \) such that \( H_i^2 = +1 \) for \( i = 1, 2, 3 \), and sign \( H_4^2 = -1 \). The coordinates \( x_i \) are related to the standard space–time coordinates \( x, y, z, t \) by the real functions \( G_t(x_1, x_2, x_3, x_4) \), \( x = G_t \), \( y = G_y \), and \( z = G_z \). In terms of the standard coordinates, the differential form (0.3) becomes

\[
ds^2 = dx^2 + dy^2 + dz^2 - dt^2.
\]

We will use each such differential form for the Klein–Gordon equation and the expression for the Klein–Gordon equation in these coordinates. We also write out the separation equations, identifying their solutions as much as possible, and we compute the three commuting operators \( L_i \) \( (i = 1, 2, 3) \) whose eigenvalues are the separation constants. Each of these three operators is written as a symmetric second order operator in the enveloping algebra of the Poincaré symmetry group \( E(3,1) \) of the Klein–Gordon equation (0.2).

When \( \lambda = -m^2 \), \( m > 0 \), Eq. (0.2) becomes

\[
(\Box + m^2) \psi(x) = 0, \quad x = (t, x, y, z);
\]

the relativistic equation describing a free neutral scalar particle with mass \( m \). In the standard field-theoretic treatments of (0.5), one expresses a positive-energy solution \( \psi \) in terms of its Fourier transform

\[
\psi(x) = \frac{1}{(2\pi)^{3/2}} \int \int \exp(-iw_{k}x \cdot y_{k} \cdot z_{k}) f(k) \, dm(k),
\]

where the integration surface is the hyperboloid \( \sum_{k=1}^{3} k_k^2 - k_0^2 = m^2 \), \( k_0 > 0 \). The Lebesgue measurable functions \( f(k) \), such that

\[
\int \int \int \int |f(k)|^2 \, dm(k) < \infty,
\]

form a Hilbert space \( H_m \) with inner product

\[
\langle f_1, f_2 \rangle = \int \int \int f_1(k) \bar{f}_2(k) \, dm(k), \quad f_1, f_2 \in H_m.
\]

The mapping (0.6) then induces a Hilbert space structure on the solution space of (0.5) given by

\[
\langle \psi_1, \psi_2 \rangle = \int \int \int (\bar{\psi}_1(x) \, \bar{\psi}_2(x) - \bar{\psi}_2(x) \, \bar{\psi}_1(x)) \, dx \, dy \, dz
\]

(independent of \( t \)), where \( \psi_j \) is related to \( f_j \in H_m \) by (0.6). The natural action of the connected Poincaré group \( E(3,1) \) on the functions \( \psi \) induces an action on the transform space \( H_m \) which is well known to be unitary and irreducible.\(^3\)

In studies of this physical system it is obviously of interest to construct various orthonormal bases for \( H_m \) particularly bases which correspond to separable solutions of (0.5). However, with few exceptions, only the plane wave basis (corresponding to separation in
Cartesian coordinates) is employed in the published literature. Here we show explicitly that every orthogonal separable coordinate system for \((0, 5)\) has the property that the associated separated solutions are characterized as simultaneous eigenfunctions of a commuting triplet of second-order symmetry operators from the enveloping algebra of \(E(3, 1)\). The corresponding operators acting on the domain of \(C^\infty\) functions with compact support in \(H_m\) are obviously symmetric. These operators can then be extended to a commuting triplet of self-adjoint operators on \(H_m\) (however, in some cases the deficiency indices are equal but nonzero, so that the extension is not unique. Furthermore, in a few cases the deficiency indices of some operators are unequal. This difficulty can be removed by extending the Hilbert space to include the negative energy solutions.) The spectral theorem for commuting sets of self-adjoint operators thus implies the existence of a basis for \(H_m\) which is a (generalized) eigenbasis of the commuting operators. Mapping the eigenbasis to the solution space of \((0, 5)\) via \((0, 6)\), we see that the basis eigenfunctions are separable solutions of \((0, 5)\). The spectral resolutions of the defining self-adjoint operators as computed in \(H_m\) can then be used to derive expansion theorems and special function identities for solutions of \((0, 5)\). Our characterization of orthogonal separable systems in terms of commuting second-order operators in the enveloping algebra which act within a unitary irreducible representation of \(E(3, 1)\) is an essential part of this program.

The paper is arranged as follows. In Sec. 1 we present the necessary details concerning the generators of the Poincaré group. In addition we give a preliminary discussion concerning the arrangement and computation of the coordinate systems. In Sec. 2 we extend the work of Eisenhart to consider orthogonal differential forms in four variables and then compute all the inequivalent classes of differential forms. In Sec. 3 we give the coordinate systems, separation equations, and operators defining the separation constants.

I. SOME PROPERTIES OF THE POINCARE GROUP \(E(3, 1)\)

Here we briefly present those properties of the Poincaré group \(E(3, 1)\) that are relevant to this article. For more details concerning this group the reader is referred to paper 3 of this series and Refs. 6, 7, and 8. The Poincaré group consists of all proper real linear transformations which preserve the differential form \((0, 4)\). The group is the semidirect product of the group of translations \(T_4\) in the space and time coordinate and the group of proper real Lorentz transformations \(SO(3, 1)\), i.e.,

\[
E(3, 1) = T_4 \times SO(3, 1).
\]

The Lie algebra is ten-dimensional with basis elements:

1. Translations
   \[
P_0 = \partial_0; \quad P_i = \partial_i, \quad i = 1, 2, 3, 4.
   \]

2. Pure Lorentz transformations
   \[
   N_\alpha = t \partial_\alpha + x_\beta \partial_\alpha, \quad N_i = t \partial_i + x_\beta \partial_i + x_\gamma \partial_i; \quad N_0 = t \partial_0 + x_\beta \partial_0 + x_\gamma \partial_0.
   \]

3. Rotations
   \[
   M_4 = y \partial_4 - z \partial_3, \quad M_3 = x \partial_3 - z \partial_2, \quad M_3 = x \partial_2 - y \partial_1.
   \]

These generators satisfy the commutation relations

\[
[M_i, M_j] = \epsilon_{ijk} M_k, \quad [N_i, N_j] = \epsilon_{ijk} N_k, \quad [N_i, M_j] = \epsilon_{ijk} M_k,
\]

where \(i, j, k = 1, 2, 3\),

\[
[P_\alpha, N_j] = \delta_{\alpha j} P_0, \quad [P_\alpha, M_j] = \epsilon_{ijk} M_k,
\]

for \(j = 1, 2, 3\), and

\[
[P_\alpha, P_\beta] = 0,
\]

for all \(i, j\).

On the Hilbert space \(H_m\) (Eqs. (0, 7) and (0, 8)) the Lie algebra generators are

\[
P_0 = -i \theta_0, \quad P_i = i \theta_i, \quad N_\alpha = k_0 \theta_\alpha, \quad i = 1, 2, 3, \quad M_4 = k_2 \theta_3 - k_3 \theta_2, \quad M_3 = k_1 \theta_3 - k_3 \theta_1, \quad M_3 = k_1 \theta_2 - k_2 \theta_1,
\]

In addition to the real Poincaré group \(E(3, 1)\) we will also consider its complexification \(E(4, 0)\). This is the group of proper complex transformations which preserve the differential form

\[
ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2,
\]

where \(x_4 \in \mathbb{C}\), \(i = 1, 2, 3, 4\). The group \(E(4, 0)\) is the semidirect product of the translation group \(T_4\) and \(SO(4, \mathbb{C})\), i.e.,

\[
E(4, 0) = T_4 \times SO(4, \mathbb{C}).
\]

The Lie algebra is ten-dimensional with basis elements:

1. Translations \(P_\alpha = \partial_\alpha\), \(i = 1, 2, 3, 4\),

2. Rotations \(I_{ij} = z_j \partial_i - z_i \partial_j\), with \(i, j = 1, 2, 3, 4\) and \(i \neq j\).

These basis elements satisfy

\[
[I_{ij}, I_{kl}] = \delta_{ik} I_{jl} - \delta_{jk} I_{il} - \delta_{kl} I_{ij} + \delta_{jl} I_{ik}, \quad [P_\alpha, P_\beta] = 0,
\]

II. ORTHOGONAL SEPARABLE DIFFERENTIAL FORMS FOR THE KLEIN-GORDON EQUATION AND ITS COMPLEXIFICATION

In this section we classify the possible orthogonal differential forms which enable \((0, 2)\) or its complexification

\[
\sum_{i=1}^3 \partial_\alpha x_\alpha \frac{\partial}{\partial x_\alpha} = \lambda \frac{\partial}{\partial x_4}
\]

to be solved by separation of variables. By this we mean a classification of all choices of new variables \(x_1, x_2, x_3, x_4\), such that \(I = G_1, \quad x = G_2, \quad y = G_3, \quad z = G_4\).

In the case of the Klein–Gordon equation, the real functions \(G_i\) \((i = 1, 2, 3, 4)\) are real differentiable functions of the real variables \(x_i\) \((j = 1, 2, 3, 4)\). In order
that the new coordinates $x_i$ be orthogonal we have the additional requirement that
\[ ds^2 = dx_1^2 + dx_2^2 + dx_3^2 - dt^2 = \sum_{i=1}^4 H_i^2 dx_i^2, \]
where $H_i^2 = +$ for $i = 1, 2, 3$ and $H_4^2 = -$.

In the case of the complexified Klein–Gordon equation, the functions $C_1$ ($i = 1, 2, 3, 4$) are analytic functions of the complex variables $x_i$. The requirement of orthogonality is the same as in the real case but with no restrictions on the signs of the metric coefficients. The coordinate systems fall into five broad classes, whose general features we now summarize. Details of the derivations are given in Ref. 9.

A. Coordinate systems of class I

These correspond to coordinate systems giving the differential form
\[ ds^2 = \frac{(x_1 - x_2)}{4} \left[ \frac{dx_1^2}{x_1^2} - \frac{dx_2^2}{x_2^2} \right] + \epsilon x_1 x_2(dx^2 + dy^2), \quad \epsilon = \pm, \]
where $x, y$ can be replaced by one of the four possible coordinate systems in the Euclidean plane (in the case of the real Klein–Gordon equation). In the case of the complexified equation, $x$ and $y$ can be replaced by one of the various possible coordinate systems for all the complex Euclidean plane. The separable solutions of (0.2) for coordinate systems of this type assume the typical form
\[ \Psi = e^{\alpha x + \beta y} D_{x_i}(\alpha + \beta x, \sqrt{N_{x_i}}) \times D_{x_i}(\alpha - \beta x, \sqrt{N_{x_i}}) E_{x_i}(x_i), \]
where $\alpha = \sqrt{x_1}$, $\beta = \sqrt{x_2}$, $\tan x = (\lambda + l_1)/(\lambda - l_1)$. Here $x_1$ and $x_2$ correspond to the appropriate choice of coordinates in the Euclidean plane and $\phi(x_1, y) = E_i (x_3) E_i (x_4)$ is a solution of
\[ [(N_1 + M_1)^2 + (N_2 - M_2)^2] \phi(x_1, y) = \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial y^2} = l_1 \phi \]
and $x = x(x_3, x_4)$, $y = y(x_3, x_4)$. Furthermore, $D = S, C$ and $E_{x_3}, E_{x_4}$ are Mathieu functions.

B. Coordinate systems of class II

These correspond to systems giving the differential form
\[ ds^2 = \frac{(x_1 - x_2)}{4} \left[ \frac{dx_1^2}{x_1^2} - \frac{dx_2^2}{x_2^2} \right] + x_1 x_2 dx^2, \]
where $dx^2$ is one of the differential forms associated with the two-dimensional sphere or the two-dimensional single or double sheeted hyperboloids. The separable solutions of (0.2) for systems of this type appear as
\[ \Psi = (x_1 x_2)^{1/4} M_{x_1 x_2}(x_1 x_2)^{1/2} \left( \frac{x_1}{x_2} \right)^{1/2} \left( \frac{x_2}{x_1} \right)^{1/2} \left( \frac{x_2}{x_1} \right)^{3/2} dx_1 dx_2, \]
where $E_2 (x_3) E_2 (x_4) = \phi$ is a solution of
\[ (N_1^2 + N_2^2 - M_1^2) \phi = (j + 1) \phi \]
and the coordinates $x_1, x_2$ are one of the nine possible types for which this equation admits a separation of variables. Here $M_{x_1 x_2}$ is a Whittaker function.

C. Coordinate systems of class III

These correspond to systems giving the differential form
\[ ds^2 = \frac{(x_1 - x_2)}{4} \left[ \frac{dx_1^2}{x_1(x_1 - 1)} - \frac{dx_2^2}{x_2(x_2 - 1)} \right] + x_1 x_2 d\omega^2, \]
with $d\omega^2$ as in Class II. The separable solutions of (0.2) for systems of this type assume the typical form
\[ \Psi = (x_1 x_2)^{1/4} P_{x_1}^{1/2} (\sqrt{1 - x_1}, \lambda) P_{x_2}^{1/2} (\sqrt{1 - x_2}, \lambda) \times E_2 (x_3) E_2 (x_4), \]
where $E_2 (x_3) E_2 (x_4)$ is as in Class II and $P_{x_1}^{1/2}$ is a spheroidal function.

D. Coordinate systems of class IV

These correspond to systems giving a differential form
\[ ds^2 = \frac{(x_1 - x_2)}{4} \left[ \frac{dx_1^2}{x_1(x_1 - 1)} - \frac{dx_2^2}{x_2(x_2 - 1)} \right] + x_1 x_2 dx_1^2 + (x_1 - 1) (x_2 - 1) dx_1^2. \]
The separation equations are
\[ \frac{d^2 E_{x_1}}{dx_1^2} - \lambda x_{x_1} E_{x_1} = 0 \quad (i = 1, 2) \]
\[ \frac{d^2 E_{x_2}}{dx_2^2} - \lambda x_{x_2} E_{x_2} = 0 \quad (i = 1, 2). \]

E. Coordinate systems of class V

These correspond to systems with a differential form
\[ ds^2 = (x_1 - \mu) (x_2 - \mu) (x_3 - \mu) (x_4 - \mu) dx_1^2 + \sum_{i=1}^4 \left( \frac{x_i - x_j}{x_i - x_j} \right) \left( \frac{x_i - x_k}{x_i - x_k} \right) dx_i^2, \]
where $i, j, k = 2, 3, 4$ are distinct, $f(x)$ is a polynomial such that $1 \leq \text{deg}(f(x)) < 3$, and $x = \mu$ is a root of $f(x)$. The separation equations are
\[ \frac{d^2 E_{x_i}}{dx_i^2} - \lambda (x_i - \mu)^3 + l_1 (x_i - \mu)^3 + l_2 (x_i - \mu) + l_3 E_{x_i} = 0, \]
where $i = 2, 3, 4$ and
\[ (\mu - \mu') (\mu - \mu'') d^2 E_{x_i} = -l_2 E_{x_i}. \]
where $\mu'$ and $\mu''$ are the other roots of $f(x)$ with multiplicity included and $\text{deg}(f(x)) = 3$. Similar separation
equations exist in the variable $x_1$ when $\deg f(x) = 2$
and $1$.

**F. Coordinate systems of class VI**

These correspond to systems giving differential forms

$$
\frac{4}{2f(x_i)} \left( \left( x_i - x_j \right) (x_i - x_k) (x_i - x_l) \right) \frac{d^2 E_1}{d x_i},
$$

(2.15)

where $i, j, k, l = 1, 2, 3, 4$ are distinct and $f(x)$ is a polynomial of degree less than or equal to 4. The separation equations are

$$
4 \sqrt{f(x_i)} \frac{d}{dx_i} \left( \sqrt{f(x_i)} \frac{dE_1}{dx_i} \right) + \left( \lambda x_i^2 + l_1 x_i + l_2 x_i^2 + l_3 \right) E_i = 0,
$$

$i = 1, 2, 3, 4$.

The remaining coordinate systems correspond to group reductions of the type $E(3, 1) \supset T_1 \otimes E(2, 1)$

$\supset T_1 \otimes O(2, 1)$, $E(3, 1) \supset T_1 \otimes E(3) \supset T_1 \otimes O(3)$, and $E(3, 1) \supset O(5, 1)$ in the case of the Klein–Gordon equation and $E(4, 4) \supset T_1 \otimes E(4, 4) \supset T_1 \otimes O(4, 4)$, $E(4, 4) \supset O(4, 4)$ in the case of its complexification. These systems have been derived elsewhere and we make no further evaluation of them.

**III. ORTHOGONAL SEPARABLE COORDINATE SYSTEMS FOR THE KLEIN–GORDON EQUATION AND ITS COMPLEXIFICATION**

In this section we supplement Sec. 2 by giving the coordinates in space–time corresponding to the differential forms presented there. In addition we give the three operators, $L_i$, $L_2$, and $L_3$ whose eigenvalues are the three separation constants $l_1$, $l_2$, and $l_3$. These operators are expressed as symmetric second order operators in the enveloping algebra of the Poincaré group or its complexification. Due to the large number of possible systems we group the coordinate systems corresponding to the differential forms of Sec. 2 into classes of systems with similar properties and make an explicit count of the number of distinct coordinate systems inequivalent under the Poincaré group. We also list the systems which separate for the complexified equation only (denoted by the symbol $\mathbb{C}$), bearing in mind that distinct real systems may be equivalent in the complex case.

**A. Coordinate systems of class I**

A suitable choice of coordinates (2.3) with $\epsilon = -$ and sign $(x_3 x_4)$ is $+$ is

$$
\begin{align*}
(1) & \quad (t - x^2) = x_1 x_2, \\
(2) & \quad (t^2 - x^2) = x_1 + x_2 x_3 (x_1 + x_2), \\
y = \sqrt{x_1 x_2 x_3}, \quad z = \sqrt{x_1 x_2 x_4},
\end{align*}
$$

(3.1)

In terms of these coordinates the Klein–Gordon equation assumes the form

$$
\psi = \frac{4}{(x_1 - x_2)} \left[ \frac{\partial}{\partial x_1} \left( \lambda x_1^2 + \frac{\partial \psi}{\partial x_1} \right) - \frac{\partial}{\partial x_2} \left( \lambda x_2^2 + \frac{\partial \psi}{\partial x_2} \right) \right],
$$

$$
- \frac{1}{x_1 x_2} \left( \lambda x_1^2 + \frac{\partial \psi}{\partial x_1} \right) - \frac{\partial \psi}{\partial x_2} = \lambda \psi.
$$

(3.2)

The separation equations for the solution $\psi$

$$
E_1(x_1) E_2(x_2) E_3(x_3) E_4(x_4)
$$

are

$$
\frac{d^2 E_1}{dx_1^2} = (l_1 - l_2) E_2, \quad \frac{d^2 E_2}{dx_2^2} = l_1 E_1,
$$

$$
4 \frac{d}{dx_i} \left( \frac{dE_i}{dx_i} \right) + \left( \frac{l_1}{x_i} - \lambda x_i + l_3 \right) E_i = 0,
$$

(3.3)

where $i = 1, 2$. The three operators whose eigenvalues are the separation constants are

$$
L_1 = (x_1 + x_2) + (x_3 + x_4), \quad L_2 = (x_1 - x_2), \quad L_3 = (x_3 + x_4).
$$

(3.4)

A typical solution for the Klein–Gordon equation (0.2) is

$$
\psi = e^{i(r \pm \omega t)} D_x \left[ a + \frac{1}{2} x \sqrt{-\lambda} \right] D_x \left[ b + \frac{1}{2} x \sqrt{-\lambda} \right]
$$

$$
\times \exp \left[ (l_1 - l_2)^{1/2} x_3 + (l_3)^{1/2} x_4 \right],
$$

(3.5)

where $D = iC, S$ and the variables are defined as in (2.4).

If $\epsilon = +$, then the corresponding coordinates are obtained from (3.1) via the transformations $S=
(1, x_4, x_3, x_2, x_1)$ and $x_3 \to ix_3, x_4 \to ix_4$.

(2) If sign $(x_3 x_4) = -$ and $\epsilon = +$, the appropriate choice of space–time coordinates is obtained from (3.1) via the transformation $T = (t, x_4, x_3, x_2, x_1)$ and $x_3 \to -ix_3, x_4 \to -ix_4$.

The remaining coordinate systems in this class are obtained by regarding $x_3, x_4$ (as given in (3.1) for coordinate systems (1)–(4)) as Cartesian coordinates in a Euclidean plane. This is the plane whose corresponding $E(2)$ Lie algebra has generators $P_1 = N_1 + M_1$, $P_2 = N_1 + M_2$, and $M = M_1 + M_2$ with commutation relations

$$
\left[ P_1, M \right] = P_1, \quad \left[ P_2, \bar{M} \right] = \bar{P}_1, \quad \left[ P_1, P_2 \right] = 0.
$$

(3.6)

The new coordinates are then obtained by choosing polar, parabolic, and elliptic coordinates in the $x_3, x_4$ plane. The three possible types of coordinates resulting from each of these three choices are obtained by the same substitutions as used to find all the systems (1)–(2), i.e., we have two inequivalent pairs of coordinate systems in each case. In all cases the operator $L_3$ is given by its counterpart in systems (1)–(2), and the separation equations in the variables $x_1, x_2$ are as in (3.3). For each case we need only give the transformation $x_3 \to f(x_3, x_4)$, $x_4 \to g(x_3, x_4)$ specifying the change in coordinates together with the operators $L_1, L_2$.

The transformation to plane polar coordinates is given by the following.

$$
(3) \quad x_3 \to \sqrt{x_3 \cos x_4}, x_4 \to \sqrt{x_3 \sin x_4}
$$

(3.7)

The $x_3, x_4$ dependent part of the separable solution is typically

$$
E_3(x_3) E_4(x_4) C_{+1/2} \left( \sqrt{-1} x_3 \right) \exp \left[ \pm (l_3 x_4)^{1/2} \right],
$$

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where \( C_\nu(z) \) is a solution of Bessel's equation. The basis defining operators are

\[
L_1 = (N_2 + M_2)^2 + (N_3 - M_3)^2, \quad L_2 = M_2^2. \tag{3.8}
\]

(5)–(6) The transformation to parabolic coordinates in the plane is given by

\[
x_3 \to \frac{1}{2}(x_3 + x_4), \quad x_4 \to \sqrt{-x_3 x_4}, \quad x_4 < 0 < x_3.
\]

The \( x_3, x_4 \) part of the separable solution is typically

\[
E_3(x_3) E_4(x_4) = D(x_3) D(x_4) \left[ z^{1/2}(x_3^2 + 1) \right] ^{1/2} \left[ z^{1/2}(x_4^2 + 1) \right] ^{1/2}.
\]

where \( D(z) \) is a parabolic cylinder function.\(^{12}\) The basis defining operators are

\[
L_1 = (N_2 + M_2)^2 + (N_3 - M_3)^2, \quad \tag{3.11}
L_2 = (N_2 - M_2) M_2 + M_4 (N_3 - M_3).
\]

(7)–(8) The transformation to elliptic coordinates in the plane is given by

\[
x_3 \to c \sqrt{x_3 x_4}, \quad x_4 \to -c \sqrt{x_3 (1 - x_3)}, \tag{3.12}
\]

with \( 0 < x_3 < 1 < x_4 \).

The \( x_3, x_4 \) part of the separable solution is typically

\[
E_3(x_3) E_4(x_4) = \left( \frac{C_\nu(x_3, h_3) c_c \eta_3, h_3)}{C_\nu(x_4, h_4) s_c \eta_4, h_4} \right)
\]

where \( x_3 = c \cos \eta_3, x_4 = x_3 \sin \eta_3 = h_3 c \sin \eta_3, h_3 = l_1 c^2 / 4, \) and \( l_2 = -\lambda / (h_3) \). \( \nu = 0, 1, 2, \ldots \). The functions \( C_\nu(x, h) \) and \( s_c \) are periodic Mathieu functions.\(^{11}\) The basis defining operators are

\[
L_1 = (N_2 + M_2)^2 - (N_3 - M_3)^2, \quad \tag{3.14}
L_2 = M_2^2 + \frac{1}{2} c_l^2 [(N_2 - M_2)^2 - (N_3 - M_3)^2].
\]

For the remaining systems of Class I we have to consider the complexified Klein–Gordon equation.

(9) [?] A suitable choice of coordinates is

\[
(x_1 - i x_2)^2 = x_1 x_2, \quad (x_1 + i x_2)^2 = x_1 x_2 + 2 x_3 x_4 (x_3 + x_4) x_3 x_4), \tag{3.15}
\]

\[
x_2 + i x_4 = 2i \sqrt{x_3 x_4} (x_3 + x_4), \quad x_3 - i x_4 = i \sqrt{x_3 x_4} (x_3 - x_4).
\]

The complexified Klein–Gordon equation has the same form as (3.2) with \( \Delta_2 \psi = (\partial_2 + \partial_4) \psi \) replaced by \( 1 / 4 (x_3 - x_4) \partial_3 (x_3 - x_4) \). The separation equations in the variables \( x_3, x_4 \) are

\[
d^2 E_i / dx_i^2 + (-4 i / x_1 + x_2) E_i = 0, \tag{3.16}
\]

where \( i = 3, 4 \). A typical solution of this equation is

\[
E_i = \left( x_1 + \frac{i}{2} x_1 \right)^{-1/3} \frac{C_\nu(x_1, h_1)}{3^{1/3}} \left( x_1 + \frac{i}{2} x_1 \right)^{-1/2}, \tag{3.17}
\]

where \( C_\nu(x) \) is a solution of Bessel's equation. The separation equations in the variables \( x_1, x_2 \) are as in (3.3) with \( l_1 + l_2 \) replaced by \( l_1 \). The basis defining operators are

\[
L_1 = (x_3 + x_4)^2 + (x_1 + x_2)^2, \quad L_2 = (x_1 + x_2)^2, \quad L_3 = (x_1 - x_2)^2, \quad \tag{3.18}
\]

\[
L_4 = (x_1 + x_2)^2 + (x_1 - x_2)^2.
\]

The complexified Klein–Gordon equation has the same form as (3.2) with \( \Delta_2 \psi = (\partial_2 + \partial_4) \psi \) replaced by

\[
\frac{4}{(x_3 - x_4)} \left[ \frac{\partial}{\partial x_3} \left( \frac{\partial \psi}{\partial x_3} \right) - \frac{\partial}{\partial x_4} \left( \frac{\partial \psi}{\partial x_4} \right) \right].
\]

The separation equations in the variables \( x_3, x_4 \) are

\[
x_i \frac{d^2 E_i}{dx_i^2} + \left( \frac{d E_i}{dx_i} \right) = 0, \tag{3.19}
\]

where \( i = 3, 4 \). A typical solution of this equation is

\[
E_i = C_\nu(x_1 + x_2)^{1/2}. \tag{3.20}
\]

The separation equations in the variables \( x_3, x_4 \) are as in system (9). The operators \( L_3, L_4 \) also have the same number of solutions (9). The remaining operator is

\[
L_1 = -\delta_3 + (l_3 + l_4 + (l_3 + l_4))^2. \tag{3.21}
\]

B. Coordinate systems of class II

(11)–(13) In analogy to our treatment of Class I we treat one of the coordinate systems in detail and give the transformations from which the remaining coordinate systems can be obtained. A suitable choice of coordinates of type (2.6) with sign \( (x_1 x_2) = + \) and \( x_3 < 0 \) is

\[
x = x_1 x_2 \cos x_3, \quad y = x_1 x_2 \sin x_3, \quad z = x_3 (x_1 + x_2), \tag{3.22}
\]

The Klein–Gordon equation assumes the form

\[
\square \psi = \frac{4}{(x_1 - x_3)} \left[ \frac{\partial^2 \psi}{\partial x_1^2} - \frac{\partial^2 \psi}{\partial x_3^2} \right] - \frac{1}{x_1 x_2} \left[ \frac{\partial^2 \psi}{\partial x_1^2} - \frac{\partial^2 \psi}{\partial x_3^2} \right] = \lambda \psi. \tag{3.23}
\]

The separation equations are

\[
4 \sqrt{-x_3} \frac{d}{dx_3} \left( x_1 x_2 \frac{d E_1}{dx_3} \right) - \frac{4}{x_3} E_3 = l_1 E_3, \quad \frac{d^2 E_3}{dx_3^2} = l_3 E_3, \tag{3.24}
\]

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where \( i = 1, 2 \). The three defining operators are

\[
L_1 = N_i + N_j - M_i, \quad L_2 = M_i, \\
L_3 = P_\theta N_i + N_j P_\theta - 2M_i P_\theta + 2M_i P_\theta.
\]

(3.26)

A typical solution of the Klein–Gordon equation is

\[
\Psi = (x_1 x_2)^{1/4} M_{4(1)}^{1/2} \tilde{x}_1^{1/2} \tilde{z}_1 (x \tilde{x}_2)^{1/2} \tilde{z}_2 \times M_{4(1)}^{1/2} \tilde{x}_1^{1/2} \tilde{z}_1 \{ t \tilde{x}_2^{1/2} \tilde{z}_2 \} \\
\times P^1 \tilde{\tau} \left( \sqrt{1 - \gamma} \right) \exp (i t^2)^{1/2} \tilde{x}_4,
\]

(3.27)

where \( P^\sigma \tilde{\tau} \) is a Legendre function. There is a further coordinate system of type (11) obtained by allowing the parameters \( x_i \) to vary in the ranges \( \text{sign}(x_1 x_2) = -1 \), \( x_3 > 1 \).

(12) Coordinate systems of this type correspond to the ranges, \( \text{sign}(x_1 x_2) = +1 \), \( 0 < x_3 < 1 \), and can be obtained from systems of type (11) via the transformation

\[
(\tilde{t}, \tilde{x}_1, \tilde{x}_2, \tilde{t}, \tilde{z}_1, \tilde{z}_2), \quad \{ t, x_1, x_2, z, x_3, x_4 \}
\]

(3.19) and (3.22) we get the appropriate coordinates and basis operators. If \( \text{sign}(x_1 x_2) = -1 \), \( x_3 < 0 \), if and we make the two transformations \( x_4 \to x_4 \) and \( (\tilde{t}, \tilde{x}_1, \tilde{x}_2, \tilde{t}, \tilde{z}_1, \tilde{z}_2) \to (\tilde{y}_1, \tilde{y}_2) \), then making the substitutions \( x_4 \to -x_4 \) and \( (t, x_1, x_2, z) \to (y_1, y_2, t, z) \) we get another set of coordinates.

We can extract the essential features of the remaining distinct coordinates of Class II from the three systems already described. There are two kinds of coordinates:

\[
l = \sqrt{x_1 x_2} t_1, \quad x = \sqrt{x_1 x_2} t_2, \\
y = \sqrt{x_1 x_2} t_3, \quad z = \frac{1}{2} (x_1 + x_2).
\]

(3.28)

If \( \text{sign}(x_1 x_2) = +1 \), then the vector \( \tau = (\tau_1, \tau_2, \tau_3) \) is parametrized by one of the nine orthogonal separable coordinate systems on the double sheeted hyperboloid \( [t, \tau] = \tau_1^2 - \tau_2^2 - \tau_3^2 = 1 \). If \( \text{sign}(x_1 x_2) = -1 \), then the vector \( \tau_1, \tau_2, \tau_3 \) is parametrized by one of the nine classes of orthogonal separable coordinate systems on the single sheeted hyperboloid \( [t, \tau] = -1 \).

\[
l = \frac{1}{2} (x_1 + x_2), \quad x = \sqrt{x_1 x_2} t_1, \\
y = \sqrt{x_1 x_2} t_2, \quad z = \sqrt{x_1 x_2} t_3
\]

(3.29)

parametrized by one of the two orthogonal separable coordinate systems on the sphere \( \tau_1^2 + \tau_2^2 + \tau_3^2 = 1 \).

For the remaining coordinate systems of Class II we need only give the 3-vector \( (\tau_1, \tau_2, \tau_3) \) in terms of the coordinates \( x_3, x_4 \), appearing in the corresponding differential form of Sec. 2. In addition we give the operator \( L_2 \) specifying each of the separable bases together with a typical solution for \( E_2(x_3) E_2(x_4) \). We note here that coordinate systems already given correspond to the following \( \tau \) vectors.

\[
\tau = (\cos \vartheta, \sin \vartheta \cos \phi, \sinh \vartheta \sin \phi), \quad [\tau, \tau] = -1, \\
-\infty < a < \infty, \quad 0 < \phi < 2\pi, \quad (x_3 = \sinh^2 \vartheta).
\]

(3.30)

\[
\tau = (\cos \theta, \sin \theta \cos \phi, \sin \theta \sin \phi), \quad \tau_1^2 + \tau_2^2 + \tau_3^2 = 1, \\
0 < \theta < \pi, \quad 0 < \phi < 2\pi, \quad (x_3 = \sin^2 \theta).
\]

(3.31)

\[
\tau = (\cosh \vartheta, \sinh \vartheta \cos \phi, \sinh \vartheta \sin \phi), \quad [\tau, \tau] = 1, \\
-\infty < a < \infty, \quad 0 < \phi < 2\pi, \quad (x_3 = \cosh^2 \vartheta).
\]

(3.32)

We now proceed to the remaining coordinate systems of Class II.

(14) The corresponding choices of the vector \( \tau \) are

\[
\tau_1 = x_3, \quad \tau_2 = \sqrt{x_1 x_2}, \quad \tau_3 = (1/\sqrt{x_1 x_2}), \\
[\tau, \tau] = 1, \quad (x_3 > 0).
\]

(3.33)

(a) \( \tau = v \times x_3 \), \( [\tau, \tau] = 1 \), \( x_3 > 0 \).

(b) The coordinates corresponding to the single sheeted hyperboloid \( [t, \tau] = -1 \) are obtained from (3.29) via the substitution \( t \to -it \) with \( x_3 < 0 \). The operator \( L_2 \) for this coordinate system is

\[
L_2 = N_2 + M_2^2, \\
\text{a typical solution for the } x_3, x_4 \text{ dependent part of the solution of (0.2) is}
\]

\[
E_2(x_3) E_2(x_4) = x_3^{1/4} K_{4/2} (-i t^2)^{1/2} \tilde{x}_4.
\]

(3.35)

where \( K_\nu (z) \) is a Macdonald function.

(15) The corresponding choice of the vector \( \tau \) is

\[
\tau_1 = x_3 x_4 / a, \quad \tau_2 = (x_3 - 1)(x_4 - 1) / (a - 1), \\
\tau_3 = (x_3 - a) (a - x_4) / (a - 1), \quad [\tau, \tau] = 1, \\
0 < x_3 < x_4 < a.
\]

(3.36)

(15b) The coordinates on the single sheeted hyperboloid \( [t, \tau] = -1 \) are obtained from (3.32), the substitution \( t \to -it \) with \( x_3 < 0 < x_4 < a \). The operator \( L_2 \) for this coordinate system is

\[
L_2 = N_3^2 + a N_3^2, \\
\text{a typical solution for the } x_3, x_4 \text{ dependent part of the solution of (0.2) is}
\]

\[
L_2 (x_3) L_2 (x_4), \\
\text{where } L_j (x) \text{ is a solution of Lamé's equation}
\]

\[
d^j L_j / dz^j + \frac{1}{2} \left( \frac{1}{z - a} + \frac{1}{z - 1} + \frac{1}{z} \right) dL_j / dz \\
+ \frac{1}{4(z - a)(z - 1)} L_j = 0.
\]

(3.37)

(16a) This coordinate system is obtained from (15a).
via the transformation \((\tau_1, \tau_2, \tau_3) \rightarrow (i\tau_2, i\tau_1, \tau_3)\) and \(x_3 < 0 < 1 < a < x_4\).

(16b) This coordinate system is related to (15b) in the same way as (16a) and \(1 < x_3 x_4 < a\), or \(x_3 x_4 > a\).

(17) Finally, the one system on the sphere is obtained from (15a) via the substitution \((\tau_1, \tau_2, \tau_3) \rightarrow (\tau_1, i\tau_2, i\tau_3)\)
and \(0 < x_2 < 1 < x_3 < a\).

(18) A suitable choice of coordinates on the double sheeted hyperboloid is:

\[
(18a) \quad (r_1 + i r_2)^2 = 2(x_3 - a)(x_4 - a)/a(a - b),
\]

\[
\tau_1^2 = x_3 x_4/a, \quad |\tau_1| = 1,
\]  
(3.40)
and \(x_3 < 0 < x_4\).

(18b) The coordinates on the single sheeted hyperboloid are obtained from those of (3.36) by the substitution \(\tau \rightarrow i\tau\). The operator \(L_2\) is

\[
L_2 = \alpha(M_1^2 - N_1^2) + \beta(M_2 N_2 + N_2 M_2)
\]  
(3.41)
and a typical solution for the \(x_3, x_4\) dependent part of the solution of (2.9) is

\[
\bar{L}_{ij}\bar{e}(x) = \bar{L}_{j\bar{e}}(x),
\]

(3.42)
where \(\bar{L}_{ij}\bar{e}(x)\) is a solution of

\[
\frac{d^2 \bar{L}_{ij}}{dx^2} + \frac{1}{2} \left[ \frac{1}{2} \left( \frac{1}{x - a} + \frac{1}{x - b} \right) + \frac{1}{2} \right] \frac{d\bar{L}_{ij}}{dx} = 0.
\]  
(3.43)

(19) A suitable choice of coordinates on the double sheeted hyperboloid is:

\[
(19a) \quad \tau_1 = \tau_2 = \sqrt{x_3 x_4}, \quad \tau_3 = \sqrt{(1 - x_3) x_4 - 1}, \quad |\tau_3| = 1
\]  
(3.44)
and \(x_3 < 0 < x_4\).

(19b) The corresponding coordinates on the single sheeted hyperboloid are obtained via the substitution \(\tau \rightarrow i\tau\) with \(x_3, x_4 < 0, 0 < x_3 x_4 < 1, x_3 x_4 > 1\). The operator for this system is

\[
L_2 = N_1^2 + (N_2 + M_2)^2
\]  
(3.45)
and a typical solution for the \(x_3, x_4\) dependent part is

\[
E_3(x_3) E_4(x_4) = P_3^{1/2} x_3^{1/2} P_4^{1/2} x_4^{1/2} (x_3 - x_4),
\]

(3.46)
where \(P_\nu^\pm (x)\) is a Legendre function.

(20a) This system is obtained from (19a) via the transformation \((\tau_1, \tau_2, \tau_3) \rightarrow (i\tau_2, i\tau_1, \tau_3)\) and \(x_3 < 0 < 1 < x_4\).

(20b) The coordinates on the single sheeted hyperboloid are obtained from (20a) via the substitution \(\tau \rightarrow i\tau\) with \(x_3 < 0 < x_4 < 1\).

(21) A suitable choice of coordinates on the double sheeted hyperboloid is:

\[
(21a) \quad \tau_1 = \tau_2 = \sqrt{x_3 x_4}, \quad \tau_3 = \frac{1}{2} \left[ \frac{(-x_3 x_4)^{1/2}}{x_3} - \frac{(-x_3 x_4)^{1/2}}{x_4} \right], \quad |\tau_3| = 1
\]  
(3.47)
and \(x_3 < 0 < x_4\).

(21b) The coordinates on the single sheeted hyperboloid are obtained via the substitution \(\tau \rightarrow i\tau\) with \(x_3, x_4 < 0\) or \(x_3, x_4 > 0\). The operator for this system is

\[
L_2 = N_1^2 + M_2^2 + (N_1 - M_2)^2
\]  
(3.48)
and a typical \(x_3, x_4\) dependent part of the solution is

\[
E_3(x_3) E_4(x_4) = (-x_3 x_4)^{1/2} C_{\tau^1/2} (\sqrt{x_3} x_4) C_{\tau^1/2} (\sqrt{x_4} x_3).
\]

C. Coordinate systems of class III

These systems are similar to systems of Class II in that the various different types are specified by the various choices of separable coordinate systems on the manifolds \([\tau_1, \tau_2, \tau_3]\) and \([\tau_1^2 + \tau_2^2 + \tau_3^2 = 1]\), where \(\tau = (\tau_1, \tau_2, \tau_3)\). We examine in detail one system, then discuss the general form of the coordinates in this class.

(22) A suitable choice of coordinates with \(x_3 < 0, x_1, x_2 > 0, 0 < x_3 x_4 < 1, x_3, x_4 > 1\) is

\[
l = \sqrt{x_3 x_4 (1 - x_3)}, \quad x = \sqrt{x_1 x_2 x_3}, \quad y = \sqrt{x_1 x_2 x_3}
\]  
(3.49)
The Klein–Gordon equation becomes

\[
\n
\frac{\partial^2 \psi}{\partial x_1^2} \frac{\partial^2 \psi}{\partial x_2^2} = \left( \frac{x_3 - 1}{x_2} \right) \frac{\partial^2 \psi}{\partial x_2^2} \left( \frac{x_3 - 1}{x_2} \right) \frac{\partial^2 \psi}{\partial x_2^2} = 0.
\]

(3.50)
The separation equations in the variables \(x_3, x_4\) are as in (3.25). The corresponding equations for the variables \(x_1, x_2\) are

\[
4 \left( \frac{x_1 - 1}{x_1} \right)^{1/2} \frac{d}{dx_1} \left( \frac{x_1 - 1}{x_1} \right) \frac{dE_1}{dx_1} = 0
\]  
(3.51)
where \(i = 1, 2\). The three defining operators are

\[
L_1 = N_1^2 + M_2^2, \quad L_2 = M_2^2
\]  
(3.52)
A typical solution of the Klein–Gordon equation is

\[
\psi(x_1, x_2)^{1/2} P_\nu^{1/2} (\sqrt{1 - x_3}, x_3 - \lambda) P_\nu^{1/2} (\sqrt{1 - x_3}, x_3 - \lambda)
\]

\[
\times P_\nu^{1/2} (\sqrt{1 - x_3}, x_3 - \lambda) \exp(\pm(x_3^{1/2})) x_3^{1/2}.
\]

(3.53)
There is a further system obtained by allowing the \(x_i\) to vary in the ranges \(x_1 < 0 < 1 < x_3 < x_4 > 0\).

(22b) Systems of this type correspond to the ranges \(x_1 < 0 < 1 < x_3, x_3 < 0\); and \(0 < x_1 < 1 < x_3, x_3 > 1\). These systems are related to (22a) via the transformation \(T: (l_1, x_1, y_1) \rightarrow (l_2, y_2, x_2, z_2)\).

(23a) Systems of this type correspond to the ranges
\[ L_1 = M_1^2, \quad L_2 = N_1^2, \]
\[ L_3 = M_1^2 + M_2^2 + N_1^2 - N_2^2 - N_3^2 + \frac{1}{2} \left[ P_1^2 - P_2^2 - P_3^2 \right]. \]

(3.56)

Coordinates of this type are generalizations of spheroidal coordinates in three dimensions.

(62) \((t, x, y, z) \rightarrow (y, x, z, t), \quad x_1 > t > x_2 > 0, \quad x_2^3, x_1^2 < 0.\)

(63) \((t, x, y, z) \rightarrow (x, t, y, z), \quad x_1, x_2 > 1, \quad 1 > t > x_2 > 0, \quad 0 > x_1 > x_2, \quad x_1^2 > 0, x_2^3 > 0.\)

(64) \((t, x, y, z) \rightarrow (x, t, iy, iz), \quad x_1 > 1 > x_2, \quad x_2^3, x_1^2 > 0.\)

E. Coordinate systems of class V

(65) \(\mu = 0, \quad \rho = \frac{(x_2 - a)(x_3 - a)(x_4 - a)}{a(1 - a)}, \quad \gamma^2 = \frac{x_2 x_4 x_4}{a} \cos^2 x_1, \quad \alpha = \frac{x_2 x_4}{a} \sin^2 x_1, \quad \beta = \frac{(x_3 - 1)(x_4 - 1)}{(1 - a)}.\)

(3.57)

In these terms of these coordinates the Klein–Gordon equation becomes

\[ \Box \psi = \frac{1}{(x_2 - x_3)(x_4 - x_4)} \left( \frac{\partial^2 \psi}{\partial x_2^2} + \frac{1}{x_2} \frac{\partial \psi}{\partial x_2} \right) + \frac{1}{(x_3 - x_3)(x_4 - x_4)} \left( \frac{\partial^2 \psi}{\partial x_3^2} + \frac{1}{x_3} \frac{\partial \psi}{\partial x_3} \right) + \frac{1}{x_2 x_3 x_4} \left( \frac{\partial^2 \psi}{\partial x_1^2} + \frac{1}{x_1} \frac{\partial \psi}{\partial x_1} \right) = \lambda \psi, \]

(3.58)

where

\[ \frac{\partial}{\partial x_2} = 2x_2 \sqrt{(x_2 - a)(x_2 - 1)} \frac{\partial}{\partial x_2}. \]

The three defining operators are

\[ L_1 = -P_2^2 + (a + 1)(P_1^2 + P_3^2), \quad L_2 = a(P_1^2 + P_3^2) - N_2^2 - N_3^2 + (a + 1) M_2^2, \]
\[ L_3 = -M_1^2. \]

(3.59)

The coordinates \(x_2, x_3\), and \(x_4\) can vary in the ranges \(x_4 > a > 1 > x_3 > 0, a > x_4 > x_3 > 1 > x_2 > 0, 1 > x_2 > x_3 > 0 > x_4\), and \(1 > x_2, x_3, x_4 > 0\) with \(x_1^2 > 0\) in all cases. For the remaining systems we give the appropriate transformation of the space–time coordinates which relates the system in question to (65).

(66) \((t, x, y, z) \rightarrow (it, ix, iy, iz)\)

(67) \((t, x, y, z) \rightarrow (ix, iy, it)\)

(68) \((t, x, y, z) \rightarrow (iz, x, y, it)\)

(69) \((t, x, y, z) \rightarrow (iz, iy, it)\)

(70) \((t, x, y, z) \rightarrow (iy, it, iz)\)

D. Coordinate systems of class IV

(61) If \(x_1 > 1 > x_2 > 0, x_1^3, x_2^2 > 0\), a suitable choice of coordinates is

\[ t = \sqrt{(x_1 - 1)(x_1 - x_2)} \sinh x_4, \quad x = \sqrt{(x_1 - 1)(1 - x_2)} \cosh x_4, \quad y = \sqrt{x_2^2 x_2} \cosh x_3, \quad z = \sqrt{x_1^2 x_2} \sinh x_3. \]

(5.34)

The Klein–Gordon equation assumes the form

\[ \Box \psi = \frac{1}{4(x_1 - x_2)} \left[ \frac{\partial}{\partial x_1} \left( x_1(x_1 - 1) \frac{\partial}{\partial x_1} \right) \right. \]
\[ - \frac{\partial}{\partial x_2} \left( x_2(x_2 - 1) \frac{\partial}{\partial x_2} \right) + \frac{1}{x_1 x_2} \frac{\partial^2 \psi}{\partial x_1^2} + \frac{1}{(x_1 - 1)(x_1 - x_2)} \frac{\partial^2 \psi}{\partial x_1^2} = \lambda \psi. \]

(3.55)

The separation equations are (2.12). The three defining operators are

\[ L_1 = M_1^2, \quad L_2 = N_1^2, \]
\[ L_3 = M_1^2 + M_2^2 + N_1^2 - N_2^2 - N_3^2 + \frac{1}{2} \left[ P_1^2 - P_2^2 - P_3^2 \right]. \]
(71) \( (l, x, y, z) \rightarrow (y, ix, t, iz) \)

(72) \( (l, x, y, z) \rightarrow (y, l, t, iz) \)

(73) \( (l, x, y, z) \rightarrow (iy, l, ix, iz) \)

(74) \(-\) (81)

(74) This type corresponds to the choice \( \mu = 1 \) and

\[
\rho^2 = \frac{(x_2 - a)(x_3 - a)(x_4 - 1)}{a(1 - a)}
\]

\[
x^2 = \frac{(x_2 - 1)(x_3 - 1)(x_4 - 1)}{(1 - a)} \cos^2 x_1,
\]

\[
y^2 = \frac{(x_2 - 1)(x_3 - 1)(x_4 - 1)}{(1 - a)} \sin^2 x_1,
\]

\[
z^2 = \frac{x_2 x_3 x_4}{a}.
\]

The three operators are

\[
L_1 = P_0^2 + (a - 2)(P_1^2 + P_2^2) + (a - 1) P_3^2,
\]

\[
+ M_1^2 + M_2^2 + M_3^2 - N_1^2 - N_2^2 - N_3^2,
\]

\[
L_2 = (a - 1)(P_1^2 + P_2^2 - M_1^2 - M_2^2) - N_1^2 - N_2^2 + (a - 2) M_3^2,
\]

\[
L_3 = (1 - a) M_3^2.
\]

The coordinates can vary in the ranges

\[ x_2, x_3, x_4 > 0; \quad x_2 > 0 > x_3, x_4; \quad \text{with } x_1^2 > 0. \]

(83) The systems of this type are obtained from (82) via the transformation \( T \). The coordinates vary in the ranges

\[ x_2, x_3 > 0 > x_4; \quad 0 > x_2, x_3, x_4; \quad \text{with } x_1^2 > 0. \]

(84) \(-\) (87)

(84) This type corresponds to \( f(x) = (x - 1) x^2, \quad \mu = 0 \), and

\[
(l - t)^2 = x_2 x_3 x_4 t,
\]

\[
(l^2 - x^2) = -(x_2 x_3 + x_4 x_5) + x_2 x_3 x_4 (1 + x_1^2),
\]

\[
y^2 = x_2 x_3 x_4 t, \quad z^2 = (x_2 - 1)(x_3 - 1)(x_4 - 1).
\]

In terms of these coordinates the Klein–Gordon equation becomes (3.65) with \( a = 1 \) and \( \partial^2 / \partial x_j \)

\[ = 2 x_j \sqrt{x_j} (x_j - 1) (\partial / \partial x_j). \]

The three operators are

\[
L_1 = 2 P_0 (P_0 + P_1) + P_2^2 + M_1^2 + M_2^2 + M_3^2 - N_1^2 - N_2^2 - N_3^2,
\]

\[
L_2 = (P_0 + P_1)^2 - N_1^2 - 2 N_2^2 + (N_3 + M_1)^2 - N_4^2 - M_2 N_3,
\]

\[
L_3 = (N_2 - M_3)^2.
\]

(85) \(-\) (88)

(85) \( (l, x, y, z) \rightarrow (l i, t, i y, t) \)

(86) \( (l, x, y, z) \rightarrow (-x, l, i y, t) \)

(87) \( (l, x, y, z) \rightarrow (i x, t, i y, z) \)

(88) \(-\) (90)

(88) \( (l, x, y, z) \rightarrow (x i, t, y, t) \)

(89) \( (l, x, y, z) \rightarrow (-x, i, y, t) \)

(90) \( (l, x, y, z) \rightarrow (-i, x, y, t) \)

(91) \(-\) (92)

(91) \( (l, x, y, z) \rightarrow (x, i, y, i) \)

(92) \( (l, x, y, z) \rightarrow (x i, t, y, i) \)

This type corresponds to \( f(x) = (x - a)(x - b), \quad a = b^* \), \( \alpha, \beta \in \mathbb{R}, \quad \mu = 0 \):

\[
(z + t)^2 = \frac{2(x_1 - a)(x_2 - a)(x_3 - a)}{(a - b)}
\]

\[
x^2 = \frac{x_2 x_3}{a} \cos^2 x_1,
\]

\[
y^2 = \frac{x_2 x_3}{a} \sin^2 x_1.
\]

The three bases defining operators are

\[
L_1 = 2 \alpha (P_0^2 + P_1^2) + \alpha (P_2^2 - P_3^2) - 2 \beta P_0 P_3
\]

\[
+ (M_1^2 + M_2^2 + M_3^2 - N_1^2 - N_2^2 - N_3^2),
\]

\[
L_2 = \alpha (N_1^2 + N_2^2 + M_1^2) - (a^2 + b^2) (P_0^2 + P_1^2)
\]

\[
+ \beta (N_1 M_1 - N_1 M_2), \quad L_3 = ab M_2^2,
\]

where \( [A, B] = AB + BA \).
(91) This system corresponds to \( f(x) = x^2 \), \( \mu = 0 \), and
\[
(l - x)^2 = x_2 x_4 x_1, \quad 2y(l - x) = x_2 x_3 + x_4 x_1 + x_5 x_4,
\]
\[
x^2 + y^2 + z^2 - \ell^2 = x_5 + x_3 + x_4, \quad z^2 = x_5 x_3 x_4 x_1.
\]

(3.68)

In terms of these coordinates the Klein–Gordon equation assumes the form (3.58) with \( \alpha = 1 \) and \( \partial^2 / \partial y_j \nu_j = 2x_5^2 (\partial / \partial x_5) \).

The three defining operators are
\[
L_1 = -2P_9 + P_3, \quad L_2 = (P_9 + P_1)^2 + (P_3 + P_1)^2 - M_3^2 - M_2^2,
\]
\[
L_3 = (P_9 + P_3)^2.
\]

The coordinates \( x_j \) \( (j = 2, 3, 4) \) vary in the ranges \( x_1 > 0 > x_3 > x_5 \); with \( x_4^2 > 0 \).

(92) \((l, x, y, z) \rightarrow (l, i x, i y, i z)\)

(93) \(- (96)\)

(93) This system corresponds to \( f(x) = x(x - 1), \mu = 0 \), and
\[
l = \frac{1}{2}(x_1 + x_4 + x_3), \quad x^2 = x_2 x_3 x_4 \cos^2 x_1,
\]
\[
y^2 = x_3 x_4 \sin^2 x_1, \quad z^2 = -(x_2 - 1)(x_3 - 1)(x_4 - 1).
\]

(3.69)

In terms of these coordinates the Klein–Gordon equation becomes (3.58) with \( \alpha = 1 \) and \( \partial / \partial \nu_j = 2x_5^2 (\partial / \partial x_5) \).

The three defining operators are
\[
L_1 = -2P_9 + P_3, \quad L_2 = (P_9 + P_1)^2 + (P_3 + P_1)^2 - M_3^2 - M_2^2,
\]
\[
L_3 = (P_9 + P_3)^2.
\]

The coordinates \( x_2, x_3, \) and \( x_4 \) vary in the ranges \( x_2 > 0 > x_3 > x_4, \) with \( x_1^2 > 0 \).

(94) \((l, x, y, z) \rightarrow (y, l, i x, i z)\)

(95) \((l, x, y, z) \rightarrow (x, i x, i y, l)\)

(96) \((l, x, y, z) \rightarrow (y, i x, i y, l)\)

(97) \(- (98)\)

(97) This system corresponds to \( f(x) = x^2 \), \( \mu = 0 \), and
\[
l - x)^2 = x_2 x_3 x_4, \quad x^2 = x_2 x_3 x_4 + x_4 x_1 + x_3 x_4 x_2^2,
\]
\[
y^2 = x_3 x_4 x_2 x_1, \quad z = \frac{1}{2}(x_1 + x_3 + x_4).
\]

(3.71)

The Klein–Gordon equation assumes the form (3.58) with \( \alpha = 0 \) and \( \partial^2 / \partial y_j \nu_j = 2x_5^2 (\partial / \partial x_5) \).

The three defining operators are
\[
L_1 = -2P_9 + P_3, \quad L_2 = (P_9 + P_1)^2 + (P_3 + P_1)^2 - M_3^2 - M_2^2,
\]
\[
L_3 = (P_9 + P_3)^2.
\]

(98) \((l, x, y, z) \rightarrow (l x, i l, y, z)\)

(99) \(- (100)\)

(99) This system corresponds to \( f(x) = x, \mu = 0 \), and
\[
2(l - x) = x_4 x_2 + x_3 x_4 + x_3 x_1 - \frac{1}{2}(x_1^2 + x_2^2 + x_3^2),
\]
\[
2(x - l) = x_1 + x_3 + x_4,
\]
\[
x^2 = -x_2 x_3 x_4 \cos^2 x, \quad z^2 = x_2 x_3 x_4 \sin^2 x.
\]

(3.73)

In terms of these coordinates the Klein–Gordon equation becomes (3.58) with \( \alpha = 1 \) and \( \partial / \partial \nu_j = 2x_5^2 (\partial / \partial x_5) \).

The three operators are
\[
L_1 = \{N_9 + P_3\} - \{P_1, N_4 - M_1\} - \{P_9, N_3 + N_1\} + \{M_9 + M_3 + M_1\},
\]
\[
L_2 = (N_9 - M_1)^2 + (N_3 + N_1)^2,
\]
\[
L_3 = M_9^2.
\]

(3.74)

and the variables are such that \( \text{sign}(x_2 x_3 x_4) = -1 \) and \( x_1^2 > 0 \).

(100) \((l, x, y, z) \rightarrow (x, l x, i y, l)\)

F. Coordinate systems of class VII

These systems correspond to the various kinds of purely elliptical coordinates for which the Klein–Gordon equation is separable. The differential form is (2.15) where \( f(x) \) is at most a fourth order polynomial in \( x \).

(101) \(- (108)\)

(101) This type corresponds to \( f(x) = (x - a)(x - b)(x - c) \), \( \alpha = b = 1 \), and
\[
x^2 = \frac{x_2 x_3 x_4 x_5}{ab}, \quad y^2 = \frac{(x_1 - 1)(x_2 - 1)(x_3 - 1)(x_4 - 1)}{(a - 1)(b - 1)},
\]
\[
z^2 = \frac{(x_2 - b)(x_3 - b)(x_4 - b)}{(a - b)(b - 1)}, \quad
\]
\[
\frac{z^2}{z^2} = \frac{(x_2 - a)(x_3 - a)(x_4 - a)}{(a - b)(b - 1)}.
\]

(3.75)

The Klein–Gordon equation becomes
\[
\sum_i \frac{1}{(x_i - x_j)(x_i - x_k)} \frac{\partial^2}{\partial x_i^2} - \frac{\partial}{\partial x_i} = \lambda \delta_i,
\]

where \( i, j, k \) are not equal and
\[
\sum_i \frac{\partial}{\partial x_i} = 2 \sqrt{(x_i - a)(x_i - b)}(x_i - 1) x_i \frac{\partial}{\partial x_i}.
\]

(3.76)

The three operators are
\[
L_1 = -N_9^2 - N_3^2 + N_4^2 - M_3^2 + M_4^2 + M_3^2 + (a + b + 1) P_3^2 - (a + 1) P_1^2 - (a + b + 1) P_3^2,
\]
\[
L_2 = (a + b) N_9^2 + (a + 1) N_3^2 + (a + 1) N_1^2 - a M_3^2 - M_4^2 - b M_4^2 - ab M_3^2 - ab P_3^2,
\]
\[
L_3 = ab N_9^2 + a M_3^2 + b M_4^2 + ab P_3^2.
\]

(3.77)

For coordinates of this type the variables \( x_2 \) can lie in the ranges \( x_2 > a > x_4 > b > x_3 > 1 > x_4 \).

(102) \((l, x, y, z) \rightarrow (l x, i l, y, z)\)
(103) \((t, x, y, z) \rightarrow (iz, x, y, it)\)

(104) \((t, x, y, z) \rightarrow (y, x, it, z)\)

(105) \((t, x, y, z) \rightarrow (it, ix, iy, iz)\)

(106) \((t, x, y, z) \rightarrow (s, ix, iy, t)\)

(107) \((t, x, y, z) \rightarrow (t, iy, ty, iz)\)

(108) \((t, x, y, z) \rightarrow (y, ix, t, iz)\)

(109) \((t, x, y, z) \rightarrow (x, iy, iy, sz)\)

(109) This type corresponds to \(f(x) = (x - a)(x - b)(x - 0)x\), \(a = b^* = a + b\), and

\[
(x + it)^2 = \frac{2(x_1 - b)(x_2 - b)(x_3 - b)(x_4 - b)}{(b - a)(b - 1)b},
\]

\[
y^2 = \frac{(x_1 - 1)(x_2 - 1)(x_3 - 1)(x_4 - 1)}{(b - a)(b - 1)} \tag{3.76}
\]

\[
z^2 = \frac{y^2(x_1 x_2 x_3 x_4)}{ab}.
\]

The Klein–Gordon equation becomes (3.76). The three operators are

\[
L_1 = \frac{N^2_1 + \frac{N^2_2}{2} + \frac{N^2_3}{2} - \frac{M^2_1}{2} - \frac{M^2_2}{2} - \frac{M^2_3}{2}}{2a}
\]

\[
L_2 = \frac{(x_1 - a)(x_2 - a)(x_3 - a)(x_4 - a)}{a^2(a - 1)},
\]

\[
z^2 = \frac{(x_1 - 1)(x_2 - 1)(x_3 - 1)(x_4 - 1)}{(b - a)(b - 1)} \tag{3.79}
\]

The variables \(x_i\) lie in the ranges \(x_1 > 1 \Rightarrow x_2 > 0 \Rightarrow x_3, x_4 \Rightarrow x_2 > 0, x_3, x_4 \Rightarrow 0\).

(110) \((t, x, y, z) \rightarrow (i, t, iy, iy, iz)\)

(111) \((t, x, y, z) \rightarrow (x_1, iy, iy, z)\)

(111) This type corresponds to \(f(x) = (x - a)(x - 1)x^2\), \(a > 1\), and

\[
(t + it)^2 = x_1 y_1 x_2 y_2 x_3 x_4 / a,
\]

\[
n(t^2 - x^2) = -(x_1 y_1 x_2 y_2 x_3 x_4 / a + 1)x_1 y_1 x_2 y_2 x_3 x_4 / a^2,
\]

\[
y^2 = \frac{(x_1 - 1)(x_2 - 1)(x_3 - 1)(x_4 - 1)}{(a - 1)} \tag{3.80}
\]

\[
z^2 = \frac{(x_1 - a)(x_2 - a)(x_3 - a)(x_4 - a)}{a^2(a - 1)}
\]

The Klein–Gordon equation becomes (3.76) with

\[
\frac{\partial}{\partial v_1} = 2x_1 \sqrt{(x_1 - a)(x_1 - 1)} \frac{\partial}{\partial x_1}.
\]

The three operators are

\[
L_1 = -N^2_1 - N^2_2 - N^2_3 - M^2_1 + M^2_2 + M^2_3
\]

\[
+ 2a(N_2^2 / 2 - P_2^2) + (P_0 - P_1)^2 + aP_1^2,
\]

\[
L_2 = -2M^2_2 + [N_2, M_2] - [N_2 + M_2]^2
\]

\[
+ a(N_2^2 - M_2^2) + (a + 1)N^2_1 + (a + 1)(P_0 - P_1)^2,
\]

\[
L_3 = -a(N_2 + M_2)^2 + aN^2_1 - (N_2 - M_2)^2 + (P_0 - P_1)^2.
\]

\[
(3.81)
\]

The variables \(x_i\) lie in the ranges

\[
x_1 > a > x_2 > 1 \Rightarrow x_3, x_4 \Rightarrow 0;
\]

\[
x_1 > a > x_2 > 1 \Rightarrow 0 \Rightarrow x_3, x_4;
\]

\[
x_1 > a > x_2, x_3, x_4 > 1; \ x_1, x_2, x_3 > a > x_4 > 1.
\]

(112) \((i, t, iy, y, z)\)

(113) \((x, t, iy, iy, iz)\)

(114) \((t, x, y, z) \rightarrow (i, t, iy, iy, iz)\)

(115) \([\mathfrak{g}]\)

This type corresponds to \(f(x) = (x - 1)x^2\) and

\[
(z_1 + P_1)^2 = (x_1 y_1 x_2 y_2 x_3 x_4 / a + 1)x_1 y_1 x_2 y_2 x_3 x_4 / a^2
\]

\[
+ (x_1 - 1)(x_2 - 1)(x_3 - 1)(x_4 - 1) \tag{3.79}
\]

\[
x_1 > a > x_2 > 1 \Rightarrow x_3, x_4 \Rightarrow 0;
\]

\[
x_1 > a > x_2 > 1 \Rightarrow 0 \Rightarrow x_3, x_4;
\]

\[
x_1 > a > x_2, x_3, x_4 > 1; \ x_1, x_2, x_3 > a > x_4 > 1.
\]

(116) \((t, x, y, z) \rightarrow (it, iy, iy, iz)\)

(116) This type corresponds to \(f(x) = (x - 1)x^2\) and

\[
(x - 1)^2 = x_1 y_1 x_2 y_2 x_3 x_4 / a,
\]

\[
y(x - l) = -(x_1 y_1 x_2 y_2 x_3 x_4 / a + 1)x_1 y_1 x_2 y_2 x_3 x_4 / a^2,
\]

\[
y^2 = \frac{(x_1 - 1)(x_2 - 1)(x_3 - 1)(x_4 - 1)}{(a - 1)} \tag{3.80}
\]

\[
z^2 = \frac{(x_1 - a)(x_2 - a)(x_3 - a)(x_4 - a)}{a^2(a - 1)}
\]

The Klein–Gordon equation becomes (3.76) with

\[
\frac{\partial}{\partial v_1} = 2x_1 \sqrt{(x_1 - a)(x_1 - 1)} \frac{\partial}{\partial x_1}.
\]

The three operators are

\[
L_1 = -N^2_1 - N^2_2 - N^2_3 - M^2_1 + M^2_2 + M^2_3
\]

\[
+ 2a(P_0^2 - P_2^2) + (P_0 - P_1)^2 + aP_1^2,
\]

\[
L_2 = -2M^2_2 + [N_2, M_2] - [N_2 + M_2]^2
\]

\[
+ a(N_2^2 - M_2^2) + (a + 1)N^2_1 + (a + 1)(P_0 - P_1)^2,
\]

\[
L_3 = -a(N_2 + M_2)^2 + aN^2_1 - (N_2 - M_2)^2 + (P_0 - P_1)^2.
\]

\[
(3.83)
\]

The Klein–Gordon equation becomes (3.76). The three operators are

\[
L_1 = -N^2_1 - N^2_2 - N^2_3 - M^2_1 + M^2_2 + M^2_3 + 2P_2(P_0 - P_1),
\]

\[
L_2 = -N^2_2 - N^2_3 - M_2^2 - [N_2, M_2] + (P_0 - P_1)^2,
\]

\[
L_3 = -(P_0 + P_1)^2 + [N_1, N_2 - M_1] - (N_2 + M_2)^2.
\]

\[
(3.85)
\]
The coordinates \( x_1 \) vary in the ranges \( x_1 > 1 > x_2 > 0 > x_3, x_4 > 0 \), and \( x_1, x_2, x_3, x_4 > 0 \).

(117) \((t, x, y, z) \rightarrow (it, ix, iy, iz)\)

(118) \([G]\)

This type corresponds to \( f(x) = x^4 \) and

\[
(z_1 + iz_2)^2 = -x_1 x_2 x_3 x_4,
\]

\[
2(z_1 + iz_2)(z_3 + iz_4) = x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4,
\]

\[
(x_1 + iz_2)(z_3 - iz_4) + (z_3 + iz_4)^2 = -x_1 x_2 - x_1 x_3 - x_2 x_4 - x_3 x_4,
\]

\[
z_1^2 + z_2^2 + z_3^2 + z_4^2 = x_1 + x_2 + x_3 + x_4. \tag{3.86}\]

The Klein–Gordon equation assumes the form (3.76) with \( \partial^2 / \partial \nu_j = 2x_j \sqrt{x_j - 1} \).

The three operators are

\[
L_1 = - \left( p_x + i p_y \right)^2 + \left( p_x + i p_y \right) \left( p_x - i p_y \right) + \frac{\alpha}{t} + \frac{\beta}{t} + \frac{\beta^2}{t} + \frac{\alpha^2}{t} + \frac{\beta^2}{t},
\]

\[
L_2 = \frac{i}{2} \left[ \left( 1 + i \beta \right) \left( 1 + i \beta \right) \right] \left( p_x + i p_y \right) \left( p_x - i p_y \right),
\]

\[
L_3 = \left( p_x + i p_y \right) \left( p_x + i p_y \right) - \left( p_x + i p_y \right) \left( p_x - i p_y \right). \tag{3.87}\]

(119) – (122)

(119) This type corresponds to \( f(x) = (x - a)(x - 1) \) and

\[
\frac{\partial}{\partial \nu_j} = 2 \sqrt{x_j - 1} \left( \partial / \partial x_j \right). \tag{3.88}\]

The coordinates \( x_1 \) can vary in the ranges \( x_1 > 1 > x_2 > 0 > x_3 \), and \( x_1, x_2, x_3, x_4 > 0 \).

\[
\frac{\partial}{\partial \nu_j} = 2 \sqrt{x_j - 1} \left( \partial / \partial x_j \right). \tag{3.89}\]

(120) \((t, x, y, z) \rightarrow (x, ix, iy, iz)\)

(121) \((l, x, y, z) \rightarrow (y, iy, l, iz)\)

(122) \((l, x, y, z) \rightarrow (x, iy, iy, iz)\)

(123) This type corresponds to \( f(x) = (x - a)(x - b) \) and

\[
y = \frac{1}{2} \left( x_1 + x_2 + x_3 + x_4 \right),
\]

\[
(t + ix)^2 = 2(x_1 - a)(x_2 - a)(x_3 - a)(x_4 - a)(a - b),
\]

\[
x^2 = -x_1 x_2 x_3 x_4 / a b, \quad a^2 = \alpha + \beta, \quad \alpha, \beta \in \mathbb{R}. \tag{3.90}\]

The Klein–Gordon equation becomes (3.76) with \( \partial / \partial \nu_j = 2x_j \sqrt{x_j - 1} \left( \partial / \partial x_j \right). \)

The three operators are

\[
L_1 = \left[ p_x + i p_y \right] \left( p_x - i p_y \right) + 2 \alpha \left( p_x^2 - p_y^2 \right) + 2 \beta \left( p_x + p_y \right),
\]

\[
L_2 = -2 \alpha \left( p_x + i p_y \right) + \alpha \left( p_0 \right) \left( p_x - i p_y \right) - 2 \beta \left( p_x + i p_y \right) \left( p_x - i p_y \right) + 2 \alpha^2 \left( p_x^2 - p_y^2 \right) + 2 \beta^2 \left( p_0 \right),
\]

\[
L_3 = \left( \alpha^2 + \beta^2 \right) \left( p_x + i p_y \right) + \left( \alpha^2 + \beta^2 \right) \left( p_x - i p_y \right) + 2 \alpha^2 \left( p_0 \right) + 2 \beta^2 \left( p_0 \right). \tag{3.91}\]

The coordinates \( x_1, x_2, x_3 > 0 > x_4; \quad x_1 > 0 > x_2, x_3, x_4. \)

(124) – (125)

(124) This type corresponds to \( f(x) = (x - 1)^2 \) and

\[
(t - x)^2 = -x_1 x_2 x_3 x_4, \quad z = \frac{1}{2} \left( x_1 + x_2 + x_3 + x_4 \right),
\]

\[
(t^2 - x^2) = x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4,
\]

\[
y^2 = -x_1 x_2 x_3 x_4 / 2, \quad x_1 > 1 > x_2, x_3, x_4. \tag{3.92}\]

The Klein–Gordon equation becomes (3.76) with \( \partial / \partial \nu_j = 2x_j \sqrt{x_j - 1} \left( \partial / \partial x_j \right). \)

The three operators are

\[
L_1 = \left[ p_0, p_x \right] + \left[ p_1, p_y \right] + \left[ p_0, p_x \right] + \left[ p_1, p_y \right] + \left[ p_0, p_1 \right] + \left[ p_y, p_0 \right] + \left[ p_x, p_0 \right] + \left[ p_y, p_x \right],
\]

\[
L_2 = -2 \left( p_0 \right) \left( p_x + i p_y \right) - 2 \left( p_0 \right) \left( p_x - i p_y \right) + 2 \left( p_0 \right) \left( p_x + i p_y \right) \left( p_x - i p_y \right) + 2 \left( p_0 \right) \left( p_x + i p_y \right) \left( p_x - i p_y \right),
\]

\[
L_3 = \left( p_0 \right) + \left( p_x + i p_y \right) + \left( p_x - i p_y \right). \tag{3.93}\]

The coordinates \( x_1 \) can vary in the ranges

\[
x_1 > 1 > x_2, x_3 > 0 > x_4; \quad x_1 > 1 > x_2, x_3, x_4; \quad x_1 > x_2, x_3 > x_4 > 0. \tag{125}\]

(126) This type corresponds to \( f(x) = x^3 \) and

\[
(t - x)^2 = -x_1 x_2 x_3 x_4, \quad z = \frac{1}{2} \left( x_1 + x_2 + x_3 + x_4 \right),
\]

\[
(t^2 - x^2) = x_1 x_2 x_3 x_4 + x_1 x_2 x_3 x_4 + x_1 x_2 x_3 x_4 + x_1 x_2 x_3 x_4, \tag{3.94}\]

\[
y^2 = x_1 x_2 x_3 x_4 / 2, \quad x_1, x_2, x_3, x_4 > 0. \]

The Klein–Gordon equation assumes the form (3.76) with \( \partial / \partial \nu_j = 2x_j \sqrt{x_j - 1} \left( \partial / \partial x_j \right). \)

The three operators are
\[ L_1 = \{P_1, M_2\} - \{P_9, N_2\} - \{P_2, M_4\}, \]
\[ L_2 = \{P_3, N_2 + M_2\} + \{M_3, P_4 + P_1\} + M_1^2 - N_1^2 - 2P_0P_1 - P_0^2, \]
\[ L_3 = \{P_0 + P_1, N_3 + M_2\} - (N_2 - M_3)^2. \]  \hspace{1cm} (3.95)

The coordinates \( x_i \) vary in the ranges \( x_i > 0 \) for \( x_1, x_2, x_3 \) or \( x_1, x_2, x_3 > 0 > x_4 \).

(127)—(128)

This type corresponds to \( f(x) = x(x - 1) \) and
\[ 2(l - x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2 + x_4^2) \]
\[ - (x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4), \]
\[ + (x_1 + x_2 + x_3 + x_4), \]
\[ 2(l + x) = x_1 + x_2 + x_3 + x_4. \]
\[ y^2 = (x_1 - 1)(x_2 - 1)(x_3 - 1)(x_4 - 1), \]
\[ z^2 = -x_1x_2x_3x_4. \]  \hspace{1cm} (3.96)

In terms of these coordinates the Klein–Gordon equation assumes the form (3.76) with \( \partial / \partial y = 2x_1(\partial / \partial x_1) \).

The three operators are
\[ 4L_1 = \{N_3 + M_2, P_3\} - \{N_2 - M_3, P_2\} + \frac{1}{2}(N_1, P_0 + P_1) - \frac{1}{2}(P_0 - P_1)^2 + P_1^2 + \frac{1}{2}P_0(P_0 + P_1), \]
\[ 4L_2 = \{N_3 + M_2, P_3\} - \{M_3, P_4 + P_1\} + \frac{1}{2}(N_1, P_0 + P_1) + \frac{1}{2}(N_2 - M_3)^2 + \frac{1}{2}(N_1, M_2, P_4) + \frac{1}{2}(N_2, P_3, P_2), \]
\[ 4L_3 = -P_1^2 - M_1^2 - N_1^2 + M_2, P_3^2, + \{N_3, P_2\}. \]  \hspace{1cm} (3.97)

The coordinates \( x_i \) vary in the ranges
\[ x_1, x_2 > 1 > x_3 > 0 > x_4; \]
\[ 1 > x_1 > 0 > x_2, x_3, x_4; \]
\[ x_1, x_2 > 1 > x_3 > 0 > x_4; \]
\[ x_1 > 0 > x_2, x_3, x_4; \]
\[ x_1 > 0 > x_2, x_3, x_4; \]
\[ 1 > x_1 > 0 > x_2, x_3, x_4; \]
\[ 1 > x_1, x_2, x_3 > 0 > x_4. \]

(128) \[(l, x, y, z) = (x, t, iyt, iz)\]

(129) \[C\]

This type corresponds to \( f(x) = x^2 \) and
\[ 2(iz_1 - x_1) = x_1 + x_2 + x_3 + x_4, \]
\[ 2(iz_1 + x_1) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2 + x_4^2) \]
\[ - (x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4), \]
\[ (x_1 - iz_1)^2 = x_1x_2x_3x_4, \]
\[ (x_1^2 + x_1^2) = -(x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4). \]  \hspace{1cm} (3.98)

The Klein–Gordon equation assumes the form (3.76) with \( \partial / \partial y = 2x_1(\partial / \partial x_1) \).

The three operators are
\[ 4L_1 = \{[P_1 + i[H_{12}, P_2], P_3\} + \{L_2, [P_4, H_{12}, P_1\} \]
\[ + \frac{1}{2}[L_{12}, P_1 + iP_2 + [P_2, P_2^* - 2P_0P_1 + P_0^2, \]
\[ + (L_{12} + iL_{12}) - (L_{12} + iL_{12})^2, \]
\[ 4L_2 = \{L_1, H_{12}, P_1\} + \{L_2 + iL_{12}, P_3\} \]
\[ + (L_{12} + iL_{12})^2, \]
\[ + \frac{1}{2}(L_{12} + iL_{12})^2, \]
\[ + \frac{1}{2}(L_{12} + iL_{12})^2. \]  \hspace{1cm} (3.99)

(130) This type corresponds to \( f(x) = x \) and
\[ 2(x - l) = 1 - x_1 - x_2 - x_3 - x_4, \]
\[ 2y - (x - l)^2 = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 \]
\[ + x_1x_2x_3x_4 + x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4, \]
\[ + (x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4), \]
\[ - (x_1 + x_2 + x_3 + x_4) + 1, \]
\[ z^2 = -x_1x_2x_3x_4. \]  \hspace{1cm} (3.100)

The Klein–Gordon equation assumes the form (3.76) with \( \partial / \partial y = 2x_1(\partial / \partial x_1) \).

We have not yet determined the three operators which describe this system. The coordinates vary in the ranges \( x_1, x_2, x_3 > 0 > x_4 \) and \( x_1 > 0 > x_2, x_3, x_4 \).

(131) \[C\]

This type corresponds to \( f(x) = 1 \) and
\[ 2(x_1 + iz_4) = - (x_1 + x_2 + x_3 + x_4), \]
\[ 2(z_1 + iz_4 + (z_1 + iz_4)^2, \]
\[ = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4, \]
\[ x_3 = -iz_4), + 2(z_1 + iz_4)(z_1 + iz_4), \]
\[ = -x_1x_2x_3 - x_1x_2x_4 - x_1x_3x_4 - x_2x_3x_4, \]
\[ L_4 = -iz_4 + (z_1 + iz_4)(z_1 - iz_4), \]
\[ + (z_1 + iz_4)^2 = x_1x_2x_3x_4. \]  \hspace{1cm} (3.101)

The Klein–Gordon equation assumes the form (3.76) with \( \partial / \partial y = 2x_1(\partial / \partial x_1) \).

We have not yet determined the operators which describe this system. This compiles the list of orthogonal coordinates for which the Klein–Gordon equation separates. As was mentioned earlier we have only given those systems which are genuinely new in that they have not been derived elsewhere before. For the wave equation \( \lambda = 0 \) we have found 125 such coordinate systems. In addition there are 34 radial coordinate systems corresponding to the group reduction \( E(3, 1) \cong SO(3, 1) \) \( \cong \{L_1, L_2\} \) where \( [L_1, L_2] = 0 \) and \( L_1, L_2 \) are second order elements in the enveloping algebra of \( SO(3, 1) \). Similarly there are 55 coordinate systems belonging to reductions of the type \( E(3, 1) \cong E(2, 1) \cong \{L_1, L_2\} \), 11 coordinate systems belonging to reductions of the type \( E(3, 1) \cong E(3) \cong \{L_1, L_2\} \) and 36 coordinate systems belonging to reductions of the type \( E(3, 1) \cong E(2) \cong \{L_1, L_2\} \), where in this last case \( L_1 \) and \( L_2 \) are second order elements in the enveloping algebras of \( E(2) \) and \( E(1, 1) \), respectively. We have a total of 261 coordinate systems in which the Klein–Gordon equation admits separation of variables.