Lie theory and the wave equation in space–time. 4. The Klein–Gordon equation and the Poincaré group

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A detailed classification is made of all orthogonal coordinate systems for which the Klein–Gordon equation in space–time, $\psi_{tt}-\Delta_3\psi=\lambda\psi$, admits a separation of variables. We show that the Klein–Gordon equation is separable in 261 orthogonal coordinate systems. In each case the coordinate systems presented are characterized in terms of three symmetric second order commuting operators in the enveloping algebra of the Poincaré group. This paper also consitutes an important step in the study of separation of variables for the wave equation in space–time $\psi_{tt}-\Delta_3\psi=0$, and its relation to the underlying conformal symmetry group O(4,2) of this equation.

INTRODUCTION

In this paper we continue[1–3] an investigation of the connection between separation of variables for the wave equation in space—time

$$\psi_{tt}-\Delta_3\psi=0, \qquad (0.1)$$

and the O(4, 2) symmetry group of this equation. Here we study all the orthogonal coordinate systems for which the Klein—Gordon equation

$$\Box\psi=\psi_{tt}-\Delta_3\psi=\lambda\psi, \quad \lambda\neq 0 \qquad (0.2)$$

admits a separation of variables. [By simply setting $\lambda=0$ in our results we will obtain orthogonal separable systems for (0.1).] The method used to compute all such coordinate systems is an adaptation of that used by Eisenhart[4] in the case of the Helmholtz equation in three-dimensional Euclidean space. The work of Eisenhart enables us to classify all distinct orthogonal differential forms

$$ds^2=\sum_{i=1}^{4} H_i^2\,dx_i^2, \qquad (0.3)$$

and hence coordinate systems for which (0.2) admits a separation of variables. In (0.3) the H_i^2 are real functions of the new variables x_i such that sign $H_i^2=+$ for $i=1,2,3$, and sign $H_4^2=-$. The coordinates x_i are related to the standard space—time coordinates x,y,z,t by the real functions $G_i(x_1,x_2,x_3,x_4)$, where $t=G_1$, $x=G_2$, $y=G_3$, and $z=G_4$. In terms of the standard coordinates, the differential form (0.3) becomes

$$ds^2=dx^2+dy^2+dz^2-dt^2. \qquad (0.4)$$

With each such differential form we give the associated space—time coordinate functions G_i and the expression for the Klein—Gordon equation in these coordinates. We also write out the separation equations, identifying their solutions as much as possible, and we compute the three commuting operators L_i $(i=1,2,3)$ whose eigenvalues are the separation constants. Each of these three operators is written as a symmetric second order operator in the enveloping algebra of the Poincaré symmetry group E(3, 1) of the Klein—Gordon equation (0.2).

When $\lambda=-m^2$, $m>0$, Eq. (0.2) becomes

$$(\Box+m^2)\psi(X)=0, \quad X=(t,x,y,z); \qquad (0.5)$$

the relativistic equation describing a free neutral scalar particle with mass m. In the standard field-theoretic treatments of (0.5),[5] one expresses a positive-energy solution ψ in terms of its Fourier transform

$$\psi(X)=\frac{1}{(2\pi)^{3/2}}\times\iiint\frac{\exp[-i(tk_0-xk_1-yk_2-zk_3)]}{\sqrt{2}}f(\mathbf{k})\,dm(\mathbf{k}), \qquad (0.6)$$

where the integration surface is the hyperboloid $k_0^2-k_1^2-k_2^2-k_3^2=m^2$, $k_0>0$. The Lebesgue measurable functions $f(\mathbf{k})$, such that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f(\mathbf{k})|^2\,dm(\mathbf{k})<\infty, \quad dm(\mathbf{k})=dk_1\,dk_2\,dk_3/k_0, \qquad (0.7)$$

form a Hilbert space H_m with inner product

$$\langle f_1,f_2\rangle=\iiint_{-\infty}^{\infty} f_1(k)\overline{f_2}(k)\,dm(k), \quad f_1,f_2\in H_m. \qquad (0.8)$$

The mapping (0.6) then induces a Hilbert space structure on the solution space of (0.5) given by

$$(\psi_1,\psi_2)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\Psi_1(x)\,\partial_t\overline{\psi}_2(x)-(\partial_t\Psi_1(x))\,\overline{\psi}_2(x)]\,dx\,dy\,dz \qquad (0.9)$$

(independent of t), where ψ_j is related to $f_j\in H_m$ by (0.6). The natural action of the connected Poincaré group E(3, 1) on the functions ψ induces an action on the transform space H_m which is well known to be unitary and irreducible.[5]

In studies of this physical system it is obviously of interest to construct various orthonormal bases for H_m, particularly bases which correspond to separable solutions of (0.5). However, with few exceptions, only the plane wave basis (corresponding to separation in

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Cartesian coordinates) is employed in the published literature. Here we show explicitly that every orthogonal separable coordinate system for (0.5) has the property that the associated separated solutions are characterized as simultaneous eigenfunctions of a commuting triplet of second-order symmetry operators from the enveloping algebra of E(3,1). The corresponding operators acting on the domain of C^∞ functions with compact support in H_m are obviously symmetric. These operators can then be extended to a commuting triplet of self-adjoint operators on H_m. (However, in some cases the deficiency indices are equal but nonzero, so that the extension is not unique. Furthermore, in a few cases the deficiency indices of some operators are unequal. This difficulty can be removed by extending the Hilbert space to include the negative energy solutions.) The spectral theorem for commuting sets of self-adjoint operators thus implies the existence of a basis for H_m which is a (generalized) eigenbasis of the commuting operators. Mapping the eigenbasis to the solution space of (0.5) via (0.6), we see that the basis eigenfunctions are separable solutions of (0.5). The spectral resolutions of the defining self-adjoint operators as computed in H_m can then be used to derive expansion theorems and special function identities for solutions of (0.5). Our characterization of orthogonal separable systems in terms of commuting second-order operators in the enveloping algebra which act within a unitary irreducible representation of E(3,1) is an essential part of this program.

The paper is arranged as follows. In Sec. 1 we present the necessary details concerning the generators of the Poincaré group. In addition we give a preliminary discussion concerning the arrangement and computation of the coordinate systems. In Sec. 2 we extend the work of Eisenhart to consider orthogonal differential forms in four variables and then compute all the inequivalent classes of differential forms. In Sec. 3 we give the coordinate systems, separation equations, and operators defining the separation constants.

I. SOME PROPERTIES OF THE POINCARE GROUP E(3,1)

Here we briefly present those properties of the Poincaré group E(3,1) that are relevant to this article. For more details concerning this group the reader is referred to paper 3 of this series and Refs. 6, 7, and 8. The Poincaré group consists of all proper real linear transformations which preserve the differential form (0.4). The group is the semidirect product of the group of translations T_4 in the space and time coordinate and the group of proper real Lorentz transformations SO(3,1), i.e.

$$E(3,1)=T_4\times SO(3,1).$$

The Lie algebra is ten-dimensional with basis elements:

1. Translations

$$P_0=\partial_t;\quad P_1=\partial_x,\quad P_2=\partial_y,\quad P_3=\partial_z;$$

2. Pure Lorentz transformations

$$N_1=t\partial_x+x\partial_t,\quad N_2=t\partial_y+y\partial_t,\quad N_3=t\partial_z+z\partial_t;$$

3. Rotations

$$M_1=y\partial_z-z\partial_y,\quad M_2=x\partial_z-z\partial_x,\quad M_3=x\partial_y-y\partial_x.$$

These generators satisfy the commutation relations

$$[M_i,M_j]=\epsilon_{ijk}M_k,\quad [M_iN_j]=\epsilon_{ijk}N_k,\quad [N_i,N_j]=-\epsilon_{ijk}M_k,$$
$$[P_i,N_j]=\delta_{ij}P_0,\quad [P_i,M_j]=\epsilon_{ijk}P_k,$$

where $i,j,k=1,2,3$;

$$[P_0,N_j]=P_j,\quad [P_0,M_j]=0,$$

for $j=1,2,3$, and

$$[P_i,P_j]=0,$$

for all i,j.

On the Hilbert space H_m [Eqs. (0.7) and (0.8)] the Lie algebra generators are

$$P_0=-ik_0,\quad P_j=ik_j,\quad N_j=k_0\partial_{k_j},\quad i=1,2,3,$$
$$M_1=k_2\partial_{k_3}-k_3\partial_{k_2},\quad M_2=k_1\partial_{k_3}-k_3\partial_{k_1},$$
$$M_3=k_1\partial_{k_2}-k_2\partial_{k_1}.$$

In addition to the real Poincaré group E(3,1) we will also consider its complexification E(4,ℂ). This is the group of proper complex transformations which preserve the differential form

$$ds^2=dz_1^2+dz_2^2+dz_3^2+dz_4^2,$$

where $z_i\in\mathbb{C}$, $i=1,2,3,4$. The group E(4,ℂ) is the semidirect product of the translation group T_4 and SO(4,ℂ), i.e.,

$$E(4,\mathbb{C})=T_4\times SO(4,\mathbb{C}).$$

The Lie algebra is ten-dimensional with basis elements:

1. Translations $P_i=\partial_{z_i}$, $i=1,2,3,4$,
2. Rotations $I_{ij}=z_i\partial_{z_j}-z_j\partial_{z_i}$,

with $i,j=1,2,3,4$ and $i\neq j$.

These basis elements satisfy

$$[I_{kl},I_{st}]=\delta_{ls}I_{kt}-\delta_{ks}I_{lt}-\delta_{lt}I_{ks}+\delta_{kt}I_{ls},$$
$$[P_i,P_j]=0,$$
$$[P_i,I_{kl}]=\delta_{ik}P_l-\delta_{il}P_k.$$

II. ORTHOGONAL SEPARABLE DIFFERENTIAL FORMS FOR THE KLEIN-GORDON EQUATION AND ITS COMPLEXIFICATION

In this section we classify the possible orthogonal differential forms which enable (0.2) or its complexification

$$\sum_{i=1}^{4}\partial_{z_iz_i}\psi=\lambda\psi \tag{2.1}$$

to be solved by separation of variables. By this we mean a classification of all choices of new variables x_1, x_2, x_3, x_4, such that $t=G_1$, $x=G_2$, $y=G_3$, and $z=G_4$.

In the case of the Klein–Gordon equation, the real functions G_i $(i=1,2,3,4)$ are real differentiable functions of the real variables x_i $(i=1,2,3,4)$. In order that the new coordinates x_i be orthogonal we have the additional requirement that

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2 = \sum_{i=1}^{4} H_i^2 \, dx_i^2, \tag{2.2}$$

where sign $H_i^2 = +$ for $i = 1, 2, 3$ and sign $H_4^2 = -$.

In the case of the complexified Klein–Gordon equation, the functions G_i $(i = 1, 2, 3, 4)$ are analytic functions of the complex variables x_i. The requirement of orthogonality is the same as in the real case but with no restrictions on the signs of the metric coefficients. The coordinate systems fall into five broad classes, whose general features we now summarize. Details of the derivations are given in Ref. 9.

A. Coordinate systems of class I

These correspond to coordinate systems giving the differential form

$$ds^2 = \frac{(x_1 - x_2)}{4} \left[\frac{dx_1^2}{x_1^2} - \frac{dx_2^2}{x_2^2}\right] + \epsilon x_1 x_2 (dx^2 + dy^2), \quad \epsilon = \pm, \tag{2.3}$$

where x, y can be replaced by one of the four possible coordinate systems in the Euclidean plane (in the case of the real Klein–Gordon equation). In the case of the complexified equation, x and y can be replaced by one of the various possible coordinate systems for all the complex Euclidean plane.[10] The separable solutions of (0.2) for coordinate systems of this type assume the typical form

$$\Psi = e^{-(a+b)} De_\nu(a + \tfrac{1}{2}x, \sqrt{-\lambda l_1}) \times De_\nu(b + \tfrac{1}{2}x, \sqrt{-\lambda l_1}) \, E_3(x_3) \, E_4(x_4), \tag{2.4}$$

where $e^a = \sqrt{x_1}$, $e^b = \sqrt{x_2}$, $\tanh x = (\lambda + l_1)/(\lambda - l_1)$. Here x_3 and x_4 correspond to the appropriate choice of coordinates in the Euclidean plane and $\phi(x, y) = E_3(x_3) E_4(x_4)$ is a solution of

$$[(N_2 + M_3)^2 + (N_3 - M_2)^2] \, \phi(x, y) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = l_1 \phi \tag{2.5}$$

and $x = x(x_3, x_4)$, $y = y(x_3, x_4)$. Furthermore, $D = S, C$ and Se_ν, Ce_ν are Mathieu functions.[11]

B. Coordinate systems of class II

These correspond to systems giving the differential form

$$ds^2 = \frac{(x_1 - x_2)}{4} \left[\frac{dx_1^2}{x_1} - \frac{dx_2^2}{x_2}\right] + x_1 x_2 \, d\omega^2, \tag{2.6}$$

where $d\omega^2$ is one of the differential forms associated with the two-dimensional sphere or the two-dimensional single or double sheeted hyperboloids. The separable solutions of (0.2) for systems of this type appear as

$$\Psi = (x_1 x_2)^{-1/4} M_{\pm i(\lambda l_3)^{1/2}/2, 1/2(j+1/2)}(\pm i x_1 / 2\sqrt{\lambda}) \times M_{\pm i(\lambda l_3)^{1/2}/2, 1/2(j+1/2)}(\pm i x_2 / 2\sqrt{\lambda}) \, E_3(x_3) \, E_4(x_4), \tag{2.7}$$

where $E_3(x_3) E_4(x_4) = \phi$ is a solution of

$$(N_1^2 + N_2^2 - M_3^2) \, \phi = j(j+1) \, \phi \tag{2.8}$$

and the coordinates x_3, x_4 are one of the nine possible types for which this equation admits a separation of variables. Here $M_{\alpha,\mu}$ is a Whittaker function.[12]

C. Coordinate systems of class III

These correspond to systems giving the differential form

$$ds^2 = \frac{(x_1 - x_2)}{4} \left[\frac{dx_1^2}{x_1(x_1 - 1)} - \frac{dx_2^2}{x_2(x_2 - 1)}\right] + x_1 x_2 \, d\omega^2, \tag{2.9}$$

with $d\omega^2$ as in Class II. The separable solutions of (0.2) for systems of this type assume the typical form

$$\Psi = (x_1 x_2)^{1/4} Ps_\nu^{j+1/2}(\sqrt{1 - x_1}, \lambda) \, Ps_\nu^{j+1/2}(\sqrt{1 - x_2}, \lambda) \times E_3(x_3) \, E_4(x_4), \tag{2.10}$$

where $E_3(x_3) E_4(x_4)$ is as in Class II and Ps_ν^μ is a spheroidal function.[11]

D. Coordinate systems of class IV

These correspond to systems giving a differential form

$$ds^3 = \frac{(x_1 - x_2)}{4} \left[\frac{dx_1^2}{x_1(x_1 - 1)} - \frac{dx_2^2}{x_2(x_2 - 1)}\right] + x_1 x_2 \, dx_3^2 + (x_1 - 1)(x_2 - 1) \, dx_4^2. \tag{2.11}$$

The separation equations are

$$4 \frac{d}{dx_i} \left(x_i (x_i - 1) \frac{dE_i}{dx_i}\right) - \left(\frac{l_1}{x_i} + \frac{l_2}{x_i - 1} + \lambda x_i + l_3\right) E_i = 0 \quad (i = 1, 2) \tag{2.12}$$

$$\frac{d^2 E_3}{dx_3^2} = l_1 E_3, \quad \frac{d^2 E_4}{dx_4^2} = l_2 E_4.$$

E. Coordinate systems of class V

These correspond to systems with a differential form

$$ds^2 = (x_2 - \mu)(x_3 - \mu)(x_4 - \mu) \, dx_1^2 + \sum_{i=1}^{4} \frac{(x_i - x_j)(x_i - x_j)}{4 f(x_i)} \, dx_i^2, \tag{2.13}$$

where $i, j, k = 2, 3, 4$ are distinct, $f(x)$ is a polynomial such that $1 \le \deg f(x) \le 3$, and $x = \mu$ is a root of $f(x)$. The separation equations are

$$4 \left(\frac{f(x_i)}{(x_i - \mu)}\right)^{1/2} \frac{d}{dx_i} \left(\sqrt{(x_i - \mu) f(x_i)} \, \frac{dE_i}{dx_i}\right) + [-\lambda (x_i - \mu)^3 + l_1 (x_i - \mu)^2 + l_2 (x_i - \mu) + l_3] \, E_i = 0, \tag{2.14}$$

where $i = 2, 3, 4$ and

$$(\mu - \mu')(\mu - \mu'') \frac{d^2 E_1}{dx_1^2} = -l_3 E_1,$$

where μ' and μ'' are the other roots of $f(x)$ with multiplicity included and $\deg f(x) = 3$. Similar separation equations exist in the variable x_1 when $\deg f(x)=2$ and 1.

F. Coordinate systems of class VI

These correspond to systems giving differential forms

$$ds^2=\sum_{i=1}^{4}(x_i-x_j)(x_i-x_k)(x_i-x_l)\frac{dx_i^2}{4f(x_i)}, \qquad (2.15)$$

where $i,j,k,l=1,2,3,4$ are distinct and $f(x)$ is a polynomial of degree less than or equal to 4. The separation equations are

$$4\sqrt{f(x_i)}\frac{d}{dx_i}\left(\sqrt{f(x_i)}\frac{dE_i}{dx_i}\right)+[-\lambda x_i^3+l_1x_i^2+l_2x_i+l_3]E_i=0,$$

$i=1,2,3,4.$

The remaining coordinate systems correspond to group reductions of the type $E(3,1)\supset T_1\otimes E(2,1)\supset T_1\otimes O(2,1)$, $E(3,1)\supset T_1\otimes E(3)\supset T_1\otimes O(3)$, and $E(3,1)\supset O(3,1)$ in the case of the Klein–Gordon equation and $E(4,\mathbb{C})\supset T_1\otimes E(3,\mathbb{C})\supset T_1\otimes O(3,\mathbb{C})$, $E(4,\mathbb{C})\supset O(4,\mathbb{C})$ in the case of its complexification. These systems have been derived elsewhere[13] and we make no further evaluation of them.

III. ORTHOGONAL SEPARABLE COORDINATE SYSTEMS FOR THE KLEIN-GORDON EQUATION AND ITS COMPLEXIFICATION

In this section we supplement Sec. 2 by giving the coordinates in space–time corresponding to the differential forms presented there. In addition we give the three operators, L_1, L_2, and L_3 whose eigenvalues are the three separation constants l_1, l_2, and l_3. These operators are expressed as symmetric second order operators in the enveloping algebra of the Poincaré group or its complexification. Due to the large number of possible systems we group the coordinate systems corresponding to the differential forms of Sec. 2 into classes of systems with similar properties and make an explicit count of the number of distinct coordinate systems inequivalent under the Poincaré group. We also list the systems which separate for the complexified equation only (denoted by the symbol $\mathbb{C}$), bearing in mind that distinct real systems may be equivalent in the complex case.

A. Coordinate systems of class I

A suitable choice of coordinates (2.3) with $\epsilon=-$ and sign $(x_1x_2)=+$ is

$$\begin{aligned}(1)\quad &(t-x)^2=x_1x_2,\\ &(t^2-x^2)=x_1+x_2+x_1x_2(x_3^2+x_4^2),\\ &y=\sqrt{x_1x_2}\,x_3,\quad z=\sqrt{x_1x_2}\,x_4.\end{aligned}\qquad (3.1)$$

In terms of these coordinates the Klein–Gordon equation assumes the form

$$\Box\Psi=\frac{4}{(x_1-x_2)}\left[\frac{\partial}{\partial x_1}\left(x_1^2\frac{\partial\Psi}{\partial x_1}\right)-\frac{\partial}{\partial x_2}\left(x_2^2\frac{\partial\Psi}{\partial x_2}\right)\right], \\ -\frac{1}{x_1x_2}\left(\frac{\partial^2\Psi}{\partial x_3^2}+\frac{\partial^2\Psi}{\partial x_4^2}\right)=\lambda\Psi. \qquad (3.2)$$

The separation equations for the solution $\Psi=E_1(x_1)E_2(x_2)E_3(x_3)E_4(x_4)$ are

$$\begin{aligned}&\frac{d^2E_3}{dx_3^2}=(l_1-l_2)E_3,\quad \frac{d^2E_4}{dx_4^2}=l_2E_4,\\ &4\frac{d}{dx_i}\left(x_i^2\frac{dE_i}{dx_i}\right)+\left(\frac{l_1}{x_i}-\lambda x_i+l_3\right)E_i=0,\end{aligned}\qquad (3.3)$$

where $i=1,2$. The three operators whose eigenvalues are the separation constants are

$$\begin{aligned}&L_1=(N_2+M_3)^2+(N_3-M_2)^2,\quad L_2=(N_3-M_2)^2,\\ &L_3=(P_0+P_1)^2+M_1^2+M_2^2+M_3^2-N_1^2-N_2^2-N_3^2.\end{aligned}\qquad (3.4)$$

A typical solution for the Klein–Gordon equation (0.2) is

$$\Psi=e^{-(a+b)}De_\nu(a+\tfrac{1}{2}x,\sqrt{-\lambda l_1})\,De_\nu(b+\tfrac{1}{2}x_1\sqrt{-\lambda l_1})$$
$$\times\exp[(l_1-l_2)^{1/2}x_3+(l_2)^{1/2}x_4], \qquad (3.5)$$

where $D=C,S$ and the variables are defined as in (2.4).

If $\epsilon=+$, then the corresponding coordinates are obtained from (3.1) via the transformations S: $(t,x,yz)\to(x,t,iy,iz)$ and $x_3\to ix_3$, $x_4\to ix_4$.

(2) If sign $(x_1x_2)=-$ and $\epsilon=+$, the appropriate choice of space–time coordinates is obtained from (3.1) via the transformation $T:(t,x,y,z)\to(it,ix,iy,iz)$. This transformation also gives the operators describing this second type of system when applied to formulas (3.4). If $\epsilon=-$, then the corresponding coordinates can be obtained from those for which $\epsilon=+$ by the transformations S and $x_3\to ix_3$, $x_4\to ix_4$.

The remaining coordinate systems in this class are obtained by regarding x_3,x_4 [as given in (3.1) for coordinate systems (1)–(2)] as Cartesian coordinates in a Euclidean plane. This is the plane whose corresponding E(2) Lie algebra has generators $\bar{P}_1=N_2+M_3$, $\bar{P}_2=N_3+M_2$, and $\bar{M}=M_1$ with commutation relations

$$[\bar{P}_1,\bar{M}]=\bar{P}_2,\quad[\bar{P}_2,\bar{M}]=\bar{P}_1,\quad[\bar{P}_1,\bar{P}_2]=0. \qquad (3.6)$$

The new coordinates are then obtained by choosing polar, parabolic, and elliptic coordinates in the x_3,x_4 plane. The three possible types of coordinates resulting from each of these three choices are obtained by the same substitutions as used to find all the systems (1)–(2), i.e., we have two inequivalent pairs of coordinate systems in each case. In all cases the operator L_3 is given by its counterpart in systems (1)–(2), and the separation equations in the variables x_1,x_2 are as in (3.3). For each case we need only give the transformation $x_3\to f(x_3,x_4)$, $x_4\to g(x_3,x_4)$ specifying the change in coordinates together with the operators L_1,L_2.

The transformation to plane polar coordinates is given by the following.

$$(3)-(4)\quad x_3\to\sqrt{x_3}\cos x_4,\quad x_4\to\sqrt{x_3}\sin x_4 \qquad (3.7)$$

The x_3,x_4 dependent part of the separable solution is typically

$$E_3(x_3)E_4(x_4)=C_{(-l_3)^{1/2}}(\sqrt{-l_1x_3})\exp[\pm(l_2x_4)^{1/2}],$$

where $C_\nu(z)$ is a solution of Bessel's equation. The basis defining operators are

$$L_1=(N_2+M_3)^2+(N_3-M_2)^2, \quad L_2=M_1^2. \tag{3.8}$$

(5)—(6) The transformation to parabolic coordinates in the plane is given by

$$x_3\to\tfrac{1}{2}(x_3+x_4), \quad x_4\to\sqrt{-x_3x_4}, \quad x_4<0<x_3. \tag{3.9}$$

The x_3, x_4 part of the separable solution is typically

$$E_3(x_3)E_4(x_4)=D_{[l_2-(l_1)^{1/2}]/(l_1)^{1/2}}[\pm\sqrt{-l_1x_3}(1+i)] \times D_{[-l_2-(l_1)^{1/2}]/(l_1)^{1/2}}[\pm\sqrt{l_4x_4}(1+i)], \tag{3.10}$$

where $D_\nu(z)$ is a parabolic cylinder function.[12] The basis defining operators are

$$L_1=(N_2+M_3)^2+(N_3-M_2)^2,$$
$$L_2=(N_3-M_2)M_1+M_1(N_3-M_2). \tag{3.11}$$

(7)—(8) The transformation to elliptic coordinates in the plane is given by

$$x_3\to c\sqrt{x_3x_4}, \quad x_4\to c\sqrt{(x_4-1)(1-x_3)} \tag{3.12}$$

with $0<x_3<1<x_4$.

The x_3, x_4 part of the separable solution is typically

$$E_3(x_3)E_4(x_4)=\begin{cases}Ce_\nu(\xi,h^2)\,ce_\nu(\eta,h^2),\\ Se_\nu(\xi,h^2)\,se_\nu(\eta,h^2),\end{cases} \tag{3.13}$$

where $x_4=\cosh^2\xi$, $x_3=\cos^2\eta$, $h^2=-l_1c^2/4$, and $l_2=-\lambda_\nu(h^2)$, $\nu=0,1,2,\cdots$. The functions $ce_\nu(z,h^2)$, $se_\nu(z,h^2)$ are periodic Mathieu functions.[11] The basis defining operators are

$$L_1=(N_2+M_3)^2+(N_3-M_2)^2,$$
$$L_2=M_1^2+\tfrac{1}{2}c^2[(N_2+M_3)^2-(N_3-M_2)^2]. \tag{3.14}$$

For the remaining systems of Class I we have to consider the complexified Klein—Gordon equation.

(9) [$\mathbb{C}$] A suitable choice of coordinates is

$$(z_1-iz_2)^2=x_1x_2,$$
$$(z_1^2+z_2^2)=x_1+x_2+2x_1x_2(x_3+x_4)(x_3-x_4)^2,$$
$$z_3+iz_4=2i\sqrt{x_1x_2(x_3+x_4)},$$
$$z_3-iz_4=i\sqrt{x_1x_2(x_3-x_4)^2}. \tag{3.15}$$

The complexified Klein—Gordon equation has the same form as (3.2) with $\Delta_2\psi=(\partial_{33}+\partial_{44})\psi$ replaced by $[1/4(x_3-x_4)](\partial_{33}-\partial_{44})\psi$. The separation equations in the variables x_3, x_4 are

$$\frac{d^2E_i}{dx_i^2}+(-4l_1x_i+l_2)E_i=0, \tag{3.16}$$

where $i=3,4$. A typical solution of this equation is

$$E_i=\left(x_i+\frac{l_2}{2\sqrt{-l_1}}\right)C_{1/3}\left[\frac{4\sqrt{-l_1}}{3}\left(x_i+\frac{l_2}{2\sqrt{-l_1}}\right)^{3/2}\right], \tag{3.17}$$

where $C_\nu(z)$ is a solution of Bessel's equation. The separation equations in the variables x_1, x_2 are as in (3.3) with l_1+l_2 replaced by l_1. The basis defining operators are

$$L_1=(I_{32}+iI_{31})^2+(I_{42}+iI_{41})^2,$$
$$L_2=\{I_{43},I_{32}+I_{42}+i(I_{31}+I_{41})\}+[I_{32}-I_{42}+i(I_{31}-I_{41})]^2,$$
$$L_3=(P_1-iP_2)^2+I_{12}^2+I_{13}^2+I_{14}^2+I_{23}^2+I_{24}^2+I_{34}^2. \tag{3.18}$$

(10) [$\mathbb{C}$] A suitable choice of coordinates is

$$(z_1-iz_2)^2=x_1x_2,$$
$$(z_1^2+z_2^2)=x_1+x_2+2x_1x_2(x_3+x_4),$$
$$(z_3+iz_4)=i\sqrt{x_1x_2}\left[\left(\frac{x_3}{x_4}\right)^{1/2}+\left(\frac{x_4}{x_3}\right)^{1/2}\right],$$
$$(z_3-iz_4)=-i\sqrt{x_1x_2x_3x_4}. \tag{3.19}$$

The complexified Klein—Gordon equation has the same form as (3.2) with $\Delta_2\psi=(\partial_{33}+\partial_{44})\psi$ replaced by

$$\frac{4}{(x_3-x_4)}\left[x_3\frac{\partial}{\partial x_3}\left(x_3\frac{\partial\psi}{\partial x_3}\right)-x_4\frac{\partial}{\partial x_4}\left(x_4\frac{\partial\psi}{\partial x_4}\right)\right].$$

The separation equations in the variables x_3, x_4 are

$$x_i\frac{d}{dx_i}\left(x_i\frac{dE_i}{dx_i}\right)+(\lambda-4l_1x_i+l_1)E_i=0, \tag{3.20}$$

where $i=3,4$. A typical solution of this equation is

$$E_i=C_\nu(2\sqrt{l_1x_3})\,C_\nu(2\sqrt{-l_1x_4}). \tag{3.21}$$

The separation equations in the variables x_1, x_2 are as in coordinate system (9). The operators L_1 and L_3 also are the same as in system (9). The remaining operator is

$$L_2=-I_{34}^2+[I_{32}+I_{14}+i(I_{31}+I_{42})]^2. \tag{3.22}$$

B. Coordinate systems of class II

(11)—(13) In analogy to our treatment of Class I we treat one of the coordinate systems in detail and give the transformations from which the remaining coordinate systems can be obtained. A suitable choice of coordinates of type (2.6) with sign $(x_1x_2)=+$ and $x_3<0$ is

(11)

$$t=\sqrt{x_1x_2(1-x_3)}, \quad x=\sqrt{-x_1x_2x_3}\cos x_4,$$
$$y=\sqrt{-x_1x_2x_3}\sin x_4, \quad z=\tfrac{1}{2}(x_1+x_2). \tag{3.23}$$

The Klein—Gordon equation assumes the form

$$\Box\psi=\frac{4}{(x_1-x_2)}\left[\sqrt{x_2}\frac{\partial}{\partial x_2}\left(\sqrt{x_2}\frac{\partial\psi}{\partial x_2}\right)-\sqrt{x_1}\frac{\partial}{\partial x_1}\left(\sqrt{x_1}\frac{\partial\psi}{\partial x_1}\right)\right]-\frac{1}{x_1x_2}\left(4\sqrt{1-x_3}\frac{\partial}{\partial x_3}\left(x_3\sqrt{1-x_3}\frac{\partial\psi}{\partial x_3}\right)-\frac{1}{x_3}\frac{\partial^2\psi}{\partial x_4^2}\right)=\lambda\Psi. \tag{3.24}$$

The separation equations are

$$4\sqrt{1-x_3}\frac{d}{dx_3}\left(x_3\sqrt{1-x_3}\frac{dE_3}{dx_3}\right)-\frac{l_2}{x_3}E_3=l_1E_3,$$
$$\frac{d^2E_4}{dx_4^2}=l_2E_4,$$

$$4\sqrt{x_i}\ \frac{d}{dx_i}\left(\sqrt{x_i}\ \frac{dE_i}{dx_i}\right) + -\left(\frac{l_1}{x_i} + \lambda x_i - l_3\right) E_i = 0, \tag{3.25}$$

where $i=1,2$. The three defining operators are

$$L_1 = N_1^2 + N_2^2 - M_3^2, \quad L_2 = M_3^2,$$
$$L_3 = P_0 N_3 + N_3 P_0 - 2M_2P_2 + 2M_1P_1. \tag{3.26}$$

A typical solution of the Klein–Gordon equation is

$$\Psi = (x_1x_2)^{-1/4} M_{\pm i(\lambda l_3)^{1/2}/2,\,1/2(j+1/2)}(\pm ix_1/2\sqrt{\lambda})$$
$$\times M_{\pm i(\lambda l_3)^{1/2}/2,\,1/2(j+1/2)}(\pm ix_2/2\sqrt{\lambda})$$
$$\times P_j^{i\sqrt{l_2}}(\sqrt{1-x_3})\exp[\pm(l_2)^{1/2}x_4], \tag{3.27}$$

where $P_\mu^\nu(z)$ is a Legendre function. There is a further coordinate system of type (11) obtained by allowing the parameters x_i to vary in the ranges $\text{sign}(x_1x_2) = -$, $x_3 > 1$.

(12) Coordinate systems of this type correspond to the ranges, sign $(x_1x_2) = +$, $0 \le x_3 < 1$, and can be obtained from systems of type (11) via the transformation $(t,x,y,z) \to (z, ix, iy, t)$.

(13) If sign $(x_1x_2) = +$, $x_3 > 1$ or sign $(x_1x_2) = -$, $x_3 < 0$, and if we make the two transformations $x_4 \to ix_4$ and $(t,x,y,z) \to (iy, it, ix, z)$ in (3.19) and (3.22) we get the appropriate coordinates and basis operators. If sign $(x_1x_2) = -$ and $0 < x_3 < 1$, then making the substitutions $x_4 \to ix_4$ and $(t,x,y,z) \to (iy, ix, t, z)$ we get another set of coordinates.

We can extract the essential features of the remaining distinct coordinates of Class II from the three systems already described. There are two kinds of coordinates:

[i] $$t = \sqrt{x_1x_2}\,\tau_1, \quad x = \sqrt{x_1x_2}\,\tau_2,$$
$$y = \sqrt{x_1x_2}\,\tau_3, \quad z = \tfrac{1}{2}(x_1 + x_2). \tag{3.28}$$

If sign $(x_1x_2) = +$, then the vector $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$ is parametrized by one of the nine orthogonal separable coordinate systems on the double sheeted hyperboloid $[\boldsymbol{\tau},\boldsymbol{\tau}] = \tau_1^2 - \tau_2^2 - \tau_3^2 = 1$. If sign $(x_1x_2) = -$, then the vector (τ_1,τ_2,τ_3) is parametrized by one of the nine classes of orthogonal separable coordinate systems on the single sheeted hyperboloid $[\boldsymbol{\tau},\boldsymbol{\tau}] = -1$.

[ii] $$t = \tfrac{1}{2}(x_1 + x_2), \quad x = \sqrt{x_1x_2}\,\tau_1,$$
$$y = \sqrt{x_1x_2}\,\tau_2, \quad z = \sqrt{x_1x_2}\,\tau_3 \tag{3.29}$$

parametrized by one of the two orthogonal separable coordinate systems on the sphere $\tau_1^2 + \tau_2^2 + \tau_3^2 = 1$.

For the remaining coordinate systems of Class II we need only give the 3-vector (τ_1,τ_2,τ_3) in terms of the coordinates x_3, x_4, appearing in the corresponding differential form of Sec. 2. In addition we give the operator L_2 specifying each of the separable bases together with a typical solution for $E_3(x_3)E_4(x_4)$. We note here that coordinate systems already given correspond to the following $\boldsymbol{\tau}$ vectors.

(11) $$\boldsymbol{\tau} = (\cosh a, \sinh a\cos\phi, \sinh a\sin\phi), \quad [\boldsymbol{\tau},\boldsymbol{\tau}] = 1,$$
$$-\infty < a < \infty, \quad 0 \le \phi < 2\pi \quad (x_3 = -\sinh^2 a),$$

$$\boldsymbol{\tau} = (\sinh a, \cosh a\cos\phi, \cosh a\sin\phi), \quad [\boldsymbol{\tau},\boldsymbol{\tau}] = -1,$$
$$-\infty < a < \infty, \quad 0 \le \phi < 2\pi \quad (x_3 = \cosh^2 a). \tag{3.30}$$

(12) $$\boldsymbol{\tau} = (\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi), \quad \tau_1^2 + \tau_2^2 + \tau_3^2 = 1,$$
$$0 \le \theta < \pi, \quad 0 \le \phi < 2\pi \quad (x_3 = \sin^2\theta). \tag{3.31}$$

(13) $$\boldsymbol{\tau} = (\cosh a\cosh b, \cosh a\sinh b, \sinh a), \quad [\boldsymbol{\tau},\boldsymbol{\tau}] = 1,$$
$$-\infty < a, b < \infty \quad (x_3 = \cosh^2 a),$$
$$\boldsymbol{\tau} = (\sinh a\cosh b, \sinh a\sinh b, \cosh a), \quad [\boldsymbol{\tau},\boldsymbol{\tau}] = -1,$$
$$(x_3 = -\sinh^2 a),$$
$$\boldsymbol{\tau} = (\sin\theta\sinh b, \sin\theta\cosh b, \cos\theta), \quad [\boldsymbol{\tau},\boldsymbol{\tau}] = -1,$$
$$(x_3 = \sin^2\theta). \tag{3.32}$$

We now proceed to the remaining coordinate systems of Class II.

(14) The corresponding choices of the vector $\boldsymbol{\tau}$ are

(a) $$\tau_1 + \tau_2 = \sqrt{x_3}, \quad \tau_1 - \tau_2 = (1/\sqrt{x_3}\,x_4^2,$$
$$\tau_3 = x_4\sqrt{x_3}, \quad [\boldsymbol{\tau},\boldsymbol{\tau}] = 1, \quad x_4, x_3 > 0. \tag{3.33}$$

(b) The coordinates corresponding to the single sheeted hyperboloid $[\boldsymbol{\tau},\boldsymbol{\tau}] = -1$ are obtained from (3.29) via the substitution $\tau \to i\boldsymbol{\tau}$ with $x_3 < 0$. The operator L_2 for this coordinate system is

$$L_2 = (N_2 + M_3)^2, \tag{3.34}$$

and a typical solution for the x_3, x_4 dependent part of the solution of (0.2) is

$$E_3(x_3)E_4(x_4) = x_3^{-1/4}K_{j+1/2}(\sqrt{-l_2/x_3})\exp[(l_2)^{1/2}x_4], \tag{3.35}$$

where $K_\nu(z)$ is a Macdonald function.

(15)–(17) The corresponding choice of the vector $\boldsymbol{\tau}$ is

(15a) $$\tau_1^2 = x_3x_4/a, \quad \tau_2^2 = (x_3 - 1)(x_4 - 1)/(a-1),$$
$$\tau_3^2 = (x_3 - a)(a - x_4)/a(a-1), \quad [\boldsymbol{\tau},\boldsymbol{\tau}] = 1,$$
$$1 < x_3 < a < x_4. \tag{3.36}$$

(15b) The coordinates on the single sheeted hyperboloid $[\boldsymbol{\tau},\boldsymbol{\tau}] = -1$ are obtained from (3.32), via the substitution $\boldsymbol{\tau} \to i\boldsymbol{\tau}$ with $x_3 < 0 < 1 < x_4 < a$. The operator L_2 for this coordinate system is

$$L_2 = N_1^2 + aN_2^2 \tag{3.37}$$

and a typical solution for the x_3, x_4 dependent part of the solution of (0.2) is

$$L_{jl_2}(x_3)L_{jl_2}(x_4), \tag{3.38}$$

where $L_{jl}(z)$ is a solution of Lamé's equation

$$\frac{d^2L_{jl}}{dz^2} + \frac{1}{2}\left(\frac{1}{z-a} + \frac{1}{z-1} + \frac{1}{z}\right)\frac{dL_{jl}}{dz}$$
$$+ \frac{(l_2 - j(j+1)z)L_{jl}}{4(z-a)(z-1)z} = 0. \tag{3.39}$$

(16a) This coordinate system is obtained from (15a) via the transformation $(\tau_1, \tau_2, \tau_3) \to (i\tau_2, i\tau_1, \tau_3)$ and $x_3 < 0 < 1 < a < x_4$.

(16b) This coordinate system is related to (15b) in the same way as (16a) and $1 < x_3x_4 < a$, or $x_3x_4 > a$.

(17) Finally, the one system on the sphere is obtained from (15a) via the substitution $(\tau_1\tau_2\tau_3) \to (\tau_1, i\tau_2, i\tau_3)$ and $0 < x_3 < 1 < x_4 < a$.

(18) A suitable choice of coordinates on the double sheeted hyperboloid is:

(18a) $$(\tau_1 + i\tau_2)^2 = 2(x_3 - a)(x_4 - a)/a(a-b),$$
$$\tau_3^2 = x_3x_4/ab, \quad [\boldsymbol{\tau}, \boldsymbol{\tau}] = 1, \qquad (3.40)$$

and $x_3 < 0 < x_4$.

(18b) The coordinates on the single sheeted hyperboloid are obtained from those of (3.36) by the substitution $\boldsymbol{\tau} \to i\boldsymbol{\tau}$. The operator L_2 is

$$L_2 = \alpha(M_1^2 - N_2^2) + \beta(M_1N_2 + N_2M_1) \qquad (3.41)$$

and a typical solution for the x_3, x_4 dependent part of the solution of (0.2) is

$$\tilde{L}_{jl_2}(x_3)\,\tilde{L}_{jl_2}(x_4), \qquad (3.42)$$

where $\tilde{L}_{jl_2}(z)$ is a solution of

$$\frac{d^2\tilde{L}_{jl_2}}{dz^2} + \frac{1}{2}\left(\frac{1}{z-a} + \frac{1}{z-b} + \frac{1}{z}\right)\frac{d\tilde{L}_{jl_2}}{dz} + \frac{[l_2 - j(j+1)z]\,\tilde{L}_{jl_2}}{4(z-a)(z-b)z} = 0. \qquad (3.43)$$

(19)—(20) A suitable choice of coordinates on the double sheeted hyperboloid is:

(19a) $$\tau_1 + \tau_2 = \sqrt{-x_3x_4},$$
$$\tau_1 - \tau_2 = \sqrt{-x_3/x_4} + \sqrt{-x_4/x_3} - \sqrt{-x_3x_4},$$
$$\tau_3 = \sqrt{(1-x_3)(x_4-1)}, \quad [\boldsymbol{\tau}, \boldsymbol{\tau}] = 1 \qquad (3.44)$$

and $x_3 < 0 < 1 < x_4$.

(19b) The corresponding coordinates on the single sheeted hyperboloid are obtained via the substitution $\boldsymbol{\tau} \to i\boldsymbol{\tau}$ with $x_3, x_4 < 0$, $0 < x_3, x_4 < 1$, $x_3, x_4 > 1$. The operator for this system is

$$L_2 = N_1^2 - (N_2 + M_3)^2 \qquad (3.45)$$

and a typical solution for the x_3, x_4 dependent part is

$$E_3(x_3)\,E_4(x_4) = P_j^{(l_2)^{1/2}}(\sqrt{1-x_3})\,P_j^{(l_2)^{1/2}}(\sqrt{1-x_4}), \qquad (3.46)$$

where $P_\mu^\nu(z)$ is a Legendre function.

(20a) This system is obtained from (19a) via the transformation $(\tau_1\tau_2\tau_3) \to (i\tau_2, i\tau_1, \tau_3)$ and $x_3 < 0 < 1 < x_4$.

(20b) The coordinates on the single sheeted hyperboloid are obtained from (20a) via the substitution $\boldsymbol{\tau} \to i\boldsymbol{\tau}$ with $x_3 < 0 < x_4 < 1$.

(21) A suitable choice of coordinates on the double sheeted hyperboloid is:

(21a) $$\tau_1 + \tau_2 = \sqrt{x_3x_4}, \quad \tau_1 - \tau_2 = (x_3 - x_4)^2/4(-x_3x_4)^{3/2},$$
$$\tau_3 = \frac{1}{2}\left[\left(\frac{-x_3}{x_4}\right)^{1/2} - \left(\frac{-x_4}{x_3}\right)^{1/2}\right], \quad [\boldsymbol{\tau}, \boldsymbol{\tau}] = 1, \qquad (3.47)$$

and $x_4 < 0 < x_3$.

(21b) The coordinates on the single sheeted hyperboloid are obtained via the substitution $\boldsymbol{\tau} \to i\boldsymbol{\tau}$ with $x_3, x_4 < 0$ or $x_3, x_4 > 0$. The operator for this system is

$$L_2 = N_1(N_2 - M_3) + (N_2 - M_3)\,N_1 \qquad (3.48)$$

and a typical x_3, x_4 dependent part of the solution is

$$E_3(x_3)\,E_4(x_4) = (-x_3x_4)^{-1/4}C_{j+1/2}(\sqrt{l_2/x_3})\,C_{j+1/2}(\sqrt{-l_2/x_4}).$$

C. Coordinate systems of class III

These systems are similar to systems of Class II in that the various different types are specified by the various choices of separable coordinate systems on the manifolds $[\boldsymbol{\tau}, \boldsymbol{\tau}] = \pm 1$ and $\tau_1^2 + \tau_2^2 + \tau_3^2 = 1$, where $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$. We examine in detail one system, then discuss the general form of the coordinates in this class.

(22)—(25)

(22a) A suitable choice of coordinates with $x_3 < 0$, $x_1, x_2 > 1$; $0 < x_1, x_2 < 1$; or $x_1, x_2 < 0$ is

$$t = \sqrt{x_1x_2(1-x_3)}, \quad x = \sqrt{-x_1x_2x_3}\,\cos x_4,$$
$$y = \sqrt{-x_1x_2x_3}\,\sin x_4, \quad x = \sqrt{(1-x_1)(1-x_2)}. \qquad (3.49)$$

The Klein—Gordon equation becomes

$$\Box\psi = \frac{4}{(x_1 - x_2)}\left[\left(\frac{x_1-1}{x_1}\right)^{1/2}\frac{\partial}{\partial x_1}\left(x_1\sqrt{x_1(x_1-1)}\,\frac{\partial\psi}{\partial x_1}\right) - \left(\frac{x_2-1}{x_2}\right)^{1/2}\frac{\partial}{\partial x_2}\left(x_2\sqrt{x_2(x_2-1)}\,\frac{\partial\psi}{\partial x_2}\right)\right]$$
$$- \frac{1}{x_1x_2}\left[4\sqrt{1-x_3}\,\frac{\partial}{\partial x_3}\left(x_3\sqrt{1-x_3}\,\frac{\partial\psi}{\partial x_3}\right) - \frac{1}{x_3}\,\frac{\partial^2\psi}{\partial x_4^2}\right]$$
$$= \lambda\psi. \qquad (3.50)$$

The separation equations in the variables x_3, x_4 are as in (3.25). The corresponding equations for the variables x_1, x_2 are

$$4\left(\frac{x_i-1}{x_i}\right)^{1/2}\frac{d}{dx_i}\left(x_i\sqrt{x_i(x_i-1)}\,\frac{dE_i}{dx_i}\right) + \left(\frac{l_1}{x_i} - \lambda x_i + l_3\right)E_i = 0, \qquad (3.51)$$

where $i = 1, 2$. The three defining operators are

$$L_1 = N_1^2 + N_2^2 - M_3^2, \quad L_2 = M_3^2,$$
$$L_3 = P_0^2 - P_1^2 - P_2^2 + M_2^2 + M_1^2 - N_3^2. \qquad (3.52)$$

A typical solution of the Klein—Gordon equation is

$$\psi = (x_1x_2)^{1/4}\,Ps_\nu^{j+1/2}(\sqrt{1-x_1}, -\lambda)\,Ps_\nu^{j+1/2}(\sqrt{1-x_2}, -\lambda)$$
$$\times P_j^{(l_2)^{1/2}}(\sqrt{1-x_3})\,\exp[\pm(l_2)^{1/2}x_4]. \qquad (3.53)$$

There is a further system obtained by allowing the x_i to vary in the ranges $x_1 < 0 < 1 < x_2$, $1 > x_3 > 0$.

(22b) Systems of this type correspond to the ranges $x_1 < 0 < 1 < x_2$, $x_3 < 0$; and $0 < x_1 < 1 < x_2$, $x_3 > 1$. These systems are related to (22a) via the transformation T: $(t, x, y, z) \to (it, ix, iy, iz)$.

(23a) Systems of this type correspond to the ranges

$x_1, x_2 < 0$; $0 < x_1, x_2 < 1$; $x_1, x_2 > 1$; $0 < x_3 < 1$. These systems are related to (22a) via the transformation $(t, x, y, z) \to (z, ix, iy, t)$.

(23b) In this case we have the ranges $x_1 < 0 < 1 < x_2$ and $0 < x_3 < 1$. This is related to (23a) via T.

As for Class II, systems of Class III are of six different kinds:

[i] $t = \sqrt{x_1 x_2}\,\tau_1$, $x = \sqrt{x_1 x_2}\,\tau_2$, $y = \sqrt{x_1 x_2}\,\tau_3$,
$z = \sqrt{(1-x_1)(1-x_2)}$, $[\boldsymbol{\tau}, \boldsymbol{\tau}] = 1$,
and $x_1, x_2 < 0$; $0 < x_1, x_2 < 1$]; $x_1 x_2 > 1$.

[ii] $t = \sqrt{-x_1 x_2}\,\tau_1$, $x = \sqrt{-x_1 x_2}\,\tau_2$, $y = \sqrt{-x_1 x_2}\,\tau_3$,
$z = \sqrt{(1-x_1)(x_2-1)}$, $[\boldsymbol{\tau}, \boldsymbol{\tau}] = 1$,
and $x_1 < 0 < 1 < x_2$.

[iii] $t = \sqrt{-x_1 x_2}\,\tau_1$, $x = \sqrt{-x_1 x_2}\,\tau_2$, $y = \sqrt{-x_1 x_2}\,\tau_3$,
$z = \sqrt{(1-x_1)(1-x_2)}$, $[\boldsymbol{\tau}, \boldsymbol{\tau}] = -1$,
and $x_1 < 0 < x_2 < 1$.

[iv] $t = \sqrt{x_1 x_2}\,\tau_1$, $x = \sqrt{x_1 x_2}\,\tau_2$, $y = \sqrt{x_1 x_2}\,\tau_3$,
$z = \sqrt{(1-x_1)(x_2-1)}$, $[\boldsymbol{\tau}, \boldsymbol{\tau}] = -1$,
and $0 < x_1 < 1 < x_2$.

[v] $t = \sqrt{(1-x_1)(1-x_2)}$, $x = \sqrt{x_1 x_2}\,\tau_1$,
$y = \sqrt{x_1 x_2}\,\tau_2$, $z = \sqrt{x_1 x_2}\,\tau_2$, $\tau_1^2 + \tau_2^2 + \tau_3^2 = 1$,
and $x_1 x_2 < 0$; $0 < x_1, x_2 < 1$; $x_1, x_2 > 1$.

[vi] $t = \sqrt{(1-x_1)(x_2-1)}$, $x = \sqrt{-x_1 x_2}\,\tau_1$,
$y = \sqrt{-x_1 x_2}\,\tau_2$, $z = \sqrt{-x_1 x_2}\,\tau_3$, $\tau_1^2 + \tau_2^2 + \tau_3^2 = 1$,
and $x_1 < 0 < 1 < x_2$.

The remaining coordinate systems of Class III can be obtained from these kinds by replacing $\boldsymbol{\tau}$ with the possible separable coordinate systems on the manifolds $[\boldsymbol{\tau}, \boldsymbol{\tau}] = \pm 1$ and $\tau_1^2 + \tau_2^2 + \tau_3^2 = 1$, exactly as for Class II. Systems (26)—(60) are of this type.[9]

D. Coordinate systems of class IV

(61) If $x_1 > 1 > x_2 > 0$, $x_3^2, x_4^2 > 0$, a suitable choice of coordinates is

$$t = \sqrt{(x_1-1)(1-x_2)}\,\sinh x_4, \quad x = \sqrt{(x_1-1)(1-x_2)}\,\cosh x_4,$$
$$y = \sqrt{x_1 x_2}\,\cos x_3, \quad z = \sqrt{x_1 x_2}\,\sin x_3. \tag{3.54}$$

The Klein—Gordon equation assumes the form

$$\Box\psi = \frac{1}{4(x_1-x_2)}\left[\frac{\partial}{\partial x_1}\left(x_1(x_1-1)\frac{\partial}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left(x_2(x_2-1)\frac{\partial\psi}{\partial x_2}\right)\right] + \frac{1}{x_1 x_2}\frac{\partial^2\psi}{\partial x_3^2} + \frac{1}{(x_1-1)(x_2-1)}\frac{\partial^2\psi}{\partial x_4^2} = \lambda\psi. \tag{3.55}$$

The separation equations are (2.12). The three defining operators are

$$L_1 = M_1^2, \quad L_2 = N_1^2,$$
$$L_3 = M_1^2 + M_2^2 + M_3^2 - N_1^2 - N_2^2 - N_3^2 + \tfrac{1}{2}[P_1^2 - P_0^2 - P_2^2 - P_3^2]. \tag{3.56}$$

Coordinates of this type are generalizations of spheroidal coordinates in three dimensions.

(62) $(t, x, y, z) \to (iy, z, x, it)$, $x_1 > 1 > x_2 > 0$, $x_3^2, x_4^2 < 0$.

(63) $(t, x, y, z) \to (ix, it, y, z)$,
$x_1, x_2 > 1$, $1 > x_1, x_2 > 0$, $0 > x_1, x_2$, $x_3^2, x_4^2 > 0$.

(64) $(t, x, y, z) \to (x, t, iy, iz)$, $x_1 > 1 > 0 > x_2$, $x_3^2, x_4^2 > 0$.

E. Coordinate systems of class V

(65)—(73)

(65) This first type corresponds to $f(x) = 4(x-a)(x-1)x$, $\mu = 0$, and

$$t^2 = \frac{(x_2-a)(x_3-a)(x_4-a)}{a(1-a)}, \quad x^2 = \frac{x_2 x_3 x_4}{a}\cos^2 x_1,$$
$$y^2 = \frac{x_2 x_3 x_4}{a}\sin^2 x_1, \quad z^2 = \frac{(x_2-1)(x_3-1)(x_4-1)}{(1-a)}. \tag{3.57}$$

In terms of these coordinates the Klein—Gordon equation becomes

$$\Box\psi = \frac{1}{(x_2-x_3)(x_2-x_4)x_2}\frac{\partial^2\psi}{\partial\nu_2^2} + \frac{1}{(x_3-x_2)(x_3-x_4)x_3}\frac{\partial^2\psi}{\partial\nu_3^2} + \frac{1}{(x_4-x_2)(x_4-x_3)x_4}\frac{\partial^2\psi}{\partial\nu_4^2} + \frac{a}{x_2x_3x_4}\frac{\partial^2\psi}{\partial x_1^2} = \lambda\psi, \tag{3.58}$$

where

$$\frac{\partial}{\partial\nu_j} = 2x_j\sqrt{(x_j-a)(x_j-1)}\,\frac{\partial}{\partial x_j}.$$

The three defining operators are

$$L_1 = -P_0^2 + (a+1)(P_1^2 + P_2^2) + aP_3^2 + M_1^2 + M_2^2 + M_3^2 - N_1^2 - N_2^2 - N_3^2,$$
$$L_2 = a(P_1^2 + P_2^2 + M_1^2 + M_2^2) - N_1^2 - N_2^2 + (a+1)M_3^2,$$
$$L_3 = -aM_3^2. \tag{3.59}$$

The coordinates x_2, x_3, and x_4 can vary in the ranges $x_2, x_3 > a > 1 > x_4 > 0$; $a > x_2, x_3 > 1 > x_4 > 0$; $1 > x_2 > 0 > x_3, x_4$; and $1 > x_2, x_3, x_4 > 0$ with $x_1^2 > 0$ in all cases. For the remaining systems we give the appropriate transformation of the space—time coordinates which relates the system in question to (65).

(66) $(t, x, y, z) \to (it, ix, iy, iz)$

(67) $(t, x, y, z) \to (z, ix, iy, t)$

(68) $(t, x, y, z) \to (iz, x, y, it)$

(69) $(t, x, y, z) \to (iy, x, it, z)$

(70) $(t, x, y, z) \to (iy, it, x, z)$

(71) $(t,x,y,z)\to(y,ix,t,iz)$

(72) $(t,x,y,z)\to(y,t,ix,iz)$

(73) $(t,x,y,z)\to(iy,t,ix,iz)$

(74)—(81)

(74) This type corresponds to the choice $\mu=1$ and

$$t^2=\frac{(x_2-a)(x_3-a)(x_4-1)}{a(1-a)},$$
$$x^2=\frac{(x_2-1)(x_3-1)(x_4-1)}{(1-a)}\cos^2 x_1, \qquad (3.60)$$
$$y^2=\frac{(x_2-1)(x_3-1)(x_4-1)}{(1-a)}\sin^2 x_1, \quad z^2=\frac{x_2x_3x_4}{a}.$$

The three operators are

$$L_1=P_0^2+(a-2)(P_1^2+P_2^2)+(a-1)P_3^2,$$
$$+M_1^2+M_2^2+M_3^2-N_1^2-N_2^2-N_3^2,$$
$$L_2=(a-1)(P_1^2+P_2^2-M_1^2-M_2^2)-N_1^2-N_2^2+(a-2)M_3^2,$$
$$L_3=(1-a)M_3^2. \qquad (3.61)$$

The coordinates can vary in the ranges

$x_2,x_3>a>x_4>1$; $x_2,x_3>a>1>x_4>0$;

$a>x_2,x_3>1>x_4>0$; $1>x_2>0>x_3,x_4$;

$1>x_2,x_3,x_4>0$ where $x_1^2>0$ in all cases. The remaining systems are specified by the transformation of space—time coordinates which relates them to (74). The various possibilities are:

(75) $(t,x,y,z)\to(iz,x,y,it)$

(76) $(t,x,y,z)\to(z,ix,iy,t)$

(77) $(t,x,y,z)\to(it,ix,iy,iz)$

(78) $(t,x,y,z)\to(iy,x,it,z)$

(79) $(t,x,y,z)\to(y,ix,t,iz)$

(80) $(t,x,y,z)\to(y,t,ix,iz)$

(81) $(t,x,y,x)\to(iy,it,x,z)$

(82)—(86)

(82) This type corresponds to $f(x)=(x-a)(x-b)x$, $a=b^*=\alpha+i\beta$, $\alpha,\beta\in\mathbb{R}$, $\mu=0$:

$$(z+it)^2=\frac{2(x_2-a)(x_3-a)(x_4-a)}{a(a-b)},$$
$$x^2=\frac{x_2x_3x_4}{ab}\cos^2 x_1, \quad y^2=\frac{x_2x_3x_4}{ab}\sin^2 x_1. \qquad (3.62)$$

The three basis defining operators are

$$L_1=2\alpha(P_1^2+P_2^2)+\alpha(P_3^2-P_0^2)-2\beta P_0P_3$$
$$+M_1^2+M_2^2+M_3^2-N_1^2-N_2^2-N_3^2,$$
$$L_2=\alpha(N_1^2+N_2^2-M_1^2-M_2^2)-(\alpha^2+\beta^2)(P_1^2+P_2^2)$$
$$+\beta(\{N_2,M_1\}-\{N_1,M_2\}), \quad L_3=abM_3^2, \qquad (3.63)$$

where $\{A,B\}=AB+BA$.

The coordinates x_j $(j=2,3,4)$ can vary in the ranges $x_2,x_3,x_4>0$; $x_2>0>x_3,x_4$; with $x_1^2>0$.

(83) The systems of this type are obtained from (82) via the transformation T. The coordinates vary in the ranges

$x_2,x_3>0>x_4$; $0>x_2,x_3,x_4$; with $x_1^2>0$.

(84)—(87)

(84) This type corresponds to $f(x)=(x-1)x^2$, $\mu=0$, and

$$(t-x)^2=x_2x_3x_4,$$
$$(t^2-x^2)=-(x_4x_2+x_4x_3+x_2x_3)+x_4x_2x_3(1+x_1^2),$$
$$y^2=x_1x_2x_3x_4, \quad z^2=(x_2-1)(x_3-1)(x_4-1). \qquad (3.64)$$

In terms of these coordinates the Klein—Gordon equation becomes (3.58) with $a=1$ and $\partial/\partial\nu_j=2x_j\sqrt{x_j(x_j-1)}(\partial/\partial x_j)$. The three operators are

$$L_1=2P_0(P_0+P_1)+P_2^2+M_1^2+M_2^2+M_3^2-N_1^2-N_2^2-N_3^2,$$
$$L_2=(P_0+P_1)^2-N_1^2-2N_2^2+(N_3+M_2)^2-N_2M_3-M_3N_2,$$
$$L_3=(N_2-M_3)^2. \qquad (3.65)$$

The coordinates x_2, x_3, and x_4 can vary in the ranges $x_2,x_3,x_4>1$; $x_2>1>x_3,x_4>0$; with $x_1^2>0$.

(85) $(t,x,y,z)\to(it,ix,iy,iz)$

(86) $(t,x,y,z)\to(x,t,iy,iz)$

(87) $(t,x,y,z)\to(ix,it,y,z)$

(88)—(90)

(88) This type corresponds to $f(x)=(x-1)x^2$, $\mu=1$, and

$$(t-z)^2=x_2x_3x_4,$$
$$(t^2-z^2)=-(x_1x_2+x_1x_3+x_2x_3)+x_1x_2x_3,$$
$$x^2=(x_2-1)(x_3-1)(x_4-1)\cos^2 x_1 \qquad (3.66)$$
$$y^2=(x_2-1)(x_3-1)(x_4-1)\sin^2 x_1.$$

In terms of these coordinates the Klein—Gordon equation becomes (3.58) with $1-a$ replaced by 1 and $\partial/\partial\nu_j=2(x_j-1)\sqrt{x_j(x_j-1)}(\partial/\partial x_j)$.

The basis defining operators are

$$L_1=-2(P_1^2+P_2^2)+2P_0(P_0+P_3)+M_1^2+M_2^2+M_3^2$$
$$-N_1^2-N_2^2-N_3^2,$$
$$L_2=-(P_1^2+P_2^2)+2(M_3^2-N_1^2-N_2^2)+\{N_2,M_1\}-\{N_1,M_2\},$$
$$L_3=M_3^2. \qquad (3.67)$$

The coordinates x_2,x_3,x_4 vary in the ranges $x_2,x_3,x_4>1$; $x_2>1>x_3,x_4>0$; $x_2>1>0>x_3,x_4$; $x_2>1>x_3>0>x_4$; with $x_1^2>0$.

(89) $(t,x,y,z)\to(it,ix,iy,iz)$

(90) $(t,x,y,z)\to(z,ix,iy,t)$

(91)—(92)

(91) This system corresponds to $f(x)=x^3$, $\mu=0$, and

$$(t-x)^2=x_2x_3x_4,\quad 2y(t-x)=x_2x_3+x_2x_4+x_3x_4,$$
$$x^2+y^2+z^2-t^2=x_2+x_3+x_4,\quad z^2=x_1^2x_2x_3x_4. \tag{3.68}$$

In terms of these coordinates the Klein—Gordon equation assumes the form (3.58) with $a=1$ and $\partial/\partial\nu_j=2x_j^2(\partial/\partial x_j)$.

The three defining operators are

$$L_1=-2P_2(P_0+P_1)+M_1^2+M_2^2+M_3^2-N_1^2-N_2^2-N_3^2,$$
$$L_2=-(P_0+P_1)^2+\{M_3-N_2,N_1\}-\{M_2+N_3,M_1\},$$
$$L_3=(N_3+M_2)^2.$$

The coordinates x_j $(j=2,3,4)$ vary in the ranges $x_2,x_3>0>x_4$; and $x_2>0>x_3,x_4$; with $x_1^2>0$.

(92) $(t,x,y,z)\to(it,ix,iy,iz)$

(93)—(96)

(93) This system corresponds to $f(x)=x(x-1)$, $\mu=0$, and

$$t=\tfrac{1}{2}(x_2+x_3+x_4),\quad x^2=x_2x_3x_4\cos^2x_1,$$
$$y^2=x_2x_3x_4\sin^2x_1,\quad z^2=-(x_2-1)(x_3-1)(x_4-1). \tag{3.69}$$

In terms of these coordinates the Klein—Gordon equation becomes (3.58) with $a=1$ and $\partial/\partial\nu_j=2x_j\sqrt{x_j-1}(\partial/\partial x_j)$.

The three operators are

$$L_1=\{P_3,N_3\}-\{P_1,N_1\}-\{P_2,N_2\}+P_0^2+P_3^2,$$
$$L_2=\{N_1,P_1\}+\{N_2,P_2\}+P_1^2+P_2^2-M_1^2-M_2^2-M_3^2, \tag{3.70}$$
$$L_3=-M_3^2.$$

The coordinates x_2, x_3, and x_4 vary in the ranges $x_2,x_3>1>x_4>0$; $1>x_2,x_3,x_4>0$; $1>x_2>0>x_3,x_4$; with $x_1^2>0$.

(94) $(t,x,y,z)\to(y,t,ix,iz)$

(95) $(t,x,y,z)\to(z,ix,iy,t)$

(96) $(t,x,y,z)\to(y,ix,t,iz)$

(97)—(98)

(97) This system corresponds to $f(x)=x^2$, $\mu=0$, and

$$(t-x)^2=x_2x_3x_4,$$
$$t^2-x^2=x_2x_3+x_2x_4+x_3x_4+x_2x_3x_4x_1^2, \tag{3.71}$$
$$y^2=x_2x_3x_4x_1^2,\quad z=\tfrac{1}{2}(x_2+x_3+x_4).$$

The Klein—Gordon equation assumes the form (3.58) with $a=0$ and $\partial/\partial\nu_j=2x_j^{3/2}(\partial/\partial x_j)$.

The three operators are

$$L_1=-\{P_0+P_1,N_3-M_2\}-\{P_2,M_1\}+(P_0+P_1)^2,$$
$$L_2=\{P_0+P_1,N_3+M_2\}+N_1^2+N_2^2-M_3^2, \tag{3.72}$$
$$L_3=(N_2-M_3)^2.$$

(98) $(t,x,y,z)\to(ix,it,y,z)$

(99)—(100)

(99) This system corresponds to $f(x)=x$, $\mu=0$, and

$$2(t-z)=x_2x_3+x_2x_4+x_3x_4-\tfrac{1}{2}(x_2^2+x_3^2+x_4^2),$$
$$2(z-t)=x_2+x_3+x_4, \tag{3.73}$$
$$x^2=-x_2x_3x_4\cos^2x_1,\quad y^2=-x_2x_3x_4\sin^2x_1.$$

In terms of these coordinates the Klein—Gordon equation becomes (3.58) with $a=1$ and $\partial/\partial\nu_j=2\sqrt{x_j}(\partial/\partial x_j)$.

The three operators are

$$L_1=\{N_3,P_0+P_3\}-\{P_1,N_1-M_2\}-\{P_2,N_2+M_2\}-\tfrac{1}{4}(P_0-P_3)^2,$$
$$L_2=\tfrac{1}{2}\{P_1,N_1+M_2\}+\tfrac{1}{2}\{P_2,N_2-M_1\}+(N_1+M_2)^2+(N_2-M_1)^2,$$
$$L_3=M_3^2, \tag{3.74}$$

and the variables are such that sign $(x_2x_3x_4)=-1$ and $x_1^2>0$.

(100) $(t,x,y,z)\to(z,ix,iy,t)$.

F. Coordinate systems of class VII

These systems correspond to the various kinds of purely elliptical coordinates for which the Klein—Gordon equation is separable. The differential form is (2.15) where $f(x)$ is at most a fourth order polynomial in x.

(101)—(108)

(101) This type corresponds to $f(x)=(x-a)(x-b)(x-1)x$, $a>b>1$, and

$$t^2=-\frac{x_1x_2x_3x_4}{ab},\quad x^2=-\frac{(x_1-1)(x_2-1)(x_3-1)(x_4-1)}{(a-1)(b-1)}$$
$$y^2=\frac{(x_1-b)(x_2-b)(x_3-b)(x_4-b)}{(a-b)(b-1)b},$$
$$z^2=-\frac{(x_1-a)(x_2-a)(x_3-a)(x_4-a)}{(a-b)(a-1)a}. \tag{3.75}$$

The Klein—Gordon equation becomes

$$\Box\psi=\sum_{i=1}^{4}\frac{1}{(x_i-x_j)(x_i-x_k)(x_i-x_l)}\frac{\partial^2\psi}{\partial\nu_i^2}=\lambda\psi,$$

where i,j,k,l are not equal and

$$\frac{\partial}{\partial\nu_i}=2\sqrt{(x_i-a)(x_i-b)(x_i-1)x_i}\;\frac{\partial}{\partial x_i}. \tag{3.76}$$

The three operators are

$$L_1=-N_1^2-N_2^2-N_3^2+M_1^2+M_2^2+M_3^2+(b+1)P_3^2+(a+1)P_2^2+(a+b)P_1^2-(a+b+1)P_0^2,$$
$$L_2=(a+b)N_1^2+(a+1)N_2^2+(b+1)N_3^2-bM_2^2-aM_3^2-M_1^2-bP_3^2-aP_2^2-abP_1^2+(a+b+ab)P_0^2,$$
$$L_3=abN_1^2+aN_2^2+bN_3^2+abP_0^2. \tag{3.77}$$

For coordinates of this type the variables x_i can lie in the ranges $x_1>a>x_2>b>x_3>1>0>x_4$.

(102) $(t,x,y,z)\to(ix,it,y,z)$

(103) $(t, x, y, z) \to (iz, x, y, it)$

(104) $(t, x, y, z) \to (iy, x, it, z)$

(105) $(t, x, y, z) \to (it, ix, iy, iz)$

(106) $(t, x, y, z) \to (z, ix, iy, t)$

(107) $(t, x, y, z) \to (x, t, iy, iz)$

(108) $(t, x, y, z) \to (y, ix, t, iz)$

(109)—(110)

(109) This type corresponds to $f(x) = (x-a)(x-b)(x-1)x$, $a = b^* = \alpha + i\beta$, and

$$(x+it)^2 = -\frac{2(x_1-b)(x_2-b)(x_3-b)(x_4-b)}{(b-a)(b-1)b},$$
$$y^2 = -\frac{(x_1-1)(x_2-1)(x_3-1)(x_4-1)}{(a-1)(b-1)} \qquad (3.78)$$
$$z^2 = \frac{x_1x_2x_3x_4}{ab}.$$

The Klein—Gordon equation becomes (3.76). The three operators are

$$L_1 = N_1^2 + N_2^2 + N_3^2 - M_1^2 - M_2^2 - M_3^2$$
$$+ (\alpha+1)(P_1^2 - P_0^2) + 2\beta P_0 P_1 + 2\alpha P_2^2 - (2\alpha+1) P_3^2,$$
$$L_2 = -2\alpha M_1^2 + (\alpha+1)(N_3^2 - M_2^2) - \beta\{M_2, N_3\}$$
$$+ \alpha(N_2^2 - M_3^2) + \beta\{M_3, N_2\} + N_1^2$$
$$+ \alpha(P_0^2 - P_1^2) - 2\beta P_0 P_1 - (\alpha^2+\beta^2) P_2^2$$
$$- (2\alpha + \alpha^2 + \beta^2) P_3^2,$$
$$L_3 = -(\alpha^2+\beta^2) M_1^2 + \alpha(N_3^2 - M_2^2) - \beta\{M_2, N_3\}. \qquad (3.79)$$

The variables x_i lie in the ranges $x_1 > 1 > x_2 > 0 > x_3, x_4$; $x_1 > 1 > x_2, x_3, x_4 > 0$.

(110) $(t, x, y, z) \to (it, ix, iy, iz)$

(111)—(114)

(111) This type corresponds to $f(x) = (x-a)(x-1)x^2$, $a > 1$, and

$$(t+x)^2 = x_1x_2x_3x_4/a,$$
$$(t^2 - x^2) = -(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_4 + x_2x_3x_4)$$
$$+ (a+1)x_1x_2x_3x_4/a^2,$$
$$y^2 = \frac{(x_1-1)(x_2-1)(x_3-1)(x_4-1)}{(a-1)},$$
$$z^2 = -\frac{(x_1-a)(x_2-a)(x_3-a)(x_4-a)}{a^2(a-1)} \qquad (3.80)$$

The Klein—Gordon equation becomes (3.76) with

$$\frac{\partial}{\partial \nu_i} = 2x_i\sqrt{(x_i-a)(x_i-1)}\,\frac{\partial}{\partial x_i}.$$

The three operators are

$$L_1 = -N_1^2 - N_2^2 - N_3^2 + M_1^2 + M_2^2 + M_3^2$$
$$+ 2a(P_0^2 - P_1^2) + (P_0 - P_1)^2 + aP_2^2,$$
$$L_2 = -2M_2^2 + \{N_3, M_2\} - (N_2 + M_3)^2$$
$$+ a(N_2^2 - M_1^2) + (a+1)N_1^2 + (a+1)(P_0 - P_1)^2,$$
$$L_3 = -a(N_2 + M_3)^2 + aN_1^2 - (N_3 - M_2)^2 + (P_0 - P_1)^2. \qquad (3.81)$$

The variables x_i lie in the ranges

$x_1 > a > x_2 > 1 > x_3, x_4 > 0$;

$x_1 > a > x_2 > 1 > 0 > x_3, x_4$;

$x_1 > a > x_2, x_3, x_4 > 1$; $x_1, x_2, x_3 > a > x_4 > 1$.

(112) $(t, x, y, z) \to (ix, it, y, z)$

(113) $(t, x, y, z) \to (x, t, iy, iz)$

(114) $(t, x, y, z) \to (it, ix, iy, iz)$

(115) [$\mathbb{C}$]

This type corresponds to $f(x) = (x-1)^2x^2$ and

$$(iz_1 - z_2)^2 = x_1x_2x_3x_4,$$
$$z_1^2 + z_2^2 = (x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4) - 2x_1x_2x_3x_4,$$
$$(iz_3 + z_4)^2 = -(x_1-1)(x_2-1)(x_3-1)(x_4-1),$$
$$z_3^2 + z_4^2 = 2x_1x_2x_3x_4 + (x_1 + x_2 + x_3 + x_4)$$
$$- (x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4) - 2. \qquad (3.82)$$

The Klein—Gordon equation becomes (3.76) with $\partial/\partial\nu_j = 2x_j(x_j - 1)(\partial/\partial x_j)$.

The operators are

$$L_1 = I_{12}^2 + I_{13}^2 + I_{14}^2 + I_{23}^2 + I_{24}^2 + I_{34}^2$$
$$+ (iP_1 + P_2)^2 - \tfrac{1}{4}(P_3 - iP_2)^2,$$
$$L_2 = -(iI_{13} + I_{23})^2 - (iI_{14} + I_{24})^2$$
$$+ (iI_{14} - I_{13})^2 - (I_{34} - iI_{23})^2$$
$$- I_{14}^2 - I_{13}^2 - I_{23}^2 - I_{24}^2 - 2I_{12}^2$$
$$+ 2(iP_1 + P_2)^2,$$
$$L_3 = -(I_{24} + iI_{23} + I_{31} + iI_{14})^2 + (iI_{14} + I_{24})^2$$
$$+ (I_{31} + iI_{23})^2 - I_{12}^2 + (iP_1 + P_2)^2. \qquad (3.83)$$

(116)—(117)

(116) This type corresponds to $f(x) = (x-1)x^3$ and

$$(x-t)^2 = x_1x_2x_3x_4,$$
$$y(x-t) = -(x_2x_3x_4 + x_1x_2x_3 + x_1x_3x_4 + x_1x_2x_4)$$
$$+ x_1x_2x_3x_4,$$
$$y^2 + x^2 - t^2 = -(x_1x_2 + x_1x_3 + x_1x_2 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4)$$
$$+ x_1x_2x_3x_4,$$
$$z^2 = -(x_1-1)(x_2-1)(x_3-1)(x_4-1). \qquad (3.84)$$

The Klein—Gordon equation becomes (3.76). The operators are

$$L_1 = -N_1^2 - N_2^2 - N_3^2 + M_1^2 + M_2^2 + M_3^2 + 2P_2(P_0 + P_1),$$
$$L_2 = N_2^2 + N_3^2 - M_1^2 - \{M_1, N_3 + M_2\} + (P_0 + P_1)^2,$$
$$L_3 = -(P_0 + P_1)^2 + \{N_1, N_2 - M_1\} - (N_3 + M_2)^2. \qquad (3.85)$$

The coordinates x_i vary in the ranges $x_1 > 1 > x_2 > 0 > x_3, x_4$; $x_1 > 1 > x_2, x_3, x_4 > 0$, and $x_1, x_2, x_3 > 1 > x_4 > 0$.

(117) $(t,x,y,z) \to (it, ix, iy, iz)$

(118) [$\mathbb{C}$]

This type corresponds to $f(x) = x^4$ and

$$(z_1 + iz_2)^2 = -x_1x_2x_3x_4,$$
$$2(z_1+iz_2)(z_3+iz_4) = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4,$$
$$(z_1+iz_2)(z_3-iz_4) + (z_3+iz_4)^2 = -x_1x_2 - x_1x_3 - x_1x_4 - x_2x_3 - x_2x_4 - x_3x_4,$$
$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = x_1 + x_2 + x_3 + x_4. \quad (3.86)$$

The Klein—Gordon equation assumes the from (3.76) with $\partial/\partial\nu_j = 2x_j^2(\partial/\partial x_j)$.

The three operators are

$$L_1 = -(P_3 + iP_4)^2 + (P_1 + iP_2)(P_3 - iP_4) + I_{12}^2 + I_{13}^2 + I_{14}^2 + I_{23}^2 + I_{24}^2 + I_{34}^2$$
$$L_2 = \tfrac{1}{2}\{I_{32} + I_{14} + i(I_{13} + I_{24}), (I_{43} + I_{12})\} + (I_{42} + iI_{23})^2 - (I_{13} + iI_{14})^2 + 2(P_3 + iP_4)(P_1 + iP_2),$$
$$L_3 = \{I_{32} + I_{41} + i(I_{13} + I_{42}), I_{21}\} - (P_1 + iP_2)^2. \quad (3.87)$$

(119)—(122)

(119) This type corresponds to $f(x) = (x-a)(x-1)x$ and $= (x-a)(x-1)x]$:

$$t = \tfrac{1}{2}(x_1 + x_2 + x_3 + x_4),$$
$$x^2 = (x_1-a)(x_2-a)(x_3-a)(x_4-a)/a(a-1),$$
$$y^2 = (x_1-1)(x_2-1)(x_3-1)(x_4-1)/(1-a), \quad (3.88)$$
$$z^2 = x_1x_2x_3x_4/a.$$

The Klein—Gordon equation becomes (3.76) with

$$\frac{\partial}{\partial\nu_j} = 2\sqrt{(x_j-a)(x_j-1)x_j}\,\frac{\partial}{\partial x_j}.$$

The operators are

$$L_1 = -\{P_3, N_3\} - \{P_2, N_2\} - \{P_1, N_1\} + (a+1)P_0^2 + aP_1^2 + P_2^2,$$
$$L_2 = (a+1)\{P_3, N_3\} + \{P_1, N_1\} + a\{P_2, N_1\} - aP_0^2 + M_1^2 + M_2^2 + M_3^2 - (a+1)(P_1^2 + P_2^2) - P_3^2,$$
$$L_3 = -a\{P_3, N_3\} - a^2P_3^2 + M_2^2 + a^2M_1^2. \quad (3.89)$$

The coordinates x_i vary in the ranges

$x_1, x_2 > a > x_3 > 1 > x_4 > 0$;

$a > x_1 > 1 > x_2, x_3, x_4 > 0$;

$a > x_1 > 1 > x_2 > 0 > x_3, x_4$.

(120) $(t,x,y,z) \to (z, ix, ix, iy, t)$

(121) $(t,x,y,z) \to (y, ix, t, iz)$

(122) $(t,x,y,z) \to (x, t, iy, iz)$

(123) This type corresponds to $f(x) = (x-a)(x-b)x$ and

$$y = \tfrac{1}{2}(x_1 + x_2 + x_3 + x_4),$$
$$(t+ix)^2 = 2(x_1-a)(x_2-a)(x_3-a)(x_4-a)/a(a-b),$$
$$z^2 = -x_1x_2x_3x_4/ab, \quad a = b^* = \alpha + i\beta, \quad \alpha, \beta \in \mathbb{R}. \quad (3.90)$$

The Klein—Gordon equation becomes (3.76) with

$$\frac{\partial}{\partial\nu_j} = 2\sqrt{(x_j-a)(x_j-b)x_j}\,\frac{\partial}{\partial x_j}.$$

The three operators are

$$L_1 = \{P_3, M_1\} + \{P_1, M_3\} - \{P_0, N_2\} + 2\alpha P_2^2 + \alpha(P_0^2 - P_1^2) + 2\beta P_0P_1,$$
$$L_2 = -2\alpha\{P_3, M_1\} + \alpha[\{P_0, N_2\} - \{P_1, M_3\}] - 2\beta\{P_1, N_3\} + M_2^2 - N_1^2 - N_2^2 - (\alpha^2 + \beta^2)P_2^2 + P_3^2 + 2\alpha(P_1^2 - P_0^2),$$
$$L_3 = (\alpha^2 + \beta^2)\{P_3, M_1\} + (\alpha^2 + \beta^2)^2P_3^2 + (\alpha^2 - \beta^2)(M_2^2 - N_3^2) + \alpha\beta\{M_2, N_3\}. \quad (3.91)$$

The coordinates vary in the ranges $x_1, x_2, x_3 > 0 > x_4$; $x_1 > 0 > x_2, x_3, x_4$.

(124)—(125)

(124) This type corresponds to $f(x) = (x-1)x^2$ and

$$(t-x)^2 = -x_1x_2x_3x_4, \quad z = \tfrac{1}{2}(x_1 + x_2 + x_3 + x_4),$$
$$(t^2 - x^2) = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 - x_1x_2x_3x_4,$$
$$y^2 = -(x_1-1)(x_2-1)(x_3-1)(x_4-1). \quad (3.92)$$

The Klein—Gordon equation becomes (3.76) with $\partial/\partial\nu_j = 2x_j\sqrt{x_j - 1}(\partial/\partial x_j)$.

The three operators are

$$L_1 = -\{P_0, N_3\} + \{P_1, M_2\} + \{P_2, M_1\} + P_3^2 - P_2^2 + (P_0 + P_1)^2,$$
$$L_2 = -2P_1N_3 - 2P_0M_2 - 2\{P_1, M_2\} + M_3^2 - N_1^2 - N_2^2 + P_2^2 - 2(P_0 + P_1)P_0,$$
$$L_3 = \{P_0 + P_1, N_3 + M_2\} - N_2^2. \quad (3.93)$$

The coordinates x_i can vary in the ranges

$x_1 > 1 > x_2, x_3 > 0 > x_4$; $x_1 > 1 > 0 > x_2, x_3, x_4$;

$x_1, x_2, x_3 > 1 > x_4 > 0$.

(125) $(t,x,y,z) \to (ix, it, y, z)$

(126) This type corresponds to $f(x) = x^3$ and

$$(x-t)^2 = -x_1x_2x_3x_4, \quad z = \tfrac{1}{2}(x_1 + x_2 + x_3 + x_4),$$
$$2y(x-t) = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4, \quad (3.94)$$
$$t^2 - x^2 - y^2 = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4.$$

The Klein—Gordon equation assumes the form (3.76) with $\partial/\partial\nu_j = 2x_j^{3/2}(\partial/\partial x_j)$.

The three operators are

$$L_1=\{P_1, M_2\}-\{P_0, N_3\}-\{P_2, M_1\},$$
$$L_2=\{P_2, N_3+M_2\}+\{M_2, P_0+P_1\} + M_3^2-N_1^2-N_2^2-2P_0P_1-P_0^2, \quad (3.95)$$
$$L_3=\{P_0+P_1, N_3+M_2\}-(N_2-M_3)^2.$$

The coordinates x_i vary in the ranges $x_1>0>x_2, x_3, x_4$ or $x_1, x_2, x_3>0>x_4$.

(127)—(128)

(127) This type corresponds to $f(x)=x(x-1)$ and

$$2(t-x)=\tfrac{1}{2}(x_1^2+x_2^2+x_3^2+x_4^2) - (x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4) + (x_1+x_2+x_3+x_4),$$
$$2(t+x)=x_1+x_2+x_3+x_4,$$
$$y^2=(x_1-1)(x_2-1)(x_3-1)(x_4-1),$$
$$z^2=-x_1x_2x_3x_4. \quad (3.96)$$

In terms of these coordinates the Klein—Gordon equation assumes the form (3.76) with $\partial/\partial\nu_j = 2\sqrt{x_j(x_j-1)}(\partial/\partial x_j)$.

The three operators are

$$4L_1=\{N_3+M_2, P_3\}-\{N_2-M_3, P_2\} + \tfrac{1}{2}\{N_1, P_0+P_1\}-\tfrac{1}{4}(P_0-P_1)^2 + P_2^2+\tfrac{1}{2}P_0(P_0+P_1),$$
$$4L_2=(N_3+M_2)^2+\{M_2, P_3\}-\tfrac{1}{2}\{N_1, P_0+P_1\} + (N_2-M_3)^2+\tfrac{1}{2}\{N_2+M_3, P_2\},$$
$$4L_3=-P_3^2-M_1^2-(N_3+M_2)^2+\{N_3, P_3\}. \quad (3.97)$$

The coordinates x_i vary in the ranges

$x_1, x_2>1>x_3>0>x_4$; $1>x_1>0>x_2, x_3, x_4$;

and $1>x_1, x_2, x_3>0>x_4$.

(128) $(t, x, y, z)\to(x, t, iy, iz)$

(129) [$\mathbb{C}$]

This type corresponds to $f(x)=x^2$ and

$$2(iz_1-z_2)=x_1+x_2+x_3+x_4,$$
$$2(iz_1+z_2)=\tfrac{1}{2}(x_1^2+x_2^2+x_3^2+x_4^2) - (x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4),$$
$$(z_3-iz_4)^2=x_1x_2x_3x_4,$$
$$(z_3^2+z_4^2)=-(x_1x_2x_3+x_1x_2x_4+x_1x_3x_4+x_2x_3x_4). \quad (3.98)$$

The Klein—Gordon equation assumes the form (3.76) with $\partial/\partial\nu_j=2x_j(\partial/\partial x_j)$.

The three operators are

$$4L_1=\{I_{32}+iI_{13}, P_3\}, +\{I_{42}+iI_{14}, P_4\} + \tfrac{1}{2}\{I_{12}, P_1+iP_2\}, +(P_3-iP_4)^2-\tfrac{1}{4}(P_2+iP_1)^2,$$
$$4L_2=\{I_{23}+iI_{13}, P_3\}, +\{I_{24}+iI_{14}, P_4\} + (I_{24}+iI_{32})^2-(I_{13}+iI_{14})^2,$$
$$4L_3=I_{34}^2-[I_{31}+I_{42}+i(I_{32}+I_{14})]^2 + \tfrac{1}{2}\{I_{13}+I_{42}+i(I_{32}+I_{41}), P_4+iP_3\}. \quad (3.99)$$

(130) This type corresponds to $f(x)=x$ and

$$2(x-t)=1-x_1-x_2-x_3-x_4,$$
$$2y+(x-t)^2=x_2x_3+x_2x_4+x_2x_1 + x_3x_1+x_3x_4+x_4x_1-(x_1+x_2+x_3+x_4)+1,$$
$$2(x+t)+2y(x-t)=-(x_1x_2x_3+x_1x_2x_4+x_1x_3x_4+x_2x_3x_4) + (x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4) - (x_1+x_2+x_3+x_4)+1,$$
$$z^2=-x_1x_2x_3x_4. \quad (3.100)$$

The Klein—Gordon equation assumes the form (3.76) with $\partial/\partial\nu_j=2\sqrt{x_j}(\partial/\partial x_j)$.

We have not yet determined the three operators which describe this system. The coordinates vary in the ranges $x_1, x_2, x_3>0>x_4$ and $x_1>0>x_2, x_3, x_4$.

(131) [$\mathbb{C}$]

This type corresponds to $f(x)=1$ and

$$2(z_1+iz_2)=-(x_1+x_2+x_3+x_4),$$
$$2(z_3+iz_4)+(z_1+iz_2)^2 =x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4,$$
$$(z_3-iz_4)+2(z_1+iz_2)(z_3+iz_4) =-x_1x_2x_3-x_1x_2x_4-x_1x_3x_4-x_2x_3x_4,$$
$$(z_1-iz_2)+(z_1+iz_2)(z_3-iz_4)+(z_3+iz_4)^2=x_1x_2x_3x_4. \quad (3.101)$$

The Klein—Gordon equation assumes the form (3.76) with $\partial/\partial\nu_j=\partial/\partial x_j$.

We have not yet determined the operators which describe this system.

This comples the list of orthogonal coordinates for which the Klein—Gordon equation separates. As was mentioned earlier we have only given those systems which are genuinely new in that they have not been derived elsewhere before. For the wave equation ($\lambda=0$) we have found 125 such coordinate systems. In addition there are 34 radial coordinate systems corresponding to the group reduction $E(3,1)\supset SO(3,1)\supset\{L_1, L_2\}$ where $[L_1L_2]=0$ and L_1, L_2 are second order elements in the enveloping algebra of SO(3, 1). Similarly there are 55 coordinate systems belonging to reductions of the type $E(3,1)\supset E(2,1)\supset\{L_1, L_2\}$, 11 coordinate systems belonging to reductions of the type $E(3,1)\supset E(3)\supset\{L_1L_2\}$,[14] and 36 coordinate systems belonging to reductions of the type $E(3,1)\supset E(2)\otimes E(1,1)\supset L_1\otimes L_2$, where in this last case L_1 and L_2 are second order elements in the enveloping algebras of E(2) and E(1, 1), respectively. We have a total of 261 coordinate systems in which the Klein—Gordon equation admits separation of variables.

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