Lie theory and the wave equation in space–time. I. The Lorentz group

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In this article we begin a study of the relationship between separation of variables and the conformal symmetry group of the wave equation $\psi_{tt}-\Delta_3\psi=0$ in space–time. In this first article we make a detailed study of separation of variables for the Laplace operator on the one and two sheeted hyperboloids in Minkowski space. We then restrict ourselves to homogeneous solutions of the wave equation and the Lorentz subgroup SO(3,1) of the conformal group SO(4,2). We study the various separable bases by using the methods of integral geometry as developed by Gel'fand and Graev. In most cases we give the spectral analysis for these bases, and a number of new bases are developed in detail. Many of the special function identities derived appear to be new. This preliminary study is of importance when we subsequently study models of the Hilbert space structure for solutions of the wave equation and the Klein–Gordon equation $\psi_{tt}-\Delta_3\psi=\lambda\psi$.

INTRODUCTION

In this article we continue[1–3] the investigation of the connection between separation of variables for the principal equations of mathematical physics and the associated symmetry groups of such equations. The object of our study in this current series of articles is the wave equation in space–time,

$$\psi_{tt}-\Delta_3\psi=0. \tag{$*$}$$

The motivation for such a study stems from the inherent physical importance of this equation as well as its intrinsic mathematical interest. In this article we initiate the study with a detailed investigation of separation of variables for the Laplace operator on the one and two sheeted hyperboloids $[X,X]=-1$ and $[X,X]=1$, respectively. Here $X=(t,x,y,z)=(x_0,x_1,x_2,x_3)$ and $[X,X]=t^2-x^2-y^2-z^2$ is the usual Lorentz space–time scalar product. In doing this we are also concerned with the corresponding problem on the cone $[X,X]=0$. Here we are dealing only with the Lorentz subgroup of the SO(4, 2) symmetry group of ($*$). A detailed study of these manifolds is however of importance when the full symmetry group is utilized to study separation of variables for ($*$). This will be shown in a subsequent article where we introduce a Hilbert space structure for solutions of ($*$) and discuss various equivalent representations of this structure. The problem of separation of variable for the Laplace operator on the upper sheet of the two sheeted hyperboloid has been investigated by Olevski[4] who found 34 coordinate systems. In this article we perform harmonic analysis on the space $L^2(H_+)$ of square integrable functions on the upper sheet H_+ of the two sheeted hyperboloid for the majority of coordinate systems given by Olevski. In a number of cases we give the harmonic analysis for coordinate systems which are representative of a particular subclass of coordinates. We specifically exclude the treatment of elliptic coordinates which requires the solution of multiparameter eigenvalue problems. This analysis is performed using the methods of integral geometry as developed by Gel'fand, Graev, and Vilenkin. The resulting spectral problems are then reduced to spectral problems on the cone $[X,X]=0$. The contents of this article are arranged as follows: In Sec. 1 we present all the mathematical preliminaries and notations necessary for subsequent sections. These include the formulas for harmonic analysis on $L^2(H_+)$ and the space $L^2(H_s)$ of square integrable functions on the single sheet hyperboloid H_s. In Sec. 2 we give explicitly the 34 coordinate systems, due to Olevski, together with the pair of operators which specify each system, expressed in terms of the generators of the Lorentz group. In Sec. 3 we compute the spectral decompositions corresponding to the various coordinate systems. In the cases where this is already known the result is merely listed. Section 4 is devoted to a presentation of the appropriate basis functions on $L^2(H_+)$ and some comments on overlap functions. Finally, in Sec. 5 we compute various expansions on $L^2(H_s)$.

1. HARMONIC ANALYSIS AND THE LORENTZ GROUP

The homogeneous Lorentz group SO(3, 1) consists of those proper real linear transformations which leave $[X, X]$ invariant. The Lie algebra of SO(3, 1) is six-dimensional, and is generated by the rotation generators

$$M_1=y\partial_z-z\partial_y,\quad M_2=x\partial_z-z\partial_x,\quad M_3=x\partial_y-y\partial_x, \tag{1.1}$$

and the Lorentz transformation generators

$$K_1 = t\partial_x + x\partial_t,$$
$$K_2 = t\partial_y + y\partial_t, \quad (1.2)$$
$$K_3 = t\partial_z + z\partial_t.$$

The communtation relations are

$$[M_i, M_j] = \varepsilon_{ijk} M_k,$$
$$[M_i, K_j] = \varepsilon_{ijk} K_k, \quad (1.3)$$
$$[K_i, K_j] = -\varepsilon_{ijk} K_k.$$

The group SO(3, 1) has two Casimir operators $\Delta = \mathbf{M}^2 - \mathbf{K}^2$ and $\Delta' = \mathbf{M}\cdot\mathbf{K}$. The irreducible representations of the identity component of SO(3, 1) are labelled by two numbers. (j_0, σ), where j_0 is an integer or half-integer and σ is in general complex. If $\psi_{j_0\sigma}$ transforms according to the irreducible representation (j_0, σ) then

$$\Delta \psi_{j_0\sigma} = -[j_0^2 + \sigma(\sigma+2)]\psi_{j_0\sigma},$$
$$\Delta'\psi_{j_0\sigma} = -j_0(\sigma+1)\psi_{j_0\sigma}. \quad (1.4)$$

If the irreducible representation is in addition unitary then σ must have the one of the forms:

1. $\sigma = -1 + is, \quad s \in \mathbf{R}$. This is the principal series.
2. $-1 \leqslant \sigma \leqslant 0, \quad \sigma \in \mathbf{R}$ and $j_0 = 0, \pm 1, \cdots$.

This is the complementary series.

Further details concerning the representation theory of the Lorentz group can be found in Naimark[5] and Gel'fand *et al.*[6] We now give the basic formulas necessary for the harmonic analysis of functions defined on the spaces $L^2(H_+)$ and $L^2(H_s)$ mentioned in the introduction. These formulas are due to Gel'fand *et al.*[7]

A. Harmonic analysis on $L^2(H_+)$

The space $L^2(H_+)$ consists of functions $f(X)$ defined on the upper sheet of the hyperboloid $[X, X] = 1$, $t \geqslant 1$, satisfying

$$\int |f(X)|^2 \, dX < \infty, \quad (1.5)$$

where $dX = dxdydz/(1 + x^2 + y^2 + z^2)^{1/2}$. The harmonic analysis of a function $f(X) \in L^2(H_+)$ requires only the unitary irreducible representations $\sigma = -1 + is$ $(0 < s < \infty)$, $j_0 = 0$ corresponding to the principal series.

It is readily verified from the coordinate representation of the generators that $\Delta' = 0$. The formulas which yield the harmonic analysis of $f(X)$ are

$$f(X) = \frac{1}{(4\pi)^3}\int_0^\infty s^2\, ds \int_\Gamma F(Y; s)\,[X, Y]^{-is-1}\, dw, \quad (1.6)$$

where Γ is any contour on the $[Y, Y] = 0$ cone which intersects each generator of the cone once and dw is the differential form defined by $dY = dPdw$ where $P(Y) = 1$ is the equation of Γ and $dY = dy_1dy_2dy_3/y_0$ is the invariant measure on the cone. Here

$$F(Y, s) = \int f(X)\,[X, Y]^{is-1}\, dX. \quad (1.7)$$

The function $f(X)$ is then decomposed into components which transform according to the unitary irreducible representations $\sigma = -1 + is$,

$$0 < s < \infty, \quad j_0 = 0.$$

B. Harmonic analysis on $L^2(H_s)$

The space $L^2(H_s)$ consists of functions $f(X)$ defined on the single sheet hyperboloid $[X, X] = -1$ and satisfying

$$f(X) = f(-X), \quad (1.8a)$$

$$\int |f(X)|^2\, dX < \infty, \quad (1.8b)$$

where $dX = dxdydz/(x^2 + y^2 + z^2 - 1)^{1/2}$. The harmonic analysis of a function $f(X) \in L^2(H_s)$ requires the unitary irreducible representations

(i) $\sigma = -1 + is, \quad (0 < s < \infty), \quad j_0 = 0$, and

(ii) $\sigma = -1, \quad j_0 = 2n, \quad n = 1, 2, \cdots$.

The reason we choose the condition (1.8a) is that the harmonic analysis of a function satisfying this symmetry condition has been studied in detail by Gel'fand *et al.*[7] An example of an expansion which does not exhibit the property (1.8a) has been given by Zmuidzinas.[8] We should also mention the work of Limic *et al.*[9] who have examined the general problem of the expansion of square integrable functions defined on the transitivity surfaces of SO(p, q) in the canonical reduction. The expansion formulas for $L^2(H_s)$ are

$$f(X) = \frac{1}{2(4\pi)^3}\int_0^\infty s^2\, ds \int_\Gamma F(Y; s)\,|[X, Y]|^{-is-1}\, dw + \frac{4}{\pi^2}\sum_{n=1}^\infty n \int_\Gamma F(Y, B; 2n) e^{2in\theta}\,\delta([X, Y])\, dw,$$

with w and Γ as in (1.6). This expansion can be inverted via the formulas

$$F(Y; s) = \int f(X)\,|[X, Y]|^{is-1}\, dX,$$
$$F(Y, B; 2n) = \int f(X) e^{-2in\theta}\,\delta([X, Y])\, dX. \quad (1.10)$$

In both these formulas B is a four vector satisfying

$$[B, B] = -1, \quad [B, Y] = [Y, Y] = 0.$$

The first component of B is zero. The angle θ is given by the relation $\cos\theta = [X, B]$. For further details concerning these formulas we refer the reader to Gel'fand *et al.*[7]

2. SEPARABLE COORDINATE SYSTEMS FOR $\Delta\psi = -\sigma(\sigma+2)$ ON THE UPPER SHEET OF $[X, X]=1$

We now study the differential equation

$$(\mathbf{M}^2 - \mathbf{K}^2)\,\psi(X) = -\sigma(\sigma+2)\,\psi(X), \qquad (+)$$

where X ranges over the upper sheet of $[X, X] = 1$. As follows from (1.6), an arbitrary $f \in L^2(H_+)$ can be decomposed as an integral over solutions of (+). Furthermore, (+) arises when one looks for solutions of (*) which are homogeneous in x, y, z, t.

Equation (+) has been studied by Olevski[4] who showed that it admitted exactly 34 separable orthogonal coordinate systems. Here we give Olevski's results in a somewhat more explicit form. We also give for the first time a characterization of each separable system in terms of a pair of commuting second-order elements L_1, L_2 in the enveloping algebra of the Lie algebra of SO(3, 1). The corresponding separated solutions are eigenfunctions of L_1 and L_2 and the eigenvalues are the separation constants. (Smorodinski and Tugov[10] have computed L_1 and L_2 earlier but only in the form of differential operators.) For each set of separable coordinates $\{\rho, \nu, \eta\}$ we list the metric

$$ds^2 = H_1^2\,d\rho^2 + H_2^2\,d\nu^2 + H_3^2\,d\eta^2, \quad H_i = H_i(\rho, \nu, \eta).$$

In terms of the metric coefficients, (+) becomes

$$\frac{1}{H_1H_2H_3}\left[\partial_\rho\left(\frac{H_2H_3}{H_1}\partial_\rho\psi\right) + \partial_\nu\left(\frac{H_1H_3}{H_2}\partial_\nu\psi\right) + \partial_\eta\left(\frac{H_1H_2}{H_3}\partial_\eta\psi\right)\right] = -\sigma(\sigma+2)\psi.$$

Setting $\psi = A_1(\rho)A_2(\nu)A_3(\eta)$ in this equation, one can easily derive the ordinary differential equations satisfied by the separated solutions A_i.

1. $ds^2 = d\rho^2 + \cosh^2\rho\, d\nu^2 + \sinh^2\rho\, d\eta^2$,

$$t = \cosh\rho\cosh\nu, \quad x = \sinh\rho\cos\eta, \qquad (2.1)$$
$$y = \sinh\rho\sin\eta, \quad z = \cosh\rho\sinh\nu$$
$$-\infty < \rho < \infty, \quad -\infty < \nu < \infty, \quad 0 \leqslant \eta < 2\pi.$$

The operators are

$$L_1 = K_3^2, \quad L_2 = M_3^2.$$

2. $ds^2 = d\rho^2 + e^{-2\rho}(d\nu^2 + d\eta^2)$,

$$t = \tfrac{1}{2}[e^\rho + (1 + \nu^2 + \eta^2)\,e^{-\rho}], \quad x = e^{-\rho}\nu, \qquad (2.2)$$
$$y = e^{-\rho}\eta, \quad z = \tfrac{1}{2}[e^\rho + (-1 + \nu^2 + \eta^2)\,e^{-\rho}],$$
$$-\infty < \rho, \nu, \eta < \infty.$$

The operators are

$$L_1 = (K_1 + M_2)^2, \quad L_2 = (K_2 - M_1)^2.$$

3. $ds^2 = d\rho^2 + \sinh^2\rho(\mathrm{sn}^2(\nu,k) - \mathrm{sn}^2(\eta,k))(d\nu^2 - d\eta^2)$,

$$t = \cosh\rho, \quad x = (1/k)\sinh\rho\,\mathrm{dn}(\nu,k)\,\mathrm{dn}(\eta,k),$$
$$y = (ik/k')\sinh\rho\,\mathrm{cn}(\nu,k)\,\mathrm{cn}(\nu,k),$$
$$z = k\sinh\rho\,\mathrm{sn}(\nu,k)\,\mathrm{sn}(\eta,k), \qquad (2.3)$$
$$0 < k < 1, \quad k' = (1-k^2)^{1/2},$$
$$-\infty < \rho < \infty, \quad \nu \in [-2K, 2K],$$
$$\eta \in [-K, -K + 2iK'].$$

The operators are

$$L_1 = M_1^2 + M_2^2 + M_3^2, \quad L_2 = M_1^2 + k^2M_2^2.$$

4. $ds^2 = d\rho^2 + \cosh^2\rho\,(\mathrm{sn}^2(\nu,k) - \mathrm{sn}^2(\eta,k))\,(d\nu^2 - d\eta^2)$,

$$t = ik\cosh\rho\,\mathrm{sn}(\nu,k)\,\mathrm{sn}(\eta,k),$$
$$x = (k/k')\cosh\rho\,\mathrm{cn}(\nu,k)\,\mathrm{cn}(\eta,k), \qquad (2.4)$$
$$y = (i/k')\cosh\rho\,\mathrm{dn}(\nu,k),\,\mathrm{dn}(\eta,k), \quad z = \sinh\rho,$$
$$-\infty < \rho < \infty, \quad \nu \in [K, K + 2iK],$$
$$\eta \in [iK', iK' + 2K].$$

The operators are

$$L_1 = K_1^2 + K_2^2 - M_3^2, \quad L_2 = k^2M_3^2 + k'^2K_2^2.$$

5. Differential form as for system 4

$$t = (ik/k')\cosh\rho\,\mathrm{cn}(\nu,k)\,\mathrm{cn}(\eta,k),$$
$$x = \varepsilon ik\cosh\rho\,\mathrm{sn}(\nu,k)\,\mathrm{sn}(\eta,k),$$
$$y = \varepsilon'(i/k')\cosh\rho\,\mathrm{dn}(\nu,k)\,\mathrm{dn}(\eta,k), \qquad (2.5)$$
$$z = \sinh\rho, \quad \varepsilon, \varepsilon' = \pm,$$
$$-\infty < \rho < \infty, \quad \nu \in [iK', iK' + 2K], \quad \eta \in [0, 2iK'].$$

The operators are

$$L_1 = K_1^2 + K_2^2 - M_3^2, \quad L_2 = K_2^2 - k^2M_3^2.$$

6. $ds^2 = d\rho^2 + \tfrac{1}{4}(\nu - \eta)\cosh^2\rho\left[\dfrac{d\nu^2}{(\nu-a)(\nu-b)\nu} - \dfrac{d\eta^2}{(\eta-a)(\eta-b)\eta}\right]$.

The coordinates are given by the equations

$$(t + iy)^2 = \frac{2(\nu-a)(\eta-a)}{a(a-b)}\cosh^2\rho, \qquad (2.6)$$
$$x = \sqrt{-\nu\eta/ab}\,\cosh\rho, \quad z = \sinh\rho,$$

where $a = b^* = \alpha + i\beta$, $\alpha, \beta \in \mathbf{R}$, $-\infty < \nu < 0$, $0 < \eta < \infty$. The operators are

$$L_1 = K_1^2 + K_2^2 - M_3^2,$$
$$L_2 = \beta(M_3K_2 + K_2M_3) + \alpha K_1^2.$$

7. $ds^2 = d\rho^2 + \cosh^2\rho\left(\dfrac{1}{\cos^2\nu} - \dfrac{1}{\cosh^2\eta}\right)(d\nu^2 + d\eta^2)$,

$$t = \tfrac{1}{2}\cosh\rho\left(\frac{\cosh\eta}{\cos\nu} + \frac{\cos\nu}{\cosh\eta}\right),$$
$$x = \cosh\rho\tanh\eta\tan\nu, \qquad (2.7)$$
$$y = \cosh\rho\left[\frac{1}{\cosh\eta\cos\nu} - \frac{1}{2}\left(\frac{\cosh\eta}{\cos\nu} + \frac{\cos\nu}{\cosh\eta}\right)\right],$$
$$z = \sinh\rho,$$
$$-\infty < \rho < \infty, \quad -\infty < \eta < \infty, \quad 0 \leqslant \nu < 2\pi.$$

The operators are

$$L_1 = K_1^2 + K_2^2 - M_3^2,$$
$$L_2 = K_1^2 + K_2^2 + M_3^2 - M_3K_2 - K_2M_3.$$

8. $ds^2 = d\rho^2 + \cosh^2\rho\left(\dfrac{1}{\sinh^2\eta} + \dfrac{1}{\sin^2\nu}\right)(d\eta^2 + d\nu^2)$,

$$t=\cosh\rho\left[\frac{1}{\sinh\eta\,\sin\nu}+\frac{1}{2}\left(\frac{\sinh\eta}{\sin\nu}-\frac{\sin\nu}{\sinh\eta}\right)\right],$$
$$x=\cosh\rho\,\coth\eta\,\cot\nu,$$
$$y=\tfrac{1}{2}\cosh\rho\left(\frac{\sin\nu}{\sinh\eta}-\frac{\sinh\eta}{\sin\nu}\right), \tag{2.8}$$
$$z=\sinh\rho,\qquad -\infty<\rho<\infty,$$
$$-\infty<\eta<\infty,\qquad 0\leqslant\nu<2\pi.$$

The operators are

$$L_1=K_1^2+K_2^2-M_3^2,$$
$$L_2=-K_1^2+K_2^2+M_3^2-M_3K_2-K_2M_3.$$

9. $ds^2=d\rho^2+\cosh^2\rho\left(\frac{1}{\nu^2}+\frac{1}{\eta^2}\right)(d\nu^2+d\eta^2),$

$$t=\cosh\rho\,\frac{[(\nu^2+\eta^2)^2+4]}{8\nu\eta},\ x=\tfrac{1}{2}\cosh\rho\left(\frac{\eta}{\nu}-\frac{\nu}{\eta}\right),$$
$$y=\cosh\rho\,\frac{[-(\nu^2+\eta^2)^2+4]}{8\nu\eta},\qquad z=\sinh\rho, \tag{2.9}$$
$$-\infty<\rho<\infty,\ -\infty<\nu<\infty,\ -\infty<\eta<\infty.$$

The operators are

$$L_1=K_1^2+K_2^2-M_3^2,$$
$$L_2=K_1K_2+K_2K_1-K_1M_3-M_3K_1.$$

10. $ds^2=d\rho^2+\sinh^2\rho\,(d\nu^2+\sin^2\nu\,d\eta^2)$

$$t=\cosh\rho,\qquad x=\sinh\rho\,\sin\nu\,\cos\eta, \tag{2.10}$$
$$y=\sinh\rho\,\sin\nu\,\sin\eta,\qquad z=\sinh\rho\,\cos\nu,$$
$$-\infty<\rho<\infty,\qquad 0\leqslant\nu<\pi,\qquad 0\leqslant\eta<2\pi.$$

The operators are

$$L_1=M_1^2+M_2^2+M_3^2,\qquad L_2=M_3^2.$$

11. $ds^2=d\rho^2+\cosh^2\rho\,(d\nu^2+\sinh^2\nu\,d\eta^2),$

$$t=\cosh\rho\,\cosh\nu,\quad x=\cosh\rho\,\sinh\nu\,\cos\eta, \tag{2.11}$$
$$y=\cosh\rho\,\sinh\nu\,\sin\eta,\qquad z=\sinh\rho,$$
$$-\infty<\rho<\infty,\qquad -\infty<\nu<\infty,\qquad 0\leqslant\eta<2\pi.$$

The operators are

$$L_1=K_1^2+K_2^2-M_3^2,\qquad L_2=M_3^2.$$

12. $ds^2=d\rho^2+\cosh^2\rho\,(d\nu^2+\cosh^2\nu\,d\eta^2),$

$$t=\cosh\rho\,\cosh\nu\,\cosh\eta,$$
$$x=\cosh\rho\,\cosh\nu\,\sinh\eta, \tag{2.12}$$
$$y=\cosh\rho\,\sinh\nu,\qquad z=\sinh\rho,$$
$$-\infty<\nu,\rho,\eta<\infty.$$

The operators are

$$L_1=K_1^2+K_2^2-M_3^2,\qquad L_2=K_1^2.$$

13. $ds^2=d\rho^2+\cosh^2\rho(d\nu^2+e^{-2\nu}\,d\eta^2),$

$$t=\tfrac{1}{2}\cosh\rho\,[e^{\nu}+(1+\eta^2)\,e^{-\nu}],$$
$$x=\eta e^{-\nu}\cosh\rho, \tag{2.13}$$
$$y=\tfrac{1}{2}\cosh\rho\,[e^{\nu}+(-1+\eta^2)\,e^{-\nu}],\qquad z=\sinh\rho,$$
$$-\infty<\nu,\rho,\eta<\infty.$$

The operators are

$$L_1=K_1^2+K_2^2-M_3^2,\qquad L_2=(K_1+M_3)^2.$$

14. $ds^2=d\rho^2+e^{-2\rho}(d\nu^2+\nu^2\,d\eta^2),$

$$t=\tfrac{1}{2}[e^{\rho}+(1+\nu^2)e^{-\rho}],\qquad x=e^{-\rho}\,\nu\cos\eta,$$
$$y=e^{-\rho}\,\nu\sin\eta,\quad z=\tfrac{1}{2}[e^{\rho}+(-1+\nu^2)\,e^{-\rho}], \tag{2.14}$$
$$-\infty<\rho<\infty,\quad -\infty<\nu<\infty,\quad 0\leqslant\eta<2\pi.$$

The operators are

$$L_1=(K_1+M_2)^2+(K_2-M_1)^2,\qquad L_2=M_3^2.$$

15. $ds^2=d\rho^2+e^{-2\rho}(\cosh 2\nu-\cos 2\eta)\,(d\nu^2+d\eta^2),$

$$t=\tfrac{1}{2}[e^{\rho}+(1+\cosh^2\nu-\sin^2\eta)\,e^{-\rho}],$$
$$x=e^{-\rho}\cosh\nu\,\cos\eta,\qquad y=e^{-\rho}\sinh\nu\,\sin\eta, \tag{2.15}$$
$$z=\tfrac{1}{2}[e^{\rho}+(-1+\cosh^2\nu-\sin^2\eta)\,e^{-\rho}],$$
$$-\infty<\rho<\infty,\quad -\infty<\nu<\infty,\quad 0\leqslant\eta<2\pi.$$

The operators are

$$L_1=(K_1+M_2)^2+(K_2-M_1)^2,$$
$$L_2=M_3^2+(K_1+M_2)^2.$$

16. $ds^2=d\rho^2+e^{-2\rho}(\eta^2+\nu^2)\,(d\eta^2+d\nu^2),$

$$t=\tfrac{1}{2}[e^{\rho}+(1+\tfrac{1}{4}(\eta^2+\nu^2)^2)\,e^{-\rho}],$$
$$x_1=\tfrac{1}{2}e^{-\rho}(\eta^2-\nu^2),\qquad y=e^{-\rho}\,\eta\nu, \tag{2.16}$$
$$z=\tfrac{1}{2}[e^{\rho}+(-1+\tfrac{1}{4}(\eta^2+\nu^2)^2)\,e^{-\rho}],$$
$$-\infty<\rho<\infty,\ -\infty<\eta<\infty,\ -\infty<\nu<\infty.$$

The operators are

$$L_1=(K_1+M_2)^2+(K_2-M_1)^2,$$
$$L_2=M_3(K_1+M_2)+(K_1+M_2)\,M_3.$$

17. $ds^2=(\mathrm{sn}^2(\rho,k)-\mathrm{sn}^2(\nu,k))(d\rho^2-d\nu^2)$
$\qquad+(k^2/k'^2)\,\mathrm{cn}^2(\rho,k)\,\mathrm{cn}^2(\nu,k)\,d\eta^2,$

$$t=k\,\mathrm{sn}(\rho,k)\,\mathrm{sn}(\nu,k),$$
$$x=(k/k')\,\mathrm{cn}(\rho,k)\,\mathrm{cn}(\nu,k)\cos\eta,$$
$$y=(k/k')\,\mathrm{cn}(\rho,k)\,\mathrm{cn}(\nu,k)\sin\eta, \tag{2.17}$$
$$z=(i/k')\,\mathrm{dn}(\rho,k)\,\mathrm{dn}(\nu,k),$$
$$\rho\in[K,K+2iK'],\quad \nu\in[iK',iK'+2K],$$
$$0\leqslant\eta<2\pi.$$

The operators are

$$L_1=M_3^2,\qquad L_2=K_1^2+K_2^2+k^2K_3^2-k'^2M_3^2.$$

18. $ds^2=(\mathrm{sn}^2(\rho,k)-\mathrm{sn}^2(\nu,k))\,(d\rho^2-d\nu^2)$
$\qquad-(1/k'^2)\,\mathrm{dn}^2(\rho,k)\,\mathrm{dn}^2(\nu,k)\,d\eta^2,$

$$t=k\,\mathrm{sn}(\rho,k)\,\mathrm{sn}(\nu,k),$$
$$x=(i/k')\,\mathrm{dn}(\rho,k)\,\mathrm{dn}(\nu,k)\cos\eta,$$
$$y=(i/k')\,\mathrm{dn}(\rho,k)\,\mathrm{dn}(\nu,k)\sin\eta, \tag{2.18}$$
$$z=(k/k')\,\mathrm{cn}(\rho,k)\,\mathrm{cn}(\nu,k),$$
$$\rho\in[K,K+2iK'],\quad \nu\in[iK',iK'+2K],$$
$$0\leqslant\eta<2\pi.$$

The operators are

$$L_1=M_3^2,\qquad L_2=K_3^2+k^2(K_1^2+K_2^2)+k'^2\,M_3^2.$$

19. $ds^2 = (\mathrm{sn}^2(\rho, k) - \mathrm{sn}^2(\nu, k))(d\rho^2 - d\nu^2) + k^2 \mathrm{sn}^2(\rho, k)\, \mathrm{sn}^2(\nu, k)\, d\eta^2,$

$$t = k\, \mathrm{sn}(\rho, k)\, \mathrm{sn}(\nu, k) \cosh\eta,$$
$$x = k\, \mathrm{sn}(\rho, k)\, \mathrm{sn}(\nu, \mathrm{k}) \sinh\eta,$$
$$y = (k/k')\, \mathrm{cn}(\rho, k)\, \mathrm{cn}(\nu, k), \qquad (2.19)$$
$$z = (i/k')\, \mathrm{dn}(\rho, k)\, \mathrm{dn}(\nu, k),$$
$$\rho \in [K, K + 2iK'], \qquad \nu \in [iK', iK' + 2K],$$
$$-\infty < \eta < \infty.$$

The operators are

$$L_1 = K_1^2,$$
$$L_2 = K_2^2 - M_3^2 + k^2(K_3^2 - M_2^2) - (1 + k^2) K_1^2.$$

20. Differential form the same as in system 17 with

$$t = (ik/k')\, \mathrm{cn}(\rho, k)\, \mathrm{cn}(\nu, k) \cosh\eta,$$
$$x = (ik/k')\, \mathrm{cn}(\rho, \mathrm{k})\, \mathrm{cn}(\nu, k) \sinh\eta,$$
$$y = (i/k')\, \mathrm{dn}(\rho, k)\, \mathrm{dn}(\nu, k), \qquad (2.20)$$
$$z = ik\, \mathrm{sn}(\rho, k)\, \mathrm{sn}(\nu, k),$$
$$\rho \in [iK', iK' + 2K], \qquad \nu \in [-iK', iK'],$$
$$-\infty < \eta < \infty.$$

The operators are

$$L_1 = K_1^2, \qquad L_2 = K_3^2 - M_2^2 + k^2(K_1^2 - M_1^2).$$

21. Differential form the same as in system 18 with

$$t = (ik/k')\, \mathrm{cn}(\rho, k)\, \mathrm{cn}(\nu, k),$$
$$x = (i/k')\, \mathrm{dn}(\rho, k)\, \mathrm{dn}(\nu, k) \cos\eta,$$
$$y = (i/k')\, \mathrm{dn}(\rho, k)\, \mathrm{dn}(\nu, k) \sin\eta, \qquad (2.21)$$
$$z = ik\, \mathrm{sn}(\rho, k)\, \mathrm{sn}(\nu, k),$$
$$\rho \in [iK', iK' + 2K], \qquad \nu \in [0, 2iK'],\ 0 \leqslant \eta < 2\pi.$$

The operators are

$$L_1 = M_3^2, \qquad L_2 = K_3^2 + M_3^2 - k^2(M_2^2 + M_1^2).$$

22. $ds^2 = \frac{1}{4}(\rho - \nu)\left[\frac{d\rho^2}{(\rho - a)(\rho - b)\rho} - \frac{d\nu^2}{(\nu - a)(\nu - b)\nu}\right] - \rho\nu d\eta^2.$

The coordinates are given by the equations

$$(t + iz)^2 = 2(\rho - a)(\nu - a)/a(a - b),$$
$$x = (\sqrt{-\rho\nu/ab}) \cos\eta, \qquad (2.22)$$
$$y = (\sqrt{-\rho\nu/ab}) \sin\eta,$$

where

$$a = b^* = \alpha + i\beta, \quad \alpha, \beta \in \mathbf{R}, \quad -\infty < \nu < 0,$$
$$0 < \rho < \infty, \qquad 0 \leqslant \eta < 2\pi.$$

The operators are

$$L_1 = M_3^2,$$
$$L_2 = \alpha(K_1^2 + K_2^2 - M_1^2 - M_2^2) - \beta(K_1M_2 + M_2K_1 + K_2M_1 + M_1K_2) - 2\alpha M_3^2.$$

23. $ds^2 = \left(\frac{1}{\cos^2\rho} - \frac{1}{\cosh^2\nu}\right)(d\rho^2 + d\nu^2) + \frac{1}{\cos^2\rho \cosh^2\nu}\, d\eta^2,$

$$t = \frac{1}{2}\left(\frac{\cosh\nu}{\cos\rho} + \frac{\cos\rho}{\cosh\nu}\right) + \frac{\eta^2}{2\cosh\nu \cos\rho},$$
$$x = \frac{\eta}{\cosh\nu \cos\rho}, \quad y = \tanh\nu \tan\rho, \qquad (2.23)$$
$$z = \left[\frac{1}{\cosh\nu \cos\rho} - \frac{1}{2}\left(\frac{\cosh\nu}{\cos\rho} + \frac{\cos\rho}{\cosh\nu}\right)\right] - \frac{\eta^2}{2\cosh\nu \cos\rho},$$
$$0 \leqslant \rho < 2\pi, \quad -\infty < \nu < \infty, \quad -\infty < \eta < \infty.$$

The operators are

$$L_1 = (K_1 + M_2)^2,$$
$$L_2 = 2K_1^2 + K_2^2 + K_3^2 + M_1^2 - K_1M_2 - M_2K_1 - K_2M_1 - M_1K_2.$$

24. $ds^2 = \left(\frac{1}{\sinh^2\nu} + \frac{1}{\sin^2\rho}\right)(d\rho^2 + d\nu^2) + \frac{1}{\sinh^2\nu \sin^2\rho}\, d\eta^2,$

$$t = \left[\frac{1}{\sinh\nu \sin\rho} + \frac{1}{2}\left(\frac{\sinh\nu}{\sin\rho} - \frac{\sin\rho}{\sinh\nu}\right)\right] + \frac{\eta^2}{2\sinh\nu \sin\rho},$$
$$x = \frac{\eta}{\sinh\nu \sin\rho}, \qquad y = \cot\rho \coth\nu,$$
$$z = \frac{1}{2}\left(\frac{\sinh\nu}{\sin\rho} - \frac{\sin\rho}{\sinh\nu}\right) - \frac{\eta^2}{\sinh\nu \sin\rho}, \qquad (2.24)$$
$$0 \leqslant \rho < 2\pi, \quad -\infty < \nu < \infty, \quad -\infty < \eta < \infty.$$

The operators are

$$L_1 = (K_1 + M_2)^2,$$
$$L_2 = 2M_2^2 + M_1^2 + K_2^2 - K_3^2 - K_2M_1 - M_1K_2 - K_1M_2 - M_2K_1.$$

25. $ds^2 = \left(\frac{1}{\cos^2\rho} - \frac{1}{\cosh^2\nu}\right)(d\rho^2 + d\nu^2) + \tan^2\rho \tanh^2\nu\, d\eta^2, \qquad t = \frac{1}{2}\left(\frac{\cosh\nu}{\cos\rho} + \frac{\cos\rho}{\cosh\nu}\right),$

$$x = \tan\rho \tanh\nu \cos\eta, \qquad y = \tan\rho \tanh\nu \sin\eta,$$
$$z = \frac{1}{\cosh\nu \cos\rho} - \frac{1}{2}\left(\frac{\cosh\nu}{\cos\rho} + \frac{\cos\rho}{\cosh\nu}\right), \qquad (2.25)$$
$$0 \leqslant \rho < 2\pi, \qquad -\infty < \nu < \infty, \qquad 0 \leqslant \eta < 2\pi.$$

The operators are

$$L_1 = M_3^2, \quad L_2 = M_2^2 + M_1^2 + K_1^2 + K_2^2 + K_3^2 - M_2K_1 - K_1M_2 - M_1K_2 - K_2M_1.$$

26. $ds^2 = \left(\frac{1}{\sinh^2\nu} + \frac{1}{\sin^2\rho}\right)(d\nu^2 + d\rho^2) + \cot^2\rho \coth^2\nu\, d\eta^2,$

$$t = \frac{1}{\sinh\nu \sin\rho} + \frac{1}{2}\left(\frac{\sinh\nu}{\sin\rho} - \frac{\sin\rho}{\sinh\nu}\right), \qquad (2.26)$$
$$x = \cot\rho \coth\nu \cos\eta,$$
$$y = \cot\rho \coth\nu \sin\eta, \qquad z = \frac{1}{2}\left(\frac{\sinh\nu}{\sin\rho} - \frac{\sin\rho}{\sinh\nu}\right),$$
$$0 \leqslant \rho < 2\pi, \quad -\infty < \nu < \infty, \quad 0 \leqslant \eta < 2\pi.$$

The operators are

$L_1 = M_3^2,$

$L_2 = M_2^2 + M_1^2 + K_1^2 + K_2^2 - K_3^2 + K_1M_2 + M_2K_1 + M_1K_2 + K_2M_1.$

27. $$ds^2 = \left(\frac{1}{\rho^2} + \frac{1}{\nu^2}\right)(d\rho^2 + d\nu^2) + \frac{1}{\rho^2\nu^2}\,d\eta^2,$$

$$t = \frac{(\rho^2+\nu^2)^2+4}{8\rho\nu} + \frac{\eta^2}{2\rho\nu}, \qquad x = \frac{\eta}{\rho\nu}, \qquad (2.27)$$

$$y = \frac{1}{2}\left(\frac{\nu}{\rho} - \frac{\rho}{\nu}\right), \qquad z = \frac{-(\rho^2+\nu^2)^2+4}{8\rho\nu} - \frac{\eta^2}{2\rho\nu}.$$

The operators are

$L_1 = (K_1 + M_2)^2,$

$L_2 = M_3(K_1 + M_2) + (K_1 + M_2)M_3 + K_3(K_2 - M_1) + (K_2 - M_1)K_3.$

28. $$ds = \frac{1}{4}\left[\frac{(\rho-\eta)(\rho-\nu)}{P(\rho)}\,d\rho^2 + \frac{(\nu-\rho)(\nu-\eta)}{P(\nu)}\,d\nu^2 + \frac{(\nu-\rho)(\nu-\eta)}{P(\eta)}\,d\eta^2\right],$$

$$P(z) = (z-a)(z-b)(z-1)z,$$

$$t^2 = \frac{\rho\nu\eta}{ab}, \qquad x^2 = \frac{(\rho-1)(\nu-1)(\eta-1)}{(a-1)(b-1)},$$

$$y^2 = -\frac{(\rho-b)(\nu-b)(\eta-b)}{(a-b)(b-1)b}, \qquad (2.28)$$

$$z^2 = \frac{(\rho-a)(\nu-a)(\eta-a)}{(a-b)(a-1)a},$$

$$0 < 1 < \eta < b < \nu < a < \rho.$$

The operators are

$L_1 = ab\,K_1^2 + a\,K_2^2 + b\,K_3^2,$

$L_2 = (a+b)K_1^2 + (a+1)K_2^2 + (b+1)K_3^2 - a\,M_3^2 - b\,M_2^2 - M_1^2.$

29. Differential form as in system 28 with

$$t^2 = -\frac{(\rho-1)(\nu-1)(\eta-1)}{(a-1)(b-1)}, \qquad x^2 = -\frac{\rho\nu\eta}{ab},$$

$$y^2 = -\frac{(\rho-b)(\nu-b)(\eta-b)}{(a-b)(b-1)}, \qquad (2.29)$$

$$z^2 = \frac{(\rho-a)(\nu-a)(\eta-a)}{(a-b)(a-1)a},$$

$$\eta < 0 < 1 < b < \nu < a < \rho.$$

The operators are

$L_1 = ab\,K_1^2 - a\,M_3^2 - b\,M_2^2,$

$L_2 = (a+b)K_1^2 - (a+1)M_3^2 - (b+1)M_2^2 + a\,K_2^2 + b\,K_3^2 - M_1^2.$

30. Differential form as in system 28 with

$$a = b^* = \alpha + i\beta, \qquad \alpha, \beta \in \mathbf{R},$$

$$(x+it)^2 = \frac{2(\rho-a)(\nu-a)(\eta-a)}{(a-b)(b-1)b}, \qquad (2.30)$$

$$y^2 = \frac{(\rho-1)(\nu-1)(\eta-1)}{(a-1)(b-1)}, \qquad z^2 = -\frac{\rho\nu\eta}{ab},$$

$$\eta < 0 < \nu < 1 < \rho.$$

The operators are

$L_1 = -(\alpha^2+\beta^2)M_1^2 + \alpha(K_3^2 - M_2^2) - \beta(K_3M_2 + M_2K_3),$

$L_2 = -2\alpha M_1^2 + (\alpha+1)(K_3^2 - M_2^2) + \alpha(K_2^2 - M_3^2) + \beta(K_2M_3 + M_3K_2 - M_2K_3 - K_3M_2).$

31. Differential form as in system 28 with

$$P(z) = (z-a)(z-1)z^2,$$

$$(t+x)^2 = \rho\nu\eta/a,$$

$$(t^2 - x^2) = (1/a^2)\,[a(\rho\nu + \rho\eta + \nu\eta) - (a+1)\,\rho\nu\eta],$$

$$y^2 = -(\rho-1)(\nu-1)(\eta-1)/(a-1), \qquad (2.31)$$

$$z^2 = (\rho-a)(\nu-a)(\eta-a)/a^2(a-1),$$

$$0 < \eta < 1 < \nu < a < \rho.$$

The operators are

$L_1 = (K_3 + M_2)^2 - a(K_2 + M_3)^2 + a\,K_1^2,$

$L_2 = (a+1)K_1^2 + K_3^2 - M_2^2 + a(M_3^2 - K_2^2) + (K_2 + M_3)^2 + (K_3 + M_2)^2.$

32. Differential form as in system 28 with

$$(t+x)^2 = -\rho\nu\eta/a,$$

$$(t^2 - x^2) = (1/a^2)\,[a(\rho\nu + \nu\eta + \rho\eta) - (a+1)\,\rho\nu\eta],$$

$$y^2 = -(\rho-1)(\nu-1)(\eta-1)/(a-1), \qquad (2.32)$$

$$z^2 = (\rho-a)(\nu-a)(\eta-a)/a^2(a-1),$$

$$-\eta < 0 < 1 < \nu < a < \rho.$$

The operators are

$L_1 = -(K_3 + M_2)^2 + a(K_2 + M_3)^2 + a\,K_1^2,$

$L_2 = (a+1)\,K_1^2 + M_2^2 - K_3^2 + a(K_2^2 - M_3^2) - (K_2 + M_3)^2 - (K_3 + M_2)^2.$

33. Differential form as in system 28 with

$$P(z) = (z-a)(z+1)\,z^2,$$

$$(t+x)^2 = -\nu\rho\eta/a,$$

$$(t^2 - x^2) = -(1/a^2)[a(\rho\eta + \rho\nu + \eta\nu) - (a-1)\,\nu\rho\eta],$$

$$y^2 = (\rho-a)(\nu-a)(\eta-a)/a^2(a+1), \qquad (2.33)$$

$$z^2 = -(\rho+1)(\nu+1)(\eta+1)/(a+1),$$

$$\eta < -1 < 0 < \nu < a < \rho.$$

The operators are

$L_1 = aK_1^2 - (K_2 + M_3)^2 + a(K_3 + M_2)^2,$

$L_2 = (a-1)\,K_1^2 - (K_2 + M_3)^2 + (K_3 + M_2)^2 + M_3^2 - K_2^2 + a(M_2^2 - K_3^2).$

34. Differential form as in system 28 with

$$P(z) = (z-1)z^3, \qquad (t-x)^2 = -\nu\rho\eta,$$

$$2y(x-t) = \nu\rho + \nu\eta + \rho\eta - \nu\rho\eta,$$

$$x^2 + y^2 - t^2 = -\nu\rho\eta + \nu\rho + \nu\eta + \rho\eta - \nu - \rho - \eta$$

$$z^2 = (\nu - 1)(\rho - 1)(\eta - 1),$$
$$\eta < 0 < \nu < 1 < \rho. \tag{2.34}$$

The operators are

$$L_1 = (M_2 - K_3)^2 - K_1(K_2 - M_3) - (K_2 - M_3)K_2,$$
$$L_2 = M_2^2 - K_3^2 - M_1^2 - (M_2 - K_3)^2 - M_1(M_2 - K_3) - (M_2 - K_3) M_1.$$

3. THE SPECTRAL ANALYSIS OF SEPARABLE BASES ON $L_s^2(C)$

Following Vilenkin[21] we construct a Hilbert space $L_s^2(C)$ of homogeneous functions $f(Y)$ on the forward light cone C: $[Y\ Y] = 0$, $Y = (y_0, y_1, y_2, y_3)$, $y_0 > 0$. In particular we require that f be homogeneous of degree $\sigma = is - 1$,

$$f(\rho Y) = \rho^{is-1} f(Y), \qquad \rho > 0. \tag{3.1}$$

Let Γ be a contour on the cone C which cuts each generator exactly once. If $Y(\nu, \eta)$ is a parametrization for Γ then every Y on C can be expressed uniquely in the form

$$Y = \rho Y(\nu, \eta), \qquad \rho > 0.$$

Now the measure on C which is invariant under the identity component of SO(3, 1) is well known to be $dY = dy_1dy_2dy_3/y_0$. We define a measure dw on Γ by $dY = \rho d\rho dw$, i.e.,

$$dw = |\det A|\, \rho^{-2} y_0^{-1}\, d\nu d\eta,$$
$$A = \begin{bmatrix} y_1 & \partial_\nu y_1 & \partial_\eta y_1 \\ y_2 & \partial_\nu y_2 & \partial_\eta y_2 \\ y_3 & \partial_\nu y_3 & \partial_\eta y_3 \end{bmatrix}, \quad Y = (y_0, y_1, y_2, y_3). \tag{3.2}$$

Then $L_s^2(C)$ is the space of measurable functions $f(Y)$, in (3.1), on C such that

$$\int_\Gamma |f(Y)|^2\, dw < \infty.$$

The inner product on this space is

$$\langle f_1, f_2\rangle = \int_\Gamma f_1(Y)\bar{f}_2(Y) dw, \qquad f_i \in L_s^2(C). \tag{3.3}$$

[It is easy to verify from (3.1) and (3.2) that the value of the inner product is independent of the contour Γ.]

Note that the function

$$h(X, Y) = [X, Y]^{is-1} \tag{3.4}$$

belongs to $L_s(C)$ for each $X \in H_+$. Furthermore, the function $F(X)$ defined by

$$F(X) = \langle f, h(X, \cdot\,)\rangle = \int_\Gamma f(Y)\, [X, Y]^{-is-1}\, dw \tag{3.5}$$

is a solution of equation (+) for each $f \in L_s^2(C)$.

The action of the identity component of SO(3, 1) on the functions $F(X)$ as defined by operators (1.1) and (1.2) induces via (3.5) a corresponding action of SO(3, 1) on $L_s^2(C)$ given by

$$M_1 = y_2\partial_{y_3} - y_3\partial_{y_2}, \qquad M_2 = y_1\partial_{y_3} - y_3\partial_{y_1},$$
$$M_3 = y_1\partial_{y_2} - y_2\partial_{y_1}, \qquad K_1 = y_0\partial_{y_1} + y_1\partial_{y_0}, \tag{3.6}$$
$$K_2 = y_0\partial_{y_2} + y_2\partial_{y_0}, \qquad K_3 = y_0\partial_{y_3} + y_3\partial_{y_0}.$$

The associated group action is unitary and irreducible. More explicity, if Γ_0 is the sphere $Y_0 = (1, \xi_1, \xi_2, \xi_3)$, $\xi_1^2 + \xi_2^2 + \xi_3^2 = 1$ so that $Y = \rho Y_0$, then the action of SO(3, 1) on elements of $L_s^2(C)$ restricted to Γ_0 is

$$M_1 = -\xi_3\partial_{\xi_2}, \quad M_2 = -\xi_3\partial_{\xi_1}, \quad M_3 = \xi_1\partial_{\xi_2} - \xi_2\partial_{\xi_1},$$
$$K_1 = (is - 1)\xi_1 + (1 - \xi_1^2)\partial_{\xi_1} - \xi_1\xi_2\partial_{\xi_2},$$
$$K_2 = (is - 1)\xi_2 - \xi_1\xi_2\partial_{\xi_1} + (1 - \xi_2^2)\partial_{\xi_2}, \tag{3.7}$$
$$K_3 = (is - 1)\xi_3 - \xi_1\xi_3\partial_{\xi_1} - \xi_2\xi_3\partial_{\xi_2}.$$

(We are taking ξ_1 and ξ_2 as the independent variables on Γ_0.) If Γ is another contour related to Γ_0 by Γ: $Y = \zeta Y_0$ then the operators L_0, in (3.7), are replaced by operators $L = L_0 + (1 - is)\ \zeta^{-1}\ (\tilde{L}_0\ \zeta)$, where $\tilde{L}_0$ is the purely differential part of L_0.

The 34 commuting pairs of operators L_1, L_2 defined in the preceding section can in an obvious manner be defined on $L_s^2(C)$ as commuting pairs of self-adjoint operators and the spectral resolution can be determined explicitly. For each of these systems we list a convenient contour Γ and in the most tractable cases a basis for $L_s^2(C)$ consisting of simultaneous eigenfunctions of L_1 and L_2. In the case of new results we give a full development of their derivation.

1. The contour Γ is given by

$$Y = (\cosh\nu, \cos\eta, \sin\eta, \sinh\nu),$$
$$-\infty < \nu < \infty, \qquad 0 \leqslant \eta < 2\pi.$$

The basis functions are

$$f_{\tau m}^{(1)} = (1/2\pi)\, e^{i\tau\nu} e^{im\eta}, \qquad -\infty < \tau < \infty,$$
$$m = 0, \pm 1, \pm 2, \cdots, \tag{3.8}$$
$$\langle f_{\tau m}^{(1)}, f_{\tau' m'}^{(1)}\rangle = \delta(\tau - \tau')\, \delta_{mm'}.$$

The eigenvalues of L_1 and L_2 are $-\tau^2$ and $-m^2$, respectively.

2. $Y = (\frac{1}{2}(1 + \nu^2 + \eta^2), \nu, \eta, \frac{1}{2}(-1 + \nu^2 + \eta^2))$

$$-\infty < \nu < \infty, \qquad -\infty < \eta < \infty.$$

The basis functions are

$$f_{\tau\kappa}^{(2)} = (1/2\pi)\, e^{i\tau\nu} e^{i\kappa\eta},$$
$$-\infty < \tau < \infty, \qquad -\infty < \kappa < \infty, \tag{3.9}$$
$$\langle f_{\tau\kappa}^{(2)}, f_{\tau'\kappa'}^{(2)}\rangle = \delta(\tau - \tau')\, \delta(\kappa - \kappa').$$

The eigenvalues of L_1 and L_2 are $-\tau^2$ and $-\kappa^2$, respectively.

3. $Y = (1, (1/k')\, \mathrm{dn}(\nu, k)\, \mathrm{dn}(\eta, k), (ik/k')\, \mathrm{cn}(\nu, k) \times \mathrm{cn}(\eta, k), k\, \mathrm{sn}(\nu, k)\, \mathrm{sn}(\eta, k)),$ (3.10)

$$\nu \in [-2K, 2K], \qquad \eta \in [-K, -K + 2iK'].$$

The basis functions are

$$f_{lm}^{(3)\,pq} = E_{lm}^{pq}(\nu) E_{lm}^{pq}(\eta), \qquad l = 0, 1, 2, \cdots,$$

a product of Lamé polynomials.[12,13] Here p and q are the eigenvalues of the rotation $e^{i\pi M_3}$ and the reflection $P \times e^{i\pi M_1}$, respectively, where P is the parity operator and m is the number of zeros of the Lamé polynomials in the interval $[0, K]$. For l even, $0 \leqslant m \leqslant \frac{1}{2}l + 1$ if $p = q = +1$, and $0 \leqslant m \leqslant \frac{1}{2}l$ otherwise. If l is odd, $0 \leqslant m \leqslant$

$\frac{1}{2}(l-1)$ for $p=q=-1$, and $0 \leqslant m \leqslant \frac{1}{2}(l+1)$ otherwise. For further details on this see Ref. 13. The eigenvalues of L_1 and L_2 are $l(l+1)$ and λ_{lm}^{pq} respectively.

4. $Y_\pm = (ik\,\mathrm{sn}(\nu,k)\,\mathrm{sn}(\eta,k),\ (k/k')\,\mathrm{cn}(\nu,k)\,\mathrm{cn}(\eta,k),$
$(i/k')\,\mathrm{dn}(\nu,k)\,\mathrm{dn}(\eta,k),\ \pm 1),$
$\nu \in [K, iK+2iK'], \qquad \eta \in [iK', iK'+2K].$

The orthonormal basis functions are

$$f_{\varepsilon jm}^{(4)pq}(\nu,\eta) = C\varepsilon E_{jm}^{pq}(\nu)\,E_{jm}^{pq}(\eta), \quad \varepsilon=\pm, \tag{3.11}$$

where $C_+ = \begin{pmatrix}1\\0\end{pmatrix}, \qquad C_- = \begin{pmatrix}0\\1\end{pmatrix},$

and $j=-\frac{1}{2}+iq,\ \ 0<q<\infty$. These basis functions are a product of periodic Lamé functions.

Here $m=0,1,2,\cdots$ and p and q are eigenvalues of the operators $P\,e^{i\pi M_2}$ and $P\,e^{i\pi M_1}$ respectively. The integer m denotes the number of zeros N_{pq} of the basis functions $E_{jm}^{pq}(z)$ in the interval $[0, 2K]$ according to the table:

N_{pq}	p	q
$2m$	1	1
$2m+1$	-1	1
$2m+2$	1	-1
$2m+1$	-1	-1

The eigenvalues of L_1 and L_2 are $j(j+1)$ and λ_{jm}^{pq} respectively.

5. $Y_\pm = \left(\frac{ik}{k'}\,\mathrm{cn}(\nu,k)\,\mathrm{cn}(\eta,k),\ \varepsilon\, ik\,\mathrm{sn}(\nu,k)\,\mathrm{sn}(\eta,k),\right.$
$\left.\varepsilon' \frac{i}{k'}\,\mathrm{dn}(\nu,k)\,\mathrm{dn}(\eta,k),\ \pm 1\right),$
$\varepsilon, \varepsilon' = \pm, \qquad \nu \in [iK', iK'+2K], \qquad \eta \in [0, 2iK'].$

The basis functions are

$$f_{\pm jm}^{(5)\varepsilon\varepsilon'} = C_{\pm\varepsilon\varepsilon'}\,F_j^m(\nu,k)\,F_j^m(\eta,k), \tag{3.12}$$

where

$$C_{\pm\varepsilon\varepsilon'}C^{+}_{\pm\bar\varepsilon\bar\varepsilon'} = \delta_{\varepsilon\bar\varepsilon}\delta_{\varepsilon'\bar\varepsilon'}, \quad j=-\tfrac{1}{2}+iq, \quad 0<q<\infty.$$

The basis functions are products of Lamé Wangerin or finite Lamé functions.[14] The label $m=0,1,\cdots$ specifies the number of zeros of these functions in the interval $[iK', iK'+2K]$. The eigenvalues of L_1 and L_2 are $j(j+1)$ and λ_{jm} respectively. There is a Dirac $\delta(j-j')$ normalization on the parameter j and a Kronecker normalization on the remaining parameters.

6. $Y_\pm = \left(\mathrm{Re}\left\{\frac{2(\nu-a)(\eta-a)}{a(a-b)}\right\}^{1/2},\ \left(\frac{-\nu\eta}{ab}\right)^{1/2},\right.$
$\left.\mathrm{Im}\left\{\frac{2(\nu-a)(\eta-a)}{a(a-b)}\right\}\ \pm 1\right),$
$-\infty<\nu<0<\eta<\infty.$

7. $Y_\pm = \left(\frac{1}{2}\left(\frac{\cosh\eta}{\cos\nu}+\frac{\cos\nu}{\cosh\eta}\right),\ \tanh\eta\,\tan\nu,\right.$
$\left.\frac{1}{\cosh\eta\,\cos\nu} - \frac{1}{2}\left(\frac{\cosh\eta}{\cos\nu}+\frac{\cos\nu}{\cosh\eta}\right),\ \pm 1\right),$
$-\infty<\eta<\infty, \qquad 0\leqslant\nu<2\pi,$

8. $Y_\pm = \left(\frac{1}{\sinh\eta\,\sin\nu} + \frac{1}{2}\left(\frac{\sinh\eta}{\sin\nu} - \frac{\sin\nu}{\sinh\eta}\right),\right.$
$\left.\coth\eta\,\cot\nu,\ \frac{1}{2}\left(\frac{\sin\nu}{\sinh\eta} - \frac{\sinh\eta}{\sin\nu}\right),\ \ \pm 1\right),$
$-\infty<\nu<\infty, \qquad 0\leqslant\eta<2\pi.$

9. $Y_\pm = \left(\frac{[(\nu^2+\eta^2)^2+4]}{8\nu\eta},\ \frac{1}{2}\left\{\frac{\eta}{\nu} - \frac{\nu}{n}\right\},\right.$
$\left.\frac{[-(\nu^2+\eta^2)^2+4]}{8\nu\eta},\ \ \pm 1\right),$
$-\infty<\nu<\infty, \qquad -\infty<\eta<\infty.$

These last four coordinate systems have been treated to some extent in Ref. 13 and we refer the reader to that work for further details.

10. $Y=(1, \sin\nu\cos\eta, \sin\nu\sin\eta, \cos\nu),$
$0\leqslant\nu\leqslant\pi, \qquad 0\leqslant\eta<2\pi.$

The orthonormal basis functions are

$$f_{lm}^{(10)} = \left(\frac{2l+1}{4\pi}\,\frac{(l-|m|)!}{(l+|m|)!}\right)^{1/2} P_l^{|m|}(\cos\nu)\,e^{im\eta}, \tag{3.13}$$

$l=0,1,2,\ldots, m=-l,-l+1,\ldots,l.$ The eigenvalues of L_1 and L_2 are $l(l+1)$ and $-m^2$, respectively.

11. $Y=(\cosh\nu, \sinh\nu\cos\eta, \sinh\nu\sin\eta, \pm 1),$
$-\infty<\nu<\infty, \qquad 0\leqslant\eta<2\pi.$

The basis functions are

$$f_{\pm jm}^{(11)} = \frac{\Gamma(j+1-m)}{\Gamma(j+1)}\,P_j^m(\cosh\nu)\,e^{im\eta}, \tag{3.14}$$

where $j=-\frac{1}{2}+iq, \qquad 0<q<\infty,$

which are normalized according to

$$\langle f_{\pm jm}^{(11)}, f_{\pm j'm'}^{(11)}\rangle = \frac{2\pi}{q\tanh\pi q}\,\delta(q-q')\,\delta_{mm'}.$$

The eigenvalues of L_1 and L_2 are $-j(j+1)$ and $-m^2$, respectively.

12. $Y=(\cosh\nu\cosh\eta, \cosh\nu\sinh\eta, \sinh\nu, \pm 1),$
$-\infty<\nu, \eta<\infty.$

The basis functions are

$$f_{\pm j\tau}^{(12)} = C_\varepsilon\,\frac{\Gamma(j+1+i\tau)\,\Gamma(-j+i\tau)}{\Gamma(j+1)} \times P^{-j-1/2}_{-1/2+i\tau}(\varepsilon\tanh\nu)\,e^{i\tau\eta}, \tag{3.15}$$

where

$\varepsilon=\pm, \qquad C_+ = \begin{pmatrix}1\\0\end{pmatrix}, \qquad C_- = \begin{pmatrix}0\\1\end{pmatrix},$
$j=-\frac{1}{2}+iq,$
$0<q<\infty, \qquad$ and $\qquad -\infty<\tau<\infty.$

These basis functions are normalized according to

$$\langle f_{\pm j\tau}^{(12)\varepsilon}, f_{\pm j'\tau'}^{(12)\bar\varepsilon}\rangle = \frac{\delta(q-q')\,\delta(\tau-\tau')}{q\tanh\pi q}.$$

The eigenvalues of L_1 and L_2 are $-j(j+1)$ and $-\tau^2$, respectively.

13. $Y = (\frac{1}{2}[e^{\nu} + (1+\eta^2)e^{-\nu}], \quad \eta e^{-\nu},$
$\frac{1}{2}[e^{\nu} + (-1+\eta^2)e^{-\nu}], \pm 1),$
$-\infty < \nu < \infty, \qquad -\infty < \eta < \infty.$

The basis functions are

$$f_{j\tau}^{(13)} = \frac{1}{\Gamma(j+1)} e^{-\nu/2} K_{j+1/2}(\tau e^{-\nu})\, e^{i\tau\eta}, \qquad (3.16)$$

where

$$j = -\tfrac{1}{2} + iq, \quad 0 < q < \infty, \quad \text{and} \quad -\infty < \tau < \infty.$$

These functions are normalized according to

$$(f_{j\tau}^{(13)}, f_{j'\tau'}^{(13)}) = \frac{\delta(q-q')\,\delta(\tau-\tau')}{q\,\tanh \pi q}.$$

The eigenvalues of L_1 and L_2 are $-j(j+1)$ and $-\tau^2$, respectively.

14. $Y = (\frac{1}{2}(1+\nu^2), \nu \cos\eta, \nu \sin\eta, \frac{1}{2}(-1+\nu^2)),$
$0 \leqslant \nu < \infty, \qquad 0 \leqslant \eta < 2\pi.$

The basis functions are

$$f_{\chi m}^{(14)} = \left(\frac{\chi}{2\pi}\right)^{1/2} J_m(\chi\nu)\, e^{im\eta}, \qquad (3.17)$$

$m = 0, \pm 1, \cdots, 0 < \chi < \infty$. The eigenvalues of L_1 and L_2 are $-\chi^2$ and $-m^2$, respectively,

$$\langle f_{\chi m'}^{(14)}, f_{\chi' m'}^{(14)} \rangle = \delta(\chi - \chi')\, \delta_{mm'}.$$

15. $Y = (\frac{1}{2}(1 + \cosh^2\nu - \sin^2\eta), \cosh\nu \cos\eta,$
$\sinh\nu \sin\eta, \frac{1}{2}(-1 + \cosh^2\nu - \sin^2\eta)),$
$-\infty < \nu < \infty, \qquad 0 \leqslant \eta < 2\pi.$

The orthogonal basis functions are

$$f_{\chi n}^{(15)} = \begin{cases} Ce_n(\nu, \chi^2/4)\, ce_n(\eta, \chi^2/4), \\ Se_n(\nu, \chi^2/4)\, se_n(\eta, \chi^2/4), \end{cases} \qquad (3.17)$$

products of Mathieu functions. Here $n = 0, 1, 2, \cdots$ is the number of zeros of the periodic Mathieu functions in the interval $0 \leqslant \eta \leqslant \frac{1}{2}\pi$.

The eigenvalues of L_1 and L_2 are $-\chi^2$ and a_n (even), b_n (odd), respectively, where even and oddness refer to the periodic Mathieu functions under the interchange $\eta \to -\eta$.

16. $Y = (\frac{1}{2}[1 + \frac{1}{4}(\eta^2+\nu^2)^2], \frac{1}{2}(\eta^2 - \nu^2), \eta\nu,$
$\frac{1}{2}(-1 + \frac{1}{4}(\eta^2+\nu^2)^2)),$
$-\infty < \nu, \eta < \infty.$

The basis functions are

$$f_{\chi\lambda}^{(16)\varepsilon} = C_\varepsilon \frac{1}{\sqrt{2}\,\cosh\pi\lambda} [D_{-i\lambda-1/2}(\varepsilon\sigma\eta)\, D_{i\lambda-1/2}(\sigma\nu) + D_{-i\lambda-1/2}(-\varepsilon\sigma\eta)\, D_{i\lambda-1/2}(-\sigma\nu)], \qquad (3.18)$$

where

$$\varepsilon = \pm, \ \sigma = e^{i\pi/4}\sqrt{2\chi}, \ \text{and} \ C_+ = \begin{pmatrix}1\\0\end{pmatrix}, \ C_- = \begin{pmatrix}0\\1\end{pmatrix},$$
$$-\infty < \lambda < \infty.$$

The eigenvalues of L_1 and L_2 are $-\chi^2$ and 2λ,

$$\langle f_{\chi\lambda}^{(16)\varepsilon}, f_{\chi'\lambda'}^{(16)\varepsilon'} \rangle = \delta(\chi - \chi')\, \delta(\lambda - \lambda')\, \delta_{\varepsilon\varepsilon'}.$$

17. For this coordinate system a suitable choice of contour on the cone is

$$Y = (k\,\mathrm{sn}\nu, (ik/k')\,\mathrm{cn}\nu \cos\eta, (ik/k')\,\mathrm{cn}\nu \sin\eta, (1/k')\,\mathrm{dn}\nu),$$

where

$$\nu \in [K, K + 2iK'], \qquad 0 \leqslant \eta < 2\pi.$$

The basis functions have the form

$$\Psi = \Phi(\nu) \begin{cases} \cos m\eta \\ \sin m\eta \end{cases}$$

where $m = 0, 1, 2, \cdots$ and Φ satisfies

$$\frac{1}{\mathrm{cn}(\nu,k)} \frac{d}{d\nu}\left(\mathrm{cn}(\nu,k) \frac{d\Phi}{d\nu}\right) - \left[\frac{m^2 k'^2}{\mathrm{cn}^2(\nu,k)} + (1+s^2)\,\mathrm{cn}^2(\nu,k) - \lambda - (1+s^2)\right]\Phi = 0, \qquad (3.19)$$

where λ is an eigenvalue of L_2, the eigenvalue of L_1 being $-m^2$. If we write $\Phi(\nu) = [\mathrm{cn}(\nu,k)]^m\, \Xi(\nu)$ then Ξ satisfies the equation

$$\frac{d^2\Xi}{d\nu^2} - \frac{(2m+1)\,\mathrm{sn}(\nu,k)\,\mathrm{dn}(\nu,k)}{\mathrm{cn}(\nu,k)} \frac{d\Xi}{d\nu} + [-k^2(1+s^2+m(m+2))\,\mathrm{cn}^2(\nu,k) + H_A]\,\Xi = 0, \qquad (3.20)$$

where $H_A = k^2\lambda + k^2(1+s^2) + (k^2 - k'^2)m(m+1)$.

There are then four types of solution to this equation and the imposition of periodic boundary conditions requires λ to assume a distinct set of discrete values λ_n. We now develop the solutions.

(1) $\Xi = \sum_{r=0}^{\infty} A_r [\mathrm{cn}(\nu,k)]^{2r}$,

the boundary condition is

$$\Xi(K + iK') = \Xi(K + 2iK') = 0.$$

The recurrence relations for the coefficients A_r are

$$k^2 H_A A_0 + 4k'^2(m+1)A_1 = 0,$$

$$k^2[4(r-1)(r+m-1) + (1+s^2+m(m+2))]A_{r-1} + [(k'^2 - k^2)\,2r(2r+2m+1) - H_A]\,A_r - \frac{4k'^2}{k^2}(r+1)(r+m+1)A_{r+1} = 0, \qquad (3.21)$$

where $H_A = \lambda + (1+s^2)$. We write this solution as

$$L_{smn}^{++}(\nu,k) = \sum_{r=0}^{\infty} A_r [\mathrm{cn}(\nu,k)]^{m+2r}, \qquad n = 0, 2, 4, \cdots.$$

(2) $\Xi = \mathrm{dn}(\nu,k) \sum_{r=0}^{\infty} B_r [\mathrm{cn}(\nu,k)]^{2r}$, the boundary condition is $\Xi'(K + iK') = \Xi(K + 2iK') = 0$. The recurrence relations for the coefficients B_r are

$$k^2 H_B B_0 + 4k'^2(m+1)B_1 = 0,$$

$$k^2[(2r-1)(2r+2m+1)+(1+s^2+m(m+2))]\,B_{r-1}+[(k'^2-k^2)\times 2r(2r+2m+1)-4rk^2-H_B]\,B_r-\frac{4k'^2}{k^2}(r+1)\times(r+m+1)B_{r+1}=0, \tag{3.22}$$

$$H_B=H_A+2k^2(m+1).$$

The solutions are written

$$L^{-+}_{smn}(\nu,k)=\mathrm{dn}(\nu,k)\sum_{r=0}^{\infty}B_r[\mathrm{cn}(\nu,k)]^{m+2r},\quad n=1,3,\cdots.$$

(3) $\Xi=\mathrm{sn}(\nu,k)\,\mathrm{dn}(\nu,k)\sum_{r=0}^{\infty}C_r[\mathrm{cn}(\nu,k)]^{2r}$, the boundary condition is $\Xi(K+iK')=\Xi'(K+2iK')=0$. The recurrence relations for C_r are

$$k^2H_CC_0+4k'^2(m+1)C_1=0,$$

$$k^2[(2r-1)(2r+2m+3)+(1+s^2+m(m+2))]\,C_{r-1}+[(k'^2-k^2)\,2r(2r+2m+3)-H_C]\,C_r-\frac{4k'^2}{k^2}(r+1)(r+m+1)\,C_{r+1}=0, \tag{3.23}$$

$$H_C=H_A+2(k^2-k'^2)(m+1).$$

The solutions are written

$$L^{+-}_{smn}(\nu,k)=\mathrm{sn}(\nu,k)\,\mathrm{dn}(\nu,k)\sum_{r=0}^{\infty}C_r[\mathrm{cn}(\nu,k)]^{m+2r},$$

$$n=2,4,\cdots.$$

(4) $\Xi=\mathrm{sn}(\nu,k)\sum_{r=0}^{\infty}D_r[\mathrm{cn}(\nu,k)]^{2r}$, the boundary condition is

$$\Xi'(K+iK')=\Xi'(K+2iK')=0.$$

The recurrence relations for the D_r are

$$k^2H_DD_0+4k'^2(m+1)D_1=0,$$

$$k^2[4(r-1)(r+m)-2(2m+1)+(1+s^2+m(m+2))]\,D_{r-1}+[(k'^2-k^2)\,2r(2r+2m+1)+4rk'^2-H_D]D_r-\frac{4k'^2}{k^2}(r+1)(r+m+1)\,D_{r+1}=0, \tag{3.24}$$

$$H_D=H_A-k'^2(4m+3).$$

The solutions are written

$$L^{--}_{smn}(\nu,k)=\mathrm{sn}(\nu,k)\sum_{r=0}^{\infty}D_r[\mathrm{cn}(\nu,k)]^{m+2r},$$

$$n=1,3,5,\cdots.$$

The general solution can be written as

$$f^{(17)\varepsilon\varepsilon'}_{smn}=L^{\varepsilon\varepsilon'}_{smn}(\nu,k)\begin{cases}\cos m\eta\\ \sin m\eta\end{cases}, \tag{3.25}$$

where ε and ε' are the eigenvalues of the operators $Pe^{i\pi M_3}$ and $e^{i\pi M_3}$. The number n is the number of zeros of the basis functions Ξ in the interval $[K-iK',K+iK']$. We will call these solutions associated periodic Lamé functions of the first kind.

18. $Y=(k\,\mathrm{sn}(\nu,k),(1/k')\,\mathrm{dn}(\nu,k)\cos\eta,\ (1/k')\,\mathrm{dn}(\nu,k)\times\sin\eta,\ \varepsilon(ik/k')\,\mathrm{cn}(\nu,k))$, where

$$\varepsilon=\pm1,\quad \nu\in[K,K+iK'],\quad 0\leqslant\eta<2\pi.$$

The basis functions have the form

$$\Psi=\Phi(\nu)\begin{cases}\cos m\eta\\ \sin m\eta\end{cases},$$

where $m=0,1,2,\cdots$ and Φ satisfies

$$\frac{1}{\mathrm{dn}(\nu,k)}\frac{d}{d\nu}\left(\mathrm{dn}(\nu,k)\frac{d\Phi}{d\nu}\right)+\left[\frac{m^2k'^2}{\mathrm{dn}^2(\nu,k)}-(1+s^2)\,\mathrm{dn}^2(\nu,k)+\lambda+(1+s^2)\right]\Phi=0. \tag{3.26}$$

The solutions to an equation similar to the above, have been investigated in Ref. 15. The development of solutions to the above equation proceeds in direct analogy with the procedure used in Ref. 15 to obtain finite solutions. In the case of interest here however we have the solutions expressed as an infinite series. We denote the solutions of (3.20),

$$\Phi=[\mathrm{dn}(\nu,k)]^m\,K^{P\varepsilon}_{snm}(\mathrm{dn}(\nu,k)),$$

in analogy with the solutions in Ref. 15. The superscript $P=A$, B, C, or D indicates the form of the solution as an expansion in Jacobi elliptic functions, viz. $P=A$ corresponds to the function

$$K^{A\varepsilon}_{snm}(\mathrm{dn}(\nu,k))=\sum_{r=0}^{\infty}A_r[\mathrm{dn}(\nu,k)]^{2r}.$$

The recurrence relations for the expansion coefficients are those in Ref. 15 with $2F(2F+2)$ replaced by $(1+s^2)$ and k by k'. Similarly we have that $P=B$ gives

$$K^{B\varepsilon}_{snm}(\mathrm{dn}(\nu,k))=\mathrm{cn}(\nu,k)\sum_{r=0}^{\infty}B_r[\mathrm{dn}(\nu,k)]^{2r},$$

$P=C$ gives

$$K^{C\varepsilon}_{snm}(\mathrm{dn}(\nu,k))=\mathrm{sn}(\nu,k)\sum_{r=0}^{\infty}C_r[\mathrm{dn}(\nu,\mathrm{k})]^{2r},$$

and $P=D$ gives

$$K^{D\varepsilon}_{snm}(\mathrm{dn}(\nu,k))=\mathrm{sn}(\nu,k)\,\mathrm{cn}(\nu,\mathrm{k})\sum_{r=0}^{\infty}D_r[\mathrm{dn}(\nu,k)]^{2r}.$$

In each case the spectrum of L_2 is discrete and is labelled by the positive integer n. The basis functions are then of the form

$$f^{(18)}_{sn}=C_\varepsilon[\mathrm{dn}(\nu,k)]^m\,K^{P\varepsilon}_{snm}(\mathrm{dn}(\nu,k))\begin{cases}\cos m\eta\\ \sin m\eta\end{cases} \tag{3.27}$$

where

$$C_+=\begin{pmatrix}1\\0\end{pmatrix}\quad\text{and}\quad C_-=\begin{pmatrix}0\\1\end{pmatrix}.$$

The $K^{P\varepsilon}_{snm}$ functions we have introduced here will be called associated periodic Lamé functions of the second kind.

21. $Y=((k/k')\,\mathrm{cn}(\nu,k),\ (k/k')\,\mathrm{dn}(\nu,k)\cos\eta,$
$(k/k')\,\mathrm{dn}(\nu,k)\sin\eta,\ ik\,\mathrm{sn}(\nu,k))$,

where $\nu\in[-iK',iK']$, $0\leqslant\eta<2\pi$. The basis functions then have the form

$$\Psi = \Phi(\nu)\begin{Bmatrix}\cos m\eta\\ \sin m\eta\end{Bmatrix}, \qquad \text{with} \quad m = 0, 1, 2, \cdots$$

where Φ satisfies the same equation as in system 18. The corresponding boundary value problem is singular at both ends $\nu = \pm iK'$. The spectrum is however discrete and suitable boundary conditions are as follows.

(1) $\mathrm{dn}(\nu, k)\ \Phi(\nu)$ bounded at $\nu = \pm iK'$, $\Phi(0) = 0$. Solutions satisfying these conditions can be developed in the form

$$\Phi(\nu) = \sum_{r=0}^{\infty} A_r [\mathrm{dn}(\nu, k)]^{-1-is-2r}.$$

Such solutions are denoted by $W_{sm}^{2n}(\nu, k)$ and correspond to an eigenvalue λ_{2n} of L_2. This solution has $2n$ zeros in the interval $[-iK', iK']$. The recurrence relations for the coefficients A_r are

$$[H + (1 + k'^2)(i - s)s]A_0 + 4(1 + is)A_1 = 0,$$

$$k'^2[m^2 - (2r - 1 + is)^2]A_{r-1} + [\lambda_{2n} + 1 + s^2 + (2r + 1 + is)(2r + is)]A_r - 4(r + 1)(r + 1 + is)A_{r+1} = 0.$$

(2) $\mathrm{dn}(\nu, k)\ \Phi(\nu)$ bounded at $\nu = \pm iK'$, $\Phi'(0) = 0$. Solutions satisfying these conditions can be developed in the form

$$\Phi(\nu) = \mathrm{sn}(\nu, k)\sum_{r=0}^{\infty} B_r [\mathrm{dn}(\nu, k)]^{-2-is-2r}.$$

Such solutions are denoted by $W_{sm}^{2n+1}(\nu, k)$ and correspond to an eigenvalue λ_{2n+1} of L_2. Each such solution has $2n + 1$ zeros in the interval $[-iK', iK']$. The recurrence relations for the coefficients B_r are

$$[H' + (1 + k'^2)(i - s)s]B_0 + 4(1 + is)B_1 = 0,$$

$$k'^2[m^2 - (2r - 1 + is)^2]B_{r-1} + [H' - 4k'^2 r + (1 + k'^2)(is + 2r + 2)(is + 2r + 1)]B_r - 4(r + 1)(r + 1 + is)B_{r+1} = 0,$$

$$H' = -2k'^2(1 + is) + (1 + s^2) + \lambda_{2n+1}.$$

The complete set of basis function is

$$f_{nm}^{(21)} = W_{sm}^{n}(\nu, k)\, e^{im\eta}, \qquad m = 0, \pm 1, \cdots. \tag{3.28}$$

22. For this coordinate system suitable coordinates on the cone are given by the relations

$$(t + iz)^2 = \frac{2(\nu - a)}{a(a - b)}, \qquad x = \sqrt{\frac{-\nu}{ab}}\cos\eta,$$

$$y = \sqrt{\frac{-\nu}{ab}}\sin\eta,$$

with $-\infty < \nu < 0$.

23. $Y = (\frac{1}{2}[\cosh\nu + \eta^2/\cosh\nu],\ \ \eta/\cosh\nu,\ \ \tanh\nu,\ \ 1/\cosh\nu - \frac{1}{2}[\cosh\nu + \eta^2/\cosh\nu]),$

$-\infty < \nu < \infty, \quad -\infty < \eta < \infty.$

If we write the basis functions as $\Psi = \cosh\nu\ \Phi\ e^{i\tau\eta}$ then Φ satisfies

$$\frac{d^2\Phi}{d\nu^2} + \tanh\nu\frac{d\Phi}{d\nu} - \left[\tau^2\cosh^2\nu + \frac{s^2}{\cosh^2\nu} + \lambda\right]\Phi = 0, \tag{3.29}$$

where λ is the eigenvalue of L_2. The spectrum of λ is discrete and the corresponding eigenvalues are denoted by $\lambda_{\nu+2r}^{is}$, $r = 0, \pm 1, \cdots$. (This is the notation adopted by Meixner and Schäfke.[20]) We note that (3.29) is a form of the spheroidal equation. The basis functions are then

$$f_{\nu+2r,\tau}^{(23)} = \cosh\nu\ S_{\nu+2r}^{is(1)}(i\sinh\nu, \tau)\, e^{i\tau\eta},$$

where $-\infty < \tau < \infty$, $r = 0, \pm 1, \pm 2$, and $\nu \neq \frac{1}{2}$ (mod 1). The eigenvalues of L_1 and L_2 are $-\tau^2$ and $\lambda_{\nu+2r}^{is}$.

24. $Y = (1/\sinh\nu + \frac{1}{2}[\sinh\nu + \eta^2/\sinh\nu],\ \ \eta/\sinh\nu, \coth\nu,\ \ \frac{1}{2}[\sinh\nu - \eta^2/\sinh\nu]),$

$-\infty < \nu < \infty, \quad -\infty < \eta < \infty.$

The basis functions have the form

$$f_{\nu+2r,\tau}^{(24)} = \sinh\nu\ S_{\nu+2r}^{is(1)}(\cosh\nu, \tau)\, e^{i\tau\eta}, \tag{3.30}$$

where $-\infty < \tau < \infty$, $r = 0, \pm 1, \pm 2, \cdots$, and $\nu \neq \frac{1}{2}$(mod 1), and the eigenvalues of L_1 and L_2 are $-\tau^2$ and $\lambda_{\nu+2r}^{is}$, respectively.

25. $Y = (\frac{1}{2}\cosh\nu,\ \ \tanh\nu\cos\eta,\ \ \tanh\nu\sin\eta,\ \ 1/\cosh\nu - \frac{1}{2}\cosh\nu),$

$-\infty < \nu < \infty, \quad 0 \leqslant \eta < 2\pi.$

The basis functions are

$$f_{\kappa m}^{(25)} = (2\pi^3)^{1/2}\,\Gamma\left(\frac{m + 1 + i\kappa + is}{2}\right)\Gamma\left(\frac{m + 1 + i\kappa - is}{2}\right) \times [\Gamma(m + 1)\,\Gamma(1 + i\kappa)]^{-1}(\tanh\nu)^m(\cosh\nu)^{-i\kappa} \times {}_2F_1\left(\frac{m + 1 + i\kappa + is}{2}, \frac{m + 1 + i\kappa - is}{2}, m + 1;\ \tanh^2\nu\right)\begin{bmatrix}\cos m\eta\\ \sin m\eta\end{bmatrix}, \tag{3.31}$$

where $m = 0, 1, \cdots$ and $0 < \kappa < \infty$. The eigenvalues of the operators L_1 and L_2 are $-m^2$ and $-\kappa^2$, respectively,

$$\langle f_{\kappa m}^{(25)}, f_{\kappa' m'}^{(25)}\rangle = \delta(\kappa - \kappa')\,\delta_{mm'}.$$

26. $Y = (1/\sinh\nu + \frac{1}{2}\sinh\nu,\ \coth\nu\cos\eta,\ \coth\nu\sin\eta,\ \frac{1}{2}\sinh\nu),$

$-\infty < \nu < \infty, \qquad 0 \leqslant \eta < 2\pi.$

The basis functions are

$$f_{\kappa m}^{(26)} = \sqrt{2\pi}\,\pi^2(\tanh\nu)^{1+i(\kappa+s)}(\cosh\nu)^{-i\kappa} \times \frac{\Gamma(\frac{1}{2}[m + 1 + i(s + \kappa)])}{\Gamma(\frac{1}{2}[m + 1 - i(s + \kappa)])} \times \frac{\Gamma(\frac{1}{2}[m + 1 + i(\kappa - s)])\,\Gamma(-i\kappa)}{\Gamma(\frac{1}{2}[m + 1 + i(s - \kappa)])\,\Gamma(1 + i\kappa)} \times {}_2F_1\left[\frac{-m + 1 + i(\kappa + s)}{2},\right.$$

$$\frac{m-1+i(\kappa+s)}{2};\quad 1+i\kappa;$$

$$\left.\frac{1}{\cosh^2\nu}\right)\begin{bmatrix}\cos m\eta\\ \sin m\eta\end{bmatrix},$$

where $m = 0, 1, 2 \cdots$ and $0 < \kappa < \infty$. The eigenvalues of the operators L_1 and L_2 are $-m^2$ and $-\kappa^2$ respectively where $m = 0, 1, \cdots$ and $0 < \kappa < \infty$. The normalization is the same as for system 25.

27. $$Y = \left(\frac{\nu^4+4}{8\nu} + \frac{\eta^2}{2\nu}, \frac{\eta}{\nu}, \frac{1}{2}\nu, \frac{-\nu^4+4}{8\nu} - \frac{\eta^2}{2\nu}\right),$$

$$-\infty < \nu < \infty, \qquad -\infty < \eta < \infty.$$

28. From this point on the spectral problems that have to be solved involve more than one eigenvalue simultaneously. We therefore give only the coordinates on the cone in these cases,

$$t^2 = \frac{\nu\eta}{ab}, \qquad x^2 = \frac{(\nu-1)(\eta-1)}{(a-1)(b-1)},$$

$$y^2 = \frac{(\nu-b)(\eta-b)}{(a-b)(b-1)b}, \qquad z^2 = \frac{(\nu-a)(\eta-a)}{(a-b)(a-1)a},$$

$$1 < \eta < b < \nu < a.$$

29. $$t^2 = -\frac{(\nu-1)(\eta-1)}{(a-1)(b-1)}, \qquad x^2 = -\frac{\nu\eta}{ab},$$

$$y^2 = -\frac{(\nu-b)(\eta-b)}{(a-b)(b-1)b},$$

$$z^2 = \frac{(\nu-a)(\eta-a)}{(a-b)(a-1)a}, \qquad \eta < 0 < 1 < b < \nu < a.$$

30. $$(x+it)^2 = \frac{2(\nu-a)(\eta-a)}{(a-b)(b-1)b},$$

$$y^2 = \frac{(\nu-1)(\eta-1)}{(a-1)(b-1)}, \qquad z^2 = -\frac{\nu\eta}{ab},$$

$$\eta < 0 < \nu < 1.$$

31. $$(t+x)^2 = \frac{\nu\eta}{a}, \qquad (t^2-x^2) = \frac{\nu+\eta}{a} - \frac{(a+1)\nu\eta}{a^2},$$

$$y^2 = -\frac{(\nu-1)(\eta-1)}{(a-1)},$$

$$z^2 = \frac{(\nu-a)(\eta-a)}{a^2(a-1)}, \qquad 0 < \eta < 1 < \nu < a.$$

32. $$(t+x)^2 = -\frac{\nu\eta}{a}, \quad (t^2-x^2) = \frac{\nu+\eta}{a} - \frac{(a+1)\nu\eta}{a^2},$$

$$y^2 = -\frac{(\nu-1)(\eta-1)}{(a-1)},$$

$$z^2 = \frac{(\nu-a)(\eta-a)}{a^2(a-1)}, \qquad -\eta<0<1<\nu<a.$$

33. $$(t+x)^2 = -\frac{\nu\eta}{a},$$

$$(t^2-x^2) = -\frac{(\nu+\eta)}{a} + \frac{(a-1)\nu\eta}{a^2},$$

$$y^2 = \frac{(\nu-a)(\eta-a)}{a^2(a+1)}, \quad z^2 = -\frac{(\nu+1)(\eta+1)}{(a+1)},$$

$$\eta < -1 < 0 < \nu < a.$$

34. $$(t-x)^2 = -\nu\eta, \quad 2y(x-t) = \nu+\eta-\nu\eta,$$

$$z^2 = (\nu-1)(\eta-1), \qquad \eta < 0 < \nu < 1.$$

4. SEPARABLE BASIS FUNCTIONS ON $L^2(H_+)$

In this section we present the basis functions for $L^2(H_+)$ corresponding to the coordinate systems presented in Sec. 2. This is done using (3.5) and the spectral resolution of the operators L_1 and L_2 computed in the previous section. The basis functions are listed with the minimum of duplication necessary. Some of these basis functions which correspond to subgroup reductions were given by Vilenkin and Smorodinski[16] (see also Kalnins[17]). In each case the integral is relatively easy to evaluate because we know in advance the form of the separated solutions in the appropriate coordinate system such that variables separate in the integral. The 34 integral identities (3.5) corresponding to the separable coordinates for $\Delta\psi = \sigma(\sigma+2)\psi$ are nontrivial. Indeed many of the following results appear to be new.

1. $$F^{(1)}_{\tau m} = \frac{\pi\Gamma([|m|+1+i(s+\tau)]/2)}{\Gamma(|m|+1)}$$

$$\times \frac{\Gamma([|m|+1+i(s-\tau)]/2)}{\Gamma(1+is)}$$

$$\times e^{i\tau\nu}e^{im\eta}(\tanh\rho)^{|m|}(\cosh\rho)^{-1-is}$$

$$\times {}_2F_1\left(\frac{|m|+1+i(s+\tau)}{2}, \frac{|m|+1+i(s-\tau)}{2}, \quad |m|+1; \tanh^2\rho\right).$$

2. $$F^{(2)}_{\tau\kappa} = \frac{\sqrt{s\sinh\pi s}}{2\pi} e^{-\rho} K_{is}((\tau^2+\kappa^2)^{1/2} e^{-\rho}) e^{i\tau\nu}e^{i\kappa\eta}.$$

3. $$F^{(3)qq'}_{em} = \frac{\Gamma(is)}{\Gamma(is-l)\sqrt{\sinh\rho}}$$

$$\times P^{-l-1/2}_{-1/2+is}(\cosh\rho)\, E^{qq'}_{lm}(\nu)\, E^{qq'}_{lm}(\eta).$$

4. $$F^{(4)qq'}_{\varepsilon jm} = C_\varepsilon H^s_q(\rho)\, E^{qq'}_{jm}(\nu)\, E^{qq'}_{jm}(\eta),$$

where

$$H^s_q(\rho) = \frac{\Gamma(\frac{1}{2}+i(q+s))\,\Gamma(\frac{1}{2}+i(s-q))}{\Gamma(1+ip)\cosh\rho}$$

$$\times P^{-is}_{-1/2+iq}(\varepsilon\tanh\rho).$$

5. $$F^{(5)\varepsilon\varepsilon'}_{\pm jm} = C_{\pm\varepsilon\varepsilon'} H^s_q(\rho)\, F^m_j(\nu, k)\, F^m_j(\eta, k).$$

6. $$F^{(10)}_{lm} = \left(\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}\right)^{1/2} \frac{\Gamma(is)}{\Gamma(is-l)\sqrt{\sinh\rho}}$$

$$\times P^{-l-1/2}_{1/2+is}(\cosh\rho)\, P^m_l(\cos\nu)\, e^{im\eta}.$$

7. $$F^{(11)}_{\varepsilon im} = C_\varepsilon H^s_q(\rho) \frac{\Gamma(\frac{1}{2}-m+iq)}{2\pi\Gamma(\frac{1}{2}+iq)} P^m_{-1/2+iq}(\cosh\nu)\, e^{im\eta}.$$

8. $$F^{(12)}_{\varepsilon\varepsilon' jm} = C_{\varepsilon\varepsilon'} H^s_q(\rho) \frac{\Gamma(\frac{1}{2}+i(\tau+q))\,\Gamma(\frac{1}{2}+i(\tau-q))}{\Gamma(\frac{1}{2}+iq)}$$

$$\times P^{-iq}_{-1/2+i\tau}(\varepsilon'\tanh\nu)\, e^{i\tau\eta}.$$

9. $$F^{(13)}_{\varepsilon j\tau} = C_\varepsilon H^s_q(\rho) \frac{e^{-\nu/2}}{\Gamma(\frac{1}{2}+iq)} K_{iq}(\tau e^{-\nu})\, e^{i\tau\eta}.$$

10. $$F^{(14)}_{\chi m} = \sqrt{s\chi\sinh\pi s}\, e^{-\rho}K_{is}(\chi e^{-\rho})\, J_m(\chi\nu)\, e^{im\eta}.$$

11. $$F^{(15)}_{\chi n} = \sqrt{s\sinh\pi s}\, e^{-\rho}K_{is}(e^{-\rho})$$

$$\times \begin{cases} Ce_n(\nu, \chi^2/4)\, ce_n(\eta, \chi^2/4) \\ Se_n(\nu, \chi^2/4)\, se_n(\eta, \chi^2/4) \end{cases}.$$

12. $F^{(16)}_{s\chi\lambda} = C_s \dfrac{\sqrt{s}\,\sinh(\pi s/2)}{\cosh \pi\lambda} e^{-\rho} K_{is}(\chi e^{-\rho})$
$$\times [D_{-i\lambda-1/2}(\varepsilon\sigma\eta)\, D_{i\lambda-1/2}(\sigma\nu) + D_{-i\lambda-1/2}(-\varepsilon\sigma\eta)\, D_{i\lambda-1/2}(-\sigma\nu)].$$

13. $F^{(17)ss'}_{mn} = L^{ss'}_{smn}(\rho, k)\, L^{ss'}_{smn}(\nu, k) \begin{cases}\cos m\eta \\ \sin m\eta\end{cases}.$

14. $F^{(18)sP}_{mn} = [\mathrm{dn}(\nu, k)\, \mathrm{dn}(\rho, k)]^m\, K^{Ps}_{snm}(\mathrm{dn}(\nu, k))$
$$\times K^{Ps}_{snm}(\mathrm{dn}(\rho, k)) \begin{cases}\cos m\eta \\ \sin m\eta\end{cases}.$$

15. $F^{(21)}_{mn} = W^n_{sm}(\rho, k)\, W^n_{sm}(\nu, k)\, e^{im\eta}.$

16. $F^{(23)}_{\nu+2r,\tau} = T_{\nu,\tau} \cosh\nu \cos\rho\, S^{is(1)}_{\nu+2r}(i \sinh\nu, \tau)$
$$\times Ps^{is}_\nu(\sin\rho, \tau^2)\, e^{i\tau\eta},$$

where $T_{\nu,\tau}$ is a normalization constant.

17. $F^{(24)}_{\nu+2r,\tau} = t_{\nu,\tau} \sinh\nu \sin\rho\, S^{is(1)}_{\nu+2r}(\cosh\nu, \tau)$
$$\times Ps^{is}_\nu(\cos\rho, \tau^2)\, e^{i\tau\eta},$$

where $t_{\nu,\tau}$ is a normalization constant.

18. $F^{(25)}_{\kappa m} = M_{\kappa m}(\tan\rho \tanh\nu)^m (\cosh\nu)^{-i\kappa} (\cos\rho)^{1+m+is}.$
$$\times {}_2F_1\Big(\frac{m+1+i(\kappa+s)}{2}, \frac{m+1+i(\kappa-s)}{2}, m+1; \tanh^2 \nu\Big)\, {}_2F_1\Big(\frac{m+1+i(\kappa+s)}{2}, \frac{m+1+i(s-\kappa)}{2}, m+1; -\sin^2\rho\Big) \begin{cases}\cos m\eta \\ \sin m\eta\end{cases},$$

$$M_{\kappa m} = \pi^2 2^{(3m+i(s-\kappa)+2)/2} \times \frac{\Gamma(a)\Gamma(b)\Gamma(a+m-\frac12)\Gamma(1+is+m)}{\Gamma(1+i\kappa)\Gamma(\frac12 a+m+\frac14)\Gamma(\frac12 a+m+\frac34)\Gamma(1+is)} \times {}_3F_2(a, b; a+m-\tfrac12, \tfrac12 a+m+\tfrac14, \tfrac12 a+m+\tfrac34, \tfrac14)$$

where

$$a = [m+1+i(\kappa+s)]/2,$$
$$b = [m+1+i(\kappa-s)]/2.$$

19. $F^{(26)}_{\kappa m} = M'_{\kappa m}(\tanh\nu)^{1+i(\kappa+s)}(\sinh\nu)^{-i\kappa}(\cot\rho)^m$
$$\times (\sin\rho)^{1+m+is}\, {}_2F_1\Big(\frac{-m+1+i(\kappa+s)}{2}, \frac{m+1+i(\kappa+s)}{2}; 1+is; \tanh^2\nu\Big) \times {}_2F_1\Big(\frac{m+1+i(\kappa+s)}{2}, \frac{m+1+i(s-\kappa)}{2}; m+1; -\cos^2\rho\Big) \begin{cases}\cos m\eta \\ \sin m\eta\end{cases},$$

where

$$M'_{\kappa m} = 2\pi^2 N \frac{\Gamma(1+is+i\kappa/2)\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}{\Gamma(b_1)\Gamma(b_2)} \times {}_3F_2\Big(\begin{matrix}a_1, a_2, a_3 \\ b_1, b_2\end{matrix}\Big| -1\Big),$$

where

$$a_1 = \frac{-m+1+i(\kappa+s)}{2},\quad a_2 = \frac{m-1+i(s+\kappa)}{2},$$
$$a_3 = \frac{m+is}{2} + 1 + i(\kappa+s),$$
$$b_1 = 1 + i\kappa,$$
$$b_2 = \frac{m+i(s+\kappa)}{2} + 2 + 2is + i\kappa,$$

and

$$N = \frac{\Gamma([m+1+i(s+\kappa)]/2)\,\Gamma([m+1+i(\kappa-s)]/2)}{\Gamma([m+1-i(s+\kappa)]/2)\,\Gamma([m+1+i(s-\kappa)]/2)} \times \frac{\Gamma(-i\kappa)\,\Gamma(1+is+m)}{\Gamma(1+i\kappa)\,\Gamma(1+is)}.$$

Let $\{f_{\lambda\mu}\}$ be an ON basis for $L^2_s(C)$ consisting of (generalized) eigenfunctions corresponding to a commuting pair L_1, L_2,

$$L_1 f_{\lambda\mu} = \lambda f_{\lambda\mu}, \qquad L_2 f_{\lambda\mu} = \mu f_{\lambda\mu},$$

and let $\{F_{\lambda\mu}\}$ be the associated separable functions on H_+,

$$F_{\lambda\mu}(X) = \langle f_{\lambda\mu}, h(X, \cdot)\rangle. \tag{4.1}$$

It follows that

$$h(X, Y) = [X, Y]^{is-1} = \sum_{\lambda,\mu} f_{\lambda,\mu}(Y)\, \bar{F}_{\lambda,\mu}(X) \tag{4.2}$$

with convergence in $L^2_s(H_+)$ for each $X \in H_+$. A direct computation yields

$$\langle h(X_1, \cdot), h(X_2, \cdot)\rangle = 4\pi\, {}_2F_1(1-is, 1+is; \tfrac32; \tfrac12 - \tfrac12[X_1, X_2]) \tag{4.3}$$

and, from (4.2),

$$\langle h(X_1, \cdot), h(X_2, \cdot)\rangle = \sum_{\lambda,\mu} F_{\lambda,\mu}(X_2)\, \bar{F}_{\lambda,\mu}(X_1). \tag{4.4}$$

Thus, (4.3) is a bilinear generating function for products of separated solutions $F_{\lambda,\mu}$.

If $\{f'_{\alpha,\beta}\}$ is ON basis for $L^2_s(C)$ consisting of eigenfunctions of another commuting pair L'_1, L'_2 and $F'_{\alpha,\beta} = \langle f'_{\alpha,\beta}, h\rangle$ we have the pointwise convergent expansion

$$F'_{\alpha,\beta}(X) = \sum_{\lambda,\mu} C^{\alpha,\beta}_{\lambda,\mu} F_{\lambda,\mu}(X), \tag{4.5}$$

where the sum or integral is taken over the spectrum of L_1, L_2.[1] Furthermore the expansion coefficients can be computed in $L^2_s(C)$. Indeed

$$C^{\alpha,\beta}_{\lambda,\mu} = \langle f'_{\alpha,\beta}, f_{\lambda,\mu}\rangle, \tag{4.6}$$

so all overlaps can be expressed as integrals over a contour Γ on C.

A number of these coefficients can be found in the literature. In particular, systems 3 and 10 correspond to the subgroup reduction SO(3, 1) ⊃ SO(3) and the overlaps relating these systems can be found in Ref. 12. Systems 4–9 and 11–13 correspond to the subgroup reduction SO(3, 1) ⊃ SO(2, 1) and appropriate overlaps are computed in Ref. 13. Systems 2 and 14–16 correspond to the subgroup reduction SO(3, 1) ⊃ E(2) and overlaps are contained in Ref. 18. The overlaps relating systems 1 and 3 can be expressed in terms of Clebsch–Gordan coefficients for SO(2, 1).

5. EXPANSIONS ON $L^2(H_s)$

In this section we give the expansions on $L^2(H_s)$ for coordinate systems on the single sheeted hyperboloid H_s which cover one half of H_s. This is the imaginary Lobachevski space of Gelfand *et al.*[7] Only some of the coordinates given in Sec. 2 correspond in a natural way to such coordinates on H_s. The spectrum of Δ is both continuous and discrete for $L^2(H_s)$ and there are therefore two sets of basis functions for each of the coordinate systems we discuss. We now list the basis functions for the coordinate systems on $L^2(H_s)$ together with the coordinates. The orthonormalization is always Kronecker delta for discrete spectrum and Dirac delta for continuous spectrum. The discrete spectrum basis functions are obtained from (1.9) and (1.10) exactly as in the example worked out in Ref. 19.

1. $t = \frac{1}{2}[e^\rho - (1 + \nu^2 + \eta^2) e^{-\rho}], \quad x = e^{-\rho}\nu,$
$y = e^{-\rho}\eta, \quad z = \frac{1}{2}[e^\rho + (1 - \nu^2 - \eta^2) e^{-\rho}],$
$-\infty < \rho, \nu, \eta < \infty.$

The basis functions are

$$Fc_{\tau\kappa}^{(2)} = \frac{e^{-\rho}}{2\pi}\left(\frac{s}{2\sinh\pi s}\right)^{1/2}[J_{is}(\chi e^{-\rho}) + J_{-is}(\chi e^{-\rho})]\, e^{i\tau\nu}e^{i\kappa\eta},$$

where $\chi^2 = \tau^2 + \kappa^2$,

$$Fd_{\tau\kappa}^{(2)} = \frac{\sqrt{n}}{\pi}\, e^{-\rho}\, J_{2n}(\chi e^{-\rho})\, e^{i\tau\nu}e^{i\kappa\eta}.$$

The superscript refers to the system in Sec. 2 to which the coordinates correspond via analytic continuation.

2. $t = \sinh\rho, \quad x = (1/k')\cosh\rho\, \mathrm{dn}(\nu, k)\, \mathrm{dn}(\eta, k),$
$y = (ik/k')\cosh\rho\, \mathrm{cn}(\nu, k)\, \mathrm{cn}(\eta, k),$
$z = k\cosh\rho\, \mathrm{sn}(\nu, k)\, \mathrm{sn}(\eta, k),$
$-\infty < \rho < \infty, \quad \nu \in [-2K, 2K],$
$\eta \in [-K, -K + 2iK'].$

The basis functions are

$$Fc_{ln}^{(3)\epsilon\epsilon'} = \frac{4\pi s\Gamma(-is)}{\cosh\rho}\left[P_l^{is}(\tanh\rho) - \frac{\Gamma(l+1+is)}{\Gamma(l+1-is)}P_l^{-is}(\tanh\rho)\right]f_{ln}^{(3)\epsilon\epsilon'},$$

$$Fd_{ln}^{(3)\epsilon\epsilon'} = 2\left(n\frac{(2n-l)!}{(2n+l)!}\right)^{1/2}\frac{1}{\cosh\rho}P_l^{2n}(\tanh\rho) f_{ln}^{(3)\epsilon\epsilon'},$$

where for the discrete spectrum part $l = 0, 2, \ldots, 2n$.

3. $t = \sinh\rho, \quad x = \cosh\rho\, \sin\nu\, \cos\eta,$
$y = \cosh\rho\, \sin\nu\, \sin\eta, \quad z = \cosh\rho\, \cos\nu,$
$-\infty < \rho < \infty, \quad 0 \leqslant \nu \leqslant \pi, \quad 0 \leqslant \eta < 2\pi.$

The basis functions are

$$Fc_{lm}^{(10)} = \frac{4\pi s\Gamma(is)}{\cosh\rho}\left[P_l^{is}(\tanh\rho) - \frac{\Gamma(l+1+is)}{\Gamma(l+1-is)}P_l^{-is}(\tanh\rho)\right]f_{lm}^{(10)},$$

$$Fd_{lm}^{(10)} = 2\left(n\frac{(2n-l)!}{(2n+l)!}\right)^{1/2}\frac{1}{\cosh\rho}P_l^{2n}(\tanh\rho) f_{lm}^{(10)}.$$

4. $t = \frac{1}{2}[(e^\rho - (1 + \nu^2) e^{-\rho}], \quad x = e^{-\rho}\nu\cos\eta,$
$y = e^{-\rho}\nu\sin\eta, \quad z = \frac{1}{2}[e^\rho + (1 - \nu^2) e^{-\rho}],$
$-\infty < \rho < \infty, \quad 0 < \nu < \infty, \quad 0 \leqslant \eta < 2\pi.$

The basis functions are

$$Fc_{\chi m}^{(14)} = \left(\frac{s}{2\sinh\pi s}\right)^{1/2} e^{-\rho}[J_{is}(\chi e^{-\rho}) + J_{-is}(\chi e^{-\rho})] f_{\chi m}^{(14)},$$

$$Fd_{\chi m}^{(14)} = 2\sqrt{n}\, e^{-\rho}\, J_{2n}(\chi e^{-\rho}) f_{\chi m}^{(14)}.$$

5. $t = \frac{1}{2}[e^\rho - (1 + \cosh^2\nu - \sin^2\eta) e^{-\rho}],$
$x = e^{-\rho}\cosh\nu\cos\eta, \quad y = e^{-\rho}\sinh\nu\sin\eta,$
$z = \frac{1}{2}[e^\rho + (1 - \cosh^2\nu + \sin^2\eta) e^{-\rho}],$
$-\infty < \rho < \infty, \quad -\infty < \nu < \infty, \quad 0 \leqslant \eta < 2\pi.$

The basis functions are

$$Fc_{\chi n}^{(15)} = \left(\frac{s}{2\sinh\pi s}\right)^{1/2} e^{-\rho}[J_{is}(\chi e^{-\rho}) + J_{-is}(\chi e^{-\rho})] f_{\chi n}^{(15)},$$

$$Fd_{\chi n}^{(15)} = 2\sqrt{n}\, e^{-\rho}\, J_{2n}(\chi e^{-\rho}) f_{\chi n}^{(15)}.$$

6. $t = \frac{1}{2}[e^\rho - (1 + \frac{1}{4}(\eta^2 + \nu^2)^2) e^{-\rho}],$
$x = \frac{1}{2}e^{-\rho}(\eta^2 - \nu^2), \quad y = e^{-\rho}\eta\nu,$
$z = \frac{1}{2}[e^\rho + (1 - \frac{1}{4}(\eta^2 + \nu^2)^2) e^{-\rho}],$
$-\infty < \rho < \infty, \quad -\infty < \nu < \infty, \quad -\infty < \eta < \infty,$

$$Fc_{\chi\lambda}^{(16)\epsilon} = \left(\frac{s}{2\sinh\pi s}\right)^{1/2} e^{-\rho}[J_{is}(\chi e^{-\rho}) + J_{-is}(\chi e^{-\rho})] f_{\chi\lambda}^{(16)\epsilon},$$

$$Fd_{\chi\lambda}^{(16)\epsilon} = 2\sqrt{n}\, e^{-\rho}\, J_{2n}(\chi e^{-\rho}) f_{\chi\lambda}^{(16)\epsilon}.$$

7. The coordinates are as for coordinate system 17 with
$\rho \in [0, 2iK'], \quad \nu \in [iK', iK' + 2K], \quad 0 \leqslant \eta < 2\pi.$

The basis functions are

$$Fu_{mn}^{(17)} = L_{tmn}^{\epsilon\epsilon'}(\rho, k)\, L_{tmn}^{\epsilon\epsilon'}(\nu, k)\begin{bmatrix}\cos m\eta\\ \sin m\eta\end{bmatrix},$$

where $t = s$ for the continuous spectrum basis functions and $t = 2in$ for the discrete spectrum functions. These solutions are obtained by solving the recurrence relations for system (17) with s replaced by $2in$.

8. The coordinates are as in system 18 with
$\rho \in [0, 2iK'], \quad \nu \in [iK', iK' + 2K], \quad 0 \leqslant \eta < 2\pi.$

The basis functions are

$$Fu_{mn}^{(18)} = [\mathrm{dn}(\nu, k)\, \mathrm{dn}(\rho, k)]^m\, K_{tnm}^{Pe}(\mathrm{dn}(\nu, k)) \times K_{tnm}^{Pe}(\mathrm{dn}(\rho, k))\begin{bmatrix}\cos m\eta\\ \sin m\eta\end{bmatrix},$$

with t as in system 7.

9. The coordinates are as in system 19 with
$\rho \in [0, 2iK'], \quad \nu \in [iK', iK' + 2K],$
$0 \leqslant \eta < 2\pi.$

10. The coordinates are as in system 21 with
$\rho \in [0, 2iK'], \quad \nu \in [0, 2iK'], \quad 0 \leqslant \eta < 2\pi.$

The basis functions are

$$Fu^{(21)}_{nm} = W^n_{tm}(\rho, k)\, W^n_{tm}(\nu, k)\, e^{im\eta},$$

where t has the same significance as in system 7 of this section.

11. The coordinates are as in system 22 with

$$0<\rho<\infty, \qquad 0<\nu<\infty.$$

12. $t = \frac{1}{2}\left[\frac{\cosh\nu}{\sinh\rho} - \frac{\sinh\rho}{\cosh\nu}\right] + \frac{\eta^2}{2\cosh\nu\,\sinh\rho},$

$$x = \frac{\eta}{\cosh\nu\,\sinh\rho}, \qquad y = \tanh\nu\,\coth\rho,$$

$$z = \frac{1}{\cosh\nu\,\sinh\rho} + \frac{1}{2}\left[\frac{\sinh\rho}{\cosh\nu} - \frac{\cosh\nu}{\sinh\rho}\right] - \frac{\eta^2}{2\cosh\nu\,\sinh\rho},$$

$$-\infty<\rho<\infty, \quad -\infty<\nu<\infty, \quad -\infty<\eta<\infty.$$

The basis functions are

$$Fs^{(23)}_{r\tau} = K^t_{r\tau}\sinh\rho\,\cosh\nu\, S^{it(1)}_{\nu+2r}(\cosh\rho, \tau) \times S^{it(1)}_{\nu+2r}(\sinh\nu, \tau)\, e^{i\tau\eta},$$

with t as in system 7 and $K^t_{r\tau}$ a normalization constant.

13. $t = \frac{1}{\sin\nu\,\sin\rho} + \frac{1}{2}\left[\frac{\sin\nu}{\sin\rho} + \frac{\sin\rho}{\sin\nu}\right] + \frac{\eta^2}{2\sin\rho\,\sin\nu},$

$$x = \frac{\eta}{\sin\nu\ \sin\rho}, \qquad y = \cot\rho\,\cot\nu,$$

$$z = \frac{1}{2}\frac{\sin\nu}{\sin\rho} + \frac{\sin\rho}{\sin\nu} - \frac{\eta^2}{2\sin\nu\,\sin\rho},$$

$$Fs^{(24)}_{r\tau} = \kappa^t_{r\tau}\sin\nu\,\sin\rho\, Ps^{it}_\nu(\cos\nu, \tau^2) \times Ps^{it}_\nu(\cos\rho, \tau^2)\, e^{i\tau\eta},$$

with t as in system 7 and $\kappa^t_{r\tau}$ a normalization constant.

14. $t = \frac{1}{2}\left[\frac{\cosh\nu}{\sinh\rho} - \frac{\sinh\rho}{\cosh\nu}\right], \quad x = \coth\rho\,\tanh\nu\,\cos\eta,$

$$y = \coth\rho\,\tanh\nu\,\sin\eta,$$

$$z = \frac{1}{\cosh\nu\,\sinh\rho} + \frac{1}{2}\left[\frac{\sinh\rho}{\cosh\nu} - \frac{\cosh\nu}{\sinh\rho}\right],$$

$$-\infty<\rho<\infty, \qquad -\infty<\nu<\infty, \qquad 0\leqslant\eta<2\pi,$$

$$Fc^{(25)}_{\kappa m} = N_{\kappa m}(\tanh\nu)^m(\tanh\rho)^{1+i(\kappa+s)}(\cosh\nu\,\sinh\rho)^{-i\kappa} \times {}_2F_1\left(\frac{m+1+i(\kappa+s)}{2}, \frac{m+1+i(\kappa-s)}{2}; m+1; \tanh^2\nu\right) {}_2F_1\left(\frac{-m+1+i(\kappa+s)}{2}, \frac{m+1+i(\kappa+s)}{2}, 1+is; \tanh^2\rho\right)\begin{Bmatrix}\cos m\eta\\ \sin m\eta\end{Bmatrix},$$

where

$$N_{\kappa m} = 4\pi^2\, 2^{[1-m+i(s+\kappa)]/2}\frac{\Gamma(1+is+m)}{\Gamma(1+is)\,\Gamma(1+i\kappa)} \times \sum_{r=0}^{\infty}\frac{(a_1)_r(a_2)_r(a_3)_r(a_4)_r}{(b_1)_r(b_2)_r(b_3)_r}\frac{1}{r!4^r},$$

$$a_1 = \tfrac{1}{2}[m+1+i(\kappa+s)],$$

$$a_2 = \tfrac{1}{2}[m+1+i(\kappa-s), \qquad a_3 = -is,$$

$$a_4 = \tfrac{1}{2}[m-1+i(s-\kappa)], \qquad b_1 = m+1,$$

$$b_2 = \tfrac{1}{4}[m-1-i(s+\kappa)],$$

and

$$b_3 = \tfrac{1}{4}[m+1-i(s+\kappa)],$$

$$Fd^{(25)}_{lm} = (\tanh\nu\,\coth\rho)^{-1/2} d^{l+}_{|m-2n|,m+2n}(\cosh\nu) \times d^{l+}_{|m-2n|,m+2n}(i\sinh\rho)\begin{Bmatrix}\cos m\eta\\ \sin m\eta\end{Bmatrix}.$$

Here $l,m = 0, 1, 2, \cdots$ and d^{l+}_{pq} (cosh) is the matrix element of a hyperbolic rotation in the compact basis of the positive discrete series of $SL(2, \mathbf{R})$ as given by Bargmann.[11] Explicitly these functions are

$$\begin{aligned} d^{l+}_{pq}(\cosh z) &= \bar N_{pq}(\sinh z)^{-(p+q)}(\cosh z)^{p-q} \times {}_2F_1(l+1-q, -l-q; 1+q-p; -\sinh^2 z), q>p \\ &= \bar N_{pq}(\sinh z)^{-(p+q)}(\cosh z)^{q-p} \times {}_2F_1(l+1-p, -l-p; 1+p-q; -\sinh^2 z), \\ & p>q, \end{aligned}$$

where

$$\bar N_{pq} = (-1)^{p-q}\bar N_{qp} = \frac{1}{(p-q)!} \times \left(\frac{\Gamma(-l-q)\,\Gamma(l+1-q)}{\Gamma(-l-p)\,\Gamma(l+1-p)}\right)^{+1/2}$$

if $p \geqslant q$. (Note: We have at the time of writing not computed the normalization constant for discrete spectrum basis functions.)

15. $t = \frac{1}{\sin\nu\,\cos\rho} - \frac{1}{2}\left[\frac{\sin\nu}{\sin\rho} + \frac{\sin\rho}{\sin\nu}\right],$

$$x = \cot\rho\,\cot\nu\,\cos\eta, \qquad y = \cot\rho\,\cot\nu\,\sin\eta,$$

$$z = \frac{1}{2}\left[\frac{\sin\nu}{\sin\rho} + \frac{\sin\rho}{\sin\nu}\right],$$

$$Fc^{(26)}_{\kappa m} = N'_{\kappa m}(\cot\rho\,\cot\nu)^m(\sin\rho\,\sin\nu)^{1+m+is} \times {}_2F_1\left(\frac{m+1+i(\kappa+s)}{2}, \frac{m+1+i(s-\kappa)}{2}; m+1; -\cos^2\rho\right) {}_2F_1\left(\frac{m+1+i(\kappa+s)}{2}, \frac{m+1+i(s-\kappa)}{2}; m+1; -\cos^2\nu\right)\begin{Bmatrix}\cos m\eta\\ \sin m\eta\end{Bmatrix},$$

where

$$N'_{\kappa m} = 2\pi^2 N\frac{\Gamma(a_1)\,\Gamma(a_2)\,\Gamma(a_3)\,\Gamma(-m+i\kappa)}{\Gamma(b_1)\,\Gamma(b_2)} \times {}_3F_2\begin{pmatrix}a_1\ a_2\ a_3 & 1\\ b_1\ b_2 & \end{pmatrix}$$

with a_1, a_2, and b_1 as for system 12 on $L^2(H_s)$,

$$a_3 = i\kappa + \tfrac{1}{2}(is+m-1),$$

$$b_2 = 2i\kappa + \tfrac{1}{2}(is-3m-1),$$

and N is as given for system 26 on $L^2(H_+)$.

$$Fd^{(26)}_{lm} = N''_{lm}(\cos\nu\,\cos\rho)^m(\sin\nu\,\sin\rho)^{2n+1}$$

$$\times P_l^{(m,2n)}(\sin 2\nu)\, P_l^{(m,2n)}(\sin 2\rho)\begin{Bmatrix}\cos m\eta\\ \sin m\eta\end{Bmatrix},$$

where $P_l^{(\alpha,\beta)}(z)$ is a Jacobi polynomial and

$$N''_{lm} = \left(\frac{n\Gamma(m+l+1)\,\Gamma(2n+l+1)\,2^{m+2n+1}}{2\pi l!\,\Gamma(m+2n+l+2)}\right)^{1/2} \Big/ P_l^{(m,2n)}(0).$$

16. $$t = \frac{(\rho^2-\nu^2)^2+4}{8\rho\nu} + \frac{\eta^2}{2\rho\nu}, \qquad x = \frac{\eta}{\rho\nu},$$

$$y = \frac{1}{2}\left[\frac{\nu}{\rho} + \frac{\rho}{\nu}\right], \qquad z = \frac{-(\rho^2-\nu^2)^2+4}{8\rho\nu} - \frac{\eta^2}{2\rho\nu},$$

$$-\infty < \nu, \rho, \eta < \infty.$$

We have not given any mention of the coordinate systems which require the solution of multiparameter eigenvalue problems.

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