Lie theory and separation of variables. 11. The EPD equation

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We show that the Euler–Poisson–Darboux equation $\{\partial_{tt}-\partial_{rr}-[(2m+1)/r]\partial_r\}\Theta=0$ separates in exactly nine coordinate systems corresponding to nine orbits of symmetric second-order operators in the enveloping algebra of SL(2, R), the symmetry group of this equation. We employ techniques developed in earlier papers from this series and use the representation theory of SL(2, R) to derive special function identities relating the separated solutions. We also show that the complex EPD equation separates in exactly five coordinate systems corresponding to five orbits of symmetric second-order operators in the enveloping algebra of SL(2, $\mathbb{C}$).

INTRODUCTION

This paper is one of a series concerning the relationships between the symmetry group of a linear second order partial differential equation and the coordinate systems in which variables separate for that equation. The previous three papers[1] were devoted to separation of variables for the wave equation $(\partial_{tt}-\Delta_2)\psi(x)=0$. If we pass to polar coordinates,

$$x_1=r\cos\varphi,\quad x_2=r\sin\varphi,$$

and consider solutions of the form $\psi(x)=\exp(im\varphi)\Phi(t,r)$, the wave equation transforms to the Euler–Poisson–Darboux (EPD) equation

$$[\partial_{tt}-\partial_{rr}-(1/r)\,\partial_r+m^2/r^2]\,\Phi=0. \qquad (0.1)$$

Many authors write $\Phi(t,r)=r^m\Theta(t,r)$ and take the EPD equation in the form

$$(\partial_{tt}-\partial_{rr}-[(2m+1)/r]\,\partial_r)\,\Theta=0, \qquad (0.2)$$

but for our purposes (0.1) is more convenient. This equation also arises from the wave equations $(\partial_{tt}-\Delta_n)\,\psi(x)=0$, $n>2$, if one looks for spherically symmetric solutions. For $n=2$, m is usually taken to be an integer while, for $n>2$, m may be half-integral. In this paper we will treat these cases simultaneously by allowing m to be a nonnegative real number.

It follows from the results of Refs. 1 that (0.1) can be solved by separation of variables in exactly nine coordinate systems associated with nine orbits of second order operators in the enveloping algebra of SL(2, R). Here SL(2, R) is the local symmetry group of the EPD equation.

In this paper we undertake a detailed study of these coordinate systems and show how one can use the representation theory of SL(2, R) and its universal covering group to derive special function identities relating separable solutions corresponding to distinct coordinate systems.

In Sec. 1 we compute the symmetry algebra sl(2, R) of the EPD equation and show that we can introduce a Hilbert space structure on the solution space of the EPD equation such that this space transforms according to a unitary irreducible representation of the universal covering group of SL(2, R), taken from the discrete series. We also relate the space to two other models of this representation which are more useful for computational purposes.

In Secs. 2 and 3 we classify the nine possible coordinate systems such that variables separate in (0.1) and relate them to nine orbits of symmetric second order operators in the enveloping algebra of sl(2, R). We also compute the spectral resolutions of these operators. In Sec. 4 we use our earlier results to compute the separable solutions of (0.1), and in Sec. 5 we determine overlap functions relating various distinct bases.

Finally, in Sec. 6 we discuss the separation of variable problem for the complex EPD equation and show that this equation permits separation in exactly five coordinate systems corresponding to five orbits of symmetric second order operators in the enveloping algebra of sl(2, $\mathbb{C}$). We relate these results to a paper by Viswanathan,[2] which employs Weisner's method[3,4] to derive generating functions for Gegenbauer polynomials. For a slightly different approach to the complex EPD equation, see Ref. 5.

All special functions appearing in this paper are defined as in the Bateman Project.[6]

1. SYMMETRIES OF THE EPD EQUATION

The symmetry algebra of the EPD equation

$$[\partial_{tt}-\partial_{rr}-(1/r)\,\partial_r+m^2/r^2]\,\Phi(t,r)=0,$$
$$r\geq 0,\quad -\infty<t<\infty, \qquad (1.1)$$

is the set of all linear differential operators

$$L=a_1(t,r)\,\partial_t+a_2(t,r)\,\partial_r+b(t,r)$$

such that $L\Phi$ is a (local) solution of (1.1) whenever Φ is a (local) solution. By using standard techniques in Lie theory,[1,4] it is straightforward to show that this algebra is isomorphic to sl(2, R). Indeed, the operators A, B, C form a basis where

$$A=\tfrac{1}{2}[(1-t^2-r^2)\,\partial_t-2tr\partial_r-t],$$
$$B=-(\tfrac{1}{2}+t\partial_t+r\partial_r), \qquad (1.2)$$
$$C=\tfrac{1}{2}[(1+t^2+r^2)\,\partial_t+2tr\partial_r+t].$$

Note that $A+C=\partial_t$. Here, we are ignoring the trivial symmetry E of multiplication by the scalar one: $E\Phi=\Phi$. The commutation relations are

$$[A,B]=-C, \quad [A,C]=-B, \quad [B,C]=A. \qquad (1.3)$$

We can express (1.1) in terms of the Lie algebra generators with the result

$$(C^2-A^2-B^2)\,\Phi=(\tfrac{1}{4}-m^2)\,\Phi, \qquad (1.1')$$

where $C^2-A^2-B^2$ is the Casimir operator for $\mathrm{sl}(2,R)$.

By definition, $\mathrm{sl}(2,R)$ is the Lie algebra of 2×2 real traceless matrices. We choose the isomorphism between our symmetry algebra and $\mathrm{sl}(2,R)$ such that the operators A, B, C correspond to the matrices $\mathcal{A}, \mathcal{B}, \mathcal{C}$, respectively, where

$$\mathcal{A}=\begin{pmatrix}0 & \frac{1}{2}\\ \frac{1}{2} & 0\end{pmatrix}, \quad \mathcal{B}=\begin{pmatrix}\frac{1}{2} & 0\\ 0 & -\frac{1}{2}\end{pmatrix}, \quad \mathcal{C}=\begin{pmatrix}0 & \frac{1}{2}\\ -\frac{1}{2} & 0\end{pmatrix}. \qquad (1.4)$$

Then, using standard results from Lie theory,[4] we find that the operators (1.2) exponentiate to a local Lie representation of the group $\mathrm{SL}(2,R)$ by operators $T(G)$, where

$$T(G)\,\Phi(t,r)=[(\alpha+\gamma t)^2-\gamma^2r^2]^{-1/2}\times\Phi\left[\frac{(\delta t+\beta)(\alpha+\gamma t)-\gamma\delta r^2}{(\alpha+\gamma t)^2-\gamma^2r^2},\ \frac{r}{(\alpha+\gamma t)^2-\gamma^2r^2}\right],$$
$$G=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in \mathrm{SL}(2,R). \qquad (1.5)$$

Here, $\mathrm{SL}(2,R)$ is a local symmetry group of (1.1) in the sense that if Φ is a local solution of the EPD equation, then $T(G)\,\Phi$ is also a local solution.

Motivated by the connection between the EPD equation and the wave equation discussed in Refs. 1, we note that for any C^∞ function $f(k)$ with compact support in $(0,\infty)$ the corresponding function

$$\Phi(t,r)=\exp(-im\pi/2)\int_0^\infty \exp(itk)\,J_m(kr)f(k)\,dk=\cup[f] \qquad (1.6)$$

is a solution of (1.1). If we introduce the inner product

$$\langle f_1,f_2\rangle=\int_0^\infty f_1(k)\bar{f}_2(k)\,dk \qquad (1.7)$$

on the space of C^∞ functions, we find that, in terms of the corresponding solutions $\Phi_i(t,r)$ of (1.1), the inner product reads

$$(\Phi_1,\Phi_2)\equiv\langle f_1,f_2\rangle=i\int_0^\infty \Phi_1(r,t)\,\partial_t\bar{\Phi}_2(r,t)\,r\,dr$$
$$=-i\int_0^\infty \bar{\Phi}_2(r,t)\,\partial_t\Phi_1(r,t)\,r\,dr. \qquad (1.8)$$

Here, the last two integrals are actually independent of t.

It follows from this that if we complete our pre-Hilbert space of C^∞ functions f to form the Hilbert space $L_2(0,\infty)$, the space of corresponding functions $\Phi=\cup[f]$ defined formally by (1.6) will form a Hilbert space $\mathcal{H}$ of weak solutions of the EPD equation with inner product (1.8). The transformation $\cup$ determines a unitary equivalence between these Hilbert spaces.

The action of the symmetry algebra $\mathrm{sl}(2,R)$ on $\mathcal{H}$ is given formally by (1.2). Indeed, these expressions make sense when applied to the dense subspace of $\mathcal{H}$ consisting of those solutions Φ which arise from C^∞ functions f with compact support on $(0,\infty)$. Moreover, as we shall see, they define symmetric operators on this subspace. The operators $\cup^{-1}K\cup$ on $L_2(0,\infty)$ related to operators $K\in\mathrm{sl}(2,R)$ on $\mathcal{H}$ are easily determined using integration by parts:

$$A=\frac{i}{2}\,k\left(\frac{d^2}{dk^2}+\frac{1}{k}\,\frac{d}{dk}-\frac{m^2}{k^2}+1\right),$$
$$B=\frac{1}{2}+k\,\frac{d}{dk}, \qquad (1.9)$$
$$C=\frac{i}{2}\,k\left(-\frac{d^2}{dk^2}-\frac{1}{k}\,\frac{d}{dk}+\frac{m^2}{k^2}+1\right).$$

[We are using the same letter to designate corresponding operators on $\mathcal{H}$ and $L_2(0,\infty)$.] It is now straightforward to show that iA, iB, and iC are essentially self-adjoint on $L_2(0,\infty)$. Moreover, it is well known that these operators determine a unitary irreducible representation of the universal covering group of $\mathrm{SL}(2,R)$ from the discrete series.[7,8]

Indeed, it is easy to check that C has discrete spectrum $i\lambda=i(m+s+1/2)$, $s=0,1,2,\cdots$ with a corresponding ON basis for $L_2(0,\infty)$ consisting of eigenfunctions

$$f_s^{(1)}(k)=\left(\frac{2\Gamma(s+1)}{\Gamma(2m+s+1)}\right)^{1/2}(2k)^m\exp(-k)\,L_s^{(2m)}(2k),$$
$$Cf_s^{(1)}=i(m+s+\tfrac{1}{2})f_s^{(1)}, \quad \langle f_s^{(1)},f_{s'}^{(1)}\rangle=\delta_{ss'}. \qquad (1.10)$$

From this fact and the relation

$$C^2-A^2-B^2=\tfrac{1}{4}-m^2 \qquad (1.11)$$

we see that the operators (1.9) determine the irreducible representation $D^-_{m-1/2}$ from the negative discrete series.[4,7,8]

For $2m$ an integer this Lie algebra representation exponentiates to a single-valued unitary irreducible representation of $\mathrm{SL}(2,R)$ defined by unitary operators $T(G)$, where

$$T(G)f(k)=-\gamma^{-1}\exp[i(k\alpha/\gamma+\pi/2-\pi m)]\times\int_0^\infty \exp(il\delta/\gamma)\,J_{2m}((2/\gamma)\sqrt{kl})f(l)\,dl, \quad \gamma\neq 0,$$
$$T(G)f(k)=\alpha\exp(ik\alpha\beta)f(\alpha^2k), \qquad \gamma=0,$$
$$G=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in\mathrm{SL}(2,R), \quad f\in L_2(0,\infty). \qquad (1.12)$$

For $2m$ not an integer, operators (1.12) define a multiple-valued representation of $\mathrm{SL}(2,R)$. In this case we obtain a single-valued representation of the universal covering group of $\mathrm{SL}(2,R)$, and expressions (1.12) are valid only in a neighborhood of the identity element of the covering group. For a discussion of the parametrization of the covering group see Refs. 7, 9.

There is another model of $D^-_{m-1/2}$, due to Bargmann,[9] which we will also find useful. This model is defined on the Hilbert space $\mathcal{H}_m$ of all functions $\mathcal{J}(z)=\sum_{n=0}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$, analytic in the disk $|z|<1$ and such that

$$\lim_{l\to 2m+1} [(l-1)/\pi] \int_{|z|<1} |\mathcal{J}(z)|^2 (1-z\bar{z})^{l-2}\, dx\, dy < \infty, \qquad l>1, \; z=x+iy.$$

The inner product is

$$\langle \mathcal{J}, \mathcal{K}\rangle_m = \lim_{l\to 2m+1} [(l-1)/\pi] \times \int_{|z|<1} \mathcal{J}(z)\bar{\mathcal{K}}(z)(1-z\bar{z})^{l-2}\, dx\, dy \tag{1.13}$$

and a convenient ON basis is provided by the monomials

$$\mathcal{J}_s(z) = [\Gamma(2m+s+1)/\Gamma(2m+3)\, s!]^{1/2} z^s, \qquad s=0,1,2,\cdots. \tag{1.14}$$

The operators A, B, C corresponding to (1.2) and (1.9) are

$$A = \frac{i}{2}\left\{(1+z^2)\frac{d}{dz} + (2m+1)z\right\},$$
$$B = \frac{1}{2}\left\{(1-z^2)\frac{d}{dz} - (2m+1)z\right\}, \tag{1.15}$$
$$C = i\left\{z\frac{d}{dz} + m + \frac{1}{2}\right\}.$$

For more details about this representation, see Refs. 7, 9. Since $C\mathcal{J}_s = i(m+s+\frac{1}{2})\mathcal{J}_s$, it follows that the basis vectors $f_s^{(1)}(k)$ and $\mathcal{J}_s$ correspond. The unitary mapping V from $\mathcal{H}_m$ onto $L_2(0,\infty)$ which carries $\mathcal{J}_s$ to $f_s^{(1)}$ and the operators (1.15) to (1.9) is

$$V\mathcal{J}(k) = \langle \mathcal{J}, V(k,\cdot)\rangle_m, \quad \mathcal{J}\in\mathcal{H}_m,$$
$$V(k,z) = \sum_{s=0}^{\infty} f_s^{(1)}(k)\mathcal{J}_s(z) \tag{1.16}$$
$$= [2/\Gamma(2m+3)]^{1/2}(2k)^m(1-z)^{-2m-1} \times \exp(-k)\exp[-2kz/(1-z)].$$

Similarly, the unitary mapping $W=UV$ from $\mathcal{H}_m$ onto $\mathcal{H}$ is

$$W\mathcal{J}(t,r) = \langle \mathcal{J}, W(t,r,\cdot)\rangle_m, \quad \mathcal{J}\in\mathcal{H}_m,$$
$$W(t,r,z) = [2^{2m} r^m \Gamma(m+\tfrac{1}{2})\exp(+im\pi/2)/\sqrt{2\pi\Gamma(2m+3)}] \times \{[(1+it)(1-z)+2z]^2 + (1-z)^2 r^2\}^{-m-1/2}.$$

2. THE SEPARABLE COORDINATE SYSTEMS

As shown in Refs. 1 the EPD equation permits separation or R-separation of variables in nine coordinate systems corresponding exactly to the nine SL(2, R)-orbits in the space $\mathcal{S}/\{C^2-A^2-B^2\}$, where $\mathcal{S}$ is the space of symmetric second-order elements in the universal enveloping algebra of sl(2, R). A particular separated solution Φ is characterized by (1.1) and the eigenvalue equation $S\Phi=\lambda\Phi$. The eigenvalue λ is the separation constant. If two operators S, S' are on the same orbit, i.e., if $cS' = T(G)ST(G^{-1})$, where $c\in R$, $c\neq 0$ and $G\in$ SL(2, R), then the coordinate systems associated with S and S' are considered equivalent. A complete list of orbit representatives is

$$
\begin{aligned}
&1].\ C^2,\\
&2].\ (A+C)^2,\\
&3].\ B^2,\\
&4].\ 2A^2+AC+CA-aB^2, \quad a>-1,\\
&5].\ 2C^2+AC+CA+aB^2, \quad a>-1,\\
&6].\ rB^2+AC+CA, \quad 0\le r<\infty,\\
&7].\ B^2-s^2C^2, \quad 0<s^2<1,\\
&8].\ C^2+k^2B^2, \quad 0<k^2<\infty,\\
&9].\ (A+C)B+B(A+C)
\end{aligned}
\tag{2.1}
$$

(In case 4] and 5] we shall always normalize so that $a=0$).

It is clear that the operators 1]–9] are symmetric in the $L_2(0,\infty)$ model of the representation $D^-_{m-1/2}$. [Here we consider these operators as initially defined on the dense subspace of C^∞ functions with compact support in $(0,\infty)$.] In this section we will determine the spectral resolutions of six of these operators in this model. Operators 6], 7], and 8] are most conveniently studied in the $\mathcal{H}_m$ model and will be treated in the following section.

System 1] has been treated above. It is straightforward to show that the operator iC has deficiency indices (0, 0). Thus iC is essentially self-adjoint. The spectrum of the closure of C is $i(m+s+\frac{1}{2})$, $s=0,1,2,\cdots$, and each of these discrete eigenvalues has multiplicity one. The corresponding ON basis of eigenvectors is listed in (1.10).

For system 2] we have $A+C=ik$. Clearly the closure of $-i(A+C)$ has continuous spectrum covering the positive real axis and a basis of generalized eigenfunctions

$$f_\lambda^{(2)}(k) = \delta(k-\lambda), \quad 0<\lambda<\infty,$$
$$\langle f_\lambda^{(2)}, f_{\lambda'}^{(2)}\rangle = \delta(\lambda-\lambda'), \quad (A+C)f_\lambda^{(2)} = i\lambda f_\lambda^{(2)}, \tag{2.2}$$

where $\delta(\lambda)$ is the Dirac delta function.

For system 3] we have $B=\frac{1}{2}+k\,d/dk$. It is easy to show that the closure of $-iB$ is self-adjoint with continuous spectrum covering the real axis and a basis of generalized eigenfunctions

$$f_\mu^{(3)}(k) = (2\pi)^{-1/2} k^{i\mu-1/2}, \quad -\infty<\mu<\infty$$
$$\langle f_\mu^{(3)}, f_{\mu'}^{(3)}\rangle = \delta(\mu-\mu'), \quad Bf_\mu^{(3)} = i\mu f_\mu^{(3)}. \tag{2.3}$$

System 4] with $a=0$ is more interesting. Here the operator $L=2A^2+AC+CA$ is symmetric with deficiency indices (1, 1). The possible self-adjoint extensions L_α can be parametrized by the real number α, $0<\alpha\le 2$. For each α, L_α has discrete spectrum $\lambda = m^2-(\alpha+2s)^2 - \frac{1}{4}$, $s=0,1,2,\cdots$, each eigenvalue of multiplicity one, and continuous spectrum. The normalized eigenvectors $f_{\alpha,s}^{(4)}(k)$ form an ON set for $L_2(0,\infty)$:

$$f_{\alpha,s}^{(4)}(k) = [2(\alpha+2s)k^{-1}]^{1/2} J_{\alpha+2s}(k), \quad s=0,1,\cdots,$$
$$\langle f_{\alpha,s}^{(4)}, f_{\alpha,s'}^{(4)}\rangle = \delta_{ss'}, \quad L_\alpha f_{\alpha,s}^{(4)} = [m^2-(\alpha+2s)^2-\tfrac{1}{4}]f_{\alpha,s}^{(4)}. \tag{2.4}$$

The overlaps between distinct self-adjoint extensions L_α, $L_{\alpha'}$, $\alpha\neq\alpha'$, are given by

$$\langle f^{(4)}_{\alpha,s}, f^{(4)}_{\alpha',s'}\rangle = \frac{\sqrt{(\alpha+2s)(\alpha'+2s')}\,\sin\pi[(\alpha-\alpha')/2+s-s']}{\pi[(\alpha-\alpha')/2+s-s'][(\alpha+\alpha')/2+s+s']}. \quad (2.5)$$

Restricting ourselves to the case $\alpha=2$ for simplicity, we find that L_2 has continuous spectrum $\lambda=m^2+\beta-\frac{1}{4}$, $\beta\geq 0$, with corresponding generalized eigenfunctions

$$\tilde{f}^{(4)}_{2,\beta}(k)=[J_{i\sqrt{\beta}}(\gamma k)+J_{-i\sqrt{\beta}}(\gamma k)]/2\sqrt{k\,\sinh(\pi\sqrt{\beta})},$$
$$\langle \tilde{f}^{(4)}_{2,\beta}, \tilde{f}^{(4)}_{2,\beta'}\rangle=\delta(\beta-\beta').$$

The functions $\{f^{(4)}_{\alpha,s}, \tilde{f}^{(4)}_{\alpha,\beta}\}$ form a complete set for $L_2(0,\infty)$. More details can be found on pp. 93—95 of Ref. 10.

For system 5] with $a=0$ we find that the operator $M=2C^2+AC+CA$ is essentially self-adjoint. The closure of M (which we also call M) has continuous spectrum $\lambda=\frac{1}{4}-m^2-\mu^2$, $\mu\geq 0$, of multiplicity one and a basis of generalized eigenvectors.

$$f^{(5)}_{\mu}(k)=[\pi/\sqrt{2k\mu\,\sinh(\mu\pi)}]K_{i\mu}(k), \quad 0<\mu<\infty,$$
$$\langle f^{(5)}_{\mu}, f^{(5)}_{\mu'}\rangle=\delta(\mu-\mu'), \quad Mf^{(5)}_{\mu}=(\tfrac{1}{4}-m^2-\mu^2)f^{(5)}_{\mu}. \quad (2.6)$$

For system 9] we find that the operator $N=(A+C)B+B(A+C)$ has unequal deficiency indices (1, 0). However, there exists an obvious extension of N to the space $L_2(R)=L_2(-\infty,0)\oplus L_2(0,\infty)$ with deficiency indices (1, 1). Of the self-adjoint extensions of this latter operator we choose the one with continuous spectrum covering the real axis and generalized eigenfunctions:

$$f^{(9)}_{\lambda}(k)=\exp(i\lambda/k)/k\sqrt{2\pi}, \quad -\infty<\lambda<\infty,$$
$$Nf^{(9)}_{\lambda}=2\lambda f^{(9)}_{\lambda}, \quad \int_{-\infty}^{\infty} f^{(9)}_{\lambda}(k)\bar{f}^{(9)}_{\lambda'}(k)\,dk=\delta(\lambda-\lambda'). \quad (2.7)$$

Note that $\{f^{(9)}_{\lambda}\}$ satisfies the usual orthogonality relations on $L_2(R)$ but not on $L_2(0,\infty)$.

3. LAMÉ BASES

The spectral resolution of the operators 6], 7], and 8] is carried out in this section using the model of $D^-_{m-1/2}$ due to Bargmann,[9] defined on the Hilbert space $\mathcal{H}_m$ as given in Sec. 1. The reason for treating these operators in this model rather than the $L_2(0,\infty)$ Hilbert space model with SL(2, R) generators as in (1.9) is that they are second order differential operators rather than fourth order.

In fact, if we consider the functions $\mathcal{G}(z)$ defined by $\mathcal{J}(z)=z^{m-1/2}\mathcal{G}(z)$, where $\mathcal{J}(z)\in\mathcal{H}_m$, and put $z=ie^{i\theta}$ with θ complex, the generators A, B, and C acting on the functions $\mathcal{G}(z)$ have the form

$$A=-\sin\theta\frac{d}{d\theta}+(m-\tfrac{1}{2})\cos\theta,$$
$$B=-\cos\theta\frac{d}{d\theta}-(m-\tfrac{1}{2})\sin\theta, \quad C=\frac{d}{d\theta}, \quad (3.1)$$

The form of the generators (3.1) is the same as used in Ref. 11, where the bases described by the operators 1]—9] were studied for the principal series of SL(2, R). The appropriate variables which change eigenvalue equations for the operators 6], 7], and 8] to Lamé equations have been given in that article. The spectral resolution for each of these operators also follows along the lines of our previous article. We now discuss each of the three Lamé bases in the order 8], 7], and 6], treating the simplest cases first.

For the coordinate system 8] it is convenient to choose the functions $\mathcal{L}(v)$ defined by

$$\mathcal{G}(z)=[\mathrm{dn}(v,s)]^{1/2-m}\mathcal{L}(v), \quad (3.2)$$

where

$$\cos\theta=\frac{1}{(1+k^2)^{1/2}}\frac{\mathrm{sn}(v,s)}{\mathrm{dn}(v,s)}$$

and $s=k/(1+k^2)^{1/2}$. The eigenvalue equation for this coordinate system is then

$$\left(\frac{d^2}{dv^2}+s^2(m^2-\tfrac{1}{4})\mathrm{cn}^2(v,s)-\frac{\lambda}{1+k^2}\right)\mathcal{L}_{\lambda}(v)=0. \quad (3.3)$$

Suitable eigenfunctions are the Lamé Wangerin functions with boundary conditions

(i) $[\mathrm{sn}(v,s)]^{1/2}\mathcal{L}_{\lambda}(v)$ bounded at $v=iK'$ and $\mathcal{L}'_{\lambda}(K+iK')=0$.

This gives the solution $\mathcal{L}_{\lambda}(v)=F^{2p}_{m-1/2}(v,s)$ with $2p$ zeros in the interval $(iK', iK'+2K)$.

(ii) $[\mathrm{sn}(v,s)]^{1/2}\mathcal{L}_{\lambda}(v)$ bounded at $v=iK'$ and $\mathcal{L}_{\lambda}(K+iK')=0$ giving the solution $\mathcal{L}_{\lambda}(v)=F^{2p+1}_{m-1/2}(v,s)$ with $2p+1$ zeros in the interval $(iK', iK'+2K)$.

It should also be mentioned that the resulting eigenfunctions

$$\mathcal{J}^{(8)}_{\lambda}(z)=\{[s'\mathrm{sn}(v,s)+i\mathrm{cn}(v,s)]/s'\mathrm{dn}^2(v,s)\}^{m-1/2}\mathcal{L}_{\lambda}(v), \quad (3.4)$$

where $s'=(1-s^2)^{1/2}$, are analytic inside the unit circle of the complex z plane and are elements of $\mathcal{H}_m (m=1,2,3,\cdots)$.

For the coordinate system 7] we choose the functions $\mathcal{M}(v)$ defined by

$$\mathcal{G}(z)=[is'/\mathrm{cn}(v,s)]^{m-1/2}\mathcal{M}(v), \quad (3.5)$$

where

$$\cos\theta=\mathrm{dn}(v,s)/\mathrm{cn}(v,s).$$

The eigenvalue equation for this coordinate system then becomes

$$\left(\frac{d^2}{dv^2}-s^2(m^2-\tfrac{1}{4})\mathrm{sn}^2(v,s)-\lambda\right)\mathcal{M}_{\lambda}(v)=0. \quad (3.6)$$

Natural choices for eigenfunctions are the Lamé Wangerin functions with boundary conditions:

(i) $[\mathrm{sn}(v,s)]^{1/2}\mathcal{M}_{\lambda}(v)$ bounded at $v=iK$ and $\mathcal{M}_{\lambda}(K+iK')=0$ giving the solution $\mathcal{M}_{\lambda}(v)=F^{2p}_{m-1/2}(v,s)$ with $2p$ zeros in the interval $(iK', iK'+2K)$;

(ii) $[\mathrm{sn}(v,s)]^{1/2}\mathcal{M}_{\lambda}(v)$ bounded at $v=iK'$ and $\mathcal{M}_{\lambda}(K+iK')=0$ giving the solution $\mathcal{M}_{\lambda}(v)=F^{2p+1}_{m-1/2}(v,s)$ with $2p+1$ zeros in the interval $(iK', iK'+2K)$.

In each case the spectrum is discrete and the eigenvalues are labeled by the index p, $p=0,1,2,\cdots$. The resulting eigenfunctions

$$\mathcal{J}^{(7)}_{\lambda}(z)=\{s'[s'\mathrm{sn}(v,s)-\mathrm{dn}(v,s)]/\mathrm{cn}^2(v,s)\}^{m-1/2}\mathcal{M}_{\lambda}(v) \quad (3.7)$$

are analytic inside the unit circle of the complex z plane and belong to $\mathcal{H}_m$. The coordinate system 6] is the most complicated of the three under consideration. A convenient choice of function $\mathcal{N}(v)$ is

$$\mathcal{G}(z)=\left(\frac{N\text{sn}(\nu, s)\text{dn}(\nu, s)}{[-(1+r^2)^{1/2}+r+1]\text{sn}^2(\nu, s)-2r}\right)^{m-1/2}\mathcal{N}(\nu) \tag{3.8}$$

where

$$\sin\theta=\frac{2[1-(1+r^2)^{1/2}]+[(1+r^2)^{1/2}-1-r]\text{sn}^2(v, s)}{[1+r-(1+r^2)^{1/2}]\text{sn}^2(v, s)-2r}$$

and

$$s^2=\frac{(1+r^2)^{1/2}-r}{2(1+r^2)^{1/2}},\quad N=\frac{8(1+r^2)^{1/2}}{r^2}[(1+r^2)^{1/2}-1].$$

The eigenvalue equation for this coordinate system is

$$\left(\frac{d^2}{d\omega^2}-t^2(m^2-\tfrac{1}{4})\text{sn}^2(\omega, t)+\frac{(m^2-\frac{1}{4})r}{(1+r^2)^{1/2}(s-is')^2}+\frac{(1+r^2)^{1/2}\lambda}{(s-is')^2}\right)\mathcal{N}_\lambda(\omega)=0. \tag{3.9}$$

Here we have introduced the variables, $\omega=(s+is')\nu+iK'(t)$ and $t=(s+is')/(-s+is')$. The resulting equation is of the Lamé type with modulus t on the unit circle. Natural choices for eigenfunctions are the Lamé Wangerin functions with boundary conditions as for the coordinate system 7], where v is replaced by ω and r by t. The eigenfunctions are then $\mathcal{N}_\lambda(\omega)=F^p_{m-1/2}(\omega, t)$ with $p=0, 1, 2, \cdots$, and the spectrum is discrete. The corresponding eigenfunctions $\mathcal{J}_\lambda^{(6)}(z)$ are analytic in the unit circle in the complex z plane and are members of $\mathcal{H}_m$.

We see that for the discrete series $D^-_{m-1/2}$ of $SL(2, R)$ the most convenient basis eigenfunctions for the three Lamé bases are the Lamé Wangerin or finite Lamé functions. This is opposed to the situation in Ref. 11, where we dealt with the principal series of $SL(2, R)$ and the corresponding basis functions for system 8] were the periodic Lamé functions. For the discrete series $D^-_{m-1/2}$ in the Bargmann model the operator specifying system 8] is singular inside the unit disc. The operators of the other two systems are singular on the unit disc. The imposition of boundary conditions that gives Lamé Wangerin functions for these systems yields eigenfunctions in $\mathcal{H}_m$, which are analytic inside the unit disc and zero at the singular points.

4. THE TWO-VARIABLE MODEL

If $\{f_\lambda^{(j)}(k)\}$ is a basis for $L_2(0,\infty)$ consisting of eigenfunctions of the operator S, symmetric and second order in the generators (1.9) of $sl(2, R)$, then $\{F_\lambda^{(j)}(t, r)\}$ is a basis for $\mathcal{H}$ consisting of eigenfunctions of the corresponding operator S' constructed from the generators (1.2), where

$$F_\lambda^{(j)}(t, r)=\cup[f_\lambda^{(j)}],\quad j=1,\ldots,9, \tag{4.1}$$

and $\cup$ is the unitary transformation (1.6). Indeed we have the relations

$$Sf_\lambda^{(j)}=\lambda f_\lambda^{(j)},\quad S'F_\lambda^{(j)}=\lambda F_\lambda^{(j)},$$
$$\langle f_\lambda^{(j)}, f_{\lambda'}^{(j)}\rangle=(\cup f_\lambda^{(j)}, \cup f_{\lambda'}^{(j)})=(F_\lambda^{(j)}, F_{\lambda'}^{(j)})=\delta_{\lambda\lambda'}. \tag{4.2}$$

Furthermore, $F_\lambda^{(j)}(t, r)$ is a solution of the EPD equation (1.1). It follows from results proved in Refs. 1 that for fixed j there exists a coordinate system $\{u(t, r), v(t, r)\}$ such that variables separate (or R-separate) in the EPD equation and such that $F_\lambda^{(j)}(t, r)=\exp[Q(u, v)]J_\lambda(u)K_\lambda(v)$, where J_λ, K_λ are solutions of the separated second order ordinary differential equations and either $Q\equiv 0$ (separation) or $Q\not\equiv 0$ and Q cannot be expressed as a sum $Q(u, v)=q_1(u)+q_2(v)$ (R-separation). In particular the possible coordinate systems and separated equations (as well as the functions Q) are listed in Ref. 1, Paper 9.

In this section it is our aim to compute all the functions $F_\lambda^{(j)}(t, r)$ defined by (4.1) and (1.6). In general, the integrals (4.1) are rather difficult to evaluate. In particular we have not been able to find the integral for $j=5$ nor two of the three integrals needed for $j=4$ in the Bateman Project.

However, our work is enormously simplified because we know in advance the coordinates in which variables separate for (4.1). Thus we can immediately evaluate the integral as a linear combination of four terms (since J_λ and K_λ each satisfy a known second order ordinary linear differential equation). The four constants can then be determined by evaluating the integral for specially chosen values of the variables u, v. In this respect the functions $F_\lambda^{(j)}(t, r)$ listed below can also be regarded as evaluations of a number of interesting integrals related to the EPD equation.

For several cases we find that the integral (4.1) does not converge sufficiently rapidly so that differentiation under the integral sign is permitted. It is not immediately apparent in these cases that $F_\lambda^{(j)}$ is actually a solution of the EPD equation. However, it is always possible to justify our assertions by noting that if we allow t to become complex and take $\text{Im}t>0$ in (1.6), then the convergence in each integral (4.1) is sufficiently rapid that multiple differentiation with respect to r and t is permitted under the integral sign. In each case one can verify by inspection that the coordinates $u(t, r)$, $v(t, r)$ can be extended to the domain $\text{Im}t\geq 0$ and that variables still separate in the EPD equation. Finally one can evaluate the integrals (4.1) for $\text{Im}t>0$ and then use the Lebesgue dominated convergence theorem or a similar device to justify going to the limit as t becomes real through positive imaginary values.

We have the following results:

1]: $F_s^{(1)}(t, r)\equiv F_s^{(1)}(\sigma,\varphi)$

$$=\left(\frac{2(s!)}{\Gamma(2m+s+1)}\right)^{1/2}\frac{\Gamma(2m+1)}{\Gamma(m+1)}2^{-m-1/2}\times\exp[-i(m-1)\pi/2]\sqrt{\cos\sigma-\cos\varphi}\sin^m\sigma\times\exp[-i(s+m+1/2)\varphi]C_s^{(m+1/2)}(\cos\sigma),$$

$s=0, 1, 2, \cdots$.

$$t=\frac{\sin\varphi}{\cos\sigma-\cos\varphi},\quad r=\frac{\sin\sigma}{\cos\sigma-\cos\varphi},$$

$$0\le\sigma\le\pi,\quad 2\pi-\sigma>\varphi>\sigma, \tag{4.3}$$

$$e^{i\varphi}=\frac{\frac12(t^2-r^2-1)+it}{[r^2+\frac14(r^2-t^2-1)^2]^{1/2}},$$

$$e^{i\sigma}=\frac{\frac12(t^2-r^2+1)+ir}{[r^2+\frac14(r^2-t^2-1)^2]^{1/2}},$$

$$e^{Q}=\sqrt{\cos\sigma-\cos\varphi},\quad (F_s^{(1)},F_{s'}^{(1)})=\delta_{ss'}.$$

Here, $C_s^{(\nu)}(z)$ is a Gegenbauer polynomial.

2]: $F_\lambda^{(2)}(t,r)=\exp(-im\pi/2)\exp(it\lambda)J_m(\lambda r),\quad \lambda>0,$

$$u=t,\quad v=r,\quad Q\equiv0,\quad (F_\lambda^{(2)},F_{\lambda'}^{(2)})=\delta(\lambda-\lambda'). \tag{4.4}$$

3]:

$$F_\mu^{(3)}(t,r)=\exp[\pm i\pi(m+1/2+i\mu)/2](\pm t)^{-m-i\mu-1/2}$$
$$\times\frac{\exp(im\pi/2)r^m}{\Gamma(m+1)\sqrt{2\pi}}\Gamma(m+i\mu+\tfrac12) \tag{4.5}$$
$$\times{}_2F_1(i\mu/2+m/2+\tfrac14,\,i\mu/2+m/2+\tfrac34;\,m+1;\,r^2/t^2),$$
$$r<|t|,$$

$$F_\mu^{(3)}(t,r)=[\exp(-im\pi/2)/\sqrt{2\pi}]\{2^{i\mu-1/2}$$
$$[\Gamma(m/2+i\mu/2+\tfrac14)r^{-m}/\Gamma(m/2-i\mu/2+\tfrac34)]$$
$$\times{}_2F_1(i\mu/2+m/2+\tfrac14,\,i\mu/2-m/2+\tfrac14;\,\tfrac12;\,t^2/r^2)$$
$$+2^{i\mu+1/2}itr^{-i\mu-3/4}[\Gamma(m/2+i\mu/2+\tfrac34)/\Gamma(m/2+i\mu/2+\tfrac14)]$$
$$\times{}_2F_1(i\mu/2+m/2+\tfrac34;\,i\mu/2-m/2+\tfrac34;\,\tfrac32;\,t^2/r^2)\},$$
$$r>|t|,$$

Here the (+) sign holds for $t>0$ and the (−) sign for $t<0$. In this case $u=t,\ v=r/t,\ Q\equiv0$, and $(F_\mu^{(3)},F_{\mu'}^{(3)})=\delta(\mu-\mu')$.

For systems of type 4] we consider three cases.

4a]: $|t|\ge r+1$: For $t\ge1$ we have $(\nu=\alpha+2s)$

$$F_{\alpha,s}^{(4)}(t,r)=F_{\alpha,s}^{(4)}(\theta,\varphi)$$
$$=\exp[i\pi(\nu/2-m+\tfrac14)]\sqrt{2/\pi}$$
$$\times P_{\nu-1/2}^{-m}(\cosh\theta)Q_{\nu-1/2}^{m}(\cosh\varphi), \tag{4.6}$$

$s=0,1,2,\ldots,$ for $\theta\ge\varphi$, and $F_{\alpha,s}^{(4)}(\theta,\varphi)=F_{\alpha,s}^{(4)}(\varphi,\theta)$ for $\theta<\varphi$. Here P_ν^μ and Q_ν^μ are Legendre functions and

$$t=\cosh\theta\cosh\varphi,\quad r=\sinh\theta\sinh\varphi,\quad \theta,\varphi\ge0. \tag{4.7}$$

For $t\le-1$ we have $F_{\alpha,s}^{(4)}(t,r)=\exp(-im\pi)\overline{F_{\alpha,s}^{(4)}(-t,r)}$.

4b]: $|t|\le r-1$:

$$F_{\alpha,s}^{(4)}(\theta,\varphi)=\frac{\exp[i\pi(-5m/2+\nu+\frac12)]2^{\nu-m-1}\Gamma(\frac34+\nu/2-m/2)}{\pi\Gamma(\frac12+\nu+m)\Gamma(\frac34+m/2-\nu/2)}$$
$$Q_{\nu-1/2}(i\sinh\theta)Q_{\nu-1/2}(i\sinh\varphi), \tag{4.8}$$

$$t=\sinh\theta\sinh\varphi,\quad r=\cosh\theta\cosh\varphi,\quad -\infty<\theta<\infty$$

4c]: $|t|+r\le1$: For $t\ge0$,

$$F_{\alpha,s}^{(4)}(\theta,\varphi)=\frac{\sqrt{\pi\nu}\,\Gamma(\nu+m+\frac12)\exp(-im\pi/2)}{\Gamma(\nu-m+\frac12)\cos[(\pi/2)(\nu-m-\frac12)]}$$
$$\times P_{\nu-1/2}^{-m}(\cos\theta)P_{\nu-1/2}^{-m}(\cos\varphi),\quad \nu=\alpha+2s, \tag{4.9}$$

$$t=\cos\theta\cos\varphi,\quad r=\sin\theta\sin\varphi,\quad 0\le\theta,\varphi\le\pi/2.$$

For $t<0$, $F_{\alpha,s}^{(4)}(t,r)=\exp(-im\pi)\overline{F_{\alpha,s}^{(4)}(-t,r)}$

These three parametrizations do not cover the full r—t plane but, as shown in Ref. 1, Paper 9, variables do not separate in the remaining domain. We omit the computation of the continuum eigenfunctions $\tilde F_{\alpha,\beta}^{(4)}$

5]: $F_\mu^{(5)}(t,r)\equiv F_\mu^{(5)}(\theta,\varphi)$

$$=\frac{\pi^{3/2}\Gamma(\frac12-i\mu+m)\Gamma(\frac12+i\mu+m)}{2\sqrt{k\mu\sinh(\mu\pi)}}$$
$$\times P_{-1/2+i\mu}^{-m}(\cosh\theta)P_{-1/2+i\mu}^{-m}(-i\sinh\varphi),$$
$$0<\mu<\infty, \tag{4.10}$$

$$t=\cosh\theta\sinh\varphi,\quad r=\sinh\theta\cosh\varphi,\quad 0\le\theta,\varphi.$$

For $t<0$ we have $F_\mu^{(5)}(t,r)=\exp(-im\pi)\overline{F_\mu^{(5)}(-t,r)}$.

9]:

$$F_\lambda^{(9)}(t,r)=F_\lambda^{(9)}(x,X)$$
$$=\begin{cases}\sqrt{2/\pi}\,K_m(2x\sqrt{-\lambda})I_m(2X\sqrt{-\lambda}) & \text{if } t>0,\ \lambda<0,\\ i\sqrt{\pi/2}\,H_m^{(1)}(2x\sqrt{\lambda})J_m(2X\sqrt{\lambda}) & \text{if } t>0,\ \lambda>0.\end{cases} \tag{4.11}$$

Here,

$$x=\tfrac12(\sqrt{t+r}+\sqrt{t-r}),\quad X=\tfrac12(\sqrt{t+r}-\sqrt{t-r})\quad \text{if } t\pm r>0,$$
$$x=\tfrac12(\sqrt{r+t}+\sqrt{r-t}),\quad X=\tfrac12(\sqrt{r+t}-\sqrt{r-t})\quad \text{if } r\pm t>0. \tag{4.12}$$

Also $F_\lambda^{(9)}(-t,r)=\exp(-im\pi)\overline{F_\lambda^{(9)}(t,r)}$.

For cases $j=6,7,8$ it is most convenient to use the model $\mathcal{H}_m$, (1.13)–(1.15). The passage to the two variable model proceeds along similar lines as in our earlier work with Lamé bases.[11] In each case the resulting basis function is determined to within a phase. This quantity can be chosen by adopting a fixed normalization of the Lamé Wangerin functions.

8]: $F_\lambda^{(8)}(t,r)=\lambda_m^p[\mathrm{dn}(\alpha,s)\,\mathrm{dn}(\beta,s)+iss'\mathrm{cn}(\beta,s)]^{1/2}$

$$\times F_{m-1/2}^{p}(\alpha,s)F_{m-1/2}^{p}(\beta,s), \tag{4.13}$$

where

$$t=s\ \mathrm{sn}(\alpha,s)\,\mathrm{sn}(\beta,s)/R,\quad r=1/R \tag{4.14}$$

and

$$R=-i\,\mathrm{dn}(\alpha,s)\,\mathrm{dn}(\beta,s)/ss'+\mathrm{cn}(\alpha,s)\mathrm{cn}(\beta,s')$$

and the variables α,β are in the ranges $\alpha\in[0,2K]$, $\beta\in[iK',iK'+2K]$.

7]: $F_\lambda^{(7)}(t,r)=\lambda_m^p[s'\mathrm{sn}(\alpha,s)\mathrm{sn}(\beta,s)+\mathrm{cn}(\alpha,s)\mathrm{cn}(\beta,s)]^{1/2}$

$$\times F_{m-1/2}^{p}(\alpha,s)F_{m-1/2}^{p}(\beta,s), \tag{4.15}$$

where

$$t=\mathrm{dn}(\alpha,s)\mathrm{dn}(\beta,s)/ss'R,\quad r=1/R,$$

and

$$R=s[\mathrm{sn}(\alpha,s)\mathrm{sn}(\beta,s)+\mathrm{cn}(\alpha,s)\mathrm{cn}(\beta,s)/s']. \tag{4.16}$$

The variables α, β can vary in the two ranges $\alpha, \beta \in [0, 2K]$, $\alpha, \beta \in [iK', iK'+2K]$.

$$6]:\ F_\lambda^{(6)}(t,r) = \lambda_m^p[(s-is')\mathrm{dn}(\omega,t)\mathrm{dn}(\mu,r) + (s+is')\mathrm{cn}(\omega,t)\mathrm{cn}(\mu,t)]^{1/2} \times F^p_{m-1/2}(\omega,t)F^p_{m-1/2}(\mu,t), \qquad (4.17)$$

where

$$t = 2\sqrt{ss'}(s+is')\mathrm{sn}(\omega,t)\mathrm{sn}(\mu,t)/R, \quad r = 2\sqrt{ss'}/R \qquad (4.18)$$

and $R = [(s-is')\mathrm{dn}(\omega,t)\mathrm{dn}(\mu,t) + (s+is')\mathrm{cn}(\omega,t)\mathrm{cn}(\mu,t)]$, $t = (s+is')/(s-is')$.

The variables ω and μ vary in the ranges $\omega, \mu \in [-iK', iK']$, which is the line segment joining the points $-iK'$, iK' in the complex plane. (Remember K' and K are complex.) Here for the Lamé bases we have used the same notation as in Sec. 3, where the spectral analysis was performed.

5. OVERLAP FUNCTIONS

Here we compute several of the overlap functions $\langle f_\mu^{(j)}, f_\lambda^{(i)}\rangle$, which allow us to expand eigenfunctions $f_\lambda^{(j)}$ in terms of eigenfunctions $f_\lambda^{(i)}$. Since $\langle T(G)f_\lambda^{(j)}, T(G)f_\lambda^{(i)}\rangle = \langle f_\mu^{(j)}, f_\lambda^{(i)}\rangle$, the same functions allow us to expand eigenfunctions $T(G)f_\mu^{(j)}$ in terms of eigenfunctions $T(G)f_\lambda^{(i)}$. Moreover, since $\langle f_\mu^{(j)}, f_\lambda^{(i)}\rangle = (F_\mu^{(j)}, F_\lambda^{(i)})$, the overlaps allow us to expand eigenfunctions $F_\mu^{(j)}$ in $\mathcal{H}$ in terms of eigenfunctions $F_\lambda^{(i)}$. These last expansions converge in the Hilbert space sense. Pointwise convergence has to be checked separately.

However, if we choose $r \geq 0$ and $\mathrm{Im}t > 0$ in (1.6), then the function $H_{t,r}(k) = \exp(+im\pi/2)\exp(-i\bar{t}k)J_m(kr)$ belongs to $L_2[0,\infty]$ and the transformation $\cup[f]$, (1.6), can be represented as an inner product on $L_2[0,\infty]$:

$$\cup[f] = \langle f, H_{t,r}\rangle. \qquad (5.1)$$

In this case it follows easily that all of the expansion formulas

$$\mathcal{J}_\mu^{(j)}(t,r) = \int \langle f_\mu^{(j)}, f_\lambda^{(i)}\rangle \mathcal{J}_\lambda^{(i)}(t,r)\, d\lambda, \quad \mathrm{Im}t > 0,\ r \geq 0, \qquad (5.2)$$

are valid in the sense of pointwise convergence, a.e., (See the analogous arguments in Refs. 12 and 13.) In each case it is easy to verify that separation of variables persists in the domain $\mathrm{Im}t > 0$ if it holds for $\mathrm{Im}t = 0$.

Overlaps involving system 2] are especially easy to compute:

$$\langle f_\mu^{(j)}, f_\lambda^{(2)}\rangle = f_\mu^{(j)}(\lambda), \quad 0 < \lambda < \infty. \qquad (5.3)$$

In addition we list the overlaps $\langle f_\mu^{(j)}, f_s^{(1)}\rangle$ between the j] basis and the discrete basis 1]:

$$\langle f_\mu^{(3)}, f_s^{(1)}\rangle = \left(\frac{\Gamma(2m+s+1)}{(s!)\pi}\right)^{1/2} 2^m \frac{\Gamma(2m+i\mu+\frac{1}{2})}{\Gamma(2m+1)} \times {}_2F_1\left(\begin{matrix}-s,\ m+i\mu+\frac{1}{2}\\ 2m+1\end{matrix}\middle| 2\right), \qquad (5.4)$$

$$\langle f_{\alpha,q}^{(4)}, f_s^{(1)}\rangle = 2^{m-\nu+1}(1+i)^{-\nu-m-1/2}\frac{\Gamma(\nu+m+\frac{1}{2})}{\Gamma(\nu+1)}\left(\frac{\nu\Gamma(2m+s+1)}{(s!)\Gamma(2m+1)}\right)^{1/2} \times F_A\left(\nu+m+\tfrac{1}{2};\ \nu+\tfrac{1}{2}, -s;\ 2\nu+1,\ 2m+1;\ \frac{2i}{1+i}, \frac{2}{1+i}\right),$$

$$\nu = \alpha + 2q, \quad q, s = 0, 1, 2, \cdots. \qquad (5.5)$$

Here F_A is a Lauricella function.[14] The overlaps $\langle f_\mu^{(5)}, f_s^{(1)}\rangle$ and $\langle f_\lambda^{(9)}, f_s^{(1)}\rangle$, while straightforward to compute, are of a complexity similar to (5.5) and will not be listed here. [Note that the latter overlaps are not unitary since $\{f_\lambda^{(9)}\}$ is an ON basis for $L_2(R)$, not $L_2[0,\infty]$.]

The overlaps $\langle f_\mu^{(j)}, f_s^{(1)}\rangle$, $j = 6, 7, 8$, can be obtained immediately from the $\mathcal{H}_m$ models. The computation of the overlap functions between the Lamé bases 6], 7], and 8] and the basis 1] is easiest to perform by giving the recurrence formulas for these coefficients (see Ref. 11.) We consider explicitly the case of coordinate system 8], where the basis function $\mathcal{J}_\lambda^{(8)}(z)$ is even under the interchange $z \to -z$. Applying the operator $C^2 + k^2B^2$ to both sides of the identity,

$$\mathcal{J}_\lambda^{(8)}(z) = \sum_{n=0}^{\infty} a_n z^{2n}, \qquad (5.6)$$

we obtain the recurrence relation

$$k^2(2n+2)(2n+1)a_{n+1} + [4n(k^2+2)(1-m-2s) - 4\lambda - (2m-1)(2m-1+k^2)]a_n + 2k^2[2(n-1)^2 + (2m-1)(n-m)]a_{n-1} = 0, \qquad (5.7)$$

$$2k^2a_1 - [4\lambda + (2m-1)(2m-1+k^2)]a_0 = 0.$$

The normalized overlap functions b_n are then given via the relation

$$a_n = [\Gamma(2m+n+1)/\Gamma(2m+3)n!]^{1/2}b_n.$$

For the case of eigenfunctions $\mathcal{J}_\lambda^8(z)$ which are odd the analysis goes through as before by making the substitution $n \to n + \frac{1}{2}$. We should mention here that even and odd eigenfunctions $\mathcal{J}_\lambda^{(8)}(z)$ correspond to Lamé Wangerin functions with an even or odd number of zeros in the interval $(iK'(s), iK'(s) + 2K(s))$ (see Sec. 3). Similar recurrence relations for the basis eigenfunctions of system 7] can be derived by making the substitutions $k^2 \to -1/s^2$, $\lambda \to -\lambda/s^2$. The recurrence relation for 6] is somewhat more lengthy and will not be presented here.

Finally we list the interesting overlaps

$$\langle f_{\alpha,s}^{(4)}, f_\mu^{(5)}\rangle = \frac{\pi}{4}\left(\frac{\nu}{\mu\sinh\mu\pi}\right)^{1/2}\frac{\Gamma((\nu+i\mu)/2)\Gamma((\nu-i\mu)/2)}{\Gamma(\nu+1)} \times {}_2F_1\left(\begin{matrix}(\nu+i\mu)/2,\ (\nu-i\mu)/2\\ \nu+1\end{matrix};\ -1\right), \quad \nu = \alpha + 2s,$$

$$\langle f_\mu^{(3)}, f_{\alpha,s}^{(4)}\rangle = \sqrt{\nu/\pi}\,\frac{2^{i\mu-1}\Gamma((\nu+i\mu)/2)}{\Gamma((\nu-i\mu)/2+1)}, \quad \nu = \alpha + 2s.$$

As discussed in earlier papers in this series, the most general overlaps between basis functions are the mixed basis matrix elements $\langle T(G)f_\mu^{(j)}, f_\lambda^{(i)}\rangle$. The determination of these matrix elements is straightforward, though frequently the result is complicated.

6. THE COMPLEX EPD EQUATION

In the case where the variables r, t in (1.1) are complex and m is a complex constant, we can regard the EPD equation from another point of view. For the symmetry algebra we now choose the *complex* Lie algebra $\mathrm{sl}(2, \mathbb{C})$, whose action on solutions of (1.1) is given by (1.5), where now the matrix elements $\alpha, \beta, \gamma, \delta$ are allowed to be complex and constrained only by the requirement $\det G = 1$.

We can now pose the problem of determining the possible coordinate systems $\{u, v\}$ in which the complex EPD equation is separable. Here, we require that the coordinate transformation functions $u(r, t)$, $v(r, t)$ be only *complex* analytic in r, t rather than real analytic as in the case of the real EPD equation. Furthermore, we regard two coordinate systems as equivalent if one can be obtained from the other by a transformation (1.5) from the group $\mathrm{SL}(2, \mathbb{C})$. Just as in Sec. 2, we expect the equivalence classes of coordinate systems to correspond to the $\mathrm{SL}(2, \mathbb{C})$-orbits in the space $\mathcal{J}^c = \mathcal{S}^c/\{C^2 - A^2 - B^2\}^c$, where $\mathcal{S}^c$ is the space of symmetric second-order elements in the universal enveloping algebra of $\mathrm{sl}(2, \mathbb{C})$.

To determine the adjoint action of $\mathrm{SL}(2, \mathbb{C})$ on $\mathcal{S}^c$, we choose a more convenient basis for $\mathrm{sl}(2, \mathbb{C})$:

$$S_1 = iA, \quad S_2 = iB, \quad S_3 = C,$$
$$[S_1, S_2] = S_3, \quad [S_3, S_1] = S_2, \quad [S_2, S_3] = S_1. \qquad (6.1)$$

A general element Q of $\mathcal{S}^c$ can be expressed uniquely in the form

$$Q = \sum_{j,k=1}^{3} q_{jk} S_j S_k, \quad q_{jk} = q_{kj} \in \mathbb{C}. \qquad (6.2)$$

Using the well-known local isomorphism of $\mathrm{SO}(3, \mathbb{C})$ and $\mathrm{SL}(2, \mathbb{C})$, and identifying Q with the 3×3 symmetric matrix $\hat{Q} = (q_{jk})$, we see that under the adjoint representation Q transforms according to $\hat{Q} \to O^{-1}\hat{Q}O$, $O \in \mathrm{SO}(3, \mathbb{C})$. The elements of $\mathcal{J}^c$ can be identified with the matrices $\hat{Q}$ such that $\mathrm{tr}\hat{Q} = 0$, or more conveniently, we can add arbitrary multiples of the identity matrix to $\hat{Q}$. It is now a simple exercise in matrix theory to classify the orbits in $\mathcal{J}^c$ under the adjoint representation of $\mathrm{SL}(2, \mathbb{C})$. We present only the results and label the four possible orbit types by the eigenvalues of $\hat{Q}$. The symbol $\lambda(2)$ [or $\lambda(3)$] signifies that the eigenvalue λ corresponds to a generalized eigenvector of degree 2 (or 3). Every $\hat{Q} \in \mathcal{J}^c$ is conjugate under the adjoint representation to an element in the following list.

	eigenvalues	*orbit representative*
a.	λ, μ, ρ; $\lambda + \mu + \rho = 0$, $(\lambda - \mu)(\lambda - \rho)(\mu - \rho) \neq 0$	$\lambda S_1^2 + \mu S_2^2 + \rho S_3^2$
b.	$2\lambda, -\lambda, -\lambda$; $\lambda \neq 0$	$\lambda(2S_1^2 - S_2^2 - S_3^2)$
c.	$\lambda(2), -2\lambda$	$(\lambda + \tfrac{1}{2})S_1^2 + (\lambda - \tfrac{1}{2})S_2^2 - 2\lambda S_3^2 + \tfrac{1}{2}i(S_1S_2 + S_2S_1)$
d.	$0(3)$	$(S_1 + iS_2)S_3 + S_3(S_1 + iS_2)$

(6.3)

For our purposes we can add a scalar multiple of $S_1^2 + S_2^2 + S_3^2$ to any of the orbit representations without changing the element of $\mathcal{J}^c$. Then we find that any element of $\mathcal{J}^c$ is equivalent to a scalar multiple of one of the following elements.

a. $S_3^2 - k^2 S_2^2$, $k \neq 0$, 6], 7], 8],

b. S_1^2, 1], 3]

c1. $(\lambda \neq 0)$ $2S_1^2 + (1 - 6\lambda)S_3^2 + i(S_1S_2 + S_2S_1)$, 4], 5], (6.4)

c2. $(\lambda = 0)$ $S_1^2 - S_2^2 + i(S_1S_2 + S_2S_1)$, 2],

d. $(S_1 + iS_2)S_3 + S_3(S_1 + iS_2)$, 9].

Each of the nine $\mathrm{SL}(2, R)$-orbit representatives in (2.1) belongs to one of the five orbits (6.4), and we have indicated the orbit inclusions in the last column of (6.4). We see that each of our five orbit-types contains at least one of the $\mathrm{SL}(2, R)$-orbits and that some contain more than one. From these facts we infer that there are no new separable coordinate systems obtained by complexifying the EPD equation: all coordinate systems follow from an obvious analytic continuation of the systems 1]–9]. However, the systems 1] and 3], the systems 4] and 5], and the systems 6], 7], 8] are equivalent for the complex EPD equation.

A particularly interesting basis for the solutions of the complex EPD equation is that of type b:

$$\mathcal{J}_n^b(w, \tau) = \tau^{n+m+1}(1 - w^2)^{m/2} C_n^{m+1/2}(w),$$
$$B\mathcal{J}_n^b = (n + m + \tfrac{1}{2})\mathcal{J}_n^b. \qquad (6.5)$$

Here, $C_n^\nu(w)$ is a Gegenbauer function (a polynomial for $n = 0, 1, 2, \cdots$) and the complex variables w, τ are given by

$$w = t(t^2 - r^2)^{-1/2}, \quad \tau = (t^2 - r^2)^{-1/2}. \qquad (6.6)$$

In terms of the variables w, τ the local group action (1.5) of $\mathrm{SL}(2, C)$ becomes

$$T(G)\Phi(w, \tau) = \cup^{1/2}V^{1/2} \times \Phi((w + 2\beta\gamma w + \alpha\beta\tau + \gamma\delta\tau^{-1})/V;\ \tau\cup^{-1}),$$
$$\cup = [(\delta^2 + \beta^2\tau^2 + 2\beta\delta\tau w)/(\alpha^2 + \gamma^2\tau^{-2} + 2\alpha\gamma\tau^{-1}w)]^{1/2},$$
$$V = [(2w + 2\beta\gamma w + \alpha\beta\tau + \gamma\delta\tau^{-1})(2\beta\gamma w + \alpha\beta\tau + \gamma\delta\tau^{-1}) + 1]^{1/2}. \qquad (6.7)$$

Here

$$G = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}).$$

We are now in a position to apply Weisner's method[2–4] to expand solutions of the complex EPD equation in terms of the basis (6.5). Suppose $\Phi(r, t) \equiv \Phi(w, \tau)$ is a solution of the EPD equation such that $\tau^{-m-1}(1 - w^2)^{-m/2}\Phi(w, \tau)$ is analytic in τ and w in a neighborhood of $(w, \tau) = (0, 0)$. Then there exist complex constants a_n such that

$$\tau^{-m-1}(1-w^2)^{-m/2}\Phi(w,\tau)=\sum_{n=0}^{\infty} a_n C_n^{m+1/2}(w)\tau^n. \qquad (6.8)$$

This method was employed by Viswanathan[2] to derive generating functions for the Gegenbauer polynomials. [The awkward factor $\tau^{-m-1}(1-w^2)^{-m/2}$ which appears in (6.8) is due to our insistence in retaining the EPD equation. One can easily remove this factor by transforming the EPD equation to the equivalent equation for ultraspherical functions which appears in Ref. 2.] In order to derive useful results from (6.8), one characterizes a solution Φ of the EPD equation by requiring that it be an eigenfunction of a first or second order operator in the enveloping algebra of SL(2, C). As Viswanathan remarked, in practice one can compute Φ precisely in the cases where it is possible to find coordinate systems in which variables separate in the equations for Φ. The results of our paper show why this is so and exactly when separable variables exist. Once a suitable Φ is computed one can evaluate the constants a_n by choosing special values of the variables, e.g., $w=0$. Similarly one can derive expansions for $T(G)\Phi$, i.e., functions which lie on the same SL(2, C)-orbit as Φ.

According to (6.4) there are five types of orbits to consider to obtain all possible generating functions for the Gegenbauer polynomials via Weisner's method. An examination of Viswanathan's paper shows that he has found four of these orbits, omitting only the Lamé case (type a). This case can be treated by using the coordinates (4.14) for α, β complex and substituting into (6.6). The remainder of the computations follow just as those given in Ref. 2. However, the resulting identities are somewhat complex due to the fact that $\mathrm{sn}(\alpha, s)$ and $\mathrm{sn}(\beta, s)$ are rather complicated algebraic functions of w and τ.

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