Lie theory and separation of variables. 10. Nonorthogonal R-separable solutions of the wave equation $\partial_{tt}\psi = \Delta_2 \psi$

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We classify and discuss the possible nonorthogonal coordinate systems which lead to R-separable solutions of the wave equation. Each system is associated with a pair of commuting operators in the symmetry algebra so (3,2) of this equation, one operator first order and the other first or second order. Several systems appear here for the first time.

INTRODUCTION

This paper is one of a series[1–9] investigating the relationship between the symmetry groups of the principal equations of mathematical physics and the coordinate systems for which the corresponding equations admit an R-separable solution. We recall that a solution $\psi(x_1,x_2,x_3)$ of an equation in three variables is R-separable if it can be written in the form

$$\psi(x_1,x_2,x_3) = \exp[Q(x_1,x_2,x_3)]A(x_1)B(x_2)C(x_3),$$

where e^Q contains no factors which are functions of one variable. The factor e^Q is called the modulation factor. The last two papers in this series[8,9] have dealt with a study of the wave equation in two space dimensions

$$\partial_{tt}\psi = \Delta_2\psi. \qquad (*)$$

In Paper 8 [8] of this series (hereafter referred to as I) we have given a detailed treatment of the symmetry group of (*) which is locally isomorphic to O(3,2). In that article are also discussed the principal equations contained in (*) when a generator of the Lie algebra is diagonalized. The resulting coordinate systems were called semisubgroup coordinate systems. In Paper 9 [9] (hereafter referred to as II) of this series, we complemented the contents of I with a detailed study of the orthogonal R-separable solutions of (*). This was achieved using pentaspherical space and families of confocal cyclides. The methods were principally those developed by Bôcher.[10] In this work we supplement the contents of I and II by looking for R-separable solutions of (*) which correspond to coordinates which are nonorthogonal.

If

$$ds^2 = dt^2 - dx^2 - dy^2$$
$$= g^{ij}\, dx_i\, dx_j$$

is such that $g^{ij} \neq 0$ for at least one pair of indices $i \neq j$ and (*) admits an R-separable solution in the variables x_1, x_2, x_3, these coordinates constitute a nonorthogonal R-separable coordinate system. It is the purpose of this article to classify such coordinate systems. The contents of the paper are divided into three sections. In Sec. I we classify all coordinate systems in terms of their differential forms. This is done in detail by elementary and straightforward methods. The separation equations for each system we find are also given here. In Sec. II we give the coordinate systems in Minkowski space which correspond to the differential forms given in Sec. I. We also give the operators which specify the separation constants in each system. These are the operators associated with each system. Finally in Sec. III we look at the properties of coordinate systems which are specified by elements of an SL$(2,R)$ subalgebra of the symmetry group of (*). This corresponds to the SL$(2,R)$ algebra in Sec. 7 of Paper 8 of this series.

I. THE CLASSIFICATION OF SEPARABLE NONORTHOGONAL COORDINATE SYSTEMS

In this section we give a classification of the nonorthogonal coordinate systems for which (*) admits an R-separable solution. As opposed to the sophisticated methods used in II, we proceed in a straightforward manner here. These techniques have already been used previously.

We use the conditions of R-separability together with the requirement that the space be flat. The first requirement reduces to a number of special cases in which the metric g^{ij} has a prescribed form. For the space to be flat means that all the components of the Riemann curvature tensor are zero. The solution of these two constraints then gives us the list of possible nonorthogonal R-separable coordinate systems for the Laplace operator in a flat space. In each case we obtain a specific form for the metric tensor g^{ij}. Each of the nonorthogonal R-separable systems that we find corresponds to a prescribed coordinate system in Minkowski space with coordinates t, x, y. This reflects the fact that the only other candidate space satisfying the above conditions is Euclidean three-space, which does not admit nonorthogonal R-separable solutions of the Laplace operator.

A few words about our definition of R-separation are in order. More specifically we consider at first pure separation. A solution of (*) $\psi(x_1,x_2,x_3)$ in three new curvilinear variables $\mu, \nu, \rho \to x_1, x_2, x_3$ is said to be separable if $\psi = A(x_1)B(x_2)C(x_3)$ and each of the factor functions satisfies a second or first order ordinary differential equation. By a nonorthogonal coordinate system we shall mean a coordinate system for which at least one g^{ij} $(i \neq j)$ is nonzero. Here g^{ij} is the metric tensor expressing the line element $ds^2 = g^{ij}dx_i dx_j$. For such a coordinate system the wave operator has the general form

$$\Delta = \partial_{tt} - \Delta_2 = \sum_{i\neq j} a_{ij}\partial_{ij} + \sum_i a_i\partial_i, \tag{1.1}$$

where $i,j=1,2,3$ and at least one a_{ij} $(i\neq j)$ is nonzero. From this general form it follows that at least one of the separation equations must be of first order. The definition of separable coordinates for such a coordinate system that we adopt is that for at least one of the variables whose separation equation is first order the wave equation $\Delta\psi=0$ can be rewritten as a function of the single variable on one side and a function of the remaining two variables on the other side so that one variable "separates." The equation in the remaining two variables separates in the same manner. (There are other more complicated ways for variables to separate which do not fall within this definition; see Sec. III. In this sense our results are not entirely complete.) In addition the coordinate functions

$$t=F(x_i), \quad x=G(x_i), \quad y=H(x_i) \quad (i=1,2,3) \tag{1.2}$$

are real functions of the x_i only. For the case of R-separation the above definition carries over to the reduced wave equation, which results when the modulation factor e^Q is extracted. The function Q may, however, depend on the separation constants. For each coordinate system the two separation constants l_1 and l_2 are the eigenvalues of two operators $\mathcal{L}_1$ and $\mathcal{L}_2$ which are expressible as at most second order symmetric operators in the enveloping algebra of the O(3,2) symmetry group of (*).

We now proceed to the solution of our problem and examine the conditions which will permit a separable solution of (*). Recall that if we rewrite (*) in terms of the variables x_i the equation assumes the form

$$\Delta\psi = \partial_{tt}\psi - \Delta_2\psi = a_{11}\partial_{11}\psi + a_{22}\partial_{22}\psi + a_{33}\partial_{33}\psi + a_{12}\partial_{12}\psi + a_{13}\partial_{13}\psi + a_{23}\partial_{23}\psi + a_1\partial_1\psi + a_2\partial_2\psi + a_3\partial_3\psi = 0. \tag{1.3}$$

Here Δ is the Laplacian corresponding to the contravariant metric tensor g^{ij} in the differential form:

$$ds^2 = g^{ij}dx_i dx_j. \tag{1.4}$$

The expression for Δ in terms of the metric tensor and variables x_i is

$$\Delta = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x_i}\left(\sqrt{g}\, g_{ij}\frac{\partial}{\partial x_j}\right), \tag{1.5}$$

where $g=\det(g^{ij})$ and g_{ij} is the covariant metric tensor of our original contravariant tensor g^{ij}. (*Note*: In this article we prefer to write all our results in terms of the covariant variables x_1, x_2, x_3 as a matter of convenience.)

It is now the problem of separation of variables for the equation $\Delta\psi=0$ that is our concern. From expression (1.3) we find four possibilities.

(1) *All the separation equations are first order*: From (1.3) and the fact that $a_{ij}=2g_{ij}$ $(i\neq j)$ and $a_{ii}=g_{ii}$ we have $g_{11}=g_{22}=g_{33}=0$. Equation (1.3) then assumes the form

$$a_{12}\partial_{12}\psi + a_{13}\partial_{13}\psi + a_{23}\partial_{23}\psi + a_1\partial_1\psi + a_2\partial_2\psi + a_3\partial_3\psi = 0. \tag{1.6}$$

If the separation equation for the variable x_3 is $\beta\, dC/dx_3 + \gamma C = 0$, then (1.6) reduces to the form

$$a_{12}\partial_{12}\phi + b_1\partial_1\phi + b_2\partial_2\phi + b_0\phi = 0,$$

where $\phi = A(x_1)B(x_2)$. The condition that this equation admit a separation is that either $a_{12}=0$ or $b_1=0$, say. From the possible forms of the first order separation equations the condition $b_1=0$ requires $a_{13}=a_1=0$. In any case the covariant metric g_{ij} is singular and therefore inadmissable.

(2) *Exactly one separation equation is of second order*: If this equation is in the variable x_3, then $g_{11}=g_{22}=0$. The resulting equation has the form

$$a_{33}\partial_{33}\psi + a_{12}\partial_{12}\psi + a_{13}\partial_{13}\psi + a_{23}\partial_{23}\psi + a_1\partial_1\psi + a_2\partial_2\psi + a_3\partial_3\psi = 0. \tag{1.7}$$

For a separable solution of (1.7) it is necessary that either $a_{13}=0$ or $a_{23}=0$. We cannot choose $a_{12}=0$ as this would imply $g_{12}=0$ and hence a singular metric tensor.

(3) *Two of the separation equations are second order*: If these equations are in the variables x_2 and x_3, then a necessary condition for the separation of (with $a_{11}=0$ by hypothesis)

$$a_{22}\partial_{22}\psi + a_{33}\partial_{33}\psi + a_{12}\partial_{12}\psi + a_{13}\partial_{13}\psi + a_{23}\partial_{23}\psi + a_1\partial_1\psi + a_2\partial_2\psi + a_3\partial_3\psi = 0 \tag{1.8}$$

is that $a_{23}=2g_{23}=0$.

(4) *All the separation equations are second order*: This case is of no interest for this work as separation of variables now implies that $a_{ij}=0$ for $i\neq j$. This is the case that has been treated in II and corresponds to orthogonal coordinates.

We now proceed to those cases of interest by taking special choices of the contravariant metric g^{ij}. We enumerate the possibilities.

(I) *R*-separable differential forms in which one nondiagonal element of the covariant metric tensor is nonzero

(A) Pure separation

The most general such form of the metric tensor is

$$g^{ij} = \begin{bmatrix} a & h & 0 \\ h & 0 & 0 \\ 0 & 0 & c \end{bmatrix}, \tag{1.9}$$

corresponding to the covariant metric tensor

$$g_{ij} = \begin{bmatrix} 0 & 1/h & 0 \\ 1/h & -a/h^2 & 0 \\ 0 & 0 & 1/c \end{bmatrix}. \tag{1.10}$$

The wave equation assumes the form

$$a_{22}\partial_{22}\psi + a_{33}\partial_{33}\psi + a_{12}\partial_{12}\psi + a_1\partial_1\psi + a_2\partial_2\psi + a_3\partial_3\psi = 0 \tag{1.11}$$

We consider first the x_1 dependence of the metric coefficients a, h, and c. In order that the x_1 dependence separate out in an equation of the form $T(x_1)\partial_1 A(x_1) = KA(x_1)$, the coefficients in (1.11) must satisfy the constraints

$$a_{22}=F(x_1)\hat{a}_{22},\quad a_{33}=F(x_1)\hat{a}_{33},\quad a_2=F(x_1)\hat{a}_2,$$
$$a_3=F(x_1)\hat{a}_3,\quad a_{12}=G(x_1)\hat{a}_{12},\quad a_1=G(x_1)\hat{a}_1, \tag{1.12}$$

where the functions $\hat{a}_{ij}$ and $\hat{a}_i$ depend upon x_2 and x_3 only. These conditions imply $h=[1/G(x_1)]\hat{h}(x_2,x_3)$. By suitable redefinition of x_1 we can take $G=1$. The remaining conditions imply $c=\hat{c}/F(x_1)$, $a=F(x_1)\hat{a}$. There are then two cases to consider. (i) $F(x_1)=$ constant. (ii) $F(x_1)$ not a constant. In the latter case the form of $F(x_1)$ can be found from the requirement $a_2=F(x_1)\hat{a}_2$. This means $F'(x_1)\propto F^2$. We can therefore take $F=1/x_1$ without loss of generality. The two cases to be considered are then specified by

(1) $h=h(x_2,x_3)$, $a=a(x_2,x_3)$, and $c=c(x_2,x_3)$,

(2) $h=h(x_2,x_3)$, $a=a(x_2,x_3)/x_1$, and $c=c(x_2,x_3)x_1$,

and will be considered separately.

(1) The equations which ensure that the space is flat are obtained by equating the nontrivially zero components of the Riemannian curvature tensor R_{ijkl} to zero. For the case (1) these equations are

$$2R_{1221}=a_{22}-\frac{h_3^2}{2c}-\frac{a_2h_2}{h}=0, \tag{1.13a}$$

$$2R_{1331}=a_{33}-\frac{h_3a_3}{h}-\frac{aa_2c_2}{2h^2}-\frac{a_3c_3}{2c}+\frac{ah_3^2}{2h^2}=0, \tag{1.13b}$$

$$2R_{2332}=c_{22}-\frac{h_2c_2}{h}-\frac{c_2^2}{2c}=0, \tag{1.13c}$$

$$2R_{3221}=-h_{32}+\frac{h_2h_3}{h}+\frac{h_3c_2}{2c}=0, \tag{1.13d}$$

$$2R_{3112}=a_{32}-\frac{a_2h_3}{h}-\frac{c_2a_3}{2c}=0, \tag{1.13e}$$

$$2R_{2331}=h_{33}-\frac{a_2c_2}{2h}-\frac{h_3c_3}{2c}-\frac{h_3^2}{2h}=0. \tag{1.13f}$$

For this case we consider two possibilities: $c_2\neq 0$, $c_2=0$. If $c_2\neq 0$, then equation (1.3) has the form

$$-\frac{a}{h^2}\partial_{22}\phi+\frac{1}{c}\partial_{33}\phi+\left(\frac{2l}{h}+a_2\right)\partial_2\phi+a_3\partial_3\phi-\frac{lc_2}{2hc}\phi=0, \tag{1.14}$$

where $\phi=B(x_2)C(x_3)$ and $\partial_1A(x_1)=lA(x_1)$. Multiplying (1.14) by c, we obtain the separation condition

$$c_2/h=[f(x_2)+g(x_3)]r(x_3).$$

From (1.13c) we have

$$c_2/h=s(x_3)c^{1/2}$$

and

$$h=2f_2r/s^2.$$

Now $h\neq 0$ which implies $f\neq$ const. Accordingly we can define a new x_2 variable $x_2=f$ so that $h=h(x_3)$. From (1.13d) we then have $h_3=0$. Therefore, $h=1$ without loss of generality. The form of (1.14) now requires $a=a(x_2)$. Equation (1.13a) then implies $a=1$ or $a=0$. We also deduce that $c=t(x_2)u(x_3)$. By a suitable redefinition of x_3 we can take $c=c(x_2)$. From (1.13c) we then can take $c=x_2^2$. We finally obtain the two differential forms:

[1] $$ds^2=2dx_1dx_2+x_2^2dx_3^2, \tag{1.15}$$

[2] $$ds^2=dx_1^2+2dx_1dx_2+x_2^2dx_3^2. \tag{1.16}$$

If $c_2=0$, we can take $c=1$. From (1.14) we have the separation conditions $a/h=f(x_2)$, $h/c=r(x_2)s(x_3)$. From (1.13d) we have $h=t(x_2)u(x_3)$; hence $a=v(x_2)u(x_3)$. By redefinition of the variable x_2 we may take $h=u(x_3)$. From (1.13a) we then have

$$v_{22}=\alpha,\quad u_3^2=2\alpha u. \tag{1.17}$$

The general solutions of these equations are

$$v=\tfrac{1}{2}\alpha x_2^2+\beta x_2+\gamma,\quad u=(\sqrt{\alpha}/2x_3+\delta)^2, \tag{1.18}$$

where α,β,γ, and $\delta\in\mathbb{R}$, and $\alpha>0$. There are two classes of differential forms to consider:

(a) $\alpha=0$: We have the three possibilities

[3] $$ds^2=2dx_1dx_2+dx_3^2, \tag{1.19}$$

[4] $$ds^2=dx_1^2+2dx_1dx_2+dx_3^2, \tag{1.20}$$

[5] $$ds^2=x_2dx_1^2+2dx_1dx_2+dx_3^2. \tag{1.21}$$

(b) $\alpha=1$: We have with suitable redefinitions

[6] $$ds^2=x_3^2x_2^2dx_1^2+2x_3^2dx_1dx_2+dx_3^2, \tag{1.22}$$

[7] $$ds^2=x_3^2(x_2^2-1)dx_1^2+2x_3^2dx_1dx_2+dx_3^2, \tag{1.23}$$

[8] $$ds^2=x_3^2(x_2^2+1)dx_1^2+2x_3^2dx_1dx_2+dx_3^2. \tag{1.24}$$

This exhausts the list of separable differential forms in which the metric coefficients a,h, and c have no x_1 dependence.

(2) The equations requiring a flat space for the case of x_1 dependence have the form

$$x_1R_{1221}=\bar{R}_{1221}=0, \tag{1.25}$$
$$2x_1R_{1331}=2\bar{R}_{1331}+(\hat{a}_2\hat{c}+\hat{a}\hat{c}_2)/2h-\tfrac{1}{2}\hat{c}=0,$$
$$2R_{2332}/x_1=2\bar{R}_{2332}=0,\quad R_{3221}=\bar{R}_{3221}=0,$$
$$2x_1R_{3112}=2\bar{R}_{3112}+h_3/2\hat{c}=0,\quad R_{2331}=\bar{R}_{2331}=0.$$

Here the curvature tensor components $\bar{R}_{ijkl}$ are those in Eqs. (1.13) with $a\to\hat{a}$, $c\to\hat{c}$. Using arguments as for case (1), we find that the only forms of $\hat{a}$ and $\hat{c}$ compatible with the curvature equations are $\hat{a}=x_2$ and $\hat{c}=1$. This gives the separable differential form

[9] $$ds^2=(x_2/x_1)dx_1^2+2dx_1dx_2+x_1dx_3^2. \tag{1.26}$$

(B) R*-separation*

As regards the possibility of an R-separable solution for coordinate systems of the type considered in this subsection, it can be shown that there are in fact no such systems. We do not reproduce the somewhat lengthy but straightforward calculations which lead to this negative result.

(II) R-separable differential forms in which two nondiagonal elements of the covariant metric tensor are nonzero and only one separation equation is second order

(A) Pure separation

The contravariant metric g^{ij} can be chosen as

$$g^{ij} = \begin{bmatrix} 0 & h & 0 \\ h & b^2 & bc \\ 0 & bc & c^2 \end{bmatrix} . \quad (1.27)$$

The corresponding covariant metric tensor is

$$g_{ij} = \begin{bmatrix} 0 & 1/h & -b/hc \\ 1/h & 0 & 0 \\ -b/hc & 0 & 1/c^2 \end{bmatrix} . \quad (1.28)$$

The wave equation assumes the form

$$a_{33}\partial_{33}\psi + a_{12}\partial_{12}\psi + a_{13}\partial_{13}\psi + a_1\partial_1\psi + a_2\partial_2\psi + a_3\partial_3\psi = 0. \quad (1.29)$$

As before we consider first the x_1 dependence. The conditions for x_1 separation are

$$a_{12} = F(x_1)\hat{a}_{12}, \quad a_{13} = F(x_1)\hat{a}_{13}, \quad a_1 = F(x_1)\hat{a}_1,$$
$$a_{33} = G(x_1)\hat{a}_{33}, \quad a_2 = G(x_1)\hat{a}_2, \quad a_3 = G(x_1)\hat{a}_3. \quad (1.30)$$

These equations imply $h = \hat{h}(x_2, x_3)/F(x_1)$. By redefinition of x_1 we may as before take $F = 1$. If G is not a constant, then the above conditions require $G' \propto G^2$, and we can take $G = 1/x_1$. We again have two cases to consider:

(1) $h = h(x_2, x_3)$, $b = b(x_2, x_3)$, and $c = c(x_2, x_3)$,

(2) $h = h(x_2, x_3)$, $b = x_1^{1/2}\hat{b}(x_2, x_3)$, (1.31)

and $c = x_1^{1/2}\hat{c}(x_2, x_3)$.

(1) The curvature equations are

$$R_{1221} = -h_3^2/4c^2 = 0, \quad R_{1331} \equiv 0,$$
$$R_{2332} = b_3^2 + bb_{33} + cc_{22} - b_{32}c - b_3c_2 - c_3b_2 - c_{32}b$$
$$+ (h_2c/h)(b_3 - c_2) + (c_3/c)(b_2c + bc_2 - bb_3)$$
$$+ (bh_3/h)(c_2 - b_3) = 0, \quad (1.32)$$
$$R_{3221} = -\tfrac{1}{2}h_{32} + (h_3/4c)(bh_3/2h - c_2) = 0,$$
$$R_{3112} \equiv 0, \quad R_{2331} = \tfrac{1}{2}h_{33} - h_3c_3/2c^2 = 0.$$

These equations immediately give $h_3 = 0$ and by redefinition of x_2 we can take $h = 1$. Multiplying (1.29) by c^2 we have the further condition $bc = F(x_3)$. By redefinition of the variable x_3 we can take $bc = 1$.

The separation conditions $a_{33} = u(x_2)v(x_3)$ and $a_{13} = u(x_2)v(x_3)$ imply that b^2 and c^2 may be taken in the form

$$b^2 = F(x_3)/H(x_2), \quad c^2 = H(x_2)/F(x_3). \quad (1.33)$$

With this choice the only nontrivial curvature equation is $R_{2332} = 0$, and it has the form

$$2FF_{33} + F_3^2 + 2HH_{22} - H_2^2 = 0. \quad (1.34)$$

The separation equations for (1.34) are then

$$2FF_{33} + F_3^2 = \alpha, \quad 2HH_{22} - H_2^2 = -\alpha. \quad (1.35)$$

There are two cases to consider,

(a) $\alpha = 0$: In this case equations (1.35) have the general solution.

$$H = (\beta x_2 + \gamma)^2, \quad F = (\delta x_3 + \epsilon)^{2/3}. \quad (1.36)$$

This gives four possibilities for the differential form according as the constants $\beta, \gamma, \delta,$ and ϵ are or are not zero:

[10] $ds^2 = 2dx_1dx_2 + 2dx_2dx_3 + \omega dx_2^2 + \frac{1}{\omega}dx_3^2,$ (1.37)

[11] $ds^2 = 2dx_1dx_2 + 2dx_2dx_3 + \omega x_3^{2/3}dx_2^2 + \frac{dx_3^2}{\omega x_3^{2/3}},$ (1.38)

[12] $ds^2 = 2dx_1dx_2 + 2dx_2dx_3 + \frac{\omega}{x_2^2}dx_2^2 + \frac{x_2^2}{\omega}dx_3^2,$ (1.39)

[13] $ds^2 = 2dx_1dx_2 + 2dx_2dx_3 + \frac{\omega x_3^{2/3}}{x_2^2}dx_2^2 + \frac{x_2^2}{\omega x_3^{2/3}}dx_3^2.$ (1.40)

(b) $\alpha = 1$: In this case we can integrate Eqs. (1.35) at once to get the relations

$$dx_3 = F^{1/2}\,dF/(F + \beta)^{1/2}, \quad dx_2 = dH/(1 + \gamma H)^{1/2}. \quad (1.41)$$

Rather than integrate these equations further, we redefine the variables x_2 and x_3 by taking the new variables as H and F, respectively. We then distinguish four cases according as the constants β and γ are or are not zero. The resulting differential forms are

[14] $ds^2 = 2dx_1dx_2 + 2dx_2dx_3 + \frac{x_3}{x_2}dx_2^2 + \frac{x_2}{x_3}dx_3^2,$ (1.42)

[15] $ds^2 = 2dx_1dx_2 + \frac{2x_3^{1/2}}{(x_3+\beta)^{1/2}}dx_2dx_3 + \frac{x_3}{x_2}dx_2^2 + \frac{x_2}{(x_3+\beta)}dx_3^2,$ (1.43)

[16] $ds^2 = \frac{2}{(1+\gamma x_2)^{1/2}}dx_1dx_2 + \frac{2}{(1+\gamma x_2)^{1/2}}dx_2dx_3 + \frac{x_3}{x_2(1+\gamma x_2)}dx_2^2 + \frac{x_2}{x_3}dx_3^2,$ (1.44)

[17] $ds^2 = \frac{2}{(1+\gamma x_2)^{1/2}}dx_1dx_2 + \frac{2x_3^{1/2}dx_2dx_3}{[(1+\gamma x_2)(x_3+\beta)]^{1/2}} + \frac{x_3dx_2^2}{x_2(1+\gamma x_2)} + x_2\frac{dx_3^2}{(x_3+\beta)}.$ (1.45)

(2) For the case of x_1 dependence the curvature equation $R_{1331} = 0$ reduces to $c = 0$, which is inadmissable. There are therefore no solutions of interest in this class.

(B) R-separation

If we assume that ψ in (*) has an R-separable solution of the form $\psi = e^R\phi$, then the equation satisfied by ϕ has the form

$$b_{33}\partial_{33}\phi + b_{12}\partial_{12}\phi + b_{13}\partial_{13}\phi + b_1\partial_1\phi + b_2\partial_2\phi + b_3\partial_3\phi + b_0\phi = 0, \tag{1.46}$$

where the b_{ij}, b_i are related to the a_{ij}, a_i in (1.6) by the equations

$$\begin{aligned} &b_{33}=a_{33}, \quad b_{12}=a_{12}, \quad b_{13}=a_{13},\\ &b_1=a_{12}R_2+a_{13}R_3+a_1, \quad b_2=a_{12}R_1+a_2,\\ &b_3=a_{13}R_1+2a_{33}R_3+a_3,\\ &b_0=a_{12}(R_{12}+R_1R_2)+a_{13}(R_{13}+R_1R_3)+a_{33}(R_{33}+R_3^2)\\ &\quad +a_3R_3+a_2R_2+a_1R_1. \end{aligned} \tag{1.47}$$

As usual, we look at the possibilities for x_1 dependence. The conditions on the coefficients of (1.46) are

$$\begin{aligned} &b_{33}=G(x_1)\hat{b}_{33}, \quad b_2=G(x_1)\hat{b}_2, \quad b_3=G(x_1)\hat{b}_3,\\ &b_0=G(x_1)\hat{b}_0, \quad b_{12}=F(x_1)\hat{b}_{12}, \quad b_{13}=F(x_1)\hat{b}_{13},\\ &b_1=F(x_1)\hat{b}_1. \end{aligned} \tag{1.48}$$

As in the case of pure separation, $F=1$ by redefinition of the variable x_1 and, consequently, $h=h(x_2,x_3)$. The remaining conditions require, as in the case of pure separation, that $G=\text{const}$ or $G\propto x_1^{1/2}$. This latter case is inadmissable by the curvature conditions.

We may then take $h=1$ and $c=c(x_2)$. The condition $b_2 = b_2(x_2,x_3)$ requires that R have the form $x_1u(x_2,x_3) + v(x_2,x_3)$. If the x_1 dependence in (1.46) is now extracted via the separation equation $dA(x_1)/dx_1 = lA(x_1)$, the resulting equation has the form

$$b_{33}\partial_{33}\phi + (lb_{12}+b_2)\partial_2\phi + (lb_{13}+b_3)\partial_3\phi + (b_1 l + b_0)\phi = 0, \tag{1.49}$$

where $\phi = B(x_2)C(x_3)$. The separation condition $lb_{13}+b_3 = s(x_2,x_3)$ implies $u(x_2,x_3)=0$. The further condition that $c^2(lb_{13}+b_3)=t(x_3)$ requires that $R_3 = lbc$ to within a sum of functions of single variables. The only nontrivial curvature equation is

$$R_{2332} = bb_{33} + b_3^2 + cc_{22} - b_{23}c - c_2b_3 = 0, \tag{1.50}$$

which has the solution $b_3 = c_2$ so that $b = c_2x_3 + g(x_2)$ and the modulation function R has the form

$$R = \tfrac{1}{2}lcc_2x_3^2 + lcgx_3. \tag{1.51}$$

Finally from the requirement $b_1 l + b_0 = v(x_2) + w(x_3)$ we obtain the constraints

$$c^3c_{22}=\beta, \quad c^3g_2=\gamma, \tag{1.52}$$

with $\beta,\gamma \in \mathbb{R}$. The general solution of the first equation is $c=(\delta x_2^2+\epsilon)^{1/2}$. We now evaluate the possibilities depending on the values of the constants δ,ϵ:

(i) $\delta=0$ and $\epsilon=1$; then $g=\omega x_2$: The resulting metric is

$$[18] \quad ds^2 = 2dx_1\,dx_2 + 2\omega x_2\,dx_2\,dx_3 + dx_3^2 + \omega^2x_2^2\,dx_2^2, \tag{1.53}$$

and the modulation function is $R=\omega l x_2 x_3$.

(ii) $\epsilon=0$ and $\delta=1$; then $g=\omega/x_2^2$ and the differential form is

$$[19] \quad ds^2 = 2dx_1\,dx_2 + 2(x_2x_3+\omega/x_2)\,dx_2\,dx_3 + (x_3+\omega/x_2^2)^2\,dx_2^2 + x_2^2\,dx_3^2. \tag{1.54}$$

The modulation function is then $R=\tfrac{1}{2}lx_2x_3^2 + \omega l x_3/x_2$.

(iii) $\delta=\epsilon=1$; then $g=\omega$ and the differential form is

$$[20] \quad ds^2 = 2dx_1\,dx_2 + 2[x_2x_3+\omega(1+x_2^2)^{1/2}]\,dx_2\,dx_3 + [x_2x_3/(1+x_2^2)^{1/2}+\omega]^2\,dx_2^2 + (1+x_2^2)\,dx_3^2, \tag{1.55}$$

with the modulation function given by

$$R = \tfrac{1}{2}lx_2x_3^2 + l\omega x_3(1+x_2^2)^{1/2}$$

(iv) $\delta=-\epsilon=1$: In this case $g=\omega$ and the differential form is

$$[21] \quad ds^2 = 2dx_1\,dx_2 + 2[\nu x_2x_3 + \omega(|1-x_2^2|)^{1/2}]\,dx_2\,dx_3 + [\nu x_3x_2/(|1-x_2^2|)^{1/2}+\omega]^2\,dx_2^2 + |1-x_2^2|\,dx_3^2, \tag{1.56}$$

where $\nu=\operatorname{sgn}(-1+x_2^2)$. The modulation function is $R = \tfrac{1}{2}\nu l x_2 x_3^2 + l\omega x_3 |1-x_2^2|^{1/2}$. This completes our list of coordinate systems of this type.

(III) *R*-separable differential forms in which two nondiagonal elements of the covariant metric tensor are nonzero and two separation equations are of second order

Pure separation

The determination of the contravariant metric is rather involved. The wave equation for coordinate systems of this type will be taken as

$$a_{22}\partial_{22}\psi + a_{33}\partial_{33}\psi + a_{12}\partial_{12}\psi + a_{13}\partial_{13}\psi + a_1\partial_1\psi + a_2\partial_2\psi + a_3\partial_3\psi = 0. \tag{1.57}$$

The contravariant metric can then be taken to be

$$g^{ij} = \begin{bmatrix} a & f & abc/f \\ f & b^2 & bc \\ abc/f & bc & c^2 \end{bmatrix} \tag{1.58}$$

so that the components of the covariant metric tensor are

$$g_{ij} = \frac{1}{(f^2-ab^2)} \begin{bmatrix} 0 & f & -bf/c \\ f & -a & 0 \\ -bf/c & 0 & f^2/c^2 \end{bmatrix}. \tag{1.59}$$

From the conditions for separation of the x_1 variable, which we do not repeat here [these are the analogs of Eqs. (1.30)], we find

$$a=G(x_1)\hat{a}, \quad b=\hat{b}/\sqrt{G(x_1)}, \quad c=\hat{c}/\sqrt{G(x_1)}, \tag{1.60}$$

where $G=1$ or $1/x_1$. There are then two distinct cases to consider:

(1) $f=f(x_2,x_3)$, $a=a(x_2,x_3)$, $b=b(x_2,x_3)$, and $c=c(x_2,x_3)$.

(2) $f=f(x_2,x_3)$, $a=\hat{a}(x_2,x_3)/x_1$, $b=\hat{b}(x_2,x_3)x_1^{1/2}$, and $c=\hat{c}(x_2,x_3)x_1^{1/2}$.

(1) From the separation conditions $a_{12}/a_{22}=r(x_2)$ and $a_{13}/a_{33}=s(x_3)$ we have the relations $a=t(x_2)f$, $bc=u(x_3)f$, and $h=abc/f=t(x_2)u(x_3)f$. By suitable redefinition of the variables x_2 and x_3 these relations can be reduced to $a=f$, $bc=f$, $h=f$. [Note these results follow also for (2)

with a, b, and c replaced by $\hat{a}, \hat{b}$, and $\hat{c}$.] With $d=f^2/c^2$ the contravariant metric then assumes the relatively simple form

$$g^{ij}=\begin{bmatrix} f & f & f \\ f & d & f \\ f & f & f^2/d \end{bmatrix} \tag{1.61}$$

with corresponding covariant metric tensor

$$g_{ij}=\frac{1}{f(f-d)}\begin{bmatrix} 0 & f & -b \\ f & -f & 0 \\ -b & 0 & b \end{bmatrix}. \tag{1.62}$$

For this case the curvature equations are

$$2R_{1221}=f_{22}+[1/2(f-d)] \times[(2-d/f)f_2^2+f_2d_2-(d/f)f_3^2+(d/f)f_3d_3]=0, \tag{1.63a}$$

$$2R_{1331}=f_{33}+[1/2(f-d)] \times[(f/d)f_3d_3+(f^2/d^2)f_2d_2+(1-2f/d)f_2^2 +(2d-3f)f_3^2]=0, \tag{1.63b}$$

$$2R_{2332}=d_{33}-2f_{33}+(2f/d)f_{32}-(f^2/d^2)d_{22}+[1/(f-d)] \times[-(2f/d)f_2^2+(3f/d^2)(2d-f)f_2d_2+\tfrac{1}{2}(1+f/d)d_3^2 -4f_2d_3+2(2-d/f)f_2f_3+(d/f-2)f_3d_3-(f/d)d_2f_3] =0, \tag{1.63c}$$

$$2R_{3221}=f_{22}-f_{23}+[1/(f-d)][\tfrac{1}{2}f_2d_2+(d/2f-1)f_2^2 +(d/2f)f_3d_3+(2-d/f)f_2f_3-(f/2d)f_3d_2-\tfrac{1}{2}f_2d_3 +(b/2f)f_3^2]=0, \tag{1.63d}$$

$$2R_{3112}=f_{32}+[1/(f-d)] \times(f/2d)f_3d_2+(d/f-2)f_2f_3+\tfrac{1}{2}f_2d_3]=0, \tag{1.63e}$$

$$2R_{2331}=f_{33}-f_{32}+[1/(f-d)][(f/2d)f_3d_3+(d/2f-1)f_2^2 +(d/2f)f_3d_3+(2-d/f)f_2f_3-(f/2d)f_3d_2-\tfrac{1}{2}f_2d_3 +(d/2f)f_3^2]=0.$$

From these equations we deduce

$$2R_{1332}+2R_{3112}-2R_{1331}=[ff_2/d(f-d)][-(f/2d)d_2+f_2]=0. \tag{1.64}$$

There are then two possibilities: (i) $f_2=0$ or (ii) $d=r(x_3)f^2$. We consider each of these cases separately.

(i) From (1.63c) we have that $f_3d_2=0$ so that either $f_3=0$ or $d_2=0$. In the first case we can take $f=1$. Equation $R_{2332}=0$ requires

$$d_{33}+\frac{(1+d)}{2d(1-d)}d_3^2-\frac{d_{22}}{d^2}+\frac{(5d-3)}{2d(d-1)}\frac{d_2^2}{d^2}=0. \tag{1.65}$$

The separation condition $a_{33}/a_{22}=r(x_2)s(x_3)$ must also be satisfied. There are then three possibilities of this type.

a. $d=d(x_3)$: The variable x_3 can be redefined to be d via the relation

$$d_3=d^{-1/2}-d^{1/2}. \tag{1.66}$$

The corresponding differential form is

$$[22]\quad ds^2=2dx_1\,dx_2+[2x_3^{1/2}/(1-x_3)]\,(dx_1\,dx_3+dx_2\,dx_3) +dx_1^2+x_3\,dx_2^2+dx_3^2/(1-x_3)^2. \tag{1.67}$$

b. $d=d(x_2)$: The variable x_2 can be redefined to be d via the relation

$$d_2=d^{3/2}(d-1). \tag{1.68}$$

$$[23]\quad ds^2=[2/(x_2-1)x_2^{3/2}]\,(dx_1\,dx_2+dx_2\,dx_3)+2dx_1\,dx_3 +dx_1^2+dx_2^2/(x_2-1)^2x_2^2+dx_3^2/x_2. \tag{1.69}$$

c. $d=\omega$, const: The differential form is

$$[24]\quad ds^2=2dx_1\,dx_2+2dx_1\,dx_3+2dx_2\,dx_3+dx_1^2 +\omega dx_2^2+(1/\omega)\,dx_3^2. \tag{1.70}$$

In addition we must consider the case when $f_3\neq 0$ and $d_2=0$. From (1.63a) this implies $f_3=d_3$ so that $f=d+\delta$ with $\delta\neq 0$. Integrating (1.63c) once, we get $d_3=\delta d^{-1/2}+d^{1/2}$. The variable x_3 can then be redefined to be d. The resulting differential form is

$$[25]\quad ds^2=2(x_3+\delta)\,dx_1\,dx_2+2x_3^{1/2}(dx_2\,dx_3+dx_1\,dx_3) +(x_3+\delta)\,dx_1^2+x_3\,dx_2^2+dx_3^2. \tag{1.71}$$

(ii) In this case the separation condition $a_{33}/a_{22}=u(x_2)v(x_3)$ ensures that d and f can each be expressed as products of functions in each of the variables x_2 and x_3. We may therefore take $f=h(x_2)r(x_3)$, $d=h^2(x_2)s(x_3)$. If r and s are both constants, then (1.63f) implies $h_2=0$. This case has already been found and corresponds to (1.69), (1.70). For nonconstant r and $s=\text{const}=1$, (1.63a) can be put into the form

$$\frac{h_2^2}{2h}-h_{22}=\frac{1}{2(r-h)}\left(-\frac{h_2^2}{h}r+2h_2^2-\frac{h^2r_3^2}{r^2}\right). \tag{1.72}$$

For the right-hand side of (1.72) to be a function of x_2 only, we require that $h=\exp(x_2)$ and $r=\exp(x_3)$. By choosing now variables h and r the differential form becomes

$$[26]\quad ds^2=2x_3\,dx_1\,dx_2+2x_2\,dx_1\,dx_3+2dx_2\,dx_3 +x_2x_3\,dx_1^2+dx_2^2+dx_3^2. \tag{1.73}$$

It is not hard to show that this is the only form of the functions r and s which are compatible with the curvature equations.

(2) For the case of explicit x_1 dependence it can be shown by straightforward but lengthy calculations that there are no differential forms of this type. Similar remarks apply to the case of R-separation.

This concludes our derivation of the differential forms.

II. EXPLICIT COORDINATES AND R-SEPARABLE SOLUTIONS

Here we present the list of coordinates corresponding to the differential forms given in Sec. I. We also present with each coordinate system the separation equations and a representative solution where possible. We connect the listed coordinate systems with the symmetry group of (*) by giving the operators which specify the separation constants in terms of the generators of the symmetry algebra. We need only recall here the form of the generators in the coordinate representation. [For more information on the group structure associated with (*) we refer the reader to Paper 8 in this series.] The generators are

1. Translations: $P_0=\partial_t,\quad P_1=\partial_x,\quad P_2=\partial_y.$ (2.1)

2. Two-dimensional Lorentz subgroup SO(2,1):

$$M_{12}=x\,\partial_y-y\,\partial_x,\quad M_{01}=t\partial_x+x\partial_t,\quad M_{02}=t\partial_y+y\partial_t. \tag{2.2}$$

3. Dilatation: $D=-t\partial_t-x\partial_x-y\partial_y-\frac{1}{2}.$ (2.3)

4. Special conformal transformations:

$$\begin{aligned}K_0&=-t-(t^2+x^2+y^2)\partial_t-2tx\partial_x-2t_y\partial_y,\\ K_1&=x+(t^2+x^2-y^2)\partial_x+2xt\partial_t+2xy\,\partial_y,\\ K_2&=y+(t^2+y^2-x^2)\partial_y+2yt\partial_t+2yx\partial_x.\end{aligned} \tag{2.4}$$

In a number of cases we give simpler forms of the differential forms than given in Sec. I. This is achieved by making use of earlier results in this series of papers and is mentioned when it occurs. We now list the coordinate systems:

[1] $ds^2=2dx_1\,dx_2+x_2^2\,dx_3^2.$ (2.5)

The coordinates are given by

$$\begin{aligned}&t=x_1+\tfrac{1}{2}x_2x_3^2+x_2(x_3+1),\quad x=x_1+\tfrac{1}{2}x_2x_3^2+x_2x_3,\\ &y=x_2(x_3+1).\end{aligned} \tag{2.6}$$

The separation equations are

$$\frac{dA}{dx_1}=l_1A,\quad 2x_2^2\,\frac{dB}{dx_2}-x_2B=\frac{l_2}{l_1}B,\quad \frac{d^2C}{dx_3^2}+l_2C=0, \tag{2.7}$$

where $\psi=A(x_1)B(x_2)C(x_3)$ is a separable solution of (*). A typical solution is

$$\psi=\exp(l_1x_1)x_2^{1/2}\,\exp(-l_2/2l_1x_2)\begin{cases}\cos\sqrt{l_2}\,x_3\\ \sin\sqrt{l_2}\,x_3\end{cases}. \tag{2.8}$$

The operators $\mathcal{L}_1$ and $\mathcal{L}_2$ which specify this coordinate system are given in terms of the generators by

$$\mathcal{L}_1=P_0+P_1,\quad \mathcal{L}_2=(M_{12}-M_{02})^2. \tag{2.9}$$

[2] $ds^2=dx_1^2+2dx_1\,dx_2+x_2^2\,dx_3^2.$ (2.10)

The three space coordinates are given by

$$t=x_2\cosh x_3,\quad x=x_2\sinh x_3,\quad y=x_1+x_2. \tag{2.11}$$

The separation equations are

$$\begin{aligned}&\frac{dA}{dx_1}=l_1A,\quad \frac{d^2B}{dx_2^2}+\left(\frac{1}{x_2}-2l_1\right)\frac{dB}{dx_2}+\left(\frac{l_2}{x_2^2}-\frac{l_1}{x_2}\right)B=0,\\ &\frac{d^2C}{dx_3^2}=l_2C.\end{aligned} \tag{2.12}$$

A typical solution is

$$\psi=\exp(l_1x_1)\exp(l_2x_2)\,C_{\sqrt{l_2}}\,(il_1x_2)\exp(i\sqrt{l_2}\,x_3), \tag{2.13}$$

where $C_\nu(z)$ is a solution of Bessel's equation. The operators which specify the coordinate system are

$$\mathcal{L}_1=P_2,\quad \mathcal{L}_2=M_{01}^2. \tag{2.14}$$

[3] $ds^2=2dx_1\,dx_2+dx_3^2.$ (2.15)

The three space coordinates are

$$\sqrt{2}\,t=x_1+x_2,\quad \sqrt{2}\,x=x_1-x_2,\quad y=x_3. \tag{2.16}$$

The separation equations have the form

$$\frac{dA}{dx_1}=l_1A,\quad \frac{dB}{dx_2}=l_2B,\quad \frac{d^2C}{dx_3^2}=-2\,l_1l_2C. \tag{2.17}$$

A typical solution is

$$\psi=\exp(l_1x_1+l_2x_2)\begin{cases}\cos\sqrt{2l_1l_2}x_3\\ \sin\sqrt{2l_1l_2}x_3\end{cases}. \tag{2.18}$$

The operators which specify the coordinate system are

$$\mathcal{L}_1=\sqrt{2}\,(P_0+P_1),\quad \mathcal{L}_2=\sqrt{2}\,(P_0-P_1). \tag{2.19}$$

[4] $ds^2=dx_1^2+2dx_1\,dx_2+dx_3^2.$ (2.20)

The three space coordinates are

$$\begin{aligned}&\text{(i) } t=x_1,\quad x=x_1+x_2,\quad y=x_3,\\ &\text{(ii) } t=x_1+x_2,\quad x=x_2,\quad y=x_3.\end{aligned} \tag{2.21}$$

The separation equations have the form

$$\frac{dA}{dx_1}=l_1A,\quad \frac{d^2B}{dx_2^2}-2l_1\,\frac{dB}{dx_2}-l_2B=0,\quad \frac{d^2C}{dx_3^2}=\pm l_2C. \tag{2.22}$$

A typical solution is

$$\psi=\exp(l_1x_1)\exp(l_1x_2)\begin{Bmatrix}\cos\sqrt{l_2+l_1^2}x_2\\ \sin\sqrt{l_2+l_1^2}\,x_2\end{Bmatrix}\begin{cases}\cos\sqrt{\pm l_2}x_3\\ \sin\sqrt{\pm l_2}x_3\end{cases}. \tag{2.23}$$

The operators which specify the coordinate system are

$$\begin{aligned}&\text{(i)}\quad \mathcal{L}_1=P_0+P_1,\quad \mathcal{L}_2=P_2^2,\\ &\text{(ii)}\quad \mathcal{L}_1=P_0,\quad \mathcal{L}_2=P_2^2.\end{aligned} \tag{2.24}$$

[5] $ds^2=4\,\dfrac{x_2dx_1^2}{x_1^2}+4\,\dfrac{dx_1\,dx_2}{x_1}+dx_3^2.$ (2.25)

The three space coordinates are

$$t=x_2x_1-1/x_1,\quad x=x_2x_1+1/x_1,\quad y=x_3. \tag{2.26}$$

The separation equations are

$$\frac{dA}{dx_1}=l_1A,\quad x_2\,\frac{d^2B}{dx_2^2}+(1-2l_1)\,\frac{dB}{dx_2}-l_2B=0,\quad \frac{d^2C}{dx_3^2}=l_2C. \tag{2.27}$$

A typical solution is

$$\psi = x_1^{l_1} x_2^{l_1/2}\, C_{l_1}(2i\sqrt{l_2 x_2}) \begin{cases} \cos\sqrt{-l_2}\, x_3 \\ \sin\sqrt{-l_2}\, x_3 \end{cases}. \qquad (2.28)$$

The operators which specify the coordinate system are

$$\mathcal{L}_1 = M_{01}, \quad \mathcal{L}_2 = P_2^2. \qquad (2.29)$$

[6] $ds^2 = x_3^2 x_2^2 dx_1^2 + 2x_3^2 dx_1 dx_2 + dx_3^2. \qquad (2.30)$

The three space coordinates are

$$\begin{aligned} t &= x_3[\tfrac12 x_2(1-Ex_1) + (E-1/E)(1-Ex_1)(1-x_1x_2/2) \\ &\quad + 1/E\,(1-x_1x_2/2)], \\ x &= x_3[1 - 2(1-Ex_1)(1-x_1x_2/2)], \\ y &= x_3[\tfrac12 x_2(1-Ex_1) + (E+1/E)(1-Ex_1)(1-x_1x_2/2) \\ &\quad - (1/E)(1-x_1x_2/2)], \\ &E \in \mathbb{R}. \end{aligned} \qquad (2.31)$$

The separation equations are

$$\frac{dA}{dx_1} = l_1 A, \quad x_2^2 \frac{d^2B}{dx_2^2} + 2(x_2 - l_1)\frac{dB}{dx_2} - l_2 B = 0, \qquad (2.32)$$
$$x_2^2 \frac{d^2C}{dx_3^2} + 2x_3 \frac{dC}{dx_3} - l_2 C = 0.$$

A typical solution is

$$\psi = \exp[l_1(x_1 - 1/x_2)]\, C_{j+1/2}(il_1/x_2) x_2^{-1/2} \begin{cases} x_3^j \\ x_3^{-j-1} \end{cases}, \qquad (2.33)$$

where $l_2 = j(j+1)$. The operators which specify the coordinate system are

$$\mathcal{L}_1 = (E^2+1)M_{12} + (E^2-1)M_{01} - 2EM_{02}, \quad \mathcal{L}_2 = -\tfrac14 + D^2. \qquad (2.34)$$

[7] $ds^2 = x_3^2(x_2^2 - 1)dx_1^2 + 2x_3^2 dx_1 dx_2 + dx_3^2. \qquad (2.35)$

There are two alternative parametrizations in three space which correspond to the above differential form. They are

(i) $$\begin{aligned} t &= x_3[(e^{x_1}+E) + \tfrac12 e^{-x_1}(x_2+1)(e^{x_1}+E)^2 - 2e^{-x_1}(x_2+1)] \\ x &= x_3[1 - 2e^{-x_1}(x_2+1)(E+e^{x_1})], \qquad (2.36) \\ y &= x_3[(e^{x_1}+E) + \tfrac12 e^{-x_1}(x_2+1)(e^{x_1}+E)^2 + 2e^{-x_1}(x_2+1)], \end{aligned}$$

where $E \in \mathbb{R}$

(ii) $$\begin{aligned} t &= x_3\{(4/\alpha)(E - \coth\tfrac12 x_1)[1 + (E\sinh\tfrac12 x_1 - \cosh\tfrac12 x_1) \\ &\quad \times (\cosh\tfrac12 x_1 - x_2\sinh\tfrac12 x_1)] \\ &\quad + (\alpha/4)\sinh\tfrac12 x_1(\cosh\tfrac12 x_1 - x_2\sinh\tfrac12 x_1)\}, \qquad (2.37) \\ x &= x_3[1 - 2(E\sinh\tfrac12 x_1 - \cosh\tfrac12 x_1)(\cosh\tfrac12 x_1 - x_2\sinh\tfrac12 x_1)], \\ y &= x_3[(4/\alpha)(E - \coth\tfrac12 x_1)[1 + (E\sinh\tfrac12 x_1 - \cosh\tfrac12 x_1) \\ &\quad \times(\cosh\tfrac12 x_1 - x_2\sinh\tfrac12 x_1)] \\ &\quad - (\alpha/4)\sinh\tfrac12 x_1(\cosh\tfrac12 x_1 - x_2\sinh\tfrac12 x_1)], \end{aligned}$$

where $\alpha,\ E \in \mathbb{R}$ and $\alpha \neq 0$.

The separation equations are

$$\frac{dA}{dx_1} = l_1 A, \quad (1 - x_2^2)\frac{d^2B}{dx_2^2} + (2l_1 - x_2)\frac{dB}{dx_2} + l_2 B = 0, \qquad (2.38)$$
$$x_3^2 \frac{d^2C}{dx_3^2} + 2x_3\frac{dC}{dx_3} - l_2 C = 0.$$

A typical solution is

$$\psi = e^{l_1 x_1} \left(\frac{x_2 - 1}{x_2 + 1}\right)^{l_1/2} \begin{Bmatrix} P_j^{l_1}(x_2) \\ Q_j^{l_1}(x_2) \end{Bmatrix} \begin{Bmatrix} x_3^j \\ x_3^{-j-1} \end{Bmatrix}, \qquad (2.39)$$

where $l_2 = j(j+1)$. The functions $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ are Legendre functions of the first and second kind respectively. The operators which specify the coordinate system are

(i) $\mathcal{L}_1 = 2M_{02} - E(M_{12} + M_{01}), \quad \mathcal{L}_2 = -\tfrac14 + D^2,$

(ii) $\mathcal{L}_1 = (4/\alpha)(E^2-1)(M_{01}+M_{12}) + (\alpha/4)(M_{12} - M_{01}) - 2EM_{02}, \quad \mathcal{L}_2 = -\tfrac14 + D^2. \qquad (2.40)$

[8] $ds^2 = x_3^2(x_2^2+1)dx_1^2 + 2x_3^2 dx_1 dx_2 + dx_3^2. \qquad (2.41)$

The three space coordinates are given by

$$\begin{aligned} t &= x_3[-(4/\alpha)(E + \tan\tfrac12 x_1) \\ &\quad \times [1 + (\sin\tfrac12 x_1 + E\cos\tfrac12 x_1)(\sin\tfrac12 x_1 + x_2\cos\tfrac12 x_1)] \\ &\quad - (\alpha/4)\cos\tfrac12 x_1(\sin\tfrac12 x_1 + x_2\cos\tfrac12 x_1)], \\ x &= x_3[1 - 2(\sin\tfrac12 x_1 + x_2\cos\tfrac12 x_1)(\sin\tfrac12 x_1 + E\cos\tfrac12 x_1)], \qquad (2.42) \\ y &= x_3[-(4/\alpha)(E + \tan\tfrac12 x_1)[1 + (\sin\tfrac12 x_1 + E\cos\tfrac12 x_1) \\ &\quad \times (\sin\tfrac12 x_1 + x_2\cos\tfrac12 x_1)] \\ &\quad + (\alpha/4)\cos\tfrac12 x_1(\sin\tfrac12 x_1 + x_2\cos\tfrac12 x_1)], \end{aligned}$$

where $\alpha,\ E \in \mathbb{R}$ and $\alpha > 0$.

The separation equations are

$$\frac{dA}{dx_1} = l_1 A, \quad (1 + x_2^2)\frac{d^2B}{dx_2^2} + 2(x_2 - l_1)\frac{dB}{dx_2} - l_2 B = 0, \qquad (2.43)$$
$$x_3^2 \frac{d^2C}{dx_3^2} + 2x_3\frac{dC}{dx_3} - l_2 C = 0.$$

A typical solution is

$$\psi = \exp(l_1 x_1)\left(\frac{x_2 - 1}{x_2 + 1}\right)^{il_1/2} \begin{Bmatrix} P_j^{il_1}(ix_2) \\ Q_j^{il_1}(ix_2) \end{Bmatrix} \begin{Bmatrix} x_3^j \\ x_3^{-j-1} \end{Bmatrix}, \qquad (2.44)$$

where as usual $l_2 = j(j+1)$. The operators which specify the coordinate system are

$$\mathcal{L}_1 = (4/\alpha)(1+E^2)(M_{01}+M_{12}) + (\alpha/4)(M_{12} - M_{01}) + 2EM_{02}, \quad \mathcal{L}_2 = -\tfrac14 + D^2. \qquad (2.45)$$

[9] $ds^2 = (x_2/x_1)dx_1^2 + 2dx_1 dx_2 + x_1 dx_3^2. \qquad (2.46)$

The three space coordinates are given by

$$t+x=2x_2\sqrt{x_1}-\tfrac{1}{2}x_3^2\sqrt{x_1},\quad t-x=-2\sqrt{x_1},\quad y=x_3\sqrt{x_1}. \quad (2.47)$$

The separation equations are

$$x_1\frac{dA}{dx_1}=l_1A,\quad x_2\frac{d^2B}{dx_2^2}+(2l_1-\tfrac{1}{2})\frac{dB}{dx_2}+l_2B=0,\quad \frac{d^2C}{dx_3^2}=l_2C. \quad (2.48)$$

A typical solution is

$$\psi=(x_1/x_2)^{l_1}x_2^{3/4}C_{3-4l_1}(2\sqrt{l_2x_2})\begin{cases}\cos\sqrt{l_2}\,x_3\\ \sin\sqrt{l_2}\,x_3\end{cases}. \quad (2.49)$$

The operators which specify this coordinate system are

$$\mathcal{L}_1=-\tfrac{1}{2}D-\tfrac{1}{4},\quad \mathcal{L}_2=\tfrac{1}{4}(M_{12}-M_{02})^2. \quad (2.50)$$

[10] $ds^2=2dx_1\,dx_2+2dx_2\,dx_3+\omega dx_2^2+(1/\omega)\,dx_3^2.$ (2.51)

The three space coordinates are

$$t+x=2x_1+2(1-E/\sqrt{\omega})x_3+(\omega-E^2)x_2,$$
$$t-x=-x_2,\quad y=x_3/\sqrt{\omega}+Ex_2. \quad (2.52)$$

The separation equations are

$$\frac{dA}{dx_1}=l_1A,\quad \frac{dB}{dx_2}=l_2B,\quad \omega^2\frac{d^2C}{dx_3^2}-2\omega^2l_1\frac{dC}{dx_3}+2l_1l_2C=0. \quad (2.53)$$

A typical solution is

$$\psi=\exp(l_1x_1)\exp(l_2x_2)\exp(l_1x_3)\begin{cases}\cos[l_1(2l_2-\omega^2l_1)]^{1/2}x_3\\ \sin[l_1(2l_2-\omega^2l_1)]^{1/2}x_3\end{cases}. \quad (2.54)$$

The operators which specify this coordinate system are

$$\mathcal{L}_1=P_0+P_1,\quad \mathcal{L}_2=\tfrac{1}{2}(\omega-E^2)(P_0+P_1)+\tfrac{1}{2}(P_1-P_0)+EP_2. \quad (2.55)$$

[11] $ds^2=2dx_1\,dx_2+2dx_2\,dx_3+\omega x_3^{2/3}dx_2^2+dx_3^2/\omega x_3^{2/3}.$ (2.56)

The three space coordinates are

$$t+x=2x_1+2x_3+(\omega x_2-3E/2\sqrt{\omega})x_3^{2/3}-\omega^{-3/2}(\tfrac{1}{3}\omega x_2^{2/3}-E)^3,$$
$$t-x=-x_2,\quad y=(3/2\sqrt{\omega})x_3^{2/3}-\tfrac{1}{6}\omega^{3/2}x_2^2+Ex_2. \quad (2.57)$$

The separation equations are

$$\frac{dA}{dx_1}=l_1A,\quad \frac{dB}{dx_2}=l_2B,$$
$$x_3\frac{d^2C}{dx_3^2}+(\tfrac{1}{3}-2l_1x_3)\frac{dC}{dx_3}-(\tfrac{1}{3}l_1+l_2/\omega\,x_3^{1/3})C=0. \quad (2.58)$$

The operators which specify this coordinate system are

$$\mathcal{L}_1=P_0+P_1,$$
$$\mathcal{L}_2=\tfrac{1}{3}\omega^{3/2}(M_{02}-M_{12})+\tfrac{1}{2}(P_1-P_0)-\tfrac{1}{2}E^2(P_0+P_1)+EP_2. \quad (2.59)$$

[12] $ds^2=2dx_1\,dx_2+2dx_2\,dx_3+(\omega/x_2^2)\,dx_2^2+(x_2^2/\omega)\,dx_3^2.$ (2.60)

The three space coordinates are given by

$$t+x=2x_1+2x_3-x_2x_3^2/\omega-(2E/\sqrt{\omega})x_2x_3-E^2x_2-\omega/x_2,$$
$$t-x=-x_2,\quad y=x_2x_3/\sqrt{\omega}+Ex_2. \quad (2.61)$$

The separation equations are

$$\frac{dA}{dx_1}=l_1A,\quad 2x_2^2\frac{dB}{dx_2}+(l_2+x_2)B=0,$$
$$\frac{d^2C}{dx_3^2}-2l_1\frac{dC}{dx_3}-\frac{l_1l_2}{\omega}C=0. \quad (2.62)$$

A typical solution is

$$\psi=\exp(l_1x_1)x_2^{-1/2}\exp(l_2/2x_2)\exp(l_1x_3)\begin{cases}\cosh\sqrt{l_1^2+l_1l_2/\omega}\,x_3\\ \sinh\sqrt{l_1^2+l_1l_2/\omega}\,x_3.\end{cases} \quad (2.63)$$

The operators which specify this coordinate system are

$$\mathcal{L}_1=P_0+P_1,\quad \mathcal{L}_2=2\omega(P_0+P_1)+K_0+K_1. \quad (2.64)$$

[13] $ds^2=2dx_1\,dx_2+2dx_2\,dx_3+(\omega x_3^{2/3}/x_2^2)\,dx_2^2+(x_2^2/\omega x_3^{2/3})\,dx_3^2.$ (2.65)

The three space coordinates are given by

$$t+x=2x_1+2x_3-(9/4\omega)x_3^{4/3}x_2+[\omega/2x_2-(3E/\sqrt{\omega})x_2]x_3^{2/3}-E^2x_2+E\omega^{3/2}/3x_2-\omega^3/108x_2^3, \quad (2.66)$$
$$t-x=-x_2,\quad y=(3/2\sqrt{\omega})x_2x_3^{2/3}+\omega^{3/2}/6x_2+Ex_2.$$

The separation equations are

$$\frac{dA}{dx_1}=l_1A,\quad 2x_2^2\frac{dB}{dx_2}+\left(\frac{l_2}{l_1}+x_2\right)B=0,$$
$$x_3\frac{d^2C}{dx_3^2}+(\tfrac{1}{3}-2l_1x_3)\frac{dC}{dx_3}-\left(\frac{l_1}{3}+\frac{l_2}{\omega}x_3^{1/3}\right)C=0. \quad (2.67)$$

The operators which specify this coordinate system are

$$\mathcal{L}_1=P_0+P_1,\quad \mathcal{L}_2=K_0+K_1+\tfrac{2}{3}E\omega^{3/2}(P_0+P_1)-\tfrac{1}{3}\omega^{3/2}P_2, \quad (2.68)$$

[14] $ds^2=2dx_1\,dx_2+2dx_2\,dx_3+(x_3/x_2)dx_2^2+(x_2/x_3)\,dx_3^2.$ (2.69)

The three space coordinates are given by

$$t+x=2x_1-E^2x_2-4E\sqrt{x_2x_3},\quad t-x=-x_2,\quad y=2\sqrt{x_2x_3}+Ex_2. \quad (2.70)$$

The separation equations are

$$\frac{dA}{dx_1}=l_1A,\quad 2x_2\frac{dB}{dx_2}=l_2B,$$
$$x_3\frac{d^2C}{dx_3^2}+(\tfrac{1}{2}-2l_1x_3)\frac{dC}{dx_3}-l_1l_2C=0. \quad (2.71)$$

A typical solution is

$$\psi=\exp[l_1(x_1+x_3)]x_2^{l_2/2}D_{-1/2-il_2}[\pm(1+i)\sqrt{2l_1x_3}], \quad (2.72)$$

where $D_\nu(z)$ is a parabolic cylinder function. The operators which specify the coordinate system are

$$\mathcal{L}_1=P_0+P_1, \quad \mathcal{L}_2=E(M_{12}-M_{02})-M_{01}-D-\tfrac{1}{2}. \quad (2.73)$$

[15] $$ds^2=2dx_1dx_2+[2x_3^{1/2}/(x_3+\beta)^{1/2}]dx_2dx_3 + (x_3/x_2)dx_2^2+[x_2/(x_3+\beta)]dx_3^2. \quad (2.74)$$

The three space coordinates are

$$t+x=2x_1-2x_3+\sqrt{x_3(x_3+\beta)}-2\beta\ln(\sqrt{x_3+\beta}+\sqrt{x_3})-\beta\ln x_2 - E^2x_2-4Ex_2^{1/2}(x_3+\beta)^{1/2}, \quad (2.75)$$
$$t-x=-x_2, \quad y=2\sqrt{x_2(x_3+\beta)}+Ex_2.$$

The separation equations are

$$\frac{dA}{dx_1}=l_1E_1, \quad x_2\frac{dB}{dx_2}=l_2B,$$

$$(x_3+\beta)\frac{d^2C}{dx_3^2}+[\tfrac{1}{2}-2l_1\sqrt{x_3(x_3+\beta)}]\frac{dC}{dx_3} + \left\{2l_1l_2+\tfrac{1}{2}l_1\left[1-\left(\frac{x_3+\beta}{x_3}\right)^{1/2}\right]\right\}C=0. \quad (2.76)$$

The operators which specify the coordinate system are

$$\mathcal{L}_1=P_0+P_1,$$
$$\mathcal{L}_2=-\tfrac{1}{2}[\beta(P_0+P_1)-D-M_{01}-\tfrac{1}{2}+E(M_{12}-M_{02})]. \quad (2.77)$$

[16] $$ds^2=\frac{2}{(1+\gamma x_2)^{1/2}}dx_1dx_2+\frac{2}{(1+\gamma x_2)^{1/2}}dx_2dx_3 + \frac{x_3}{x_2(1+\gamma x_2)}dx_2^2+\frac{x_2}{x_3}dx_3^2. \quad (2.78)$$

The three space coordinates are given by

$$t+x=2x_1-2(1+\gamma x_2)^{1/2}(E^2/\gamma+x_3)-4E\sqrt{x_2x_3}+2x_3,$$
$$t-x=-(2/\gamma)(1+\gamma x_2)^{1/2},$$
$$y=2\sqrt{x_2x_3}+(2E/\gamma)(1+\gamma x_2)^{1/2}.$$

The separation equations are

$$\frac{dA}{dx_1}=l_1A, \quad 2l_1x_2(1+\gamma x_2)^{1/2}\frac{dB}{dx_2}+[\tfrac{1}{2}(1+\gamma x_2)^{1/2}-l_2]B=0, \quad (2.80)$$
$$x_3\frac{d^2C}{dx_3^2}+(\tfrac{1}{2}-2l_1x_3)\frac{dC}{dx_3}+(l_2-l_1)C=0.$$

The operators which specify this coordinate system are

$$\mathcal{L}_1=P_0+P_1, \quad \mathcal{L}_2=(\gamma/4)(K_0+K_1)+(1/\gamma)[(1-E^2)P_0 - (1+E^2)P_1-EP_2]. \quad (2.81)$$

[17] $$ds^2=\frac{2}{(1+\gamma x_2)^{1/2}}dx_1dx_2+\frac{2x_3^{1/2}}{\sqrt{(1+\gamma x_2)(x_3+\beta)}}dx_2dx_3 + \frac{x_3}{x_2(1+\gamma x_2)}dx_2^2+\frac{x_2}{(x_3+\beta)}dx_3^2. \quad (2.82)$$

The three space coordinates are given by

$$t+x=2x_1-(2/\gamma)(1+\gamma x_2)^{1/2}(x_3+E^2)-4E\sqrt{x_2(x_3+\beta)} -\beta\ln[(\sqrt{1+\gamma x_2}-1)/(\sqrt{1+\gamma x_2}+1)]+\sqrt{x_3(x_3+\beta)} - 2\beta\ln(\sqrt{x_3+\beta}+\sqrt{x_3})$$
$$t-x=-(2/\gamma)(1+\gamma x_2)^{1/2}, \quad y=2\sqrt{x_2(x_3+\beta)}+(2E/\gamma)\sqrt{1+\gamma x_2}. \quad (2.83)$$

The separation equations are

$$\frac{dA}{dx_1}=l_1A, \quad 2l_1x_2\sqrt{1+\gamma x_2}\frac{dB}{dx_2}+(\tfrac{1}{2}\sqrt{1+\gamma x_2}-l_2)B=0, \quad (2.84)$$

$$(x_3+\beta)\frac{d^2C}{dx_3^2}+[\tfrac{1}{2}-2l_1\sqrt{x_3(x_3+\beta)}]\frac{dC}{dx_3} + \left[l_2-\frac{1}{2}\left(\frac{x_3+\beta}{x_3}\right)^{1/2}\right]C=0.$$

The operators which specify the coordinate system are

$$\mathcal{L}_1=P_0+P_1,$$
$$\mathcal{L}_2=(\gamma/4)(K_0+K_1)+(1/\gamma)(P_0-P_1)-(2E/\gamma)P_2 + (E^2/\gamma-\beta)(P_0+P_1). \quad (2.85)$$

[18] $$ds^2=2dx_1dx_2+2\omega x_2dx_2dx_3+dx_3^2+\omega^2x_2^2dx_2^2 \quad (2.86)$$

with modulation function $R=\omega l_1x_2x_3$.

The three space variables are given by

$$t+x=2x_1, \quad t-x=-x_2, \quad y=x_3+\tfrac{1}{2}\omega x_2^2. \quad (2.87)$$

The separation equations are

$$\frac{dA}{dx_1}=l_1A, \quad 2\frac{dB}{dx_2}-(l_1^2\omega^2x_2^2+l_2)B=0, \quad (2.88)$$
$$\frac{d^2C}{dx_3^2}+(2l_1^2\omega x_3+l_1l_2)C=0.$$

A typical solution is then

$$\psi=\exp(l_1\omega x_2x_3+l_1x_1+\tfrac{1}{6}l_1\omega^2x_2^3+\tfrac{1}{2}l_2x_2)\begin{cases}\mathrm{Ai}(z)\\ \mathrm{Bi}(z)\end{cases} \quad (2.89)$$

with $z=(2l_1^2\omega)^{1/3}x_3+l_2(2l_1^2\omega)^{-1/3}$. The functions Ai$(z)$ and Bi(z) are Airy functions.

The operators which define the coordinate system are

$$\mathcal{L}_1=P_0+P_1, \quad \mathcal{L}_2=P_1-P_0+2\omega(M_{12}-M_{02}). \quad (2.90)$$

[19] $$ds^2=2dx_1dx_2+2(x_2x_3+\omega/x_2)dx_2dx_3 + (x_3+\omega/x_2^2)^2dx_2^2+x_2^2dx_3^2. \quad (2.91)$$

with modulation factor $R=+\tfrac{1}{2}l_1x_2x_3^2+\omega l_1x_3/x_2$. The three space coordinates are

$$t+x=2x_1-E^2x_2-2Ex_2x_3+2\omega E/x_2,$$
$$t-x=-x_2, \quad y=x_2x_3+Ex_2-\omega/x_2. \quad (2.92)$$

The separation equations are

$$\frac{dA}{dx_1}=l_1A,\quad 2x_2^2\frac{dB}{dx_2}+\left(x_2-\frac{\omega l_1}{x_2^2}-l_2\right)B=0,$$
$$\frac{d^2C}{dx_3^2}+(-4\omega l_1^2l_2^2x_3-l_2)C=0. \tag{2.93}$$

A typical solution is

$$\psi=x_2^{-1/2}\exp\left(\tfrac{1}{2}l_1x_2x_3^2+\frac{\omega l_1x_3}{x_2}+l_1x_1-\frac{l_2}{2l_1x_2}-\frac{\omega^2l_1}{6x_2^3}\right.$$
$$\times\begin{cases}\mathrm{Ai}(z)\\ \mathrm{Bi}(z)\end{cases}, \tag{2.94}$$

where $z=(4l_1^2\omega)^{1/3}x_3+l_2(4l_1^2\omega)^{-1/3}$.

The operators which define the coordinate system are

$$\mathcal{L}_1=P_0+P_1,\quad \mathcal{L}_2=K_1+K_0+4\omega P_2. \tag{2.95}$$

[20] $$ds^2=2dx_1dx_2+2x_2x_3dx_2dx_3+[x_2^2x_3^2/(1+x_2^2)]dx_2^2+(1+x_2^2)dx_3^2 \tag{2.96}$$

with modulation function $R=\frac{1}{2}l_1x_2x_3^2$.

The three space variables are

$$t+x=2x_1-E^2x_2-2Ex_3(1+x_2^2)^{1/2},$$
$$t-x=-x_2,\quad y=x_3(1+x_2^2)^{1/2}+Ex_2. \tag{2.97}$$

The separation equations are

$$\frac{dA}{dx_1}=l_1A,\quad 2l_1(1+x_2^2)\frac{dB}{dx_2}+(l_1x_2-l_2)B=0,$$
$$\frac{d^2C}{dx_3^2}+(l_1^2x_3^2+l_2)C=0. \tag{2.98}$$

A typical solution is then

$$\psi=(1+x_2^2)^{-1/4}\exp[\tfrac{1}{2}l_1x_2x_3^2+l_1x_1+(l_2/2l_1)\tan^{-1}x_2]$$
$$\times D_{-(1+il_2/l_1)/2}[\pm(1+i)x_3\sqrt{l_1}], \tag{2.99}$$

where $D_\nu(z)$ is a parabolic cylinder function. The operators which specify the coordinate system are

$$\mathcal{L}_1=P_0+P_1,$$
$$\mathcal{L}_2=-E^2(P_0+P_1)+2EP_2+P_1-P_0+K_1+K_0. \tag{2.100}$$

[21] $$ds^2=2dx_1dx_2+2\epsilon x_2x_3dx_2dx_3+(x_2^2x_3^2/|1-x_2^2|)dx_2^2+|1-x_2^2|dx_3^2, \tag{2.101}$$

where $\epsilon=\mathrm{sgn}(x_2^2-1)$ and the modulation function is $R=\frac{1}{2}\epsilon l_1x_2x_3^2$.

The three space coordinates are

$$t+x=2x_1-E^2x_2-2Ex_3|1-x_2^2|^{1/2},$$
$$t-x=-x_2,\quad y=x_3|1-x_2^2|^{1/2}+Ex_2. \tag{2.102}$$

The separation equations are

$$\frac{dA}{dx_1}=l_1A,\quad 2\epsilon l_1(x_2^2-1)\frac{dB}{dx_2}-(l_1x_2+l_2)B=0,$$
$$\frac{d^2C}{dx_3^2}+(-l_1^2x_3^2+l_2)C=0. \tag{2.103}$$

A typical solution is

$$\psi=\exp(\tfrac{1}{2}\epsilon l_1x_2x_3^2+l_1x_1)(x_2-1)^{\epsilon(l_2+l_1)/4l_1}$$
$$\times(x_2+1)^{\epsilon(l_2-l_1)/4l_1}D_{-(l_2+l_1)/2l_1}(\pm\sqrt{2l_1}x_3). \tag{2.104}$$

The operators which specify the coordinate system are

$$\mathcal{L}_1=P_0+P_1,$$
$$\mathcal{L}_2=-E^2(P_0+P_1)-2EP_2+P_0-P_1+K_0+K_1. \tag{2.105}$$

[22] $$ds^2=2dx_1dx_2+[2\sqrt{x_3}/(1-x_3)](dx_1dx_3+dx_2dx_3)+dx_1^2+x_3dx_2^2+dx_3^2/(1-x_3)^2. \tag{2.106}$$

The three space coordinates are given by

$$t=2\sqrt{1-x_3}\ \sinh\tfrac{1}{2}x_2,\quad x=2\sqrt{1-x_3}\ \cosh\tfrac{1}{2}x_2,$$
$$y=x_1+x_2+2\sqrt{x_3}+\ln[(\sqrt{x_3}-1)/(\sqrt{x_3}+1)]. \tag{2.107}$$

The separation equations are

$$\frac{dA}{dx_1}=l_1A,\quad \frac{d^2B}{dx_2^2}-2l_1\frac{dB}{dx_2}-l_2B=0,$$
$$(x_3-1)\frac{d^2C}{dx_3^2}+(x_3-1+2l_1\sqrt{x_3})\frac{dC}{dx_3}+\left(\frac{l_1}{2x_3}+l_2-\tfrac{1}{2}l_1\right)C=0. \tag{2.108}$$

The operators which specify this coordinate system are

$$\mathcal{L}_1=P_2,\quad \mathcal{L}_2=-\tfrac{3}{4}M_{01}^2-M_{01}P_2+P_2^2. \tag{2.109}$$

[23] $$ds^2=\frac{2}{(x_2-1)x_2^{3/2}}(dx_1dx_2+dx_2dx_3)+2dx_1dx_3+dx_1^2+\frac{dx_2^2}{(x_2-1)^2x_2^2}+\frac{dx_3^2}{x_2}. \tag{2.110}$$

The three space coordinates are given by

$$t=2\sqrt{(1/x_2)-1}\ \cosh\tfrac{1}{2}x_3,\quad x=2\sqrt{(1/x_2)-1}\ \sinh\tfrac{1}{2}x_3,$$
$$y=x_1+x_3+2x_2^{-1/2}+\ln[(\sqrt{x_2}+1)/(\sqrt{x_2-1})]. \tag{2.111}$$

The separation equations are

$$\frac{dA}{dx_1}=l_1A,$$
$$x_2^2(x_2-1)^2\frac{d^2B}{dx_2^2}+\left(3x_2^2-2x_2^3-x_2+\frac{2l_1}{x_2}\right)\frac{dB}{dx_2}$$
$$+\left(\frac{l_1}{2}+l_2-\frac{l_1}{2x_2}\right)B=0,$$
$$\frac{d^2C}{dx_3^2}-2l_1\frac{dC}{dx_3}-l_2C=0. \tag{2.112}$$

The operators which specify the coordinate system are

$$\mathcal{L}_1=P_2,\quad \mathcal{L}_2=-\tfrac{3}{4}M_{01}^2-M_{01}P_2+P_2^2. \tag{2.113}$$

[24] $$ds^2=2dx_1dx_2+2dx_1dx_3+2dx_2dx_3+dx_1^2+\omega dx_2^2+(1/\omega)dx_3^2. \tag{2.114}$$

The three space coordinates are given by

$$t=\sqrt{(\omega-1)/\omega}\,x_3,\quad x=\sqrt{\omega-1}\,x_2,\quad y=x_1+x_2+x_3. \tag{2.115}$$

The separation equations are

$$\frac{dA}{dx_1}=l_1A,\quad \frac{d^2B}{dx_2^2}-2l_1\frac{dB}{dx_2}-l_2B=0,$$
$$\frac{d^2C}{dx_3^2}-2l_1\frac{dC}{dx_3}-\frac{l_2}{\omega}C=0. \tag{2.116}$$

A typical solution is

$$\psi=\exp[l_1(x_1+x_2+x_3)]\begin{cases}\cos\sqrt{l_1^2+l_2}\,x_2\\ \sin\sqrt{l_1^2+l_2}\,x_2\end{cases}\begin{cases}\cos\sqrt{l_1^2+l_2/\omega}\,x_3\\ \sin\sqrt{l_1^2+l_2/\omega}\,x_3\end{cases}. \tag{2.117}$$

The operators which specify this coordinate system are

$$\mathcal{L}_1=P_2,\quad \mathcal{L}_2=(\omega-1)(P_0^2+P_1^2)-(\omega+1)P_2^2. \tag{2.118}$$

[25] $$ds^2=2(x_3+\delta)\,dx_1\,dx_2+2x_3^{1/2}(dx_2\,dx_3+dx_1\,dx_3)+(x_3+\delta)\,dx_1^2+x_3\,dx_2^2+dx_3^2/(x_3+\delta)^2. \tag{2.119}$$

The three space coordinates are given by

$$t=2\sqrt{\delta(x_3+\delta)}\,\sinh[(x_1+x_2+2\sqrt{x_3}-2\sqrt{\delta}\ \tan^{-1}\sqrt{x_3/\delta})2\sqrt{\delta}\,],$$
$$x=2\sqrt{\delta(x_3+\delta)}\,\cosh[(x_1+x_2+2\sqrt{x_3}-2\sqrt{\delta}\ \tan^{-1}\sqrt{x_3/\delta})2\sqrt{\delta}\,],$$
$$y=\sqrt{\delta}\ x_2. \tag{2.120}$$

The separation equations are

$$\frac{dA}{dx_1}=l_1A,\quad \frac{d^2B}{dx_2^2}-2l_1\frac{dB}{dx_2}+l_2B=0,$$
$$(x_3+\delta)\frac{d^2C}{dx_3^2}+(1-2l_1)\frac{dC}{dx_3}+\left[l_2-\frac{l_1}{2}\left(1+\frac{\delta}{x_3}\right)\right]C=0. \tag{2.121}$$

The operators which specify the coordinate system are

$$\mathcal{L}_1=M_{01},\quad \mathcal{L}_2=\delta P_2^2-M_{01}^2. \tag{2.122}$$

[26] $$ds^2=2x_3\,dx_1\,dx_2+2x_2\,dx_1\,dx_3+2dx_2\,dx_3+x_2x_3\,dx_1^2+dx_2^2+dx_3^2. \tag{2.123}$$

The three space coordinates are given by

$$t=x_2x_3\exp(x_1/2)+\exp(-x_1/2),$$
$$x=x_2x_3\exp(x_1/2)-\exp(-x_1/2),\quad y=x_2+x_3. \tag{2.124}$$

The separation equations are

$$\frac{dA}{dx_1}=l_1A,\quad x_2\frac{d^2B}{dx_2^2}+(1-2l_1)\frac{dB}{dx_2}+l_2B=0,$$
$$x_3\frac{d^2C}{dx_3^2}+(1-2l_1)\frac{dC}{dx_3}+l_2C=0. \tag{2.125}$$

A typical solution is

$$\psi=\exp(l_1x_1)\,(x_2x_3)^{l_1}C_{2l_1}(2\sqrt{l_2x_2})C_{2l_1}(2\sqrt{l_2x_3}). \tag{2.126}$$

The operators which specify the coordinate system are

$$\mathcal{L}_1=\tfrac{1}{2}M_{01},\quad 2\mathcal{L}_2=P_0M_{02}+M_{02}P_0+P_1M_{12}+M_{12}P_1. \tag{2.127}$$

III. OTHER TYPES OF SEPARATION

In this section we examine the coordinate systems associated with the diagonalization of the operator $L=\frac{1}{2}M_{12}-\frac{1}{4}(P_0-K_0)$. The algebra of (*) when L has been diagonalized is $SL(2,R)$ with basis

$$A=\tfrac{1}{2}M_{12}+\tfrac{1}{4}(P_0-K_0),\quad B=\tfrac{1}{4}M_{01}+\tfrac{1}{4}(P_2-K_2),$$
$$C=-\tfrac{1}{2}M_{02}+\tfrac{1}{4}(P_1-K_1) \tag{3.1}$$

and commutation relations

$$[A,B]=C,\quad [C,A]=B,\quad [C,B]=A. \tag{3.2}$$

The coordinate systems associated with the diagonalization of L and an additional operator from the above $SL(2,R)$ algebra are the semisubgroup coordinates of type 7 of Paper 8 of this series. In this section we give the three subgroup coordinates discussed in Paper 8 and leave open the question of whether there are any more. This will be the subject of subsequent study. The three coordinate systems we present are different from those presented in the earlier two sections in that they do not enable a separation of variables to occur explicitly in the equation. This becomes clear for the individual coordinate systems.

For the choice of variables

$$t=\frac{\sin\frac{1}{2}(\beta-\rho)}{\cos\sigma-\cos\frac{1}{2}(\beta-\rho)},\quad x=\frac{\sin\sigma\cos\frac{1}{2}(\beta+\rho)}{\cos\sigma-\cos\frac{1}{2}(\beta-\rho)},$$
$$y=\frac{\sin\sigma\sin\frac{1}{2}(\beta+\rho)}{\cos\sigma-\cos\frac{1}{2}(\beta-\rho)}, \tag{3.3}$$

and $\psi=[\cos\sigma-\cos\frac{1}{2}(\beta-\rho)]^{1/2}\exp(i\chi\beta)\,\Theta(\sigma,\rho)$, we have $L\psi=i\chi\beta\psi$, where the function $\Theta(\sigma,\rho)$ satisfies the equation

$$(A^2-B^2-C^2)\Theta=(L^2+\tfrac{1}{4})\,\Theta=(\tfrac{1}{4}-\chi^2)\Theta. \tag{3.4}$$

The diagonalization of A is easily performed in this coordinate system as $A=\partial_\rho$ when acting on the function Θ, and so for $\Theta(\sigma,\rho)=\Phi(\sigma)\exp(i\tau\rho)$ the corresponding solutions of (*) have the form

$$\psi_{\chi\tau}(\sigma,\beta,\rho)=[\cos\sigma-\cos\tfrac{1}{2}(\beta-\rho)]^{1/2}\times\exp(i\chi\beta)\exp(i\tau\rho)P^{\tau+\chi}_{\chi-\tau-1/2}(\cos\sigma). \tag{3.5}$$

In particular we note that the $SL(2,R)$ generators acting on the functions Θ have the form

$$C+iB=\exp(i\rho)(-\partial_z-i\coth z\,\partial_\rho+\chi/\sinh z+\tfrac{1}{4}\tanh\tfrac{1}{2}z),$$
$$C-iB=\exp(-i\rho)(-\partial_z+i\coth z\,\partial_\rho-\chi/\sinh z+\tfrac{1}{4}\tanh\tfrac{1}{2}z),$$
$$A=\partial_\rho, \tag{3.6}$$

where $\sin\sigma=\tanh\frac{1}{2}z$. The pure derivative parts of these operators are the same as the corresponding operators that would be obtained on the two dimensional hyperboloid parametrized by $t=(\cosh z,\sinh z\cos\rho,\sinh z\sin\rho)$. This suggests the procedure necessary for the remaining two subgroup coordinate systems which diagonalize C and $A-C$. The appropriate change of variables is given by a knowledge of the subgroup type coordinates on the hyperboloid.[11] After extraction of the appropriate modulation function, the separation of variables is achieved. The results are:

1. *The diagonalization of C*: The appropriate change of variables is $\cosh z = \cosh a \cosh b$, $\tan\rho = \tanh a \sinh b$, and the R-separation modulation function is

$$f = (\cosh a \cosh b + 1)^{1/4} \exp[i\chi \tan^{-1}(\sinh a \coth b)]. \tag{3.7}$$

The generators acting on the functions Φ, where $\Theta = f\Phi$ have the form

$$A = \sinh b\,\partial_a - \tanh a \cosh b\,\partial_b - i\chi \cosh b/\cosh a,$$
$$B = -\cosh b\,\partial_a + \tanh a \sinh b\,\partial_b + i\chi \sinh b/\cosh a, \tag{3.8}$$

Then for $\Phi = \exp(i\tau b)H(a)$ the function H satisfies

$$\left(\frac{\partial^2}{\partial a^2} + \tanh a \frac{\partial}{\partial a} + \frac{\chi\tau}{\cosh^2 a} \sinh a + \chi^2 - \tfrac{1}{4}\right)H(a) = 0 \tag{3.9}$$

with solutions

$$H(a) = P^{-1/2+\chi}_{i\sqrt{\chi\tau/2},\sqrt{\chi\tau/2}}(\cosh a),\quad Q^{-1/2+\chi}_{i\sqrt{\chi\tau/2},\sqrt{\chi\tau/2}}(\cosh a),$$

where $P^{l}_{uv}(z)$ and $Q^{l}_{uv}(z)$ are the generalized Legendre functions.[11-13]

2. *The diagonalization of $A - C$*: The appropriate change of variables is

$$\cosh z = \cosh a + \tfrac{1}{2}r^2 e^{-a},\quad \tan\rho = re^{-a}/(\sinh a + \tfrac{1}{2}r^2 e^{-a}),$$

and the modulation function is

$$f = [(\cosh a + \tfrac{1}{2}r^2 e^{-a} - 1)/(\cosh a + \tfrac{1}{2}r^2e^{-a} + 1)]^{i\chi/2}$$
$$\exp\{-\tfrac{1}{2}\tan^{-1}[r/(e^a + 1)]\}. \tag{3.10}$$

The generators acting on the functions $\Phi = f\psi$ have the form

$$B = \partial_a + r\partial_r,\quad A - C = \partial_r, \tag{3.11}$$
$$A + C = 2r\partial_a + (r^2 - e^{2a})\partial_r + \tfrac{1}{4}(2e^a - 1).$$

Then for $\Phi = \exp(i\tau r)H(a)$ the function H satisfies

$$[\partial^2/\partial a^2 - \partial/\partial a - \tau^2 e^{2a} - \tfrac{1}{4}i\tau(2e^a - 1) - \chi^2 + \tfrac{1}{4}]H(a) = 0, \tag{3.12}$$

which has solutions

$$H(a) = M_{i/4, \pm 2(\chi^2 - i\tau)^{1/2}}(2\tau e^a),$$

where $M_{\mu,\nu}(z)$ is a solution of Kummers differential equation.[14] We see that each of the subgroup types has an R-separable solution and does not fit into the scheme of Sec. I. We do not yet know if there are any more systems of these types.

The principal contribution of this article is to provide examples of R-separable solutions, which to our knowledge have not previously been exhibited. A unified group theoretical approach must be able to account for the explicit solutions and coordinate systems produced here.

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