Lie theory and separation of variables. 8. Semisubgroup coordinates for $\Psi_{tt} - \Delta_2\Psi = 0$

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We classify and study all coordinate systems which permit R-separation of variables for the wave equation in three space–time variables and such that at least one of the variables corresponds to a one-parameter symmetry group of the wave equation. We discuss 33 such systems and relate them to orbits of commuting operators in the enveloping algebra of the conformal group $SO(3,2)$.

I. INTRODUCTION

This paper is one of a series[1–7] devoted to uncovering the relationships between the symmetry group of a linear second order partial differential equation and the coordinate systems in which variables separate for that equation. Here, we study the wave equation

$$(\partial_{00} - \partial_{11} - \partial_{22})\Psi(x) = 0 \qquad (*)$$

in three space–time variables. The symmetry group of this equation is locally isomorphic to the ten-parameter group $SO(3,2)$. In Paper 9 of this series we will derive explicitly the possible orthogonal coordinate systems with respect to which variables separate or R-separate in $(*)$. (More precisely we shall list all coordinate systems obtained from confocal cyclides and their limits.[8]) We will show that each such system corresponds to a two-dimensional (commuting) subspace of the space of second order symmetric elements in the enveloping algebra of $so(3,2)$. Here, the elements of $so(3,2)$ are first order differential operators which are symmetries of $(*)$. If the commuting operators Q,S form a basis for such a subspace, then the separated solutions Ψ of $(*)$ associated with this coordinate system are characterized by the eigenvalue equations $Q\Psi = \lambda\Psi$, $S\Psi = \mu\Psi$, where the eigenvalues λ, μ are the separation constants. The group $SO(3,2)$ acts on the enveloping algebra of $so(3,2)$ via the adjoint representation and preserves the rank of operators in the enveloping algebra. In particular the infinitesimal symmetries of $(*)$ generate the identity component of $SO(3,2)$ and the symmetry I such that $I\Psi(x) = \Psi(-x)$ lies in the component not connected with the identity. Under this action the two-dimensional commuting subspaces of symmetric second order elements are decomposed into $SO(3,2)$-orbits. We regard coordinate systems attached to subspaces on the same orbit as equivalent, i.e., one such system can be obtained from any other one by an $SO(3,2)$ transformation.

Much of this paper is an introduction to the problem of separation of variables for $(*)$. Most of the detailed calculations will be presented in Paper 9 of this series and subsequent publications. In Sec. 1 we compute the symmetry algebra of $(*)$ in two different bases and by taking a Fourier transform we construct the well-known Hilbert space $\mathcal{H}_+$ of positive energy solutions of $(*)$. On $\mathcal{H}_+$ the symmetry operators of $so(3,2)$ exponentiate to yield a unitary irreducible representation of a covering group $\widetilde{SO(3,2)}$ of the identity component in $SO(3,2)$. In Sec. 2 we determine explicitly the action of $\widetilde{SO(3,2)}$ on $\mathcal{H}_+$. Most of the results of this section appear to be new.

The remainder of the paper is devoted to separation of variables. If a separable coordinate system corresponds to a subspace where there exist operators $Q = A^2$, $S = B^2$ with $[A,B] = 0$ and $A, B \in so(3,2)$, we call such coordinates *subgroup coordinates*. In this case one can diagonalize the first-order operators A, B. These systems are the best-known and easiest to find. More generally, if there exist operators Q,S such that $Q = A^2$, $[A,S] = 0$, and $A \in so(3,2)$, we call these coordinates *semisubgroup coordinates*. Here, one can diagonalize the first order operator A. If there exists no pair Q,S such that Q is a square of some $A \in so(3,2)$, we call the coordinates *nonsubgroup*. Nonsubgroup coordinates are the most intractible of all separable coordinates and appear the least frequently in applications.

A given $A \in so(3,2)$ may correspond to several (or to no) semisubgroup systems. Indeed, if Ψ satisfies both $(*)$ and $A\Psi = i\lambda\Psi$, then, since A is a symmetry of $(*)$, we can use standard Lie theory and introduce new variables y_0, y_1, y_2 such that $A = \partial_{y_0} + f(y)$ (where f may be zero) and $\Psi(y) = r(y)\exp(i\lambda y_0)\Phi_\lambda(y_1, y_2)$, where r is a fixed function satisfying $\partial_{y_0} r + fr = 0$. Then $(*)$ reduces to a second order partial differential equation (†) for Φ_λ in the two variables y_1, y_2. The possible semisubgroup systems A^2, S thus correspond to the possible coordinate systems such that the reduced equation (†) separates. In particular S corresponds to a second order symmetry of the reduced equation.

In Secs. 3–7 we examine the possible semisubgroup systems. The systems are of seven types corresponding to seven choices for A. Using the notation for elements of $so(3,2)$ introduced in Sec. 1, we find that these types are:

1]. $A = \Gamma_{45}$. Then (†) becomes the eigenvalue equation for the Laplace operator on the sphere S_2. We find two coordinate systems.[4,9]

2]. $A=P_0$ and (†) is the reduced wave equation (4.1). We find four coordinate systems.

3]. $A=P_2$ and (†) is the Klein—Gordon equation (4.5). We find 11 coordinate systems.[3,10]

4]. $A=D$ and (†) is the eigenvalue equation (4.9) for the Laplace operator on a hyperboloid. We find nine coordinate systems.[4]

5]. $A=P_0+P_1$ and (†) is the free particle Schrödinger equation (5.1). We find four coordinate systems.[5]

6]. $A=M_{12}$ and (†) is the Euler—Poisson—Darboux (EPD) equation (6.1). We find nine coordinate systems.

7]. $A=\frac{1}{2}(\Gamma_{23}-\Gamma_{45})$ and (†) is Eq. (7.1). The problem of separation of variables for coordinates of this type is currently under study. There are at least three coordinate systems.

Eliminating duplicate coordinate systems we obtain a total of 33 distinct semisubgroup systems at this writing, 27 of which have already been discussed in Refs. 1—7. The systems of types 6] and 7] are related to unitary representations of the universal covering group of $SL(2,R)$ which belong to the discrete series. They will be discussed in detail in future papers.

At this writing we have determined all $A\in so(3,2)$ such that a separable coordinate system corresponds to some commuting pair A^2,S and such that S belongs to the enveloping algebra of the symmetry algebra of the reduced equation (†) associated with A^2. However, there are some omissions on our list, due to the fact that diagonalization of A does not uniquely determine the variable y_0 which is split off to obtain (†). The systems we have omitted correspond to nonorthogonal coordinates and are such that S is not expressible in terms of the symmetry algebra of (†). These systems prove rather intractible from the group theoretical viewpoint. The proofs of the above remarks follow from the results of Sec. 8. In Paper 9 and later publications we will settle these questions by using other techniques to explicitly list all systems (orthogonal or not) such that (*) R-separates.

In only a few representative cases do we explicitly list the coordinate systems. For 27 systems the expressions are given in Refs. 1—7 and 10. In Paper 9 we will derive explicitly all orthogonal systems allowing R-separation in (*) and obtain the corresponding semisubgroup systems as special cases.

Finally, in Sec. 8 we classify the orbits in $so(3,2)$ under the adjoint action of $SO(3,2)$ to see why not every $A\in so(3,2)$ corresponds to a semisubgroup system of the form A^2,S.

The special functions appearing in this paper are all defined as in the Bateman Project.[11]

1. $SO(3,2)$ AND THE WAVE EQUATION

We are concerned with the wave equation

$$(\partial_{00}-\partial_{11}-\partial_{22})\psi(x)=0,\quad x=(x_0,x_1,x_2). \tag{1.1}$$

As usual[5] we define the symmetry algebra of (1.1) to be the set of all linear differential operators

$$L=\sum_{\alpha=0}^{2} a_\alpha(x)\partial_\alpha+b(x)$$

such that $L\psi$ is a (local) solution of (1.1) whenever ψ is a (local) solution.

It is well known that the possible operators L form a ten-dimensional Lie algebra, isomorphic to $so(3,2)$, where the commutator is the usual Lie bracket.[12] As a convenient basis for this model of $so(3,2)$ we choose the momentum operators

$$P_\alpha=\partial_\alpha,\quad \alpha=0,1,2, \tag{1.2}$$

the generators of homogeneous Lorentz transformations

$$M_{12}=x_1\partial_2-x_2\partial_1,\quad M_{01}=x_0\partial_1+x_1\partial_0,$$
$$M_{02}=x_0\partial_2+x_2\partial_0, \tag{1.3}$$

the generator of dilatations

$$D=-(\tfrac{1}{2}+x_0\partial_0+x_1\partial_1+x_2\partial_2), \tag{1.4}$$

and the generators of special conformal transformations

$$K_0=-x_0+(x\cdot x-2x_0^2)\partial_0-2x_0x_1\partial_1-2x_0x_2\partial_2,$$
$$K_1=x_1+(x\cdot x+2x_1^2)\partial_1+2x_1x_0\partial_0+2x_1x_2\partial_2, \tag{1.5}$$
$$K_2=x_2+(x\cdot x+2x_2^2)\partial_2+2x_2x_0\partial_0+2x_2x_1\partial_1,$$

where

$$x\cdot y=x_0y_0-x_1y_1-x_2y_2=x_0y_0-\mathbf{x}\cdot\mathbf{y}. \tag{1.6}$$

The commutation relations follow from (1.27) and (1.28) which will be derived later.

These symmetry operators can be exponentiated to obtain a local Lie transformation group of symmetries of (1.1).[12,13] In particular, the momentum and Lorentz operators generate the Poincaré group of symmetries,

$$\psi(x)\to\psi(\Lambda^{-1}(x-a)),\quad a=(a_0,a_1,a_2),\quad \Lambda\in SO(1,2), \tag{1.7}$$

the dilatation operator generates

$$(\exp\lambda D)\psi(x)=\exp(-\lambda/2)\psi[\exp(-\lambda)x],\quad \lambda\in R, \tag{1.8}$$

and the K_α generate the special conformal transformations

$$\exp(a_0K_0+a_1K_1+a_2K_2)\psi(x)$$
$$=[1+2x\cdot a+(a\cdot a)(x\cdot x)]^{-1/2}$$
$$\times\Psi\left(\frac{x+a(x\cdot x)}{1+2x\cdot a+(a\cdot a)(x\cdot x)}\right). \tag{1.9}$$

In addition we shall consider the inversion, space reflection, and time reflection operators,

$$R\psi(x)=(1/\sqrt{-x\cdot x})\psi(-x/x\cdot x),$$
$$S\psi(x)=\psi(x_0,-x_1,x_2), \tag{1.10}$$
$$T\psi(x)=\psi(-x_0,x_1,x_2),$$

which are not generated by the local Lie symmetries (1.2)—(1.5).

As is well-known,[12,14] by formally taking the Fourier transform in the variables x_α we can express a solution $\psi(x)$ of (1.1) in the form

$$\psi(x)=\frac{1}{4\pi}\int_{-\infty}^{\infty}\int[\exp(ik\cdot x)f(k_1,k_2)$$

$$+\exp(i\hat{k}\cdot x)\tilde{f}(k_1,k_2)]\,d\mu(\mathbf{k}), \tag{1.11}$$

where $k_0=+(k_1^2+k_2^2)^{1/2}$, $\hat{k}=(-k_0,k_1,k_2)$, and $d\mu(\mathbf{k})=dk_1\,dk_2/k_0$.

Let $\mathcal{H}=\mathcal{H}_+\oplus\mathcal{H}_-$ be the space of all ordered pairs of complex-valued functions $\{f(k_1,k_2),\tilde{f}(k_1,k_2)\}=\mathbf{F}(k_1,k_2)$ defined on R_2 such that

$$\int\int(|f|^2+|\tilde{f}|^2)\,d\mu(\mathbf{k})<\infty \tag{1.12}$$

(Lebesgue integral), and consider the indefinite inner product on $\mathcal{H}$ given by

$$\langle\mathbf{F},\mathbf{G}\rangle=\int\int(f\bar{g}-\tilde{f}\bar{\tilde{g}})\,d\mu(\mathbf{k}). \tag{1.13}$$

Then, as is well-known,[12,14] the functions ψ, Φ related to $\mathbf{F}$, $\mathbf{G}$ by (1.11) satisfy

$$\langle\Psi,\Phi\rangle\equiv\langle\mathbf{F},\mathbf{G}\rangle=2i\int_{x_0=t}\int(\psi(x)\partial_0\bar{\Phi}(x)-[\partial_0\psi(x)]\bar{\Phi}(x))dx_1\,dx_2 \tag{1.14}$$

independent of t. More precisely (1.14) can be derived from (1.13) by first considering the dense subspace of $\mathcal{H}$ consisting of C^∞ functions with compact support bounded away from (0,0) and then passing to the limit. For $\mathbf{F}\in\mathcal{H}$ the corresponding $\psi(x)$ is a solution of (1.1) in the sense of distribution theory; it may not be true that Ψ is two times continuously differentiable in each variable.

The operators (1.2)–(1.5) acting on solutions of (1.1) induce corresponding operators on $\mathcal{H}$ under which $\mathcal{H}_+$ and $\mathcal{H}_-$ are separately invariant. Indeed with repeated integrations by parts we can establish that the action of these operators on $\mathcal{H}_+$ is

$$P_0=ik_0,\quad P_j=-ik_j,\quad j=1,2, \tag{1.15}$$

$$M_{12}=k_1\partial_{k_2}-k_2\partial_{k_1},\quad M_{01}=k_0\partial_{k_1},\quad M_{02}=k_0\partial_{k_2}, \tag{1.16}$$

$$D=\tfrac{1}{2}+k_1\partial_{k_1}+k_2\partial_{k_2}, \tag{1.17}$$

$$K_0=ik_0(\partial_{k_1k_1}+\partial_{k_2k_2}), \tag{1.18}$$

$$K_1=i(k_1\partial_{k_1k_1}-k_1\partial_{k_2k_2}+2k_2\partial_{k_1k_1}+\partial_{k_1}),$$

$$K_2=i(-k_2\partial_{k_1k_1}+k_2\partial_{k_2k_2}+2k_1\partial_{k_1k_2}+\partial_{k_2}).$$

The action on $\mathcal{H}_-$ is the same except that k_0 is replaced by $-k_0$ in each of (1.15)–(1.18). Moreover, it is straightforward to verify that these operators are skew-Hermitian on $\mathcal{H}_+$ and $\mathcal{H}_-$ separately.

The induced operators S and T on $\mathcal{H}$ are

$$S\mathbf{F}(k_1,k_2)=\mathbf{F}(-k_1,k_2)=(f(-k_1,k_2),\tilde{f}(-k_1,k_2)),$$
$$T\mathbf{F}(k_1,k_2)=(\tilde{f}(k_1,k_2),f(k_1,k_2)). \tag{1.19}$$

Thus, $\mathcal{H}_+$ and $\mathcal{H}_-$ are invariant under S, but these spaces are interchanged by T. In view of this interchange property of T we will henceforth limit ourselves to consideration of elements in the Hilbert space $\mathcal{H}_+$, or what amounts to the same thing, the positive energy solutions

$$\Psi(x)=\frac{1}{4\pi}\int_{-\infty}^{\infty}\int\exp(ik\cdot x)f(k_1,k_2)\,d\mu(\mathbf{k}). \tag{1.20}$$

The inner product on $\mathcal{H}_+$ is

$$\langle f,g\rangle=\int_{-\infty}^{\infty}\int f(k_1,k_2)\bar{g}(k_1,k_2)\,d\mu(\mathbf{k}) \tag{1.21}$$

and

$$\langle\Psi,\Phi\rangle\equiv\langle f,g\rangle=4i\int_{x_0=t}\int\Psi(x)\partial_0\bar{\Phi}(x)\,dx_1\,dx_2$$

$$=-4i\int_{x_0=t}\int\bar{\Phi}(x)\partial_0\Psi(x)\,dx_1\,dx_2. \tag{1.22}$$

Furthermore, if Ψ is given by (1.20), we have

$$f(k_1,k_2)=\frac{k_0}{\pi}\int_{-\infty}^{\infty}\int\Psi(x)\exp(-ik\cdot x)\,dx_1\,dx_2. \tag{1.23}$$

By employing arguments analogous to those in Ref. 12, one can show that $\mathcal{H}_+$ is invariant under R and

$$Rf(\mathbf{k})=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int\cos\sqrt{2l\cdot k}\,f(\mathbf{l})\,d\mu(\mathbf{l}),\quad f\in\mathcal{H}_+,$$
$$R^2=E, \tag{1.24}$$

where E is the identity operator on $\mathcal{H}_+$. Clearly, R extends to a unitary self-adjoint operator on $\mathcal{H}_+$ with eigenvalues ±1. It follows from the configuration-space realization of our operators that

$$RK_jR^{-1}=P_j,\quad j=1,2,\quad RK_0R^{-1}=-P_0,\quad RDR^{-1}=-D, \tag{1.25}$$

$$RM_{\alpha\beta}R^{-1}=M_{\alpha\beta},\quad R=R^{-1}.$$

At this point it is convenient to introduce another basis for our Lie algebra of symmetry operators which clearly displays the isomorphism between this algebra and $so(3,2)$. We define $so(3,2)$ as the ten-dimensional Lie algebra of 5×5 matrices A such that $AG+GA^t=0$, where 0 is the zero matrix and

$$G=\begin{pmatrix}1&&&&0\\&1&&&\\&&1&&\\&&&-1&\\0&&&&-1\end{pmatrix}.$$

Let $\mathcal{E}_{ij}$ be the 5×5 matrix with a 1 in row i, column j and zeros elsewhere. Then the matrices

$$\Gamma_{ab}=\mathcal{E}_{ab}-\mathcal{E}_{ba}=-\Gamma_{ba},\quad a\neq b,$$
$$\Gamma_{aB}=\mathcal{E}_{aB}+\mathcal{E}_{Ba}=\Gamma_{Ba},\quad 1\leq a,b\leq3, \tag{1.26}$$
$$\Gamma_{AB}=-\mathcal{E}_{AB}+\mathcal{E}_{BA}=-\Gamma_{BA},\quad 4\leq A,B\leq5,$$

form a basis for $so(3,2)$ with commutation relations

$$[\Gamma_{ab},\Gamma_{cd}]=\delta_{bc}\Gamma_{ad}+\delta_{ad}\Gamma_{bc}+\delta_{ca}\Gamma_{db}+\delta_{db}\Gamma_{ca},$$
$$[\Gamma_{aB},\Gamma_{cd}]=-\delta_{ad}\Gamma_{cB}+\delta_{ac}\Gamma_{dB},$$
$$[\Gamma_{Ab},\Gamma_{45}]=\delta_{A5}\Gamma_{4b}-\delta_{A4}\Gamma_{5b}, \tag{1.27}$$
$$[\Gamma_{aB},\Gamma_{cD}]=\delta_{BD}\Gamma_{ac}-\delta_{ac}\Gamma_{BD}.$$

This Γ-basis is related to our other basis via

$$P_0=\Gamma_{14}+\Gamma_{45},\quad P_1=\Gamma_{12}+\Gamma_{25},\quad P_2=\Gamma_{13}+\Gamma_{35},$$

$$K_0=\Gamma_{14}-\Gamma_{45},\quad K_1=\Gamma_{12}-\Gamma_{25},\quad K_2=\Gamma_{13}-\Gamma_{35},\tag{1.28}$$
$$M_{12}=\Gamma_{23},\quad M_{01}=\Gamma_{42},\quad M_{02}=\Gamma_{43},\quad D=\Gamma_{15}.$$

Furthermore, we can set $R=-G$.

For our model of $so(3,2)$ we have

$$P_0^2-P_1^2-P_2^2=K_0^2-K_1^2-K_2^2=0,\tag{1.29}$$

where the result for the K-operators follows from (1.25). Furthermore, direct computations yield

$$\begin{aligned}&\Gamma_{12}^2+\Gamma_{13}^2+\Gamma_{23}^2=\Gamma_{45}^2+\tfrac14,\\ &M_{12}^2-M_{01}^2-M_{02}^2=-D^2+\tfrac14,\\ &\Gamma_{45}^2-\Gamma_{41}^2-\Gamma_{51}^2=\Gamma_{23}^2+\tfrac14.\end{aligned}\tag{1.30}$$

If $\{\Psi_\alpha(x)\}$ is an orthonormal (ON) basis for the Hilbert space of positive energy solution of (1.1), then (in the sense of distributions)

$$\sum_\alpha \overline{\Psi_\alpha(x)}\Psi_\alpha(x')=\Delta_+(x-x')=\frac{1}{16\pi^2}\iint_{-\infty}^{\infty}\exp[ik\cdot(x'-x)]\,d\mu(\mathbf{k}),\tag{1.31}$$

where the distribution Δ_+ defined by (1.31) has the explicit expression

$$\Delta_+(x)=\begin{cases}\dfrac{2\pi i}{(t^2-r^2)^{1/2}}, & t>r,\\[2mm] -\dfrac{2\pi i}{(t^2-r^2)^{1/2}}, & t<-r,\quad r=(x_1^2+x_2^2)^{1/2},\\[2mm] \dfrac{2\pi}{(r^2-t^2)^{1/2}}, & -r<t<r.\end{cases}\tag{1.32}$$

The computation of (1.32) is carried out in analogy with the corresponding result for four-dimensional spacetime.[15] It follows immediately that

$$\Psi(x)=\langle\Psi,\Delta_+(x'-x)\rangle,\tag{1.33}$$

where the integration is carried out over $\mathbf{x}'$.

2. THE ACTION OF THE CONFORMAL GROUP

It is well known that the representation of $so(3,2)$ on $\mathcal{H}_+$ induced by the operators (1.15)–(1.18) exponentiates to a global irreducible unitary representation of a covering group $\tilde{S}O(3,2)$ of the identity component of $SO(3,2)$.[12] The maximal connected compact subgroup of $\tilde{S}O(3,2)$ is $SO(3)\times SO(2)$, where $SO(3)$ is generated by Γ_{12}, Γ_{13}, Γ_{23}, and $SO(2)$ by Γ_{45}. We will determine the explicit action of this subgroup on $\mathcal{H}_+$ as well as the action of several other interesting subgroups of $\tilde{S}O(3,2)$.

The operators M_{01}, M_{02}, M_{12} generate a subgroup of $\tilde{S}O(3,2)$ isomorphic to $SO(2,1)$. The action of this subgroup on $\mathcal{H}_+$ is determined by

$$\begin{aligned}&(\exp\theta M_{12})f(\mathbf{k})=f(k_1\cos\theta-k_2\sin\theta,k_1\sin\theta+k_2\cos\theta),\\ &(\exp aM_{01})f(\mathbf{k})=f(k_1(a),k_2),\\ &k_1(a)=[e^a(k_1+k_0)^2-e^{-a}k_2^2]/2(k_1+k_0),\quad f\in\mathcal{H}_+.\end{aligned}\tag{2.1}$$

(The result for M_{02} follows easily from that for M_{01}.)

The P_α generate a translation subgroup of $\tilde{S}O(3,2)$:

$$(\exp\textstyle\sum a_\alpha P_\alpha)f(\mathbf{k})=\exp(ia\cdot k)f(\mathbf{k}).\tag{2.2}$$

Unitary operators of the form $\exp\sum a_\alpha K_\alpha$ are somewhat more difficult to compute explicitly. However, the subgroup $SO(2,1)$, (2.1), transforms the vector a under the adjoint action and there are only three distinct cases to consider: (1) $a=(a_0,0;0)$, $a_0\neq0$, timelike; (2) $a=(0,a_1,0)$, $a_1\neq0$, spacelike; (3) $a=(a_1,a_1,0)$, lightlike.

We start with the timelike case. Note that the quantities $f_{l_1l_2}$,

$$\begin{aligned}&f_{l_1l_2}(\mathbf{k})=\delta(k_1-l_1)\delta(k_2-l_2)k_0,\quad -\infty<l_j<\infty,\\ &P_jf_{l_1l_2}=-l_jf_{l_1l_2},\quad j=1,2,\quad P_0f_{l_1l_2}=il_0f_{l_1l_2},\end{aligned}\tag{2.3}$$

form a basis for $\mathcal{H}_+$ of generalized eigenvectors of the commuting operators P_α. The orthogonality relation is

$$\begin{aligned}&\langle f_{l_1l_2},f_{s_1s_2}\rangle=\delta(l_1-s_1)\delta(l_2-s_2)l_0,\\ &l_0=(l_1^2+l_2^2)^{1/2}.\end{aligned}\tag{2.4}$$

Thus, the quantities $g_{l_1l_2}=Rf_{l_1l_2}$,

$$g_{l_1l_2}(k)=(1/2\pi)\cos\sqrt{2l\cdot k},\tag{2.5}$$

form a basis for $\mathcal{H}_+$ of generalized eigenvectors of the commuting operators K_α:

$$\begin{aligned}&K_jg_{l_1l_2}=-il_jg_{l_1l_2},\quad K_0g_{l_1l_2}=-il_0g_{l_1l_2},\\ &\langle g_{l_1l_2},g_{s_1s_2}\rangle=\delta(l_1-s_1)\delta(l_2-s_2)l_0.\end{aligned}\tag{2.6}$$

(Here we are using the fact that R is unitary.)

The unitary operator $\exp aK_0$ takes the form

$$(\exp aK_0)f(\mathbf{s})=\iint_{-\infty}^{\infty}G(a,1,\mathbf{s})f(1)\,d\mu(1),\quad f\in\mathcal{H}_+,\tag{2.7}$$

where

$$\begin{aligned}G(a,1,\mathbf{s})&=\langle\exp(aK_0)f_1,f_{\mathbf{s}}\rangle=\langle R\exp(-aP_0)Rf_1,f_{\mathbf{s}}\rangle\\ &=\langle\exp(-aP_0)g_1,g_{\mathbf{s}}\rangle\\ &=\frac{1}{4\pi^2}\iint\exp(-iak_0)\cos\sqrt{2l\cdot k}\cos\sqrt{2s\cdot k}\,d\mu(\mathbf{k}).\end{aligned}\tag{2.8}$$

We can evaluate this integral by expanding the cosines,

$$\begin{aligned}&\cos\sqrt{2l\cdot k}=\sum_{n=-\infty}^{\infty}\exp[in(\theta-\varphi)]J_{2n}[2(l_0k_0)^{1/2}],\\ &l_1+il_2=l_0\exp(i\varphi),\quad k_1+ik_2=k_0\exp(i\theta),\end{aligned}$$

and integrating term-by-term. The result is

$$\begin{aligned}G(a,1,\mathbf{s})=&-(i/2\pi a)\exp[i(s_0+t_0)/a]\\ &\times\cos[(1/a)\sqrt{2(s_0l_0+s_1l_1+s_2l_2)}\,].\end{aligned}\tag{2.9}$$

To compute the action of $\exp aK_1$, we need a basis of generalized eigenvectors of the commuting operators M_{02} and P_1. The basis is

$$\begin{aligned}&h_{\lambda\mu}(\mathbf{k})=(1/\sqrt{2\pi})\delta(S-\mu)\exp(i\lambda T),\quad -\infty<\mu,\lambda<\infty,\\ &S=k_1,\quad T=\ln(k_0+k_2),\quad d\mu(\mathbf{k})=dS\,dT.\end{aligned}\tag{2.10}$$

Indeed,

$$\begin{aligned}&M_{02}h_{\lambda\mu}=i\lambda h_{\lambda\mu},\quad P_1h_{\lambda\mu}=-i\mu h_{\lambda\mu},\\ &\langle h_{\lambda\mu},h_{\lambda'\mu'}\rangle=\delta(\mu-\mu')\delta(\lambda-\lambda').\end{aligned}\tag{2.11}$$

A straightforward computation, using the unitarity of R, shows that the $Rh_{\lambda\mu}$ are corresponding eigenfunctions of M_{02} and K_1 satisfying the same orthogonality relations as the $h_{\lambda\mu}$. Here,

$$M_{02}(Rh_{\lambda\mu}) = i\lambda(Rh_{\lambda\mu}), \quad K_1(Rh_{\lambda\mu}) = -i\mu Rh_{\lambda\mu},$$

$$Rh_{\lambda\mu}(\mathbf{k}) = \frac{\exp(i\lambda T)}{(2\pi)^{3/2}2^{1/2}} \begin{cases} 4(\mu/S)^{i\lambda}\cosh(\lambda\pi)K_{2i\lambda}(2\sqrt{S\mu}\,), S\mu > 0, \\ 2(-\mu/S)^{i\lambda}[K_{2i\lambda}(2\exp(-\pi i/2)\sqrt{-S\mu}) \\ +K_{2i\lambda}(2\exp(\pi i/2)\sqrt{-S\mu}\,)], \quad S\mu < 0, \end{cases} \tag{2.12}$$

where $K_\lambda(z)$ is a MacDonald function. The unitary operator $\exp aK_1$ assumes the form

$$(\exp aK_1)f(\mathbf{s}) = \int\int_{-\infty}^{\infty} H(a, \mathbf{l}, \mathbf{s})f(\mathbf{l})\,d\mu(\mathbf{l}), \tag{2.13}$$

where

$$H(a, \mathbf{l}, \mathbf{s}) = \langle \exp(aK_1)f_\mathbf{l}, f_\mathbf{s}\rangle = \int\int_{-\infty}^{\infty} \langle f_\mathbf{l}, \exp(-aK_1)Rh_{\mu\lambda}\rangle$$

$$\langle Rh_{\mu\lambda}, f_\mathbf{s}\rangle\,d\mu\,d\lambda = \int_{-\infty}^{\infty}\int \exp(-i\mu a)\overline{Rh}_{\mu\lambda}(\mathbf{l})Rh_{\mu\lambda}(\mathbf{s})\,d\mu\,d\lambda$$

$$= \frac{1}{8\pi|a|}\exp[-i(s_1+l_1)/a]$$

$$\times\cos\left(\frac{s_1(l_2+l_0)-l_1(s_2+s_0)}{a(s_2+s_0)^{1/2}(l_2+l_0)^{1/2}}\right), \tag{2.14}$$

as follows from a tedious computation.

The computation of $\exp[a(K_0+K_1)]$ can most conveniently be carried out in Sec. 5, where we relate this operator to the free-particle Schrödinger equation.

The dilatation operator D generates the one-parameter group $\exp aD$,

$$(\exp aD)f(\mathbf{k}) = \exp(a/2)f(e^a\mathbf{k}). \tag{2.15}$$

We can now easily exponentiate the compact generator $\Gamma_{45} = \frac{1}{2}(P_0 - K_0)$. Indeed, the operators P_0, D, and K_0 generate an $SL(2,R)$ subgroup of $\widetilde{SO}(3,2)$. It is easy to verify the relation

$$\exp 2\theta\Gamma_{45} = \exp(\tan\theta\,P_0)\exp(-\sin\theta\cos\theta\,K_0) \times\exp(-2\ln\cos\theta\,D) \tag{2.16}$$

on $SL(2,R)$, and, evaluating the right-hand side of this expression, we find

$$(\exp 2\theta\Gamma_{45})f(\mathbf{k}) = \frac{i\csc\theta}{2\pi}\exp[-i(k_0+l_0)\cot\theta]$$

$$\times\cos[\csc\theta\sqrt{2(k_0l_0+k_1l_1+k_2l_2)}\,]f(\mathbf{l})\,d\mu(\mathbf{l}),$$

$$\theta \neq n\pi. \tag{2.17}$$

Similarly, the operators P_1, D, and K_1 generate an $SL(2,R)$ subgroup of $\widetilde{SO}(3,2)$ and one can verify the relation

$$\exp 2\theta\Gamma_{12} = \exp(\tan\theta\,P_1)\exp(\sin\theta\cos\theta\,K_1)\exp(-2\ln\cos\theta\,D),$$

$$2\Gamma_{12} = K_1 + P_1,$$

or

$$(\exp 2\theta\Gamma_{12})f(\mathbf{k}) = \frac{\exp\{-ik_1[(\sin^2\theta+1)/\sin\theta\cos\theta]\}}{8\pi|\sin\theta|}$$

$$\times\int\int\exp(-il_1\cot\theta)$$

$$\times\cos\left[\frac{k_1(l_2+l_0)-l_1(k_2+k_0)}{\sin\theta\,(k_2+k_0)^{1/2}(l_2+l_0)^{1/2}}\right]f(\mathbf{l})\,d\mu(\mathbf{l}),$$

$$\theta \neq n\pi/2. \tag{2.18}$$

The operators (2.18) together with the operators $\exp\theta M_{12}$, (2.1), determine the action of the $SO(3)$ subgroup.

3. THE LAPLACE OPERATOR

On restriction of our irreducible representation of $\widetilde{SO}(3,2)$ to the compact subgroup $SO(3)$ this representation decomposes into a direct sum of irreducible representations D_l of $SO(3)$, $\dim D_l = 2l+1$. We will determine a convenient basis for $\mathcal{H}_+$, which exhibits this decomposition. This is a basis of eigenvectors of the commuting operators Γ_{45} and Γ_{23}:

$$\Gamma_{45}f = i\lambda f, \quad \Gamma_{23}f = imf, \quad -i\Gamma_{45} = \tfrac{1}{2}k_0(-\partial_{k_1k_1} - \partial_{k_2k_2} + 1). \tag{3.1}$$

By setting $k_1 = k_0\cos\theta$, $k_2 = k_0\sin\theta$, it is easy to show that the ON basis of eigenvectors is

$$f^{(l,m)}(\mathbf{k}) = [(l-m)!/\pi(l+m)!]^{1/2}(2k_0)^m\exp(-k_0) \times L_{l-m}^{(2m)}(2k_0)\exp(im\theta), \tag{3.2}$$

$$\lambda = l + \tfrac{1}{2}, \quad l = 0, 1, 2, \cdots, \quad m = -l, -l+1, \ldots, l.$$

Here, $L_n^{(\alpha)}(z)$ is a Laguerre polynomial.

From this result and the first of Eqs. (1.30) we see that the $\{f^{(l,m)}: m = l, l-1, \ldots, -l\}$ for fixed l form an ON basis for the representation D_l of $SO(3)$. Furthermore, on restriction to $SO(3)$ our representation decomposes as

$$\sum_{l=0}^{\infty} \oplus D_l.$$

From the known recurrence relations for Laguerre polynomials we find

$$\Gamma_{15}f^{(l,m)} = \tfrac{1}{2}\sqrt{(l-m+1)(l+m+1)}\,f^{(l+1,m)} - \tfrac{1}{2}\sqrt{(l-m)(l+m)}\,f^{(l-1,m)},$$

$$\Gamma_{42}f^{(l,m)} = -\tfrac{1}{4}\sqrt{(l+m+2)(l+m+1)}\,f^{(l+1,m+1)} + \tfrac{1}{4}\sqrt{(l-m)(l-m-1)}\,f^{(l-1,m+1)} + \tfrac{1}{4}\sqrt{(l+m)(l+m-1)} \times f^{(l-1,m-1)} - \tfrac{1}{4}\sqrt{(l-m+1)(l-m+2)}\,f^{(l+1,m-1)}. \tag{3.3}$$

Using (3.1), (3.3) and taking commutators, we can compute the action of any $\Gamma_{\alpha\beta}$ on this basis.

Note the close connection between the eigenvalue equation $\Gamma_{45}f = i\lambda f$ and the quantum Kepler problem in two-dimensional space:

$$Hg = \mu g, \quad H = -\partial_{xx} - \partial_{yy} + e/r, \quad r = (x^2+y^2)^{1/2}, \quad \int\int |g|^2\,dx\,dy < \infty. \tag{3.4}$$

The two eigenvalue equations can be identified provided we set $k_1 = x\sqrt{-\mu}$, $k_2 = y\sqrt{-\mu}$, $\mu = -e^2/4\lambda^2$. The eigenvalue problems are defined on Hilbert spaces with different inner products, but from the Virial theorem[16] we see that if the energy eigenvalue μ belongs to the point spectrum of H and g is a corresponding eigenvector, then g also has finite norm in $\mathcal{H}^+$. Conversely, if f is an eigenfunction of Γ_{45}, then $\int\int |f|^2\,dx\,dy < \infty$ and f corresponds to an energy eigenvalue μ in the point spectrum of H. Since the eigenvalues λ of Γ_{45} are $\lambda = l + \frac{1}{2}$, $l = 0, 1, 2, \cdots$, it follows that the point eigenvalues of H are $\mu_l = -e^2/4(l+\frac{1}{2})^2$. Although this is a satisfying explanation of the point spectrum of H, it sheds no light on the continuous spectrum of H since Γ_{45} has only point spectrum.

Using (1.20), we can compute the corresponding ON basis of positive energy solutions $\psi^{(l,m)}(x)$ of (1.1):

$$\psi^{(l,m)}(x) = 1/4\pi \int\int_{-\infty}^{\infty} \exp(ik\cdot x) f^{(l,m)}(\mathbf{k})\, d\mu(\mathbf{k})$$

$$= [(l-m)!/4\pi(l+m)!]^{1/2} \exp[im(\alpha - \pi/2)] \quad (3.5)$$

$$\times \int_0^\infty \exp[(ix_0 - 1)k_0](2k_0)^m J_m(k_0 r) L_{l-m}^{(2m)}(2k_0)\, dk_0.$$

$x_1 = r\cos\alpha, \quad x_2 = r\sin\alpha.$

In terms of the coordinates

$$x_0 = \frac{\sin\Psi}{\cos\sigma - \cos\Psi}, \quad x_1 = \frac{\sin\sigma\cos\alpha}{\cos\sigma - \cos\Psi},$$

$$x_2 = \frac{\sin\sigma\sin\alpha}{\cos\sigma - \cos\Psi}, \quad (3.6)$$

variables R-separate in (1.1) and (3.5) to give

$$\Psi^{(l,m)} = i\sqrt{(l-m)!/8\pi(l+m)!}\sqrt{\cos\sigma - \cos\Psi}$$

$$\times \exp[-i\Psi(l+\tfrac{1}{2})] \exp[im(\alpha - \pi/2) P_l^m(\cos\sigma)$$

$$= [(-i)^{m-1}/\sqrt{4l+2}]\sqrt{\cos\sigma - \cos\Psi} \exp[-i\Psi(l+\tfrac{1}{2})] Y_l^m(\sigma,\alpha), \quad (3.7)$$

where P_l^m is an associated Legendre function and Y_l^m is a spherical harmonic. (We can always parametrize so that $\cos\sigma - \cos\Psi > 0$.) Indeed, in our three-variable model we find

$$\Gamma_{45} = -\partial_\psi + \tfrac{1}{2}\sin\psi/(\cos\sigma - \cos\psi), \quad \Gamma_{23} = \partial_\alpha. \quad (3.8)$$

Thus $\psi^{(l,m)}(x) = \sqrt{\cos\sigma - \cos\psi}\exp[-i\psi(l+\frac{1}{2})]\exp(im\alpha)g(\sigma)$, and, substituting into (1.1), we see that variables R-separate and $g(\sigma)$ is a linear combination of $P_l^m(\cos\sigma)$ and $Q_l^m(\cos\sigma)$. Evaluating the integral (3.5) for special values of the parameters, e.g., $\sigma = 0, \pi$, we establish (3.7).

For future use we point out that there is another model of our irreducible representation of $\tilde{S}O(3,2)$ in which the eigenfunctions of Γ_{45} and Γ_{23} take an especially simple form. The representation space is the Bargmann Hilbert space $\mathcal{F}_2$ consisting of all entire functions $h(z_1, z_2)$ such that[17]

$$\int_{C\times C} |h|^2 d\xi(\mathbf{z}) < \infty, \quad d\xi(\mathbf{z}) = (\exp[-(|z_1|^2 + |z_2|^2)]/\pi^2) \times dx_1\, dx_2\, dy_1\, dy_2,$$

$z_j = x_j + iy_j, \quad j = 1, 2. \quad (3.9)$

The inner product is

$$\langle f, h\rangle = \int_{C\times C} \bar{f} h\, d\xi(\mathbf{z}).$$

The carrier space for our representation is not $\mathcal{F}_2$ but the subspace $\mathcal{F}_2^+$ consisting of all $h \in \mathcal{F}_2$ such that $h(-z_1, -z_2) = h(z_1, z_2)$. The functions

$$f^{(l,m)}(z_1, z_2) = z_1^{l+m} z_2^{l-m}/\sqrt{(l+m)!\,(l-m)!},$$

$$l = 0, 1, 2, \cdots, \quad m = l, l-1, \ldots, -l, \quad (3.10)$$

form an ON basis for $\mathcal{F}_2^+$. Setting

$$\Gamma_{45} = (i/2)(z_1\partial_{z_1} + z_2\partial_{z_2} + 1), \quad \Gamma_{15} = \tfrac{1}{2}(z_1 z_2 - \partial_{z_1 z_2}),$$
$$\Gamma_{23} = (i/2)(z_1\partial_{z_1} - z_2\partial_{z_2}), \quad \Gamma_{42} = \tfrac{1}{4}(\partial_{z_1 z_1} + \partial_{z_2 z_2} - z_1^2 - z_2^2) \quad (3.11)$$

and comparing with expressions (3.3), we see that we have a new model of our representation of $\tilde{S}O(3,2)$ in which the functions $f^{(l,m)}(\mathbf{k})$ can be identified with the functions (3.10). The explicit unitary mapping U from $\mathcal{H}^+$ to $\mathcal{F}_2^+$ which commutes with the group action is

$$Uf(z_1, z_2) = \int\int_{R_2} U(\mathbf{k}, \mathbf{z}) f(\mathbf{k})\, d\mu(\mathbf{k}), \quad f \in \mathcal{H}^+, \quad (3.12)$$

where

$$U(\mathbf{k}, \mathbf{z}) = \sum_{l,m} \bar{f}^{(l,m)}(\mathbf{k}) f^{(l,m)}(\mathbf{z})$$

$$= [\exp(k_0 + z_1 + z_2)/\sqrt{\pi}] \cosh[\sqrt{2k_0} \times (z_1\exp(-i\theta/2) - z_2\exp(i\theta/2))], \quad (3.13)$$

$k_1 = k_0\cos\theta, \quad k_2 = k_0\sin\theta.$

For convenience we will list the pairs of commuting second order elements in the Lie algebra of $\tilde{S}O(3,2)$ which correspond to each coordinate system we discuss. Thus the system (3.6), (3.7) corresponds to the operators

1) $\Gamma_{45}^2, \ \Gamma_{23}^2$.

There is also a Lamé-type coordinate system related to the $SO(3)$ subgroup and determined by

2) $\Gamma_{45}^2, \ \Gamma_{12}^2 + a^2\Gamma_{13}^2, \ a \neq 0,$

which we shall not treat here. The relationships between 1) and 2) are discussed in Refs. 4 and 9. These systems correspond to the eigenvalue equation for the Laplace operator on the sphere S_2.

4. DIAGONALIZATION OF P_0, P_2, AND D

We next look for those coordinate systems permitting separation of variables in (1.1) such that the corresponding basis functions Ψ are eigenfunctions of P_0: $P_0\Psi = i\lambda\Psi$. For such systems we have $\Psi(x) = \exp(i\lambda x_0) \times \Phi(x_1, x_2)$, where

$$(\partial_{11} + \partial_{22} + \lambda^2)\Phi = 0. \quad (4.1)$$

Thus the equation for the eigenfunctions reduces to the Helmholtz equation (4.1). Now P_0 commutes with every element in the Euclidean subalgebra $\mathcal{E}(2)$ generated by P_1, P_2, and M_{12}. The Euclidean group in the plane $E(2)$ with Lie algebra $\mathcal{E}(2)$ is the symmetry group of (4.1). It is known that this equation separates in exactly four coordinate systems, each system corresponding to a symmetric second- order element in the enveloping algebra of $\mathcal{E}(2)$. See Refs. 1, 6, and 18 for discussion of these matters as well as listings of the coordinates and the related eigenfunctions. The pairs of commuting second order symmetric operators associated with these systems are

3) $P_0^2, \ P_1^2$, Cartesian,

4) $P_0^2, \ M_{12}^2$, polar,

5) $P_0^2,\ M_{12}P_2+P_2M_{12}$, parabolic cylinder,

6) $P_0^2,\ M_{12}^2+P_2^2$, elliptic.

On $\mathcal{H}^+$ the requirement $P_0f=i\lambda f$ implies $f(\mathbf{k}) = \delta(k_0-\lambda)g_\lambda(\theta)$, where $\lambda>0$, $k_1=k_0\cos\theta$, $k_2=k_0\sin\theta$. The search for the functions g_λ reduces to a study of the Hilbert space $L_2[0,2\pi]$ on which $E(2)$ acts via

$$P_1=-i\lambda\cos\theta,\quad P_2=-i\lambda\sin\theta,\quad M_{12}=\partial_\theta. \tag{4.2}$$

As is well known, these operators determine a unitary irreducible representation of $E(2)$ on $L_2[0,2\pi]$. Once the eigenfunctions $g_{\lambda\mu}(\theta)$ of the second operator in 3)—6) have been determined, the corresponding separable solutions $\Psi_{\lambda\mu}$ of (1.1) can be obtained from the relation

$$\Psi_{\lambda\mu}(x)=\frac{\exp(i\lambda x_0)}{4\pi}\int_0^{2\pi}\exp[-i\lambda(x_1\cos\theta+x_2\sin\theta)]g_{\lambda\mu}(\theta)\,d\theta. \tag{4.3}$$

The eigenfunctions $g_{\lambda\mu}$ and the corresponding integrals (4.3) have been worked out in Refs. 1 and 6. For future reference we give the basis 4):

$$P_0\Psi_{\lambda n}=i\lambda\Psi_{\lambda n},\quad M_{12}\Psi_{\lambda n}=in\Psi_{\lambda n},\quad n=0,\pm1,\pm2,\cdots,$$
$$f_{\lambda n}(k_0,\theta)=\delta(k_0-\lambda)\exp(in\theta)/\sqrt{2\pi},\quad \langle f_{\lambda n},f_{\lambda' n'}\rangle=\delta(\lambda-\lambda')\delta_{nn'}, \tag{4.4}$$
$$\Psi_{\lambda n}(x_0,r,\varphi)=[(-i)^n/\sqrt{8\pi}]\exp[i(\lambda x_0+n\varphi)]J_n(\lambda r),$$
$$x_1=r\cos\varphi,\quad x_2=r\sin\varphi.$$

Now we search for coordinate systems allowing separation of variables in (1.1) such that the basis functions Ψ are eigenfunctions of P_2: $P_2\Psi=-i\gamma\Psi$. Here we have $\Psi(x)=\exp(-i\gamma x_2)\Phi(x_0,x_1)$, where

$$(\partial_{00}-\partial_{11}+\gamma^2)\Phi=0. \tag{4.5}$$

The operator P_2 commutes with the pseudo-Euclidean subalgebra $\mathcal{E}(1,1)$ generated by P_0, P_1, and M_{01} and, indeed, the pseudo-Euclidean group $E(1,1)$ is the symmetry group of (4.5). This equation separates in nine coordinate systems associated with nine symmetric second order operators in the enveloping algebra of $\mathcal{E}(1,1)$. Details on the coordinates and basis functions are given in Refs. 1, 3, and 10. The pairs of commuting operators associated with the corresponding solutions of (1.1) are

3)′ $P_2^2,\ P_0P_1$,

7) $P_2^2,\ M_{01}^2$,

8) $P_2^2,\ M_{01}P_0+P_0M_{01}$,

9) $P_2^2,\ M_{01}^2-(P_0+P_1)^2$,

10) $P_2^2,\ M_{01}^2+(P_0+P_1)^2$,

11) $P_2^2,\ M_{01}(P_0-P_1)+(P_0-P_1)M_{01}-(P_0+P_1)^2$,

12) $P_2^2,\ M_{01}^2-P_0P_2$,

13) $P_2^2,\ M_{01}^2+P_1^2$,

14) $P_2^2,\ M_{01}^2-P_1^2$.

The case 3)′ is equivalent to 3).

On $\mathcal{H}^+$ the requirement $P_2f=-i\gamma f$ implies $f(\mathbf{k}) = \delta(k_2-\gamma)g_\gamma(\xi)$, where $-\infty<\gamma<\infty$, $k_1=|k_2|\sinh\xi$, $k_0 = |k_2|\cosh\xi$. The search for eigenfunctions reduces to a study of the Hilbert space $L_2(R)$ on which $E(1,1)$ acts via

$$P_0=i|\gamma|\cosh\xi,\quad P_1=-i|\gamma|\sinh\xi,\quad M_{01}=\partial_\xi. \tag{4.6}$$

These operators define a unitary irreducible representation of $E(1,1)$ on $L_2(R)$. After the eigenfunctions $g_{\gamma\mu}(\xi)$ of the second operator in 7)—14) have been determined, the corresponding separable solutions $\Psi_{\gamma\mu}$ of (1.1) follow from

$$\Psi_{\gamma\mu}(x)=\frac{\exp(-i\gamma x_2)}{4\pi}\int_{-\infty}^{\infty}\exp[i|\gamma|(x_0\cosh\xi-x_1\sinh\xi)]\times g_{\gamma\mu}(\xi)\,d\xi. \tag{4.7}$$

The eigenfunctions $g_{\gamma\mu}$ and the integrals (4.7) are computed in Ref. 3 for cases 7)—12) and various overlaps between these bases have been determined. Here we give only the basis 7):

$$P_2\Psi_{\gamma\mu}=-i\gamma\Psi,\quad M_{01}\Psi_{\gamma\mu}=i\mu\Psi_{\gamma\mu},\quad -\infty<\gamma,\mu<\infty,$$
$$f_{\gamma\mu}(k_2,\xi)=\delta(k_2-\gamma)\exp(i\mu\xi)/\sqrt{2\pi},$$
$$\langle f_{\gamma\mu},f_{\gamma'\mu'}\rangle=\delta(\gamma-\gamma')\delta(\mu-\mu'), \tag{4.8}$$
$$\Psi_{\gamma\mu}(x_2,\rho,\theta)=(1/\sqrt{8\pi^4})\exp[i(\mu\theta-\gamma x_2)]K_{i\mu}(i|\gamma|\rho),$$
$$x_0=\rho\cosh\theta,\quad x_1=\rho\sinh\theta.$$

Next we look for coordinate systems yielding separation of variables in (1.1) such that the basis functions Ψ are eigenfunctions of D: $D\Psi=-i\nu\Psi$. In this case we have $\Psi(x)=\rho^{i\nu-1/2}\Phi(s_0,s_1,s_2)$, where

$$x_\alpha=\rho s_\alpha,\quad \rho\geq0,\quad s_0^2-s_1^2-s_2^2=\epsilon$$

and $\epsilon=+1,-1$, or 0 depending on whether $x\cdot x>0$, <0 or $=0$. It follows from the second of Eqs. (1.30) that

$$(M_{12}^2-M_{01}^2-M_{02}^2)\Phi(s)=(\nu^2+\tfrac14)\Phi(s). \tag{4.9}$$

Now D commutes with the subalgebra $so(2,1)$ generated by M_{12}, M_{01}, and M_{02}, and in fact $SO(2,1)$ is the symmetry group of (4.9). This equation separates in nine coordinate systems associated with nine symmetric second order operators in the enveloping algebra of $so(2,1)$. The details for the case $\epsilon=+1$ are worked out in Refs. 4 and 10. The pairs of commuting operators associated with separated solutions of (1.1) are

15) $D^2,\ M_{12}^2$, spherical,

16) $D^2,\ M_{01}^2$, equidistant,

7)′ $D^2,\ (M_{12}-M_{02})^2$, horocyclic,

17) $D^2,\ M_{12}^2+a^2M_{01}^2$, elliptic,

18) $D^2,\ M_{01}^2-a^2M_{12}^2$, hyperbolic $(0<a<1)$

19) $D^2,\ -M_{12}M_{02}-M_{02}M_{12}+aM_{01}^2$, semihyperbolic $(0<a<\infty)$

20) $D^2,\ aM_{01}^2+M_{02}^2+M_{12}^2-M_{02}M_{12}-M_{12}M_{02}$, elliptic-parabolic $(0<a)$

21) $D^2,\ -aM_{01}^2+M_{02}^2+M_{12}^2-M_{02}M_{12}-M_{12}M_{02}$, hyperbolic-parabolic $(0<a)$

22) D^2, $M_{02}M_{01}+M_{01}M_{02}-M_{01}M_{12}-M_{12}M_{01}$,

semicircular-parabolic.

[System 7)′ is equivalent to 7).]

On $\mathcal{H}^+$ the requirement $Df=-i\nu f$ implies $f(\mathbf{k}) = k_0^{-i\nu-1/2}h_\nu(\theta)$, where $-\infty<\gamma<\infty$, $k_1=k_0\cos\theta$, $k_2=k_0\sin\theta$. The eigenfunction problem thus reduces to a study of the Hilbert space $L_2[0,2\pi]$ on which $SO(2,1)$ acts via

$$M_{12}=\partial_\theta,\quad M_{01}=-\sin\theta\partial_\theta-(i\nu+\tfrac{1}{2})\cos\theta,$$
$$M_{01}=\cos\theta\partial_\theta-(i\nu+\tfrac{1}{2})\sin\theta. \tag{4.10}$$

These operators define a unitary irreducible representation of $SO(2,1)$ which is single-valued and belongs to the principal series: $l=-\frac{1}{2}+i|\nu|$. Once the eigenfunctions $h_{\nu\alpha}(\theta)$ of the second operator in 15)–22) have been determined, the corresponding separable solutions $\Psi_{\nu\alpha}$ of (1.1) can be obtained from

$$\Psi_{\nu\alpha}(x)=\frac{\rho^{i\nu-1/2}}{4\pi}\Gamma(\tfrac{1}{2}-i\nu)\int_0^{2\pi}\exp[\pm i\pi(\tfrac{1}{2}-i\nu)/2]$$
$$|s_0-s_1\cos\theta-s_2\sin\theta|^{i\nu-1/2}h_{\nu\alpha}(\theta)\,d\theta, \tag{4.11}$$

where the plus sign occurs when $s_0-s_1\cos\theta-s_2\sin\theta>0$ and the minus sign occurs when this expression is <0. We list explicitly the basis 17):

$$D\Psi_{\nu m}=-i\nu\Psi_{\nu m},\quad M_{12}\Psi_{\nu m}=im\Psi_{\nu m},\quad -\infty<\nu<\infty,$$
$$m=0,\pm1,\cdots,$$

$$f_{\nu m}(k_0,\theta)=k_0^{-i\nu-1/2}\exp(im\theta)/2\pi,\quad \langle f_{\nu m},f_{\nu' m'}\rangle=\delta(\nu-\nu')\delta_{mm'},$$

$$\Psi_{\nu m}(x)=\frac{\rho^{i\nu-1/2}}{4\pi}\Gamma(-i\nu-m+\tfrac{1}{2})\exp[i\pi(\tfrac{1}{2}-i\nu)/2] \tag{4.12}$$
$$\times P^m_{i\nu-1/2}(\cosh a)\exp(im\varphi),$$

$x_0=\rho\cosh a$, $x_1=\rho\sinh a\cos\varphi$, $x_2=\rho\sinh a\sin\varphi$.

Here, the expression for $\Psi_{\nu m}(x)$ is the one valid for $x\cdot x>0$, $x_0>0$. There is a similar result for the case $x\cdot x>0$, $x_0<0$, but the expression for the case $x\cdot x<0$ is somewhat more complicated:

$$\Psi_{\nu m}(x)=\frac{\rho^{i\nu-1/2}}{\sqrt{8\pi}}\left(\frac{\exp(-i\pi/4-\nu\pi/2)+(-1)^m\exp(i\pi/4+\nu\pi/2)}{\exp(-\pi\nu)+\exp(\pi\nu)}\right)$$
$$\times(\cosh a)^{-1/2}P^{-i\nu}_{-m-1/2}(\tanh a)\exp(im\varphi), \tag{4.13}$$

$x_0=\rho\sinh a$, $x_1=\rho\cosh a\cos\varphi$,

$x_2=\rho\cosh a\sin\varphi$, $a>0$.

For $a<0$, $\Psi_{\nu m}(x)$ is equal to expression (4.13) multiplied by $(-1)^m$ and a replaced with $-a$. Finally, for $x\cdot x=0$, $x_0>0$ we have

$$\Psi_{\nu m}(x)=\frac{\exp[i\pi(\frac{1}{2}-i\nu)/2]}{4\pi\sqrt{2\pi}}\Gamma(-i\nu-m+\tfrac{1}{2})$$
$$\times\left[\exp[-(\tfrac{1}{2}+i\nu)a]\frac{\Gamma(-i\nu)}{\Gamma(\frac{1}{2}-m-i\nu)}\right.$$
$$\left.+\exp[(-\tfrac{1}{2}+i\nu)a]\frac{\Gamma(i\nu)}{\Gamma(\frac{1}{2}-m+i\nu)}\right]\exp(im\varphi) \tag{4.14}$$

$x_0=e^a$, $x_1=e^a\cos\varphi$, $x_2=e^a\sin\varphi$.

5. THE SCHRÖDINGER EQUATION

Of special interest are the coordinate systems permitting separation of variables in (1.1) such that the basis functions Ψ are eigenfunctions of P_0+P_1: $(P_0+P_1)\Psi=i\beta\Psi$. For this case we have $\Psi(x)=\exp(is\beta)\Phi(t,x_2)$, where $2s=x_0+x_1$, $2t=x_1-x_0$. The function Φ satisfies the free particle Schrödinger equation

$$(i\beta\partial_t+\partial_{x_2x_2})\Phi(t,x_2)=0. \tag{5.1}$$

This equation admits as symmetries the operators

$$\mathcal{P}=P_2,\quad \mathcal{K}_{-2}=-P_0+P_1,\quad \mathcal{E}=P_0+P_1,\quad \mathcal{B}=\tfrac{1}{2}(M_{02}-M_{12}),$$
$$\mathcal{D}=-D-M_{01},\quad \mathcal{K}_2=-\tfrac{1}{4}(K_0+K_1), \tag{5.2}$$

which all commute with $P_0+P_1=\mathcal{E}$. These operators form a basis for the six-dimensional Schrödinger algebra of (5.1). Indeed the script notation and the commutation relations for the basis agree with that found in Ref. 5, where equation (5.1) was analyzed. The pairs of commuting operators associated with separable systems for (5.1) are (deleting squares as in Ref. 5):

3)″ P_0+P_1, P_2, free particle,

23) P_0+P_1, $P_0-P_1-\frac{1}{4}K_0-\frac{1}{4}K_1$, oscillator,

24) P_0+P_1, $P_0-P_1+aM_{12}-aM_{02}$,

$a\neq0$, linear potential,

25) P_0+P_1, $D+M_{01}$, repulsive oscillator.

[The so-called free-particle coordinates 3)″ are the same as 3).] On $\mathcal{H}^+$ the requirement $(P_0+P_1)f=i\beta f$ implies $f(\mathbf{k})=u\delta(u-\beta)l_\beta(v)$, where $\beta>0$, $u=k_0-k_1$, $v=k_2$. Thus, the search for the l_β reduces to a study of the Hilbert space $L_2(R)$ on which the Schrödinger group acts via

$$\mathcal{E}=i\beta,\quad \mathcal{P}=-iv,\quad \mathcal{B}=\tfrac{1}{2}\beta\partial_v,\quad \mathcal{D}=-\tfrac{1}{2}-v\partial_v,$$
$$\mathcal{K}_{-2}=-iv^2/\beta,\quad \mathcal{K}_2=-\tfrac{1}{4}\beta\partial_{vv}. \tag{5.3}$$

As shown in Ref. 5, these operators determine an irreducible unitary representation of the Schrödinger group on $L_2(R)$. Once the eigenfunctions $l_{\beta n}(v)$ of the second operators in 23)–25) have been determined, the corresponding separable solutions $\Psi_{\beta n}$ of (1.1) can be computed from

$$\Psi_{\beta n}(x)=\frac{\exp(i\beta s)}{4\pi}\int_{-\infty}^{\infty}\exp[-i(v^2t/\beta+vx_2)]l_{\beta n}(v)\,dv. \tag{5.4}$$

For reference we list the basis 23):

$$(P_0+P_1)\Psi_{\beta n}=i\beta\Psi_{\beta n},\quad (P_0-P_1-\tfrac{1}{4}K_0-\tfrac{1}{4}K_1)\Psi_{\beta n}=i(n+\tfrac{1}{2})\Psi_{\beta n},$$

$$n=0,1,2,\cdots,\quad f_{\beta n}(u,v)=\frac{u\delta(u-\beta)}{(n!\sqrt{2\pi}\,2^n)^{1/2}}\exp(-v^2/\beta)H_n\left(v\left(\frac{2}{\beta}\right)^{1/2}\right), \tag{5.5}$$

$$\langle f_{\beta n},f_{\beta' n'}\rangle=\beta\delta(\beta-\beta')\delta_{nn'},$$

$$\Psi_{\beta n}(s,t,x_2)=\frac{\exp(i\beta s)(-i)^n}{4(n!\,2^n\pi\sqrt{2\pi})^{1/2}}\left(\frac{\beta}{1+it}\right)^{1/2}\left(\frac{1-it}{it+1}\right)^{n/2}\times\exp\left(\frac{-x_2^2\beta}{4(1+it)}\right)H_n\left(x_0\left(\frac{\beta}{2(1+t^2)}\right)^{1/2}\right).$$

Using the u,v coordinates we can easily compute $\exp[a(K_0+K_1)]$. Indeed from the expression for $\mathcal{K}_2$ in (5.3) and formula (3.8) of Ref. 5 we find

$$\{\exp[a(K_0+K_1)f]\}(k_1,k_2)$$

$$= \frac{1}{[4\pi i a(k_0 - k_1)]^{1/2}} \int_{-\infty}^{\infty} \exp\left(\frac{-(k_2 - w)^2}{4ia(k_0 - k_1)}\right)$$
$$\times f\left(\frac{w^2 - (k_0 - k_1)^2}{2(k_0 - k_1)}, w\right) dw, \quad f \in \mathcal{H}^+. \tag{5.6}$$

6. THE EPD EQUATION

Next we look for coordinate systems yielding separation of variables for (1.1) such that the basis functions Ψ are eigenfunctions of M_{12}: $M_{12}\Psi = im\Psi$. We have $\Psi(x) = \exp(im\varphi)\Phi(x_0, r)$, where

$$x_1 = r\cos\varphi, \quad x_2 = r\sin\varphi,$$

and Φ satisfies the Euler—Poisson—Darboux equation

$$\left(\partial_{00} - \partial_{rr} - \frac{1}{r}\partial r + \frac{m^2}{r^2}\right)\Phi = 0 \tag{6.1}$$

or

$$(\Gamma_{45}^2 - \Gamma_{41}^2 - \Gamma_{51}^2)\Phi = (\Gamma_{23}^2 + \tfrac{1}{4})\Phi = -(m + \tfrac{1}{2})(m - \tfrac{1}{2})\Phi \tag{6.2}$$

from the last of expressions (1.30). The symmetry group of (6.1) is $SL(2,R)$ and is generated by the symmetry operators $\Gamma_{45}, \Gamma_{41}, \Gamma_{51}$.

The coordinate systems in which (1.1) separates via (6.1) are characterized by the following pairs of commuting operators $[\Gamma_{23} = M_{12},\ \Gamma_{15} = D,\ \Gamma_{45} = \frac{1}{2}(P_0 - K_0),\ \Gamma_{14} = \frac{1}{2}(P_0 + K_0)]$:

$$\begin{aligned}
&1)' \quad \Gamma_{23}^2,\ \Gamma_{45}^2,\\
&4)' \quad \Gamma_{23}^2,\ (\Gamma_{45} + \Gamma_{14})^2,\\
&17)' \quad \Gamma_{23}^2,\ \Gamma_{15}^2,\\
&26) \quad \Gamma_{23}^2,\ 2\Gamma_{14}^2 + \Gamma_{45}\Gamma_{14} + \Gamma_{14}\Gamma_{45},\\
&27) \quad \Gamma_{23}^2,\ 2\Gamma_{45}^2 + \Gamma_{45}\Gamma_{14} + \Gamma_{14}\Gamma_{45},\\
&28) \quad \Gamma_{23}^2,\ \Gamma_{14}^2 + a(\Gamma_{45}\Gamma_{15} + \Gamma_{15}\Gamma_{45}), \quad a \neq 0,\\
&29) \quad \Gamma_{23}^2,\ \Gamma_{45}^2 + a\Gamma_{15}^2, \quad a \neq 0,\\
&30) \quad \Gamma_{23}^2,\ a\Gamma_{14}^2 + \Gamma_{15}^2, \quad a \neq 0,\\
&31) \quad \Gamma_{23}^2,\ (\Gamma_{14} + \Gamma_{45})\Gamma_{15} + \Gamma_{15}(\Gamma_{14} + \Gamma_{45}).
\end{aligned} \tag{6.3}$$

These statements will be proved and the corresponding coordinate systems derived in another publication.

On $\mathcal{H}^+$ the requirement $M_{12}f = imf$ implies $f(\mathbf{k}) = \exp(im\theta)\mathfrak{f}_m(k_0)$ where $m = 0, \pm 1, \pm 2, \cdots$, $k_1 = k_0\cos\theta$, $k_2 = k_0\sin\theta$. The eigenfunction problem reduces to a study of the Hilbert space $L_2[0,\infty]$ on which $SL(2,R)$ acts via

$$\begin{aligned}
\Gamma_{45} &= \frac{i}{2}k_0\left(-\partial_{k_0k_0} - \frac{1}{k_0}\partial_{k_0} + \frac{m^2}{k_0^2} + 1\right),\\
\Gamma_{14} &= \frac{i}{2}k_0\left(\partial_{k_0k_0} + \frac{1}{k_0}\partial_{k_0} - \frac{m^2}{k_0^2} + 1\right),\\
\Gamma_{15} &= \tfrac{1}{2} + k_0\partial_{k_0}.
\end{aligned} \tag{6.4}$$

This action is irreducible and unitary equivalent to a single-valued representation of $SL(2,R)$, not $SO(2,1)$, from the negative discrete series $D^-_{|m|-1/2}$, as can be seen from (6.2) and (3.2). Indeed, the eigenvalues of Γ_{45} in this model are $i(n + \frac{1}{2})$, $n = |m|, |m|+1, |m|+2, \cdots$. This model of D_l^- has been studied by a number of authors, e.g., Refs. 19, 20, but the connection with separation of variables in the EPD equation has not been pointed out before. In another publication we will use this and other models of the discrete series D_l^- to study the spectra of the operators (6.3) and derive special function expansions related to the EPD equation.

7. DIAGONALIZATION OF $\Gamma_{23} - \Gamma_{45}$

Finally we look for separable solutions of (1.1) such that the basis functions χ are eigenfunctions of $L = \frac{1}{2}(\Gamma_{23} - \Gamma_{45})$: $L\chi = i\kappa\chi$. Introducing the coordinates (3.6) and setting $\chi(x) = \sqrt{\cos\sigma - \cos\Psi}\,\Phi$, we find

$$\left(\frac{1}{\sin\sigma}\partial_\sigma(\sin\sigma\partial_\sigma) - \partial_{\Psi\Psi} - \tfrac{1}{4} + \frac{1}{\sin^2\sigma}\partial_{\alpha\alpha}\right)\Phi = 0. \tag{7.1}$$

Now we choose as independent coordinates σ, β, ρ where $\beta = \alpha + \Psi$, $\rho = \alpha - \Psi$. In terms of these coordinates, the induced action of L on Φ is $L = \partial_\beta$ and the solutions of $L\chi = i\kappa\chi$ are $\chi(x) = \sqrt{\cos\sigma - \cos\Psi}\ \exp(i\kappa\beta)\Theta$, where

$$[\sin^2\sigma\partial_{\sigma\sigma} + \cos\sigma\sin\sigma\partial_\sigma + (\tfrac{1}{4} - \kappa^2)\cos^2\sigma - 2i\kappa(\sin^2\sigma + 1)\partial\rho$$
$$+ \cos^2\sigma\partial_{\rho\rho} - \tfrac{1}{4}]\Theta(\sigma, \rho) = 0. \tag{7.2}$$

The symmetry algebra of (7.2) is $sl(2,R)$ with basis

$$A = \tfrac{1}{2}(\Gamma_{23} + \Gamma_{45}), \quad B = \tfrac{1}{2}(\Gamma_{24} + \Gamma_{35}), \quad C = \tfrac{1}{2}(\Gamma_{25} - \Gamma_{34}) \tag{7.3}$$

and commutation relations

$$[A,B] = C, \quad [C,A] = B, \quad [C,B] = A. \tag{7.4}$$

Moreover, it is straightforward to verify the identity

$$A^2 - B^2 - C^2 = L^2 + \tfrac{1}{4}. \tag{7.5}$$

In terms of the coordinates σ, β, ρ we also have $A = \partial_\rho$ as the action of A on the solutions of (7.2). From (3.1), (3.2) we see that the spectrum of L is given by $\kappa = \frac{1}{2}(s + \frac{1}{2})$, $s = 0, 1, 2, \cdots$, and for fixed κ the eigenvalues μ of $-iA$, $A\Theta = i\mu\Theta$, are $\mu = l + \frac{1}{4} - s/2$, $l = 0, 1, 2, \cdots$. In terms of (7.5), Eq. (7.2) becomes

$$(A^2 - B^2 - C^2)\Theta = -(-\kappa - \tfrac{1}{2})(-\kappa + \tfrac{1}{2})\Theta \tag{7.6}$$

so that the solutions of (7.2) form the basis space for a model of the representation $D^-_{-\kappa-1/2}$, (negative discrete series), of the twofold covering group of $SL(2,R)$.

The problem of separation of variables for (7.2) has not been settled yet. We will investigate this problem in a future paper to see if there are for this symmetry algebra exactly nine separable systems corresponding to the nine orbits of second-order operators just as found in Secs. 4 and 6. At the moment we know definitely only the subgroup systems:

$$\begin{aligned}
&1)' \quad L^2,\ A^2,\\
&32) \quad L^2,\ B^2,\\
&33) \quad L^2,\ (A+B)^2.
\end{aligned}$$

Except for 1)' our $\mathcal{H}_+$-model is not very convenient for systems of this type. Thus, in later publications we shall employ the model (3.11) and other models of the discrete series, such as those found in Refs. 19 and 21, to study this case. This conclude our list of semisubgroup coordinate systems in which variables separate in (1.1).

8. CONCLUDING REMARKS

In the previous sections we have classified all separable coordinate systems for the wave equation such that the defining operators take the form A^2, S where $A \in so(3,2)$, S is a second order symmetric element in the enveloping algebra of $so(3,2)$ and $[A, S]=0$. As explained in the Introduction, $SO(3,2)$ acts on the pair via the adjoint representation to generate an orbit of $SO(3,2)$—equivalent pairs of commuting second order operators. Coordinate systems corresponding to equivalent pairs of operators are considered as equivalent.

By choosing appropriate examples it is easy to show that there are elements A in $so(3,2)$ for which there is no S such that the pair A^2, S corresponds to a separable coordinate system. To see why this is the case, we classify the orbits in $so(3,2)$ under the adjoint action of $SO(3,2)$. This classification has been obtained in principle by Zassenhaus,[22] but we present the results here in a much more explicit form. Indeed, we list the possible eigenvalues of a 5×5 matrix $A \in so(3,2)$ and for each choice of eigenvalues we list a canonical form $\Gamma \in so(3,2)$ such that $\Gamma = TAT^{-1}$ for some $T\in SO(3,2)$, i.e., we list an element Γ on each $SO(3,2)$ orbit in $so(3,2)$. From the relation $A^t = -GAG$ it follows easily that $\det(A-\lambda E) = -\det(A+\lambda E)$, where $\lambda = \alpha + i\beta \in \mathbb{C}$ and E is the 5×5 identity matrix. Thus, $\lambda = 0$ is always an eigenvalue of A and, if $\lambda \neq 0$ is an eigenvalue, then so are $-\lambda$ and $\bar{\lambda}$. We use the notation $\lambda(n)$, $n=2,3,4,5$, to signify that λ corresponds to a generalized eigenvector x of rank n, i.e., n is the smallest integer m such that $(A-\lambda E)^m x = 0$.

	Possible Eigenvalue	*Canonical Form* $\Gamma \in so(3,2)$
1.	$0, \pm\lambda, \pm\bar{\lambda}$ $\lambda = \alpha + i\beta,\ \alpha, \beta \neq 0$	$\alpha(\Gamma_{24}+\Gamma_{35}) + \beta(\Gamma_{23}-\Gamma_{45})$
2.	$0, \pm\alpha, \pm\beta$ $\beta \neq 0$	$\alpha\Gamma_{24} + \beta\Gamma_{35}$
3.	$0, \pm\alpha, \pm i\beta$ $\alpha, \beta \neq 0$	$\alpha\Gamma_{14} + \beta\Gamma_{23}$
4.	$0, \pm i\alpha, \pm i\beta$ $\alpha, \beta \neq 0$	$\alpha\Gamma_{23} - \beta\Gamma_{45}$
5.	$0, \alpha(2), -\alpha(2)$	$P_0 + P_1 + \alpha(D + M_{01})$
6.	$0, i\alpha(2), -i\alpha(2)$ $\alpha \neq 0$	$2\alpha L + A + B$ [see (7.3)]
7.	$0, 0, 0, \pm i\beta$ $\beta \neq 0$	$\beta\Gamma_{23}$ or $\beta\Gamma_{45}$
8.	$O(3), i\alpha, -i\alpha$ $\alpha \neq 0$	$1/\sqrt{2}(\Gamma_{35}+\Gamma_{45}) + \alpha\Gamma_{12}$
9.	$O(3), \pm\alpha$ $\alpha \neq 0$	$1/\sqrt{2}(\Gamma_{13}+\Gamma_{15}) + \alpha\Gamma_{24}$
10.	$O(3), 0, 0$	P_0 or P_1
11.	$O(5)$	$1/\sqrt{2}(\Gamma_{13}+\Gamma_{15})$ $+\frac{1}{2}(\Gamma_{25}-\Gamma_{23}+\Gamma_{34}+\Gamma_{45})$

Suppose $A=\Gamma$ takes the form 1. It is straightforward to show that the operators $2B = \Gamma_{24}+\Gamma_{35}$ and $2L = \Gamma_{23}-\Gamma_{45}$ commute with each other and that the only nontrivial elements of $so(3,2)$ commuting with A are linear combinations of B and L. Thus the separable coordinate system associated with a pair A^2, S and of the type discussed in the last paragraphs of the Introduction is equivalent to the system associated B^2 and L^2, i.e., the system 32). Similar remarks hold for cases 2—9.

The operator of case 11 lies on the same orbit as the linear potential operator 24): $P_0 - P_1 + aM_{12} - aM_{02}$. This latter operator commutes only with a linear combination of itself and $P_0 + P_1$.

It is clear that we can use our $\mathcal{H}_+$-model to compute the spectra corresponding to pairs A^2, S and to compute overlaps between basis functions corresponding to distinct pairs, in analogy with the procedures developed in Refs. 1—7. We will present these results in forthcoming papers.

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