Lie theory and separation of variables. 7. The harmonic oscillator in elliptic coordinates and Ince polynomials

C. P. Boyer

Centro de Investigacion en Matematicas Aplicadas y en Sistemas, Universidad Nacional Autonoma de Mexico, Mexico D.F., Mexico

E. G. Kalnins and W. Miller Jr.

Centre de Recherches Mathematiques, Universite de Montreal, Montreal 101, P.Q., Canada

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As a continuation of Paper 6 we study the separable basis eigenfunctions and their relationships for the harmonic oscillator Hamiltonian in two space variables with special emphasis on products of Ince polynomials, the eigenfunctions obtained when one separates variables in elliptic coordinates. The overlaps connecting this basis to the polar and Cartesian coordinate bases are obtained by computing in a simpler Bargmann Hilbert space model of the problem. We also show that Ince polynomials are intimately connected with the representation theory of $SU(2)$, the group responsible for the eigenvalue degeneracy of the oscillator Hamiltonian.

INTRODUCTION

In Ref. 1 (hereafter referred to as 6) the authors gave a detailed investigation of the nine-parameter symmetry group $G$ (the Schrödinger group) of the equation

$$iU_t + \Delta U = 0.$$  

(1)

It was found that (1) separates in 26 coordinate systems and that with each coordinate system is associated an orbit under the action of the Galilean subgroup $G(2) \subset G$ consisting of a pair of commuting operators $(K,S)$, where $K \subset G$ the Lie algebra of $G$ and $S$ is a second-order element in the universal enveloping algebra of $G$. It was further shown that in all except five cases (which are subgroup coordinates) the first-order symmetry operator $K$ corresponds to an orbit which can be associated with one of four types of potentials: the free particle, the attractive and repulsive harmonic oscillator, and the linear potential.

The Schrödinger equation for the attractive harmonic oscillator in two space variables separates in exactly three orthogonal coordinate systems: Cartesian, polar, and elliptic. The corresponding eigenfunctions in the three systems are a product of two Hermite polynomials, a Laguerre polynomial times an exponential function, and a product of two Ince polynomials, respectively.

In this paper we examine these bases and compute the overlap functions relating different bases, with special emphasis on the Ince polynomial case. Due to the equivalence of the free particle Schrödinger equation (1) and the (time dependent) harmonic oscillator equation we have chosen to present our eigenfunctions as solutions of (1). However, all our results translate immediately to the harmonic oscillator problem.

It can be seen from Table II of 6 that to each type of potential and corresponding symmetry $S$ except the attractive harmonic oscillator there correspond two coordinate systems equivalent under $G$ though not under $G(2)$. In one of these equivalent coordinate systems labeled by superscript (1), the eigenfunctions and corresponding calculations are quite simple, while the other system affords the close connection with one of the physical potentials mentioned above. It was the existence of the "simple" systems which made the computations in 6 so easy. Now it is a remarkable fact that for the attractive harmonic oscillator, the analog of the coordinate systems of type (1) is the realization of the harmonic oscillator given several years ago by Bargmann. 2 Note that although the Bargmann transform is not a member of $G$, it is a member of the complexification $G' = C \cdot G$. It is the purpose of this work to explore fully this analogy, especially in the case of elliptic coordinates where almost all of the developments presented are new.

It is well known that the eigenvalues of the harmonic oscillator Hamiltonian are degenerate and that the group responsible for the degeneracy is $SU(2)$. In Sec. 3 we discuss the relationship between this group and the elliptic basis, developing the connection between Ince polynomials and the representation theory of $SU(2)$ in analogy to the connection between Lamé polynomials and $SU(2)$ as discussed in Ref. 4.

1. PRELIMINARIES

First we give explicitly the Lie algebra $G$ of the symmetry group $G$, as well as the spectral resolutions of the pairs corresponding to the oscillator coordinates mentioned above. For further details the reader is referred to 6. The real Lie algebra $G$ is spanned by the differential operators

$$K_i = -i\partial_i - i(x_i \hat{\partial}_x + x_x \hat{\partial}_x) - i + i/2(x_i^2 + x_x^2), \quad K = \hat{\partial}_x,$$

$$P_i = \partial_i, B_i = -i\partial_i + x_i/2, \quad i = 1, 2, \quad M = x_x \partial_x - x_x \partial_x, \quad D = x_x \partial_x + x_x \partial_x + 2i\partial_x + 1, \quad E = i.$$  

(1.1)

The coordinate systems related to the attractive harmonic oscillator are written as $O_c$, $O_r$, and $O_e$ [corresponding to Cartesian, radial (polar), and elliptic coordinates respectively], and are presented in Table I of 6. The associated pairs of operators are $(K_o - K_x, P_x^2 + B_x^2)$, $(K_o - K_x, M^2)$, and $(K_o - K_x, M^2 - P_x^2 - B_x^2)$ respectively, as listed in Table II of 6.

The spectral resolutions of these pairs as given in 6 with $L_3 = K_o - K_x$ are...
form to relate the basis states in Bargmann Hilbert space to the solutions (1.2), (1.3), and (1.4) of the free particle Schrödinger equation (8).

2. BARGMANN’S REALIZATION

Bargmann’s transformation (9) (we consider only the case of two spatial dimensions) is a unitary mapping of $L^2(R^2)$ onto the Hilbert space $\mathcal{H}(z)$ of functions $f$ of two complex variables $z = (x_1, x_2)$ completed with the norm $\|f\|$ induced by the inner product

$$
\langle g, f \rangle = \int_{R^2} d\mu(z) \overline{g(z)} f(z),
$$

with $d\mu(z) = e^{-\gamma} \sum_{n=1}^\infty \delta_{n} z^n d^n z$. The mapping is given by

$$
f(z) = \langle z | \langle z | \psi \rangle \psi \rangle,
$$

where $\psi(x) \in L^2(R^2)$ and

$$
A(x, z) = z^{1/2} \exp(-\frac{1}{2}(x^2 + z^2)) + \sqrt{z} \cdot x.
$$

The inverse mapping $A^{-1}$ is given by

$$
\phi(x) = (A^{-1}) (\psi) = \lim_{\gamma \to 0} \int_{R^2} d\mu(z) \overline{\Lambda(z, x)} g(z),
$$

for any $\alpha \in J_z$.

The composition of the two unitary maps $\exp(iK_x)$ and $A^{-1}$ will then map entire functions $f \in J_z$ onto $L^2(R^2)$ functions which are solutions of (8). This mapping is given by

$$
\exp(iK_x) \Lambda^{-1}(x, i) = \int_{R^2} d\mu(z) \overline{K(z, x)} g(z),
$$

where

$$
\overline{K(z, x)} = \frac{1}{(1 + 2it)^{\nu}} \exp\left(-\frac{1}{2}(1 - 2it) \overline{z^2} - \frac{1}{2} x^2 + \sqrt{2\overline{z} \cdot x}\right).
$$

Notice that when $t \to 0$, we recover Bargmann’s mapping (2.4) as we must. The inverse map $A \exp(-iK_x)$ with the kernel $\Lambda(z, x)$ is then obtained by complex conjugation of (2.6), viz.,

$$
A \exp(-iK_x) \psi(x) = \int_{R^2} d\mu(z) K(z, x) \overline{\psi}(x).
$$

Thus we have established the one-to-one correspondence between Bargmann’s Hilbert space of entire functions $J_z$ with the $L^2(R^2)$ solution of the free particle Schrödinger equation (8). One can also use (2.7) to construct the Lie algebra $G$ in the Bargmann realization; however, it is easier to evaluate the generators (1.1) at $t = 0$ and make the replacement $\gamma \to -\gamma$ as done in 6 and then pass to Bargmann’s realization by replacing the annihilation operator $\frac{1}{2}x_1 - \frac{1}{2}x_2$ by its analytic representation $z_0$. In this way the generators of $G$ take the form

$$
L_z = -(ix_1 \partial_{x_1} + x_2 \partial_{x_2} - 1),
$$

$$
L_x = \frac{1}{2} \left( \partial_{x_1} + \partial_{x_2} + z_1^2 + z_2^2 \right),
$$

$$
D = \partial_{x_1} - \partial_{x_2} + z_1^2 - z_2^2, \quad \beta_i = \frac{1}{2}(x_i + z_i),
$$

$$
\beta_i = \frac{1}{2}(x_i - z_i), \quad \eta = (x_1 \partial_{x_2} - x_2 \partial_{x_1}), \quad \xi = i,
$$

where the script letters correspond to the block letters in (1.1) and we have used, instead of $K_x$ and $K_x$, the combinations $L_3 = K_x - K_x$ and $L_3 = K_x + K_x$, which take a simpler form in the $(x_1, x_2)$ formalism. Indeed the harmonic oscillator Hamiltonian $i\Delta$ now appears as a
dilatation operator making its spectral resolution in $\mathcal{J}_2$ very simple. As well we can give the integrated group action of (2.8) as done in 6. However, as we use only (2.5) and (2.7), we omit this.

Now the second order operator $P_1^2 + \beta_1^2$ for the oc system takes the form

$$P_1^2 + \beta_1^2 = -(x_1^2 + 1)$$

and hence, the normalized eigenfunctions of the pair $(iL_3, P_1^2 + \beta_1^2)$ with eigenvalues $(\lambda, \mu)$ yield Bargmann’s well-known result

$$g_{\lambda, \mu}(z) = e^{i\pi \frac{1}{2} \frac{\lambda \mu}{\mu}} e^{-\pi i \frac{\lambda \mu}{\mu} \frac{1}{2}},$$

(2.9)

where $\mu = -n_1 + 1$, $\lambda = n_1 + n_2 + 1$. These functions form an ON basis in $\mathcal{J}_2$ and map onto the Oc functions (1.2) via the unitary map (2.5).

In order to treat the systems or and where it is introduced to the complex polar coordinates

$$z_1 = \rho \cos \xi, \quad \rho \in \mathbb{R}, 0 < \xi < \infty, \quad -\infty < \Im \rho < \infty$$

$$z_2 = \rho \sin \xi, \quad -\pi < \xi < \pi, \quad -\infty < \Im \rho < \infty,$$

(2.10)

In these coordinates the operators $iL_3$ and $\mathcal{H}$ take the simple form

$$iL_3 = \frac{\partial}{\partial \xi} + 1, \quad \mathcal{H} = \hbar \xi,$$

and hence the spectral resolution of the pair $(iL_3, \mathcal{H})$ with eigenvalues $(\lambda, \mu)$ yields the eigen functions

$$g_{\lambda, \mu}^*(z) = K(2^{n_1 - 1} \Gamma(n_1 + 1))^{1/2} e^{-\mu \rho^2} \cos \xi,$$

(2.11a)

$$g_{\lambda, \mu}(z) = K(2^{n_1 - 1} \Gamma(n_1 + 1))^{1/2} e^{-\mu \rho^2} \sin \xi,$$

(2.11b)

where $K, n_1, n_2$ are as in (1.3). These basis functions form an ON basis in $\mathcal{J}_2$ which map onto the Oc functions (1.2) by (2.5).

For the elliptic system we consider the spectral resolution of the pair $(iL_3, \mathcal{H})$ with eigenvalues $(\lambda, \mu)$. It is easy to see that the second of these operators gives the differential equation for Ince functions in the complex variable $\xi$, which we discuss in more detail in the next section. Suffice it now to write down the eigenfunctions (S$^n$ is an odd-parity Ince polynomial)

$$g_{\lambda, \mu}^*(z) = 2^{\nu/2} \mu C_{\lambda}(\xi),$$

(2.12a)

$$g_{\lambda, \mu}(z) = 2^{\nu/2} \mu S_{\lambda}(\xi),$$

(2.12b)

where the notation follows from (1.4) and (1.5). The functions (2.12) form an ON basis in $\mathcal{J}_2$ which map onto the functions (1.4) through the unitary map (2.5).

3. INCE POLYNOMIALS AND SU(2)

As is well known the degeneracy group for the harmonic oscillator in two spatial dimensions is SU(2). Although SU(2) is not a subgroup of $G$, a representation of its Lie algebra appears as a subalgebra of the 20-dimensional vector space of second-order elements in the enveloping algebra of $G$. Rather than give immediately the representations of the Lie algebra SU(2) in terms of these operators, we prefer to develop the abstract formalism along the lines presented by Patera and Winternitz for Lamé polynomials, establishing the connection with the preceding section at the end.

A. The algebraic approach

Denote by $\mathcal{U}$ the universal enveloping algebra of the Lie algebra $L$ [here $L = SU(2)$], $\mathcal{C}$ the center of $\mathcal{U}$, $\mathcal{U}_2$ the symmetric second-order elements of $\mathcal{U}$, and define $\mathcal{U}^{(2)} = L \oplus \mathcal{U}_2$. Let $J_i, i = 1, 2, 3$, be the standard basis for $SU(2)$. Then a general element of $\mathcal{U}^{(2)}$ can be written as

$$a_1 J_1 J_1 + a_2 J_2 J_2 + a_3 J_3 J_3,$$

where $a_1, a_2, a_3 \in \mathbb{R}$. Note that for $SU(2)$, $\mathcal{U}_2 \subset \mathcal{U}_2$. It suffices to consider only elements of the factor algebra $\mathcal{U}^{(2)}/\mathcal{C}$. Now an arbitrary element of $\mathcal{U}^{(2)}/\mathcal{C}$ can be brought to the form $J_2^2 + nJ_2 + aJ_3$, through an inner automorphism of $SU(2)$. The symmetric second order elements $\mathcal{U}_2$ have been studied by Patera and Winternitz, and they have shown the one-to-one correspondence between the two $SU(2)$ orbits and separation of variables on the sphere $S^2$. In any case a general element of $\mathcal{U}^{(2)}/\mathcal{C}$ describes an eigenvalue problem with four free parameters giving rise to special functions which have as limiting cases both Lamé polynomials and polynomials arising from the element $J_2^2 + nJ_2$, which we shall show to be Ince polynomials.

The Lie algebra $SU(2)$ with the basis of Hermitian generators $J_i$ takes the form

$$[J_i, J_j] = i\epsilon_{ijk}J_k.$$  

(3.1)

The canonical basis for the representation space is defined by

$$J_3 \psi_{\nu + 1} = (\nu + 1) \psi_{\nu}, \quad J_3 \psi_{\nu - 1} = (\nu - 1) \psi_{\nu},$$

(3.2)

$$J_2 \psi_{\nu} = \nu \psi_{\nu},$$

$$J_1 \psi_{\nu} = \mu \psi_{\nu},$$

with $J_3 = j_3 \pm \hbar/2$, where we employ Villenkin’s phase convention

$$\exp(\pm i t J_3) \psi_{\nu} = \exp(\pm i \nu \tau) \psi_{\nu}.$$  

We are interested in the eigenvalue problem defined by the operator

$$E = J_2^2 + nJ_2 + aJ_3,$$

with eigenvalue taken for later convenience to be $1/\eta$, $\nu \tau$.  

$$E \psi_{\nu,j} = \frac{1}{\eta} \psi_{\nu,j},$$

(3.4)

First we consider some symmetries of $E$ in the group of automorphisms of $SU(2)$. Now any such symmetry must map $J_3 \rightarrow J_3$ and $J_2 \rightarrow \pm J_2$. It is not difficult to see that any transformation $R$ of this type necessarily takes one of two possible forms:

(i) $R = a_1$, $a_1 \in \mathbb{C}$, $I = \text{identity in } SU(2)$,

(ii) $R = \beta \exp(-i \xi J_3), \quad \beta \in \mathbb{C}.$

From the existence of $R^*$ and Schur’s lemma it is clear that the functions $\psi_{\nu,j}$ do not completely specify a basis for an irreducible representation of $SU(2)$. We can define a complete basis by further specifying the eigenvalues of $R^*$. Furthermore, since $(R^*)^2$ is a multiple of the identity, we can take these eigenvalues to be $\pm 1$ which then determines $\beta$ to be $\exp(i \xi)$. We hereafter drop the minus superscript on $R$ and write

$$R^* \psi_{\nu,j} = \pm \psi_{\nu,j}.$$  

(3.5)
where \( R = \exp(i\eta) \exp(-i\eta J_3) \). The hermiticity of \( E \) and \( R \) then guarantees the orthogonality conditions

\[
(\psi_j^{\eta}, \psi_n^{\eta}) = \delta_{nn}, \;
\]

(3.6)

where we have properly normalized \( \psi_j^{\eta} \).

The determination of \( \psi_j^{\eta} \) is then tantamount to the determination of the overlap functions \( (\psi_j^{\eta}, \psi_n^{\eta}) \). From (3.5) we find

\[
(\psi_j^{\eta}, \psi_n^{\eta}) = \pm \exp(-i\pi r)(\psi_j^{\eta}, \psi_n^{\eta})
\]

(3.7)

and from (3.2), (3.3), and (3.4) we obtain the third-term recursion formula

\[
-\frac{q}{2\\sqrt{2}}(j + r)(j + r + 1)^{1/2}(\psi_j^{\eta}, \psi_n^{\eta}) = -(p - \frac{1}{2}n)(\psi_j^{\eta}, \psi_n^{\eta}).
\]

(3.8)

It is now convenient to introduce new coefficients \( A_r^{\eta} \) as

\[
A_r^{\eta} = \frac{\exp(i\pi(j - r)/2)(\psi_j^{\eta}, \psi_n^{\eta})}{\sqrt{(j - r)!}(j + r)!},
\]

(3.9)

where \( 0 < m < j \), and from (3.7) \( A_r^{\eta} \) can be defined for negative \( r \) as

\[
A_r^{\eta} = \pm A_r^{\eta}.
\]

We see immediately that \( A_n^{\eta} = 0 \). Upon substituting into (3.8) our recursion formula takes the form given by Arscott for Ince polynomials with \( i \) integer, viz.,

\[
\xi(j + r + 2)A_{r+2}^{\eta} + (4r^2 + 4 - \eta)A_{r+1}^{\eta} + \xi(j - r)A_r^{\eta} = 0,
\]

(3.10a)

\[
\xi(j + r + 1)A_r^{\eta} + (4 - \eta)A_{r-1}^{\eta} + 2\xi j A_r^{\eta} = 0,
\]

(3.10b)

\[
\xi(j + 1)A_{r+1}^{\eta} - \xi A_r^{\eta} = 0,
\]

(3.10c)

\[
(4r^2 - \eta)rA_{r+1}^{\eta} + 2A_r^{\eta} = 0.
\]

(3.10d)

where \( \xi = -2\pi \) and we have identified \( A_r^{\eta} \) and \( A_r^{\eta} \) with Arscott's trigonometric coefficients \( A_r^{\eta} \) and \( B_r^{\eta} \) respectively, up to normalization. Notice also that our \( r \) takes on both integer and half-integer values. Now for \( j \) half-integer we merely delete Eqs. (3.10b, c).

Moreover, Arscott's parameter \( p \) is identified with our \( 2\eta / (p + 2\eta) \). Thus even \( p \) corresponds to integer \( \Gamma \)'s irreducible representations of \( SU(2) \) and odd \( p \) to half-integer \( \Gamma \)'s.

Following Arscott, we denote the characteristic values \( \eta \) by \( a_r^{\eta}(\xi) \) and \( b_r^{\eta}(\xi) \) for \( \psi_j^{\eta} \) respectively. Now the dimension of an \( IR \) is \( (2j + 1) \), and from (3.10) we conclude that for integer \( j \) there are \( j + 1 \) even parity characteristic values \( a_r^{\eta}(\xi) \) and \( j \) odd parity characteristic values \( b_r^{\eta}(\xi) \), whereas for half-integer \( j \) there are \( (j + 1) \xi / 2 \) of each type.

From the structure of the operator \( E \) in (3.3), there is a further interesting symmetry property noticed by Arscott. Putting \( \alpha = 2\xi \) and writing the \( \xi \) dependence explicitly, i.e., \( E(\xi) = E(\xi - 2\xi J_3) \), we notice that

\[
\exp(i\pi A_\xi)E(\xi) = E(-\xi)
\]

(3.11)

and a similar relation is obtained by replacing \( J_3 \) by \( J_3 \). It follows from (3.11) that if \( a_r^{\eta}(\xi) \) or \( b_r^{\eta}(\xi) \) are characteristic values for \( E(\xi) \), then \( a_r^{\eta}(\xi - \xi) \) and \( b_r^{\eta}(\xi - \xi) \) are also characteristic values for \( E(\xi) \). Furthermore, a short computation demonstrates that

\[
\exp(i\pi A_\xi)R = R \quad \text{for integer } j,
\]

\[
\exp(i\pi A_\xi)R = -R \quad \text{for half-integer } j,
\]

(3.12)

Hence, it follows that for half-integer \( j \) the set \( \{b_r^{\eta}(\xi)\} \) is given by the set \( \{a_r^{\eta}(\xi)\} \) whereas for integer \( j \), \( a_r^{\eta}(\xi - \xi) \in \{a_r^{\eta}(\xi)\} \) and \( b_r^{\eta}(\xi - \xi) \in \{b_r^{\eta}(\xi)\} \).

The expansion of the \( \psi_j^{\eta} \) basis in terms of the canonical basis is readily obtained,

\[
\psi_j^{\eta} = \sum_r \sqrt{(j + r)!} \exp(-i\pi(r - j)/2)
\]

\[
\times A_r^{\eta}(\xi) \psi_j^{\eta}, \quad \psi_j^{\xi} = \sum_r \exp(-i\pi(r - j)/2) A_r^{\eta}(\xi) \psi_j^{\eta},
\]

(3.13)

where the sum over \( r \) runs \( r = 0, \ldots, j \) for integer \( j \) and \( r = \frac{1}{2}, \frac{3}{2}, \ldots, j \) for half-integer \( j \). From the orthonormalization condition (3.6), we find

\[
\frac{4(j + 1) A_r^{\eta}(\xi \eta) A_r^{\eta}(\xi) + 2\sum_r (j - r)! A_r^{\eta}(\xi \eta) A_r^{\eta}(\xi)}{\sqrt{(j + r)!}(j + r)!} = \delta_{\eta \xi}, \quad (j = \text{integer})
\]

(3.14a)

\[
2\sum_r (j - r)! A_r^{\eta}(\xi \eta) A_r^{\eta}(\xi) = \delta_{\eta \xi}, \quad (j = \frac{1}{2})
\]

(3.14b)

Notice that our normalization for \( A_r^{\eta}(\xi) \) is different from that of Arscott. The inverse expansion is easily obtained from (3.13):

\[
\psi_j^{\eta} = \exp[i\pi(j - r)/2] \sum_r A_r^{\eta}(\xi \eta) \psi_j^{\eta}, \quad r \neq 0,
\]

\[
\psi_j^{\eta} = 2(j + 1) \exp(i\pi r/2) \sum_r A_r^{\eta}(\xi \eta) \psi_j^{\eta}.
\]

(3.15)

From the orthonormality of the \( \psi_j^{\eta} \)'s we find

\[
\sum_r A_r^{\eta}(\xi) = \delta_{\eta \xi}, \quad (\eta \neq \xi \text{ not both 0})
\]

(3.16a)

\[
\sum_r A_r^{\eta}(\xi) A_r^{\eta}(\xi) = \frac{1}{2} (j + 1)^2
\]

(3.16b)

B. One variable model

A well-known\(^8\) model of \( SU(2) \) on the space of polynomials of degree \( 2j \) in one complex variable is given by the realization

\[
J_0 = \frac{d}{dx}, \quad J_\pm = 2j \pm z \pm \frac{d}{dx}, \quad J_3 = i - z \frac{d}{dx}.
\]

(3.17)

The canonical basis states are then realized as

\[
\psi_j(\xi ; x) = z^{j - r} \sqrt{(j - r)!}(j + r)! x^{j - r}(j + r - 1)
\]

(3.18)

In this realization the operator \( E \) [Eq. (3.3)] takes the form

\[
E = z \frac{d^2}{dx^2} + \left( \frac{d}{dx}^2 + 1 \right) \frac{d}{dx} + (j + i\alpha x).
\]

(3.19)

However, for our purposes it is more convenient to consider another one variable model of \( SU(2) \) obtained from (3.17) by a similarity transformation. Set \( z = \exp(\pi i/2) \exp(2i\xi x) \) and consider the operators \( J_1 = x \frac{d}{dx} \times J_\xi \). In the new variable \( \xi \) the generators \( J_3, J_\xi \) take the form

\[
J_3 = \frac{1}{2} \frac{d}{dx}, \quad J_\xi = -\exp(2i\xi x) \frac{1}{2} \frac{d}{dx} + i
\]

(3.20)

and the canonical basis states are
\[ \psi_j(x) = e^{i(x - 2 \pi/3)} \exp(-2izx) 2^{2n+2m+2} \Gamma(2n+2m+1)/(2\sqrt{\pi}) \]

It is easy to check that the operators (3.20) satisfy the relations (3.2) on the states (3.21). Furthermore, the operator \( E \) takes the form

\[ E = \frac{1}{4} \frac{d^2}{dx^2} - \frac{a}{2} \sin 2\xi \frac{d}{dx} + ia \cos 2\xi \]  

(3.22)

and the eigenvalue equation (3.4) becomes

\[ \psi^{(}\xi^{)} + 2i \sin 2\xi \psi^{(}\xi^{)} + (\eta - 2iJ) \cos 2\xi \psi^{(}\xi^{)} = 0, \]  

(3.23)

which is precisely what Arscott calls Ince's equation with 2\( j \) identified with Arscott's \( p \).

We construct a realization for the scalar product (3.6) which covers the complex \( \xi \) plane once and for which (3.21) forms an orthonormal basis for each integer or half-integer \( j \), viz.

\[ (f, g) = \int_\chi \int_\chi d\xi_1 d\xi_2 f(\xi_1) g(\xi_2), \]  

(3.24)

where \( \xi = \xi_1 + i \xi_2, \xi_1, \xi_2 \in \mathbb{R} \). Writing the expansion formula (3.13) explicitly with the state (3.21), we obtain

\[ \psi^{(}\xi^{)}(\xi) = 2 \sum_j A_j(\eta) \cos 2\xi \psi^{(}\eta^{)}(\xi, \eta) \]  

(3.25a)

\[ \psi^{(}\xi^{)}(\xi) = -2i \sum_j A_j(\eta) \sin 2\xi \psi^{(}\eta^{)}(\xi, \eta) \]  

(3.25b)

It is readily verified by substitution that the solutions (3.24) satisfy the differential equation (3.23) with the recursion formulas (3.10).

It is now a simple task to make the connection of our model in this section with the previous section. It can be seen that the spectral resolution of the operator \( \mathcal{H}^2 - \ell^2 \) of Sec. 2 gives exactly the differential equation (3.23) with the identification \( p = \lambda - 1 = 2j, \frac{1}{2} - \mu = \eta, \) and \( \ell = \frac{1}{2} \).

Now the Lie algebra model (3.17) has been integrated to the group \( SU(2) \) by Vilenkin, and it is a simple task to express his representation in terms of our functions \( \psi^{(}\xi^{)} \). In doing so we can express the cross-basis matrix elements of \( e^{-i\delta/2} \) in terms of a finite sum of Jacobi polynomials.

4. OVERLAP FUNCTIONS

In this section we calculate the overlap functions between the bases \( \text{o}_c, \text{o}_r, \) and \( \text{o}_e \), respectively. However, since these functions are invariant under the unitary transformations of \( G \) as well as Bargmann's transformation \( A \), they also apply to the bases \( \text{OC}_c, \text{OC}_r, \) and \( \text{OE} \) in Sec. 1. Thus we obtain expansion formulas for each one of these functions in terms of the others. These expansions involving the OE basis are probably new.

The overlap function for \( \text{oc}-\text{o}_r \) systems has been calculated for the case of three-dimensions. In the two-dimensional case here we find

\[ (\text{oc}_n \text{o}_m x \text{o}_n \text{o}_m^*) = K_n L_n^2 \Gamma(2n+2m+2)/(2\sqrt{\pi}) \]  

(4.1)

These coefficients allow us to expand the Hermite functions (1.2) in terms of the Laguerre functions (1.3) and vice-versa.

The overlap functions for the system \( \text{or}^{+} - \text{oe}^{+} \) are even easier to calculate, viz.,

\[ (\text{oe}_n \text{o}_m^* x \text{or}_n \text{o}_m^*) = K_n L_n^2 \Gamma(2n+2m+2)/(2\sqrt{\pi}) \]  

(4.2)

whereas the overlap between different parity states vanishes. These coefficients allow us to expand the functions (1.3) in terms of the functions (1.4) and vice-versa. The composition of (4.1) and (4.2) gives us the overlap functions as an infinite series

\[ (\text{oc}_n \text{o}_m^* x \text{or}_n \text{o}_m^*) = \sum_n (\text{oc}_n \text{o}_m^*) (\text{or}_n \text{o}_m^*) \]  

(3.24)

Furthermore, we can combine the above results with those of 6 to obtain further overlap functions. However, we present here only those which can be readily obtained in close form and were not given in 6, viz., for the free particle radial coordinates and harmonic oscillator elliptic coordinates:

\[ (\text{or}_n \text{m}^* x \text{oe}_n^*) = 2^{1-n^2/2} \Gamma(n+1)/2 \Gamma(n+2)/2 \]  

(4.4)

These functions allow us to expand the Bessel functions given by Eq. (4.24) in 6 in terms of the Ince polynomials (1.4) and conversely to write the functions (1.4) as an integral and sum of Bessel functions.

Similarly, for the repulsive oscillator, radial coordinates, and the harmonic oscillator, elliptic coordinates,

\[ (\text{or}_n \text{m}^* x \text{oe}_n^*) = K^2 A_m^2 \Gamma(2n+2)/(2\sqrt{\pi}) \]  

(4.5a)

\[ \Gamma((m+1-\lambda)/2) \sqrt{m+1} \]  

\[ F_n \Gamma((m+1-\lambda)/2) (m+1-\lambda)/2, 1/2, \Gamma((m+1-\lambda)/2) \sqrt{m+1} \]  

(4.5b)

whereas for the negative parity solutions we have

\[ (\text{rr}_n \text{m}^* x \text{oe}_n^*) = -i (\text{rr}_n \text{m}^* x \text{oe}_n^*) \]  

(4.5b)

Accordingly these coefficients allow us to expand the Whittaker functions, Eq. (4.38) of 6, in terms of the Ince polynomials (1.4) and conversely to express the Ince polynomials as an integral and sum over Whittaker functions.

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It is easy to see from our mapping that the subspace of polynomials in $f_2$ maps onto polynomial solutions of ($\star$) in $L_2(\mathbb{R})$. This is closely related to a recent article on generalized heat polynomials, G.G. Bilodeau, SIAM J. Math. Anal., 5, 43 (1974).