Lie theory and separation of variables. 6. The equation $iU_t + \Delta_2 U = 0$

C. P. Boyer

Centro de Investigación en Matemáticas Aplicadas y en Sistemas, Universidad Nacional de México, México,20 D.F., Mexico

E. G. Kalnins and W. Miller Jr.*

Centre de recherches mathématiques, Université de Montréal, Montréal 101, P.Q., Canada
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This paper constitutes a detailed study of the nine-parameter symmetry group of the time-dependent free particle Schrödinger equation in two space dimensions. It is shown that this equation separates in exactly 26 coordinate systems and that each system corresponds to an orbit consisting of a commuting pair of first- and second-order symmetry operators. The study yields a unified treatment of the (attractive and repulsive) harmonic oscillator, linear potential and free particle Hamiltonians in a time-dependent formalism. Use of representation theory for the symmetry group permits simple derivations of addition and expansion theorems relating various solutions of the Schrödinger equation, many of which are new.

INTRODUCTION

This paper is a continuation of a series of articles studying the connection between Lie group theory and the separation of variables in the principal equations of mathematical physics.[1–5] The group theoretic method for the description of separation of variables originated from the study of the Helmholtz equation

$$\Delta\psi = \partial_\mu \partial^\mu \psi = \lambda\psi, \quad \partial_\mu = \frac{\partial}{\partial x_\mu}, \quad \mu = 1, 2, \tag{0.1}$$

in two variables for spaces of constant (or zero) curvature. Much of this original work was done by Winternitz and co-workers[6,7] with a view to describing all possible quantum mechanical operators which can be used to label bases for the "little groups" of the Poinçaré group. This work used the earlier results of Olevskiĭ,[8] who classified all separable coordinate systems for (0.1) in two and three dimensions for spaces of constant (nonzero) curvature. In order to correlate separation of variables with the underlying symmetry group G of (0.1), it is found necessary to require that ψ be the eigenfunction of an additional basis operator L. This operator belongs to the factor space $T = S/S \cap C$, where C is the center of the universal enveloping algebra U of G and S is the set of all symmetric second order elements in U. There is then a one-to-one correspondence between equivalence classes of elements of T under the action of G and the various distinct orthogonal separable coordinate systems for (0.1). It is found that the operator L in many cases does not correspond to the Casimir operator of a Lie subgroup of G. The resulting type of basis has been termed a non-subgroup basis.[9] We should mention here that the case of the Helmholtz equation in the pseudo-Euclidean plane is somewhat more complicated. The reader is referred to Ref. 10 for further details. The correlation between separation of variables and the symmetry group of (0.1) in n dimensions can easily be extended from the two-dimensional case. A basis is now specified by an $(n-1)$tuple of mutually commuting operators $L_1, \ldots, L_{n-1}$. In addition to equivalence under the group action two such $(n-1)$tuples, $\{L_1, \ldots, L_{n-1}\}'$ and $\{L_1', \ldots, L_{n-1}'\}$ are equivalent if

$$L_i = \sum_{j=1}^{n-1} a_{ij} L_j'$$

with real nonsingular matrix (a_{ij}).

For the treatment of the Helmholtz equation in a two-dimensional space of positive constant curvature [two-dimensional sphere with symmetry group $SO(3)$], see Refs. 4, 9. The corresponding problem for negative constant curvature [upper sheet of two sheeted hyperboloid with symmetry group $SO(2,1)$], see Refs. 4, 11. Some investigations have also been made for the Helmholtz equation in three dimensions in Euclidean space,[12] on the three-dimensional sphere,[13] and on the upper sheet of the two sheeted three-dimensional hyperboloid.[14]

The present paper is a continuation of Ref. 5 which will be referred to as 5 in the following. In that paper the problem of the separation of variables for the free-particle time-dependent Schrödinger equation in one space dimension was treated in detail, i.e., the equation

$$U_{xx} + iU_t = 0. \tag{0.2}$$

The corresponding symmetry group G of this equation was taken to be that generated by the largest set of first-order partial differential operators in the variables t and x [each of which is a symmetry of (0.2)]. This group is isomorphic to the semidirect product of the three-dimensional Weyl group and $SL(2,R)$. It was found in 5 that there is a correspondence between R-separable coordinate systems for (0.2) and equivalence classes of elements of the Lie algebra of G. In this paper we extend this earlier work to the case of two space dimensions.

We present a detailed study of the free-particle time-dependent Schrödinger equation

$$u_{x_1x_1} + u_{x_2x_2} + iu_t = 0. \tag{0.3}$$

Boyer[15] has classified all equations of the form

$$u_{x_1x_1} + u_{x_2x_2} - V(x_1, x_2)\,u + iu_t = 0, \tag{0.4}$$

which admit a nontrivial symmetry algebra of first order differential operators. He has shown that (1) the maximal dimension for a symmetry algebra is nine, (2) this maximum occurs only for constant, linear, and attractive or repulsive harmonic oscillator potentials, and (3) the algebras of maximal dimension are isomorphic. Furthermore, it is known, e.g., Niederer,[16] that the oscillator and linear potential equations are actually equivalent to (0.3). In this paper we will examine the equivalence explicitly and relate it to separation of variables for (0.3).

In Sec. 1 we rederive the nine-parameter symmetry group G of (0.3). Here G is a semidirect product of the five-parameter Weyl group W and $SO(2)\otimes SL(2,R)$. We determine the global action of G and compute the orbit structure of its Lie algebra $\mathcal{G}$ (the Schrödinger algebra) under the adjoint representation. We also determine the second order symmetry operators admitted by (0.3) and show that they form a 20-dimensional vector space consisting of symmetric quadratic polynomials from $\mathcal{G}$. (This last computation was carried out in Ref. 17 for the equivalent case of the harmonic oscillator, but the results are incomplete.)

In Sec. 2 we classify the 26 possible coordinate systems such that variables separate in (0.3). In Sec. 3 we show that each such system is characterized by a G-orbit of symmetry operators, an element of which consists of a commuting pair of symmetries, one first order and one second order. Our derivation of possible coordinates which permit separation and the relation to G-orbits is new. We also show that each of these orbits can be naturally associated with exactly one of the four Hamiltonians mentioned above.

In Secs. 4 and 5 we compute the eigenbasis in a two-parameter model for a representative of each G-orbit. We also calculate the basis in the three-parameter model of functions depending on variables x_1, x_2, t and determine overlap functions relating various distinct bases. Our knowledge of the G-structure of (0.3) greatly simplifies these computations and provides many expansion theorems for functions in $L_2(R_2)$, some of which are new.

Among the special functions arising as solutions of (0.3) are Bessel, Airy, Hermite, parabolic cylinder, Mathieu, Laguerre, and Ince functions. Our group theoretic approach provides deep insight into the problem of expanding one of these functions in terms of another. Unless otherwise mentioned, all special functions are defined as in the Bateman project.[18]

1. SYMMETRIES OF THE EQUATION $iu_t + \Delta_2 u = 0$

Let X be the partial differential operator

$$X = i\partial_t + \partial_{x_1x_1} + \partial_{x_2x_2} \tag{1.1}$$

acting on the space $\mathcal{J}$ of locally C^∞ functions of the real variables x_j, t. We begin by determining the maximal symmetry algebra of the free-particle Schrödinger equation

$$iu_t + u_{x_1x_1} + u_{x_2x_2} = Xu = 0, \tag{1.2}$$

i.e., we compute all linear differential operators

$$L = a(x_j,t)\,\partial_{x_1} + b(x_j,t)\,\partial_{x_2} + c(x_j,t)\,\partial_t + d(x_j,t), \quad a,\ldots,d\in\mathcal{J}, \tag{1.3}$$

such that Lu is a solution of (1.2) whenever u is a solution. A necessary and sufficient condition for L to be a symmetry is

$$[L,X] = r(x_j,t)\,X \tag{1.4}$$

for some $r\in\mathcal{J}$.[15,19,20] Equating coefficients of $\partial_{x_ix_j}, \partial_t, \partial_{x_j}$ and 1 on both sides of (1.4), we obtain a system of differential equations for $a,\ldots,d,r$, see Refs. 15, 19 for details. Solving these equations, one finds that the allowable L form a nine-dimensional complex Lie algebra $\mathcal{G}^c$ with basis

$$K_2 = -t^2\partial_t - t(x_1\partial_{x_1} + x_2\partial_{x_2}) - t + (i/4)(x_1^2+x_2^2), \quad K_{-2} = \partial_t,$$
$$P_j = \partial_{x_j}, \quad B_j = -t\partial_{x_j} + ix_j/2, \quad M = x_1\partial_{x_2} - x_2\partial_{x_1}, \quad E = i,$$
$$D = x_1\partial_{x_1} + x_2\partial_{x_2} + 2t\partial_t + 1 \tag{1.5}$$

and commutation relations

$$[D,K_{\pm2}] = \pm 2K_{\pm2}, \quad [D,B_j] = B_j, \quad [D,P_j] = -P_j$$
$$[D,M] = 0, \quad [M,K_{\pm2}] = 0, \quad [P_j,M] = (-1)^{j+1}P_l,$$
$$[B_j,M] = (-1)^{j+1}B_l, \quad [K_2,K_{-2}] = D, \quad [K_2,B_j] = 0,$$
$$[K_{-2},B_j] = -P_j, \quad [K_{-2},P_j] = 0, \quad [P_j,K_2] = B_j,$$
$$[P_j,B_j] = \tfrac{1}{2}E, \quad [P_j,B_l] = 0, \quad j,l = 1,2,\ j\neq l, \tag{1.6}$$

with E in the center of $\mathcal{G}^c$. In the following we will study only the real Lie algebra $\mathcal{G}$ with basis (1.5), the Schrödinger algebra.

A second useful basis for $\mathcal{G}$ is given by the operators B_j, P_j, E which generate the five-dimensional Weyl algebra $\mathcal{W}$, the operator M, and the three operators L_1, L_2, L_3, where

$$L_1 = D, \quad L_2 = K_2 + K_{-2}, \quad L_3 = K_{-2} - K_2. \tag{1.7}$$

Here,

$$[L_1,L_2] = -2L_3, \quad [L_3,L_1] = 2L_2, \quad [L_2,L_3] = 2L_1 \tag{1.8}$$

so that the L_ν satisfy the commutation relations of $sl(2,R)$. It follows that $\mathcal{G}$ is the semidirect product of $sl(2,R)\oplus o(2)$ and $\mathcal{W}$. Here $o(2)$ is the one-dimensional Lie algebra spanned by M.

Using standard results from Lie theory,[21] we can exponentiate the differential operators in $\mathcal{G}$ to obtain a local Lie group G of operators acting on $\mathcal{J}$ and mapping solutions of (1.2) into solutions, the Schrödinger group. The action of the Weyl group W is given by operators

$$T(\mathbf{w},\mathbf{z},\alpha) = \exp(w_1B_1)\exp(z_1P_1)\exp(w_2B_2)\exp(z_2P_2) \times \exp(\alpha E),$$
$$\mathbf{w} = (w_1,w_2), \quad \mathbf{z} = (z_1,z_2)$$

such that

$$T(\mathbf{w},\mathbf{z},\alpha)\,T(\mathbf{w}',\mathbf{z}',\alpha') = T(\mathbf{w}+\mathbf{w}', \mathbf{z}+\mathbf{z}', \alpha+\alpha'+\tfrac{1}{2}\mathbf{w}'\cdot\mathbf{z}), \tag{1.9}$$

where

$$[T(\mathbf{w},\mathbf{z},\alpha)f](\mathbf{x},t)=\exp(i/4)(2\mathbf{x}\cdot\mathbf{w}-t\,\mathbf{w}\cdot\mathbf{w}+4\alpha)]\times f[\mathbf{x}-t\,\mathbf{w}+\mathbf{z},t],\quad f\in\mathcal{J}.$$

The action of $SO(2)$ is given by $T(\theta)=\exp\theta M$,

$$T(\theta)\,T(\theta')=T(\theta+\theta'),$$

where

$$[T(\theta)f]\,(\mathbf{x},t)=f(\mathbf{x}\Theta,t),$$

$$\Theta=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}. \qquad (1.10)$$

Finally, the action of $SL(2,R)$ is given by operators

$$[T(A)f](x,t)=\exp[i\beta(x^2+y^2)/[4(\delta+t\beta)](\delta+t\beta)^{-1}\times f[(\delta+t\beta)^{-1}\mathbf{x},(\gamma+t\alpha)/(\delta+t\beta)],\quad f\in\mathcal{J},$$

$$A=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in SL(2,R), \qquad (1.11)$$

where

$$T(A)\,T(B)=T(AB),\quad A,B\in SL(2,R).$$

The one-parameter subgroups of $SL(2,R)$ generated by $K_{\pm2},L_1,L_2,L_3$, respectively, are given by expressions (1.11) in Ppaer 5. The adjoint actions of $SO(2)$ and $SL(2,R)$ on W are

$$T^{-1}(A)\,T(\mathbf{w},\mathbf{z},\alpha)\,T(A)=T(\mathbf{w}A,\mathbf{z}A,\alpha'),$$
$$\alpha'=\alpha+\tfrac{1}{4}(\mathbf{w}\cdot\mathbf{z}-\mathbf{w}A\cdot\mathbf{z}A), \qquad (1.12)$$
$$T^{-1}(\theta)\,T(\mathbf{w},\mathbf{z},\alpha)\,T(\theta)=T(\mathbf{w}\Theta,\mathbf{z}\Theta,\alpha).$$

These identities define G as a semidirect product of $SL(2,R)\oplus SO(2)$ and W:

$$g=(A,\theta,v)\in G,\quad A\in SL(2,R),\quad \theta\in SO(2),$$
$$v=(\mathbf{w},\mathbf{z},\alpha)\in W, \qquad (1.13)$$
$$T(g)=T(A)\,T(\theta)\,T(v).$$

The group G acts on the Lie algebra $\mathcal{G}$ of differential operators K via the adjoint representation

$$K\to K^g=T(g)KT^{-1}(g)$$

and this action splits $\mathcal{G}$ into G-orbits. We will classify the orbit structure of the factor algebra $\mathcal{G}'\cong\mathcal{G}/\{E\}$, where $\{E\}$ is the center of $\mathcal{G}$. Let $K\in\mathcal{G}'$ and let A_2,A_0,A_{-2} respectively, be the coefficients corresponding to K_2,D,K_{-2} in the expansion of $K\neq0$ in terms of the basis (1.5). Setting $\alpha=A_2A_{-2}+A_0^2$, we find that α is invariant under the adjoint representation.

The following list is a complete set of orbit representatives in the sense that any $K\neq0$ lies on the same G-orbit as a real multiple of exactly one of the operators in this list:

Case 1$(\alpha<0)$: $K_{-2}-K_2+\beta^2M,\ |\beta|\neq1,\ K_{-2}-K_2+M+\gamma B_1;$

Case 2$(\alpha>0)$: $D+\beta M;$

Case 3$(\alpha=0)$: $K_2+M,\ K_2+P_1,\ K_2,\ M,\ P_1+B_2,\ P_1.$ (1.14)

We next consider the problem of determining symmetries of (1.2) which are differential operators of arbitrary finite order in x_1, x_2, and t. That is, we look for linear differential operators S of arbitrary order which map solutions of (1.2) into solutions. This is equivalent to the requirement that

$$[S,X]=RX$$

for some linear differential operator R of arbitrary finite order in x_1, x_2, and t. Since we will only apply S to solutions u of $Xu=0$, without loss of generality we can require that S contains no derivatives in t. In other words, wherever ∂_t appears in S we can replace it by $i(\partial_{x_1x_1}+\partial_{x_2x_2})$. Another way to view this is to note that if S is a symmetry operator, then so is $S'=S+QX$, where Q is an arbitrary differential operator. Moreover, $S'u=Su$ for any solution u of (1.2). There is a unique choice of Q such that S' contains no derivatives with respect to t.

With this in mind we see that only the operators P_j,B_j,E, generating the Weyl algebra and M are first order or less in the x_j. The elements $K_2=-i(B_1^2+B_2^2)$, $K_{-2}=i(P_1^2+P_2^2)$, and $D=-i(B_1P_1+P_1B_1+B_2P_2+P_2B_2)$ are second order. [These equalities are valid modulo the replacement of ∂_t by $i(\partial_{x_1x_1}+\partial_{x_2x_2})$.] More generally we can compute all symmetries S which are second order or less in x_1 and x_2:

$$S=\sum_{i,j=1}^{2}a_{ij}(x_1,x_2,t)\,\partial_{x_ix_j}+\sum_{i=1}^{2}b_i(x_1,x_2,t)\,\partial_{x_i}+c(x_1,x_2,t).$$

A tedious computation shows that such S form a 20-dimensional vector space. A basis for this space is provided by the zeroth-order operator E, the five first-order operators P_j,B_j,M and the three second-order operators $iK_{\pm2},iD$ listed above, plus the eleven second-order operators

$$B_1^2-B_2^2,\ B_1P_1-B_2P_2,\ P_1^2-P_2^2,\ B_1M+MB_1,\ B_2M+MB_2,$$
$$P_1M+MP_1,\ P_2M+MP_2,\ B_1B_2,\ P_1P_2,\ B_1P_2+B_2P_1,\ M^2. \qquad (1.15)$$

It follows that all second-order symmetries are symmetric quadratic forms in B_j, P_j, E, and M.

2. SEPARATION OF VARIABLES FOR THE EQUATION $u_{xx}+u_{yy}+iu_t=0$

In this section we examine the problem of separation of variables for Eq. (1.2). As with the similar problem for one space dimension treated in 5, we proceed directly. Let us first make the transformation of coordinates

$$x=G(v_1,v_2,v_3)\quad y=H(v_1,v_2,v_3),\quad t=F(v_1,v_2,v_3) \qquad (2.1)$$

with G, H, and F real functions of v_i $(i=1,2,3)$. Then we have for the partial derivatives

$$\partial_x=B_{11}\partial_1+B_{21}\partial_2+B_{31}\partial_3,$$
$$\partial_y=B_{12}\partial_1+B_{22}\partial_2+B_{32}\partial_3, \qquad (2.2)$$
$$\partial_t=B_{13}\partial_1+B_{23}\partial_2+B_{33}\partial_3,$$

where $B_{ij}=M_{ij}/\det A$, M_{ij} being the cofactor of the matrix

$$A=\begin{bmatrix} G_1 & H_1 & F_1 \\ G_2 & H_2 & F_2 \\ G_3 & H_3 & F_3 \end{bmatrix}, \tag{2.3}$$

(subscripts on the functions G, H, and F denote differentiation with respect to the variables v_i).

Equation (1.2) can then be written in the form

$$a_{11}\partial_{11}+a_{22}\partial_{22}+a_{33}\partial_{33}+a_{12}\partial_{12}+a_{13}\partial_{13}+a_{23}\partial_{23} + a_1\partial_1+a_1\partial_2+a_3\partial_3+i(b_1\partial_1+b_2\partial_2+b_3\partial_3)=0 \tag{2.4}$$

We now consider the possible cases for the coefficients a_{ij} $(i<j)$:

(i) $a_{ij}\neq 0$ for all $i<j$. In this case the only way to have a separable solution is for two of the solutions to be exponentials and all the remaining coefficients to be functions of the remaining variable.

(ii) $a_{12}=0$, $a_{13}, a_{23}\neq 0$. The only possible separable solution is an exponential solution in the variable v_3. The coefficients are then functions of v_1 and v_2.

(iii) $a_{12}=0, a_{13}=0, a_{23}\neq 0$. The only possible solution is an exponential solution in the variable v_3.

(iv) $a_{ij}=0(i<j)$.

Let us proceed to evaluate all coordinate systems which are of type (iv) and admit a separation of variables. We shall see that all the coordinate systems of interest arise in this case. We shall discuss the evaluation of cases (i)—(iii) at the end of this section. For the conditions $a_{ij}=0(j>i)$ we must have the relations

$$\begin{aligned} &B_{11}B_{21}+B_{12}B_{22}=0,\\ &B_{21}B_{31}+B_{22}B_{32}=0,\\ &B_{11}B_{31}+B_{12}B_{32}=0. \end{aligned} \tag{2.5}$$

These conditions may be interpreted to mean that the vectors $\mathbf{b}_1=(B_{11}, B_{12})$, $\mathbf{b}_2=(B_{21}, B_{22})$, and $\mathbf{b}_3=(B_{31}, B_{32})$ are mutually orthogonal. Therefore, there must exist a nontrivial relation of the form

$$\alpha\mathbf{b}_1+\beta\mathbf{b}_2+\gamma\mathbf{b}_3=0 \tag{2.6}$$

with $(\alpha,\beta,\gamma)\neq(0,0,0)$. Let us enumerate the possibilities for the vector (α,β,γ):

(i) α,β,γ all nonzero. This case implies B_{ij} $(j\neq 3)$ are all zero and is hence inadmissable.

(ii) $\gamma=0$, β,α nonzero. This case implies that $B_{21}=B_{31}=B_{22}=B_{32}=0$. Now, considering the conditions $B_{21}=B_{22}=0$ which can be written

$$H_1F_3-F_1H_3=G_1F_3-F_1G_3=0,$$

we see that, in order to have the partial derivative ∂_2 appear in (2.4) at all, we must have $H_1G_3-G_1H_3\neq 0$. This implies $F_1=F_3=0$; the other conditions similarly imply that $F_2=F_3=0$. The matrix A is then singular. This case is therefore inadmissable.

(iii) $\alpha,\beta=0$, γ nonzero. This case implies by reasoning as in case (ii) that $F_1=F_2=0$, so that $F=F(v_3)$ and $F\neq$ const. Accordingly we can define $F(v_3)=v_3$. For this case $\det A=G_1H_2-H_1G_2$ and we have the simplifications $M_{11}=H_2$, $M_{12}=-G_2$, $M_{21}=-H_1$, $M_{22}=G_1$. The only nontrivial constraint arising from conditions (2.5) is

(*) $H_1H_2+G_1G_2=0$.

There are two possible types of separation.

(1) There is an exponential solution in the variable v_1 and $B_{12}=0$. The conditions of separation then also require $B_{21}=0$. For a nonsingular choice of coordinates these conditions imply $H_1=0$, $G_2=0$. The condition $\partial_1(H_2/G_1)=0$ implies that $G=h(v_3)v_1+f(v_3)$. In particular the condition $\partial_1(B_{31})=0$ requires $h=$const. We can, by suitably redefining the variable v_1, take $f=0$. The corresponding problem then is equivalent to finding all separable coordinate systems for the equation $\lambda^2+u_{yy}+iu_t=0$. The only new coordinate system is then

$$x=v_1,\quad y=v_3^{1/2}v_2,\quad t=v_3. \tag{2.7}$$

If we remove the requirement $B_{12}=0$, then the coordinates which have exponential solutions will appear in separable systems of the second type (see below).

(2) These are coordinate systems for which all B_{ij}, $(i,j=1,2)$ are nonzero.

The conditions for separation for the second derivative terms are

$$\begin{aligned} &B_{11}^2+B_{12}^2=f(v_1,v_2)/h^2(v_3)\\ &B_{21}^2+B_{22}^2=g(v_1,v_2)/h^2(v_3) \end{aligned} \tag{2.8}$$

Now for functions $G=\overline{G}h$, $H=\overline{H}h$, the corresponding reduced functions $\overline{B}_{ij}$ $(i,j=1,2)$ satisfy the constraints (2.8) without the $h^2(v_3)$ term on the right-hand side. The conditions for separation in v_1, v_2 satisfied by the $\overline{B}_{ij}$ are then exactly those conditions satisfied for separation of variables in the two-dimensional Helmholtz equation in orthogonal coordinates. Therefore, to within a Euclidean motion $\overline{G},\overline{H}$, assume one of the four standard separable forms of the Helmholtz equation in two space dimensions. We can thus write

$$\begin{aligned} &G=h[\cos\alpha\,\mathcal{F}_i-\sin\alpha\,\mathcal{H}_i]+T,\\ &H=h[\sin\alpha\,\mathcal{F}_i+\cos\alpha\,\mathcal{H}_i]+U, \end{aligned} \tag{2.9}$$

where α, T, and U can be functions of v_3. The standard separable forms will be taken as:

1) Cartesian coordinates:

$$\mathcal{F}_1=v_1,\quad \mathcal{H}_1=v_2; \tag{2.10}$$

2) polar coordinates:

$$\mathcal{F}_2=v_1\cos v_2,\quad \mathcal{H}_2=v_1\sin v_2; \tag{2.11}$$

3) parabolic coordinates:

$$\mathcal{F}_3=\tfrac{1}{2}(v_1^2-v_2^2),\quad \mathcal{H}_3=v_1v_2; \tag{2.12}$$

4) elliptic coordinates:

$$\mathcal{F}_4=\cosh v_1\cos v_2,\quad \mathcal{H}_4=\sinh v_1\sin v_2. \tag{2.13}$$

The remaining conditions for separability then become

$$B_{13}=f(v_1,v_2)/h^2,\quad B_{23}=g(v_1,v_2)/h^2. \tag{2.14}$$

The form of the functions f and g is determined by the choice of $\mathcal{F}_i$ and $\mathcal{H}_i$. It follows from the general form

(2.4) that the functions h, α, T, and U have the general form

$$h=\sqrt{bv_3+c}, \quad \alpha=K\ln(bv_3+c), \quad T=av_3, \quad U=bv_3. \tag{2.15}$$

We shall now summarize our results. In each case we give the form of the functions f and g in Eqs. (2.14) and the corresponding coordinates in reduced form.

1) Cartesian coordinates. In this case $f=f(v_1)$, $g=g(v_2)$, and $K=0$. The contributions of T and U may be transformed away by using the v_3 translation properties of the Weyl group action. This process does not affect separability. The resulting coordinate systems are then

$$x=v_3^{1/2}v_1, \quad y=v_3^{1/2}, \quad t=v_3, \tag{2.16}$$

$$x=v_1, \quad y=v_2, \quad t=v_3. \tag{2.17}$$

2) Polar coordinates. In this case $f=f(v_1)$, $g=g(v_1)$, and T and U are both zero. In particular for $b\neq 0$ we have $K\neq 0$. The resulting coordinate systems are

(i) $$x=v_3^{1/2}v_1\cos(v_2+K\ln v_3), \quad t=v_3, \quad y=v_3^{1/2}v_1\sin(v_2+K\ln v_3), \tag{2.18}$$

(ii) $$x=v_1\cos v_2, \quad y=v_1\sin v_2, \quad t=v_3. \tag{2.19}$$

3) Parabolic coordinates. In this case $f=\bar{f}(v_1')/(v_1^2+v_2^2)$ and $g=\bar{g}(v_2)/(v_1^2+v_2^2)$. This implies that K, U, T, and b are all zero. We thus have only one coordinate system, viz.,

$$x=\tfrac{1}{2}(v_1^2-v_2^2), \quad y=v_1v_2, \quad t=v_3. \tag{2.20}$$

4) Elliptic coordinates. In this case $f=\bar{f}(v_1)/(\sinh^2 v_1+\cos^2 v_2)$ and $g=\bar{g}(v_2)/(\sinh^2 v_1+\cos^2 v_2)$. This implies that K, U, T are zero. The two resulting coordinate systems are

(i) $$x=v_3^{1/2}\cosh v_1\sin v_2, \quad y=v_3^{1/2}\sinh v_1\cos v_2, \quad t=v_3, \tag{2.21}$$

(ii) $$x=\cosh v_1\sin v_2, \quad y=\sinh v_1\cos v_2, \quad t=v_3. \tag{2.22}$$

This completes the list of separable coordinate systems. In particular we note that we can essentially take $K=0$ for the angular variable in the system (2.18) by redefining the variable v_2. We now seek to classify all solutions of (1.2) which admit an R-separable solution, i.e., a solution of the form $u=\exp Q(v_1,v_2,v_3)\times A(v_1)B(v_2)C(v_3)$, where Q is not expressible as a sum of functions of each of the individual variables v_i nor is it a constant. If we extract the multiplier and write down the equation for the product, we obtain an equation of the form (2.4) with new coefficients $\bar{a}_i$ and an additional constant term a_0 on the left-hand side. The possible types of R-separation can then be classified in the same manner, i.e., types (1) and (2). For solutions of type (1) we have the R-separable solutions for the corresponding equation in one space dimension found previously in Paper 5. They are:

(i) $$x=v_1, \quad y=v_2v_3+b/v_3, \quad t=v_3, \quad S=\tfrac{1}{4}v_2^2v_3-bv_2/2v_3; \tag{2.23}$$

(ii) $$x=v_1, \quad y=v_2+bv_3^2, \quad t=v_3, \quad S=bv_2v_3; \tag{2.24}$$

(iii) $$x=v_1, \quad y=v_2(1+v_3^2)^{1/2}, \quad t=v_3, \quad S=\tfrac{1}{4}v_2^2v_3; \tag{2.25}$$

(iv) $$x=v_1, \quad y=v_2(1-v_3^2)^{1/2}, \quad t=v_3 \quad (|v_3|<1), \quad S=-\tfrac{1}{4}v_2^2v_3 \tag{2.26}$$
$$x=v_1, \quad y=v_2(v_3^2-1)^{1/2}, \quad t=v_3 \quad (|v_3|>1), \quad S=\tfrac{1}{4}v_2^2v_3.$$

Here we have written the multiplier function $Q=R+iS$ and $R=0$ in each case. For R-separable solutions of type (2) we again require that G and H have the form given in (2.9) with $\alpha=0$. The coefficients of the partial derivatives ∂_1 and ∂_2 are then

$$c_1=2a_{11}R_1+a_1+i(2a_{11}S_1+b_1), \quad c_2=2a_{22}R_2+a_2+i(2a_{22}S_2+b_2), \tag{2.27}$$

respectively.

The requirement of separability implies that R is at most a sum of functions of the individual variables. We may therefore take $R=0$. We give an outline of the method for the case $\not{f}=v_1$, $\not{H}=v_2$ and then list the results for the remaining coordinate systems. From the requirement that $c_1=f(v_1)$, $c_2=g(v_2)$ we find that S can be written in the form

$$S=\tfrac{1}{4}hh'(v_1^2+v_2^2)+\tfrac{1}{2}h(T'v_1+U'v_2). \tag{2.28}$$

Then from the constraint

$$a_0=a_{11}(-S_1^2+iS_{11})+a_{22}(-S_2^2+iS_{22})+ia_1S_1+ia_2S_2-b_1S_1-b_2S_2-b_3S_3=(p(v_1)+q(v_2))/h^2+s(v_3) \tag{2.29}$$

we obtain the following set of coordinate systems:

(i) $$x=v_1v_3+a/v_3, \quad y=v_2v_3+b/v_3, \quad t=v_3, \quad a,b\geq 0, \quad S=\tfrac{1}{4}(v_1^2+v_2^2)v_3-(1/2v_3)(av_1+bv_2); \tag{2.30}$$

(ii) $$x=v_1+av_3^2, \quad y=v_2+bv_3^2, \quad t=v_3, \quad a,b\geq 0, \quad S=(av_1+bv_2)v_3; \tag{2.31}$$

(iii) $$x=v_1(1+v_3^2)^{1/2}, \quad y=v_2(1+v_3^2)^{1/2}, \quad t=v_3, \quad S=\tfrac{1}{4}(v_1^2+v_2^2)v_3; \tag{2.32}$$

(iv) $$x=v_1(1-v_3^2)^{1/2}, \quad y=v_2(1-v_3^2)^{1/2}, \quad t=v_3, \quad |v_3|<1, \quad S=-\tfrac{1}{4}(v_1^2+v_2^2)v_3, \tag{2.33}$$
$$x=v_1(v_3^2-1)^{1/2}, \quad y=v_2(v_3^2-1)^{1/2}, \quad t=v_3, \quad |v_3|>1, \quad S=\tfrac{1}{4}(v_1^2+v_2^2)v_3.$$

In the remaining three types of coordinate systems we have the following possibilities:

Polar coordinates:

(i) $$x=(1+v_3^2)^{1/2}v_1\cos v_2, \quad y=(1+v_3^2)^{1/2}v_1\sin v_2, \quad t=v_3, \quad S=\tfrac{1}{4}v_1^2v_3; \tag{2.34}$$

(ii) $$x=(1-v_3^2)^{1/2}v_1\cos v_2, \quad y=(1-v_3^2)^{1/2}v_1\sin v_2, \quad t=v_3, \quad |v_3|<1, \quad S=-\tfrac{1}{4}v_1^2v_3;$$

$$x=(v_3^2-1)^{1/2}\, v_1 \cos v_2, \quad y=(v_3^2-1)^{1/2}\, v_1 \sin v_2,$$
$$t=v_3, \quad |v_3|>1,$$
$$S=\tfrac{1}{4}v_1^2 v_3; \qquad (2.35)$$

(iii) $x=v_3 v_1 \cos v_2, \quad y=v_3 v_1 \sin v_2, \quad t=v_3$

$$S=\tfrac{1}{4}v_1^2 v_3. \qquad (2.36)$$

Parabolic coordinates:

(i) $x=\tfrac{1}{2}(v_1^2-v_2^2)\, v_3+a/v_3, \quad y=v_1 v_2 v_3, \quad t=v_3,$

$$S=\tfrac{1}{16}(v_1^2+v_2^2)^2 v_3-(a/4v_3)(v_1^2-v_2^2); \qquad (2.37)$$

(ii) $x=\tfrac{1}{2}(v_1^2-v_2^2)+a v_3^2, \quad y=v_1 v_2, \quad t=v_3,$

$$S=\tfrac{1}{2}a(v_1^2-v_2^2)\, v_3. \qquad (2.38)$$

Elliptic coordinates:

(i) $x=(1+v_3^2)^{1/2}\cosh v_1 \sin v_2, \quad y=(1+v_3^2)^{1/2}\sinh v_1 \cos v_2,$

$$t=v_3,$$
$$S=\tfrac{1}{4}v_3(\sinh^2 v_1+\cos^2 v_2); \qquad (2.39)$$

(ii) $x=(1-v_3^2)^{1/2}\cosh v_1 \cos v_2, \quad y=(1-v_3^2)^{1/2}\sinh v_1 \sin v_1,$

$$t=v_3, \quad |v_3|<1,$$
$$S=-\tfrac{1}{4}v_3(\sinh^2 v_1+\cos^2 v_2);$$
$$x=(v_3^2-1)^{1/2}\cosh v_1 \cos v_2, \; y=(v_3^2-1)^{1/2}\sinh v_1 \sin v_2,$$
$$t=v_3, \quad |v_3|>1,$$
$$S=\tfrac{1}{4}v_3(\sinh^2 v_1+\cos^2 v_2); \qquad (2.40)$$

(iii) $x=v_3 \cosh v_1 \sin v_2, \quad y=v_3 \sinh v_1 \cos v_2, \quad t=v_3,$

$$S=\tfrac{1}{4}v_3(\sinh^2 v_1+\cos^2 v_2). \qquad (2.41)$$

This completes the list of R-separable solutions of (1.2).

At this point we comment on the separable solutions of types (i)—(iii). In defining a separable coordinate system we require that in addition to admitting a separable solution, the equation in question be equivalent to three ordinary differential equations, one in each of the separation variables. For solutions of type (i)—(iii) this is not the case as we have proven. Separable solutions of types (i)—(iii) actually correspond to a change of coordinates

$$x=a_{1j}v_j, \quad y=a_{2j}v_j, \quad t=a_{3j}v_j,$$
$$\det(a_{ij})\neq 0, \quad a_{ij} \text{ constants}. \qquad (2.42)$$

We accordingly make no further comment on these cases.

The general features of the separable systems we have classified are evident from our explicit procedure. Corresponding to each system there is always a first order operator K and a second-order operator S defining the coordinate system in question. These two operators are also symmetries of (1.2), mapping solutions into solutions. The operators K and S can accordingly be expressed as first- and second-order operators, respectively, in the generators of the Lie algebra $\mathcal{G}$. The form of these basis defining operators is discussed in the next section. The notation for the coordinate systems we have introduced in Table I requires some comment. The capital letter corresponds to the type of Hamiltonian, i.e., F ⟷ free particle, L ⟷ stark effect (linear potential), O ⟷ harmonic oscillator, and R ⟷ repulsive harmonic oscillator. The small letters indicate the type of coordinates used in each of these Hamiltonians, i.e., c ⟷ Cartesian, r ⟷ radial (polar) coordinates, p ⟷ parabolic, and e ⟷ elliptic coordinates. The superscript (i) determines the coordinate

TABLE I. Separable coordinate systems for the equation $U_{xx}+U_{yy}+iU_t=O$ $(\epsilon=\operatorname{sgn}(1-v_3^2))$.

Coordinate system	Coordinates	Multiplier e^{iS}
1) $\mathrm{Fc}^{(1)}$	$x=v_1v_3, \; y=v_2v_3$	$S=(v_1^2+v_2^2)v_3/4$
2) $\mathrm{Fc}^{(2)}$	$x=v_1, \; y=v_2$	0
3) $\mathrm{Fr}^{(1)}$	$x=v_1v_3\cos v_2, \; y=v_1v_3\sin v_2$	$v_1^2v_3/4$
4) $\mathrm{Fr}^{(2)}$	$x=v_1\cos v_2, \; y=v_1\sin v_2$	0
5) $\mathrm{Fp}^{(1)}$	$x=v_3(v_1^2-v_2^2)/2, \; y=v_1v_2v_3$	$(v_1^2+v_2^2)^2 v_3/16$
6) $\mathrm{Fp}^{(2)}$	$x=(v_1^2-v_2^2)/2, \; y=v_1v_2$	0
7) $\mathrm{Fe}^{(1)}$	$x=v_3\cosh v_1\cos v_2, \; y=v_3\sinh v_1\sin v_2$	$(\sinh^2 v_1+\cos^2 v_2)v_3/4$
8) $\mathrm{Fe}^{(2)}$	$x=\cosh v_1\cos v_2, \; \sinh v_1\sin v_2$	0
9) $\mathrm{Lc}^{(1)}$	$x=v_1v_3+a/v_3, \; y=v_2v_3+b/v_3$	$(v_1^2+v_2^2)v_3/4-(1/2v_3)(av_1+bv_2)$
10) $\mathrm{Lc}^{(2)}$	$x=v_1+av_3^2, \; y=v_2+bv_3^2$	$(av_1+bv_2)v_3$
11) $\mathrm{Lp}^{(1)}$	$x=v_3(v_1^2-v_2^2)/2+a/v_3, \; y=v_1v_2v_3$	$(v_1^2+v_2^2)^2 v_3/16-(a/4v_3)(v_1^2-v_2^2)$
12) $\mathrm{Lp}^{(2)}$	$x=(v_1^2-v_2^2)/2+av_3^2, \; y=v_1v_2$	$av_3(v_1^2-v_2^2)/2$
13) Oc	$x=v_1(1+v_3^2)^{1/2}, \; y=v_2(1+v_3^2)^{1/2}$	$(v_1^2+v_2^2)v_3/4$
14) Or	$x=(1+v_3^2)^{1/2}v_1\cos v_2, \; y=(1+v_3^2)^{1/2}v_1\sin v_2$	$v_1^2v_3/4$
15) Oe	$x=(1+v_3^2)^{1/2}\cosh v_1\cos v_2, \; y=(1+v_3^2)^{1/2}\sinh v_1\sin v_2$	$(\sinh^2 v_1+\cos^2 v_2)v_3/4$
16) $\mathrm{Rc}^{(1)}$	$x=v_1v_3^{1/2}, \; y=v_2v_3^{1/2}$	0
17) $\mathrm{Rc}^{(2)}$	$x=v_1(\lvert v_3^2-1\rvert)^{1/2}, \; y=v_2(\lvert v_3^2-1\rvert)^{1/2}$	$\epsilon(v_1^2+v_2^2)v_3/4$
18) $\mathrm{Rr}^{(1)}$	$x=v_1v_3^{1/2}\cos v_2, \; y=v_2v_3^{1/2}\sin v_2$	0
19) $\mathrm{Rr}^{(2)}$	$x=(\lvert v_3^2-1\rvert)^{1/2}v_1\cos v_2, \; y=(\lvert v_3^2-1\rvert)^{1/2}v_1\sin v_2$	$\epsilon v_1^2v_3/4$
20) $\mathrm{Re}^{(1)}$	$x=v_3^{1/2}\cosh v_1\cos v_2, \; y=v_3^{1/2}, \; y=v_3^{1/2}\sinh v_1\sin v_2$	0
21) $\mathrm{Re}^{(2)}$	$x=(\lvert v_3^2-1\rvert)^{1/2}\cosh v_1\cos v_2, \; y=(\lvert v_3^2-1\rvert)^{1/2}\sinh v_1\sin v_2$	$\epsilon(\sinh^2 v_1+\cos^2 v_2)v_3/4$
22) L1	$x=v_1, \; y=v_2v_3+b/v_3$	$v_3v_2^2/4-bv_2/2v_3$
23) L2	$x=v_1, \; y=v_2+av_3^2$	av_2v_3
24) O1	$x=v_1, \; y=v_2(1+v_3^2)^{1/2}$	$v_3v_1^2/4$
25) R1	$x=v_1, \; y=v_2v_3^{1/2}$	0
26) R2	$x=v_1, \; y=v_2(\lvert v_3^2-1\rvert)^{1/2}$	$\epsilon v_2^2v_3/4$

system which is the simpler of two which lie on the same orbit from the point of view of the spectral analysis in a given basis.

3. THE OPERATOR CHARACTERIZATION OF VARIABLE SEPARATION

From the method of the preceding section we see that corresponding to every separation of variables for Eq. (1.2) we can find a pair of commuting differential operators K, S such that:

1) K and S are symmetries of (1.2);

2) K is first order in x_1, x_2, t and contains a term in ∂_t (except for the subgroup coordinates);

3) S is second order in x_1, x_2 and contains no terms in ∂_t.

The separation of variables is then characterized by the simultaneous equations

$$Xu=0, \quad Ku=i\lambda u, \quad Su=\mu u. \tag{3.1}$$

In particular, the eigenvalues λ, μ are the usual separation constants.

It follows from the results of Sec. 1 that K lies in the symmetry algebra $\mathcal{G}'$ while S can be expressed as a symmetric quadratic form in B_j, P_j, E, and M. Thus the possible coordinate systems in which (1.2) separates can always be characterized by eigenfunction equations for operators at most second order in the enveloping algebra of $\mathcal{G}$. From the results of Sec. 2 it is straightforward to determine the operators K, S associated with each coordinate system. This information is listed in Table II.

TABLE II. Symmetry operators associated with variable separation

Coordinate system	1st-order symmetry K	2nd-order symmetry S
1) $\mathrm{Fc}^{(1)}$	K_2	B_1^2
2) $\mathrm{Fc}^{(2)}$	K_{-2}	P_1^2
3) $\mathrm{Fr}^{(1)}$	K_2	M^2
4) $\mathrm{Fr}^{(2)}$	K_{-2}	M^2
5) $\mathrm{Fp}^{(1)}$	K_2	B_2M+MB_2
6) $\mathrm{Fp}^{(2)}$	K_{-2}	P_2M+MP_2
7) $\mathrm{Fe}^{(1)}$	K_2	$M^2-B_2^2$
8) $\mathrm{Fe}^{(2)}$	K_{-2}	$M^2-P_2^2$
9) $\mathrm{Lc}^{(1)}$	$K_2+2aP_1+2bP_2$	$B_2^2+2bP_2E$
10) $\mathrm{Lc}^{(2)}$	$K_{-2}-2aB_1-2bB_2$	$P_1^2+2aB_1E$
11) $\mathrm{Lp}^{(1)}$	K_2+aP_1	$B_2M+MB_2+aP_2^2$
12) $\mathrm{Lp}^{(2)}$	$K_{-2}+2aB_1$	$P_2M+MP_2+2aB_2^2$
13) Oc	$K_{-2}-K_2$	$P_1^2+B_1^2$
14) Or	$K_{-2}-K_2$	M^2
15) Oe	$K_{-2}-K_2$	$M^2-P_2^2-B_2^2$
16) $\mathrm{Rc}^{(1)}$	D	$B_1P_1+P_1B_1$
17) $\mathrm{Rc}^{(2)}$	$K_{-2}+K_2$	$P_1^2-B_1^2$
18) $\mathrm{Rr}^{(1)}$	D	M^2
19) $\mathrm{Rr}^{(2)}$	$K_{-2}+K_2$	M^2
20) $\mathrm{Re}^{(1)}$	D	$M^2+(B_2P_2+P_2B_2)/2$
21) $\mathrm{Re}^{(2)}$	$K_{-2}+K_2$	$M_2-P_2^2+B_2^2$
22) L1	P_1	$B_2^2-2bP_2E$
23) L2	P_1	$P_2^2+2aB_2E$
24) O1	P_1	$P_2^2+B_2^2$
25) R1	P_1	$F_2P_2+P_2B_2$
26) R2	P_1	$P_2^2-B_2^2$

The above 26 coordinate systems were classified up to equivalence under the Gallilean group $G(2) \subset G$. However, from another point of view we can regard two coordinate systems as equivalent if the first can be transformed to the second under the action of some $g \in G$. In terms of operators, the system described by K, S is equivalent to the system described by K', S' if, under the adjoint action of G on the enveloping algebra of $\mathcal{G}$, the two-dimensional space spanned by K, S can be mapped onto the two-dimensional space spanned by K', S'. Under this more general equivalence relation not all of the above coordinate systems are inequivalent. Indeed the systems denoted $\mathrm{Ab}^{(1)}$ and $\mathrm{Ab}^{(2)}$ lie on the same two-dimensional orbits so that there are only 17 equivalence classes of orbits.

We can describe these equivalences in terms of the operator $J=\exp\frac{1}{4}\pi(K_2-K_{-2})$:

$$Jf(\mathbf{x},t)=[\sqrt{2}/(1+t)]\exp[\tfrac{1}{4}i(1+t)^{-1}\mathbf{x}\cdot\mathbf{x}] \times f[\sqrt{2}(1+t)^{-1}\mathbf{x},(t-1)/(1+t)], \quad f\in\mathcal{J}. \tag{3.2}$$

Note that $J^2=\exp\frac{1}{2}\pi(K_2-K_{-2})$, and

$$\begin{aligned} J^2 f(\mathbf{x},t) &= t^{-1}\exp[(i/4t)\mathbf{x}\cdot\mathbf{x}]f(t^{-1}\mathbf{x},-t^{-1}), \\ J^4 f(\mathbf{x},t) &= -f(-\mathbf{x},t), \\ J^8 f(\mathbf{x},t) &= f(\mathbf{x},t). \end{aligned} \tag{3.3}$$

It is easy to show that $J(K_{-2}+K_2)J^{-1}=D$, and, checking the adjoint action of J on second-order operators, we can verify that the three coordinate systems $\mathrm{Rc}^{(2)}$, $\mathrm{Rr}^{(2)}$, $\mathrm{Re}^{(2)}$ are equivalent under J to the three systems $\mathrm{Rc}^{(1)}$, $\mathrm{Rr}^{(1)}$, $\mathrm{Re}^{(1)}$ respectively.

Denoting the adjoint action of J^2 on $K\in\mathcal{G}$ by $K'=J^2KJ^{-2}$, we find $P_j'=-B_j$, $B_j'=P_j$, $K_{-2}'=-K_2$, $K_2'=-K_{-2}$, $D'=-D$, $M'=M$, $E'=E$ so that the six pairs of the form $\mathrm{Fa}^{(1)}$, $\mathrm{Fa}^{(2)}$ or $\mathrm{La}^{(1)}$, $\mathrm{La}^{(2)}$ are equivalent under J^2.

4. TWO- AND THREE-VARIABLE MODELS

We next demonstrate that the operators (1.5) can be interpreted as a Lie algebra of skew-Hermitian operators on the Hilbert space $L_2(R_2)$ of complex-valued Lebesgue square-integrable functions on the real line. This is accomplished by considering t as a fixed parameter and replacing ∂_t by $i(\partial_{x_1x_1}+\partial_{x_2x_2})$ in expressions (1.5). It is then clear that the resulting operators multiplied by i and restricted to the domain of C^∞-functions on R_2 with compact support are essentially self-adjoint. In fact these operators are real linear combinations of the operators

$$\begin{aligned} &\mathcal{K}_2=\tfrac{1}{4}i(x_1^2+x_2^2), \quad \mathcal{K}_{-2}=i(\partial_{x_1x_1}+\partial_{x_2x_2}), \quad \mathcal{P}_j=\partial_{x_j}, \\ &\mathcal{B}_j=\tfrac{1}{2}ix_j, \quad \mathcal{M}=x_1\partial x_2-x_2\partial_{x_1}, \quad \mathcal{E}=i, \\ &\mathcal{D}=x_1\partial_{x_1}+x_2\partial_{x_2}+1, \end{aligned} \tag{4.1}$$

which are well known to be essentially skew-adjoint. Note that when the parameter $t=0$ the operators (1.5) reduce to (4.1). Thus the script operators (4.1) satisfy the same commutation relations (1.6) as do the block operators (1.5). More specifically we have the identities

$$(\exp t\mathcal{K}_{-2})\mathcal{P}_j[\exp(-t\mathcal{K}_{-2})] = P_j, \qquad (4.2)$$
$$(\exp t\mathcal{K}_{-2})\mathcal{B}_j[\exp(-t\mathcal{K}_{-2})] = B_j,$$

with similar expressions relating the other script and block operators.

If $f \in L_2(R_2)$, then $u(t) = (\exp t\mathcal{K}_{-2})f$ satisfies $u_t = \mathcal{K}_{-2}u$ or $iu_t = -\Delta_2 u$ (for almost every t) wherever f is in the domain of $\mathcal{K}_{-2}$, and $u(0) = f$. Also the unitary operators $\exp\alpha K = \exp(t\mathcal{K}_{-2})(\exp\alpha\mathcal{K})\exp(-t\mathcal{K}_{-2})$ map u into $v = (\exp\alpha K)u$ which also satisfies $v_t = \mathcal{K}_{-2}v$ for each linear combination $\mathcal{K}$ of the operators (4.1). Thus the operators $\exp\alpha K$ are symmetries of (1.2).

We will see later that the operators (4.1) generate a global unitary irreducible representation of the group G on $L_2(R_2)$. Assuming this here, we let $U(g)$, $g \in G$, be the corresponding unitary operators and set $T(g) = (\exp t\mathcal{K}_{-2})U(g)[\exp(-t\mathcal{K}_{-2})]$. It is then easy to show that the $T(g)$ are unitary symmetries of (1.2) with associated infinitesimal operators $K = (\exp t\mathcal{K}_{-2})\mathcal{K} \times [\exp(-t\mathcal{K}_{-2})]$.

Next consider the operator $\mathcal{L}_3 = \mathcal{K}_{-2} - \mathcal{K}_2 = i(\Delta_2 - \frac{1}{4}(x_1^2 + x_2^2)) \in \mathcal{G}$. If $f \in L_2(R_2)$, then $u(t) = (\exp t\mathcal{L}_3)f$ satisfies $u_t = \mathcal{L}_3 u$ or

$$iu_t = -\Delta_2 u + \tfrac{1}{4}(x_1^2 + x_2^2)u \qquad (4.3)$$

and $u(0) = f$. Similarly the unitary operators $V(g) = (\exp t\mathcal{L}_3)U(g)[\exp(-t\mathcal{L}_3)]$ are symmetries of (4.3), the Schrödinger equation for the harmonic oscillator, and the associated infinitesimal operators $(\exp t\mathcal{L}_3)\mathcal{K}[\exp(-t\mathcal{L}_3)]$ can be expressed as first-order differential operators in t and x. Analogous statements hold for the operator $\mathcal{L}_2 = \mathcal{K}_{-2} + \mathcal{K}_2 = i(\Delta_2 + \frac{1}{2}(x_1^2 + x_2^2))$ with associated equation $u_t = \mathcal{L}_2 u$,

$$iu_t = -\Delta_2 u - \tfrac{1}{4}(x_1^2 + x_2^2)u, \qquad (4.4)$$

(Schrödinger equation for the repulsive oscillator) and the operator $\mathcal{K}_{-2} - \mathcal{B}_1 = i(\Delta_2 - x_1/2)$ with associated equation $u_t = (\mathcal{K}_{-2} - \mathcal{B}_1)u$,

$$iu_t = -\Delta_2 u + \tfrac{1}{2}x_1 u \qquad (4.5)$$

(linear potential).

These remarks show explicitly the equivalence of equations (1.2), (4.3)–(4.5). Through we have chosen to start with Eq. (1.2) in this paper, an analysis of any of the other equations would have led us to the same results.

From Table II we see that, except for the subgroup coordinates (22)–(26) which were essentially discussed in 5, every separable coordinate system corresponds to a G-orbit which contains exactly one of the Hamiltonian operators $i\mathcal{K}_{-2}$, $i\mathcal{L}_3$, $i\mathcal{L}_2$, or $i(\mathcal{K}_{-2} - \mathcal{B}_1)$. Thus each coordinate system is naturally associated with one of these four Hamiltonians.

Consider a pair of commuting self-adjoint operators $i\mathcal{K}, \mathcal{S}$, where $\mathcal{K} \in \mathcal{G}$ and $\mathcal{S}$ is a symmetric quadratic operator in the enveloping algebra of $\mathcal{G}$. These operators have a common spectral resolution, i.e., there is a complete set of (generalized) eigenvectors $f_{\lambda,\mu}(\mathbf{x})$ in $L_2(R_2)$ with

$$i\mathcal{K}f_{\lambda,\mu} = \lambda f_{\lambda,\mu}, \quad \mathcal{S}f_{\lambda,\mu} = \mu f_{\lambda,\mu}, \quad (f_{\lambda,\mu}, f_{\lambda',\mu'}) = \delta_{\lambda\lambda'}\delta_{\mu\mu'}, \qquad (4.6)$$

where

$$(h_1, h_2) = \iint_{-\infty}^{\infty} h_1(\mathbf{x})\bar{h}_2(\mathbf{x})\,dx_1\,dx_2, \quad h_j \in L_2(R_2). \qquad (4.7)$$

Now suppose $i\mathcal{K}'$, $\mathcal{S}'$ are another pair of commuting self-adjoint operators on the same G-orbit as $i\mathcal{K}, \mathcal{S}$. Then by renormalizing these operators if necessary, it follows that there is a $g \in G$ such that

$$\mathcal{K}' = U(g)\mathcal{K}U(g^{-1}), \quad \mathcal{S}' = U(g)\mathcal{S}U(g^{-1}).$$

Thus the spectral resolution of the primed pair is identical to that for the unprimed pair. Indeed for $f'_{\lambda,\mu} = U(g)f_{\lambda,\mu}$ we have

$$i\mathcal{K}'f'_{\lambda,\mu} = \lambda f'_{\lambda,\mu}, \quad \mathcal{S}'f'_{\lambda,\mu} = \mu f'_{\lambda,\mu}$$
$$(f'_{\lambda,\mu}, f'_{\lambda',\mu'}) = \delta_{\lambda\lambda'}\delta_{\mu\mu'}, \qquad (4.8)$$

and the $f'_{\lambda,\mu}$ form a complete ON set in $L_2(R_2)$.

In the following we will frequently need the spectral resolution of a pair $i\mathcal{K}$, $\mathcal{S}$, where $i\mathcal{K}$ is one of the four Hamiltonians listed above. However, in many cases we will be able to use the unitary symmetry operators $U(g)$ to construct an equivalent pair $i\mathcal{K}'$, $\mathcal{S}'$ whose spectral resolution is much simpler to compute. This information will then provide the spectral resolution of the original pair.

As a special case of these remarks consider the operator $\mathcal{K}_{-2} = i\Delta_2$. If $\{f_{\lambda,\mu}\}$ is the basis (4.6) of generalized eigenvectors for the pair $\mathcal{K}$, $\mathcal{S}$, then $\{f'_{\lambda,\mu}(t) = [\exp(t\mathcal{K} - 2)]f_\lambda\}$ is the corresponding basis of generalized eigenvectors for the block operators

$$K = (\exp t\mathcal{K}_{-2})\mathcal{K}[\exp(-t\mathcal{K}_{-2})], \quad S = (\exp t\mathcal{K}_{-2})\mathcal{S}[\exp(-t\mathcal{K}_{-2})]$$

and the $f'_{\lambda,\mu}(t)$ satisfy Eq. (1.2). Similar remarks hold for the other Hamiltonians. This clarifies the relationship between the two- and three-variable models of $\mathcal{G}$.

We now explicitly compute the spectral resolutions of the pairs of commuting operators listed in Table II. We being with the Oc orbit, i.e., by determining the spectral resolution of the pair $\mathcal{L}_3 = \mathcal{K}_{-2} - \mathcal{K}_2, \mathcal{P}_1^2 + \mathcal{B}_1^2$. Equations (4.6) are

$$[-\Delta_2 + \tfrac{1}{4}(x_1^2 + x_2^2)]f = \lambda f, \quad (\partial_{x_1 x_1} - \tfrac{1}{4}x_1^2)f = \mu f,$$

and the well-known normalized eigenfunctions are

$$f_{\lambda,\mu} = \mathrm{oc}_{n,m}(\mathbf{x}) = (2^{m+n}\pi n!\,m!)^{-1/2}\exp[-(x_1^2 + x_2^2)/4]$$
$$\times H_n(x_1/\sqrt{2})H_m(x_2/\sqrt{2}),$$
$$\mu = -n - \tfrac{1}{2}, \quad \lambda + \mu = m + \tfrac{1}{2}. \qquad (4.9)$$
$$(\mathrm{oc}_{n',m'}, \mathrm{oc}_{n,m}) = \delta_{nn'}\delta_{mm'},$$

where $H_n(x)$ is a Hermite polynomial.

At this point one can easily show in a manner analogous to that presented in 5, Sec. 3, that the operators (4.1) exponentiate to yield a global unitary irreducible representation of G. Indeed from the known recurrence formulas for the Hermite polynomials one can see that the operators $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ acting on the oc-basis define a unitary representation of $sl(2,R)$ which is a direct sum of representations from the discrete series, and the $\mathcal{W}$-operators define a unitary irreducible representation of $\mathcal{W}$. As follows from the work of Bargmann,[22,23] this

Lie algebra representation extends to a global representation of G, irreducible since its restriction to W is already irreducible.

We now compute the unitary operators $U(g)$ on $L_2(R_2)$. The operators

$$U(\mathbf{w},\mathbf{z},\alpha)=\exp(w_1\mathcal{B}_1)\exp(z_1\mathcal{P}_1)\exp(w_2\mathcal{B}_2)\exp(z_2\mathcal{P}_2)\times\exp(\alpha\mathcal{E})$$

defining the irreducible representation of W are

$$[U(\mathbf{w},\mathbf{z},\alpha)f](\mathbf{x})=\exp[i(\alpha+\tfrac{1}{2}\mathbf{w}\cdot\mathbf{x})f(\mathbf{x}+\mathbf{z})],\quad f\in L_2(R_2). \tag{4.10}$$

The operator $U(\theta)=\exp(\theta\mathcal{M})$ is

$$[U(\theta)f](\mathbf{x})=f(\mathbf{x}\Theta)$$

where Θ is given by (1.10). The operators $U(A)$, $A\in SL(2,R)$, are more difficult. From Ref. 24, we find

$$(\exp a\mathcal{K}_{-2})f(\mathbf{x})=\text{l.i.m.}\ \frac{1}{4\pi i a}\times\iint_{-\infty}^{\infty}\exp[-(\mathbf{x}-\mathbf{y})^2/4ia]\,f(\mathbf{y})\,dy_1\,dy_2. \tag{4.11}$$

(In the following we will drop the l.i.m. symbol.)

Also

$$(\exp b\mathcal{K}_2)f(\mathbf{x})=\exp(ib\,\mathbf{x}\cdot\mathbf{x}/4)\,f(\mathbf{x}),$$

$$(\exp c\mathcal{D})f(\mathbf{x})=e^c f(e^c\mathbf{x}). \tag{4.12}$$

Using group multiplication in $SL(2,R)$, we find

$$\exp\phi\mathcal{L}_2=\exp(\tanh\phi\mathcal{K}_2)\exp(\sinh\phi\cosh\phi\mathcal{K}_{-2})\times\exp[-\ln(\cosh\phi)\mathcal{K}_3]$$

so that

$$(\exp\phi\mathcal{L}_2)f(\mathbf{x})=\frac{\exp(i\coth\phi\,\mathbf{x}\cdot\mathbf{x}/4)}{4\pi i\sinh\phi}\iint_{-\infty}^{\infty}\times\exp\frac{i}{4}\left(-\frac{2}{\sinh\phi}\,\mathbf{x}\cdot\mathbf{y}+\cosh\phi\,\mathbf{y}\cdot\mathbf{y}\right)\times f(\mathbf{y})\,dy_1\,dy_2. \tag{4.13}$$

Similar computations yield

$$(\exp\theta\mathcal{L}_3)f(\mathbf{x})=\frac{\exp(i\cot\theta\,\mathbf{x}\cdot\mathbf{x}/4)}{4\pi i\sin\theta}\times\iint_{-\infty}^{\infty}\exp\frac{i}{4}\left(-\frac{2}{\sin\theta}\,\mathbf{x}\cdot\mathbf{y}+\cot\theta\,\mathbf{y}\cdot\mathbf{y}\right)\times f(y)\,dy_1\,dy_1, \tag{4.14}$$

$$\exp\rho(\mathcal{K}_{-2}+a\mathcal{B}_1)f(\mathbf{x})=\exp i\,\frac{(a\rho x_1/2-a^2\rho^3/12)}{4\pi i}\times\iint_{-\infty}^{\infty}\exp\frac{i}{4\rho}\left[(x_1-a\rho^2-y_1)^2+(x_2-y_2)^2\right]f(\mathbf{y})\,dy_1\,dy_2. \tag{4.15}$$

From (4.11) it follows that the basis functions $\mathrm{oc}_{n,m}(\mathbf{x})$ map to the ON basis functions $\mathrm{Oc}_{n,m}(\mathbf{x},t)=\exp t\mathcal{K}_{-2})\,\mathrm{oc}_{n,m}(\mathbf{x})$ or

$$\mathrm{Oc}_{n,m}(x,t)=(2^{m+n+1}\pi n!\,m!)^{-1/2}\exp[i\pi(m+n-1)/2]\times\exp[-\tfrac{1}{4}(v_1^2+v_2^2)(1-iv_3)]\left(\frac{v_3+i}{v_3-i}\right)^{(m+n)/2}\times(v_3-i)^{-1}H_m(v_1/\sqrt{2})H_n(v_2/\sqrt{2}), \tag{4.16}$$

where

$$x_1=v_1(1+v_3^2)^{1/2},\quad x_2=v_2(1+v_3^2)^{1/2},\quad t=v_3.$$

The functions (4.16) are those corresponding to the separable coordinate system Oc in Table I.

Next we compute the spectral resolution for the system Or:

$$i(\mathcal{K}_{-2}-\mathcal{K}_2)f=\lambda f,\quad \mathcal{M}^2 f=\mu f.$$

The basis of eigenvectors is

$$\mathrm{or}^+_{n,m}(\mathbf{x})=[m!/2^m\pi(n+m)!]^{1/2}\exp(-r^2/4)\,r^m L_n^m(\tfrac{1}{2}r^2)\times\cos m\theta,$$

$$\mathrm{or}^-_{n,m}(\mathbf{x})=\tan m\theta\,\mathrm{or}^+_{n,m}(\mathbf{x}), \tag{4.17}$$

where $m\geq 1, n\geq 0$ and $x_1=r\cos\theta$, $x_2=r\sin\theta$. The eigenvalues λ,μ are related to m,n via $\mu=-m^2$, $\lambda=2n+m+1$. For $m=0$ we get

$$\mathrm{or}^+_{n,0}(\mathbf{x})=(2/\pi n!)^{1/2}\exp(-r^2/4)\,L_n(\tfrac{1}{2}r^2),$$

where $L_n^\alpha(r)$ is a generalized Laguerre polynomial. The orthogonality relations are

$$(\mathrm{or}^{\epsilon'}_{n',m'},\mathrm{or}^{\epsilon}_{n,m})=\delta_{\epsilon\epsilon'}\delta_{n'n}\delta_{m'm},\epsilon,\epsilon'=\pm.$$

The three-variable basis functions $\mathrm{Or}_{n,m}(\mathbf{x},t)=(\exp t\mathcal{K}_{-2})\,\mathrm{or}_{n,m}(\mathbf{x})$ are

$$\mathrm{Or}^+_{n,m}(\mathbf{x},t)=K\left(\frac{m!}{\pi^3 2^m(n+m)!}\right)^{1/2}\frac{(-1)^{m+n}}{2^{2m}}\frac{(v_3+i)^{m/2+n}}{(v_3-i)^{m/2+n+1}}\times\exp[\tfrac{1}{4}v_1^2(it-1)]L_n^m(\tfrac{1}{2}v_1^2)\cos mv_2, \tag{4.18}$$

$$\mathrm{Or}^-_{n,m}(\mathbf{x},t)=\tan mv_2\,\mathrm{Or}^+_{n,m}(\mathbf{x},t),\quad m\geq 1$$

for $m=0$, $K=\sqrt{2}$; otherwise $K=1$. Also $x_1=(1+v_3^2)^{1/2}v_1\cos v_2$, $x_2=(1+v_3^2)^{1/2}v_1\sin v_2$, $t=v_3$.

For the system Oe,

$$i(\mathcal{K}_{-2}-\mathcal{K}_2)f=\lambda f,\quad (\mathcal{M}^2-\mathcal{P}_2^2-\mathcal{B}_2^2)f=\mu f,$$

we obtain the ON basis

$$\mathrm{oe}^+_{n,m}(\mathbf{x})=(1/\pi)\,\mathrm{hc}_p^m(i\xi,\tfrac{1}{2})\,\mathrm{hc}_p^m(\eta,\tfrac{1}{2}),$$
$$\mathrm{oe}^-_{n,m}(\mathbf{x})=(1/\pi)\,\mathrm{hs}_p^m(i\xi,\tfrac{1}{2})\,\mathrm{hs}_p^m(\eta,\tfrac{1}{2}), \tag{4.19}$$

where

$$\mathrm{hc}_p^m(\eta,\tfrac{1}{2})=\exp(-\cos 2\eta/8)\,C_p^m(\eta,\tfrac{1}{2}),$$
$$\mathrm{hs}_p^m(\eta,\tfrac{1}{2})=\exp(-\cos 2\eta/8)\,S_p^m(\eta,\tfrac{1}{2}),$$
$$0\leq m\leq p<\infty,\ (-1)^{m-p}=1,$$

$$x_1=\cosh\xi\cos\eta,\quad x_2=\sinh\xi\sin\eta.$$

The eigenvalues λ and μ are related to p and m via $\lambda=p+1$, $\mu=\frac{1}{2}\lambda+a_p^m(\frac{1}{2})$ or $\mu=\frac{1}{2}\lambda+b_p^m(\frac{1}{2})$.

The orthogonality relations are

$$(\mathrm{oe}^{\epsilon'}_{n'm'},\mathrm{oe}^{\epsilon}_{nm})=\delta_{\epsilon\epsilon'}\delta_{n'n}\delta_{m'm},\quad \epsilon,\epsilon'=\pm.$$

The functions $C_p^m(\eta,\xi), S_p^m(\eta,\xi)$ are Ince polynomials.[25,26] They are polynomial solutions of period 2π of the Whittaker—Hill equation. This equation has been investigated in detail by Arscott,[26] and it is his notation for the solutions and eigenvalues that we use. The three-variable basis functions $\mathrm{Oe}_{n,m}(\mathbf{x},t)=(\exp t\mathcal{K}_{-2})\,\mathrm{oe}_{n,m}(\mathbf{x})$ are

$$\mathrm{Oe}^{+}_{n,m}(\mathbf{x},t)=(\lambda^{m+}_{p}/\pi)\exp[(i/4)\,v_3(\sinh^2 v_1+\cos^2 v_2)]\\ \times(v_3-i)^{p/2+1}(v_3+i)^{-p/2}\,\mathrm{hc}^{m}_{p}(iv_1,\tfrac12)\,\mathrm{hc}^{m}_{p}(v_2,\tfrac12)$$

where (4.20)

$$x_1=(1+v_3^2)^{1/2}\cosh v_1\cos v_2,\;\; x_2=(1+v_3^2)^{1/2}\times\sinh v_1\sin v_2,\quad t=v_3.$$

The expression for $\mathrm{Oe}^{-}_{n,m}(\mathbf{x},t)$ is as above except that we now have a new constant of modulus unity λ^{m-}_{p} and the functions $\mathrm{hc}^{m}_{p}(\eta,\xi)$ are replaced by $\mathrm{hs}^{m}_{p}(\eta,\xi)$. The constants $\lambda^{m\pm}_{p}$ are in principle calculable from a knowledge of the explicit form of the Ince polynomials. They can always be calculated by inserting special values of the parameters v_i. Accordingly we make no further comment on their determination.

In the remaining cases there are always two coordinate systems associated with each orbit. For simplicity we shall always treat the coordinate system with superscript (1). The corresponding results for system (2) follow immediately upon application of the operators J or J^2, (3.2), (3.3).

The Fc system is defined by equations

$$i\mathcal{K}_2 f=-\tfrac14\gamma^2 f,\;\; \mathcal{B}_1 f=\tfrac12 i\gamma\cos\alpha\, f$$

and has a basis of generalized eigenvectors

$$\mathrm{fc}_{\gamma,\alpha}(\mathbf{x})=[\delta(r-\gamma)/\sqrt{r}\,]\,\delta(\theta-\alpha),\;\;(\mathrm{fc}_{\gamma,\alpha},\mathrm{fc}_{\gamma',\alpha'})=\delta(\gamma-\gamma')\,\delta(\alpha-\alpha'),$$

$$x_1=r\cos\theta,\quad x_2=r\sin\theta,\quad 0\le\alpha<2\pi,0\le\gamma. \tag{4.21}$$

The basis functions $\mathrm{Fc}_{\gamma,\alpha}(\mathbf{x},t)=(\exp t\mathcal{K}_{-2})\,\mathrm{fc}_{\gamma,\alpha}(x)$ are

$$\mathrm{Fc}_{\gamma,\alpha}(\mathbf{x},t)=\frac{\sqrt{\gamma}}{4\pi i t}\;\exp\frac{i}{4t}\;\Big[(x_1-\gamma\cos\alpha)^2+(x_2-\gamma\sin\alpha)^2\Big]. \tag{4.22}$$

The Fr system is defined by

$$i\mathcal{K}_2 f=-\tfrac14\gamma^2 f,\quad i\mathcal{M} f=-mf$$

with basis

$$\mathrm{fr}_{\gamma,m}(\mathbf{x})=\frac{\delta(r-\gamma)}{\sqrt{r}}\;\frac{\exp(im\theta)}{\sqrt{2\pi}},\quad(\mathrm{fr}_{\gamma,m},\mathrm{fr}_{\gamma',m'})=\delta(\gamma-\gamma')\,\delta_{mm'}. \tag{4.23}$$

Here $0\le\gamma$, $m=0,\pm1,\ldots$, and r,θ are polar coordinates. The three-variable basis functions are

$$\mathrm{Fr}_{\gamma,m}(\mathbf{x},t)=\left(\frac{\gamma}{2\pi}\right)^{1/2}\exp\frac{i}{4t}\,(r^2+\gamma^2)\;\frac{i^{m-1}}{2t}\times\exp(im\theta)\,J_m\left(\frac{-r\gamma}{2t}\right), \tag{4.24}$$

where $J_m(z)$ is a Bessel function.

The Fp system is determined by equations

$$i\mathcal{K}_2 f=-\tfrac14\gamma^2 f,\;(\mathcal{B}_2\mathcal{M}+\mathcal{M}\mathcal{B}_2)f=-\mu f$$

with basis

$$\mathrm{fp}^{+}_{\gamma,\mu}(\mathbf{x})=(1/\sqrt{2\pi})(1+\cos\theta)^{i\mu/2-1/4}(1-\cos\theta)^{i\mu/2-1/4}\times[\delta(r-\gamma)/\sqrt{r}\,],\quad -\pi\le\theta<0,$$
$$0,\quad 0\le\theta\le\pi,$$

$$\mathrm{fp}^{-}_{\gamma,\mu}(\mathbf{x})=\mathrm{fp}^{-}_{\gamma,\mu}(r,\theta)=\mathrm{fp}^{+}_{\gamma,\mu}(r,-\theta). \tag{4.25}$$

Here r,θ are polar coordinates, $0\le\gamma,-\infty<\mu<\infty$, and the spectrum is continuous of multiplicity two.[1] The orthogonality relations are

$$(\mathrm{fp}^{\pm}_{\gamma,\mu},\mathrm{fp}^{\pm}_{\gamma',\mu'})=\delta(\gamma-\gamma')\,\delta(\mu-\mu'),$$
$$(\mathrm{fp}^{\pm}_{\gamma,\mu},\mathrm{fp}^{\mp}_{\gamma',\mu'})=0.$$

The three-variable basis functions are

$$\mathrm{Fp}^{\pm}_{\gamma,\mu}(\mathbf{x},t)=\frac{i\sqrt{\gamma}\exp(i\gamma^2/4t)}{2^3\pi t\cos(i\mu\pi)}\exp\left(\frac{i}{16t}\,(\xi^2+\eta^2)^2\right)\times\Bigg[D_{-i\mu/2-1/2}\left(\frac{\sigma\xi}{\sqrt{t}}\right)\times D_{i\mu/2-1/2}\left(\frac{\sigma\eta}{\sqrt{t}}\right)+D_{-i\mu/2-1/2}\left(\frac{-\sigma\xi}{\sqrt{t}}\right)D_{-i\mu/2-1/2}\left(\frac{-\sigma\eta}{\sqrt{t}}\right)\Bigg],$$

$$\mathrm{Fp}^{-}_{\gamma,\mu}(x_1,x_2,t)=\mathrm{Fp}^{+}_{\gamma,\mu}(x_1,-x_2,t), \tag{4.26}$$

where $\sigma=\exp(i\pi/4)\sqrt{\gamma}$ and ξ,η are parabolic coordinates

$$2x_1=\xi^2-\eta^2,\quad x_2=\xi\eta.$$

The Fe system is defined by equations

$$i\mathcal{K}_2 f=-\gamma^2 f,\quad(\mathcal{M}^2+4\mathcal{B}_1^2-4\mathcal{B}_2^2)f=-\mu f,$$

[equivalent to (7) in Table II]. The basis functions are

$$\mathrm{fe}_{\gamma,n}(\mathbf{x})=\frac{\delta(r-\gamma)}{\sqrt{r\pi}}\begin{cases}\mathrm{ce}_n(\theta,\gamma^2/2), & n=0,1,2,\ldots,\\ \mathrm{se}_{-n}(\theta,\gamma^2/2), & n=-1,-2,\ldots,\end{cases} \tag{4.27}$$

$$0\le\gamma,\quad(\mathrm{fe}_{\gamma,n},\mathrm{fe}_{\gamma',n'})=\delta(\gamma-\gamma')\,\delta_{nn'},$$

where $\mathrm{ce}_n(\theta,q),\mathrm{se}_n(\theta,q)$ are the periodic Mathieu functions of integral order and r,θ are polar coordinates. The eigenvalues $\mu=\mu_n$ are discrete and all of multiplicity one. The basis functions

$$\mathrm{Fe}_{\gamma,n}(\mathbf{x},t)=(\exp t\mathcal{K}_{-2})\,\mathrm{fe}_{\gamma,n}(\mathbf{x})$$

are

$$\mathrm{Fe}_{\gamma,n}(\mathbf{x},t)=\frac{A_{\gamma,n}}{4\pi i\tau}\left(\frac{\gamma}{\pi}\right)^{1/2}\exp[i\tau(\cos^2\sigma+\sinh^2\rho+\gamma^2)]\times\begin{cases}\mathrm{ce}_n(\sigma,\gamma^2/2)\,\mathrm{Ce}_n(\rho,\gamma^2/2), & n=0,1,2,\ldots,\\ \mathrm{se}_{-n}(\sigma,\gamma^2/2)\,\mathrm{Se}_n(\rho,\gamma^2/2), & n=-1,-2,\ldots,\end{cases} \tag{4.28}$$

where $A_{\gamma,n}$ is a normalization constant, $\mathrm{Se}_n(\rho,q)$ and $\mathrm{Ce}_n(\rho,q)$ are modified Mathieu functions, and

$$x_1=-2\tau\cosh\rho\cos\sigma,\quad x_2=-2\tau\sinh\rho\sin\sigma,\quad t=\tau.$$

The Lc system (transformed so that $b=0$) can be defined by equations

$$i(\mathcal{K}_2+a\mathcal{P}_1)f=\lambda f,\;\;\mathcal{B}_2^2 f=-\tfrac14\rho^2 f$$

with basis functions

$$\mathrm{lc}_{\lambda,\rho'}(\mathbf{x})=\frac{\delta(x_2-\rho)}{\sqrt{2\pi|a|}}\exp\left[-\frac{i}{a}\left(\lambda x_1+\frac{\rho^2}{4}\,x_1+\frac{x_1^3}{12}\right)\right],$$

$$(\mathrm{lc}_{\lambda,\rho},\mathrm{lc}_{\lambda',\rho'})=\delta(\lambda-\lambda')\,\delta(\rho-\rho'),\quad-\infty<\lambda,\rho<\infty. \tag{4.29}$$

The three-variable basis functions are

$$\mathrm{Lc}_{\lambda,\rho}(\mathbf{x},t)=\frac{(9a)^{-1/3}}{8iv_3\sqrt{2\pi|a|}}\exp\left[i\left((v_1^2+v_2^2)\frac{v_3}{4}-\frac{av_1}{v_3}-\rho\frac{v_2}{2}-\frac{a^2}{3v_3^3}-\frac{\lambda}{v_3}\right)\right]\mathrm{Ai}\left[(36a)^{-1/3}\left(\frac{v_1}{a}+\frac{\lambda}{a}+\frac{\rho^2}{4a}\right)\right], \tag{4.30}$$

where $\mathrm{Ai}(z)$ is an Airy function. Here,

$$x_1=v_1v_3+a/v_3,\quad x_2=v_2v_3,\quad t=v_3.$$

The Lp system is defined by

$$i(K_2+aP_1)f=\lambda f,\quad (B_2M+MB_2+aP_2^2)f=\mu f$$

with basis functions

$$\mathrm{lp}_{\lambda,n}(\mathbf{x})=(1/\sqrt{2\pi|a|})\,h_n(x_2)\exp[-(i/a)(\lambda x_1+\tfrac14 x_1x_2^2+\tfrac{1}{12}x_1^3)],$$
$$-\infty<\lambda<\infty,\ n=0,1,2,\ldots,$$

$$(\mathrm{lp}_{\lambda,n},\mathrm{lp}_{\lambda',n'})=\delta(\lambda-\lambda')\delta_{nn'}. \tag{4.31}$$

Here $h_n(x)$ is a solution of

$$h''-\left(\frac{\mu}{a}+\frac{\lambda x^2}{a^2}+\frac{x^4}{4a^2}\right)h=0 \tag{4.32}$$

such that

$$\int_{-\infty}^{\infty}|h_n(x)|^2\,dx=1. \tag{4.33}$$

The eigenvalues $\mu=\mu_n$ of (4.32) subject to condition (4.33) are discrete,[27] with multiplicity one, and we assume them ordered so that $\mu_0<\mu_1<\mu_2<\cdots$. Here $h_n(x)$ is either even or odd for each value of n.

Denote a general solution of (4.32) by $h_{\mu,\lambda,a}(x)$. Then it is straightforward to show that the basis functions $\mathrm{Lp}_{\lambda,n}(\mathbf{x},t)=(\exp tK_{-2})\,\mathrm{lp}_{\lambda,n}(\mathbf{x})$ are

$$\mathrm{Lp}_{\lambda,n}(\mathbf{x},t)=\frac{C_{\lambda,n}}{v_3}\times\exp\left[i\left((v_1^2+v_2^2)^2\frac{v_3}{16}-\frac{a}{4v_3}(v_1^2-v_2^2)\frac{a^2}{12v_3^2}-\frac{\lambda}{v_3}\right)\right]\times h_{2\mu_n,\lambda,a/2}(v_1)\,h_{2\mu_n,\lambda,a/2}(iv_2), \tag{4.34}$$

where the two h functions have the same parity as $h_n(x)$ and $C_{\lambda,n}$ is a normalization constant. Also

$$x_1=(v_1^2-v_2^2)\frac{v_3}{2}+\frac{a}{v_3},\quad x_2=v_1v_2v_3,\quad t=v_3.$$

The Rc system is defined by the equations

$$Df=\rho f,\quad (B_1P_1+P_1B_1)f=\mu f$$

with basis functions

$$\mathrm{rc}_{\lambda\mu}^{\epsilon\epsilon'}(\mathbf{x})=\frac{1}{2\pi}\,x_{1\epsilon}^{-i\lambda-1/2}x_{2\epsilon'}^{-i\mu-1/2}$$

$$-\infty<\lambda<\infty,\quad -\infty<\mu<\infty,\ \epsilon,\epsilon'=\pm,\quad \lambda=\rho-\mu$$

where

$$x_+^\lambda=\begin{cases}x^\lambda, & x>0\\ 0, & x<0\end{cases},\quad \lambda=\rho-\mu, \tag{4.35}$$

and similarly for x_-^λ. The orthogonality relations are

$$(\mathrm{rc}_{\lambda'\mu'}^{\bar\epsilon\bar\epsilon'},\mathrm{rc}_{\lambda\mu}^{\epsilon\epsilon'})=\delta_{\bar\epsilon\epsilon}\delta_{\bar\epsilon'\epsilon'}\delta(\lambda'-\lambda)\,\delta(\mu'-\mu).$$

The three-variable basis functions are

$$\mathrm{Rc}_{\lambda\mu}^{++}=\frac{1}{8\pi^2 iv_3}\,[\exp(i\pi/4)\sqrt{2v_3}\,]^{-i(\lambda+\mu)+1}\,\Gamma(-i\lambda+\tfrac12)\,\Gamma(-i\mu+\tfrac12)\times\exp[i(v_1^2+v_2^2)/8]D_{i\lambda-1/2}(-v_1/\sqrt{2i})\,D_{i\mu-1/2}(-v_2/\sqrt{2i}), \tag{4.36a}$$

where $x_1=v_3^{1/2}v_1,\ x_1=v_3^{1/2}v_2,\ t=v_3$. The remaining three-variable basis functions are given by

$$\mathrm{Rc}_{\lambda\mu}^{++}(v_1,v_2)=(-1)^{1-i(\lambda+\mu)}\mathrm{Rc}_{\lambda\mu}^{--}(-v_1,-v_2)=(-1)^{1/2-i\lambda}\mathrm{Rc}_{\lambda\mu}^{-+}(-v_1,v_2)=(-1)^{1/2-i\mu}\mathrm{Rc}_{\lambda\mu}^{+-}(v_1,-v_2). \tag{4.36b}$$

The Rr system is defined by the equations

$$Df=\rho f,\quad Mf=imf.$$

The eigenfunctions are then

$$\mathrm{rr}_{\rho m}(\mathbf{x})=(1/2\pi)\,r^{i\rho-1}\exp(im\theta), \tag{4.37}$$
$$-\infty<\rho<\infty,\quad m=0,\pm1,\cdots,\ x_1=r\cos\theta,\ x_2=r\sin\theta,$$

satisfying the orthogonality relations

$$(\mathrm{rr}_{\rho'm'},\mathrm{rr}_{\rho m})=\delta_{m'm}\,\delta(\rho'-\rho).$$

The three variable basis functions are

$$\mathrm{Rr}_{\rho m}(\mathbf{x},t)=\frac{2}{i\pi\sqrt{v_3}}(2\sqrt{iv_3})^{1+i\rho}\frac{\Gamma(m/2+(1+i\rho)/2)}{m!}\times v_1^{-1}\exp(iv_1^2/8)\,M_{i\rho/2,m/2}(iv_1^2/4)\exp(imv_2), \tag{4.38}$$

where $M_{v,\mu}(z)$ is a solution of Whittaker's equation and $x_1=v_3^{1/2}v_1\cos v_2,\ x_2=v_3^{1/2}v_1\sin v_2,\ t=v_3$.

The Re system is defined by the equations

$$Df=i\lambda f,\ [M^2+\tfrac12(B_2P_2+P_2B_2)]f=\mu f.$$

The orthonormalized eigenfunctions are then

$$\mathrm{re}_{\lambda m}^{+}(\mathbf{x})=(1/\sqrt{2\pi})\,r^{i\lambda-1}\mathrm{Gc}_m(\theta,\tfrac14,-\lambda),$$
$$\mathrm{re}_{\lambda m}^{-}(\mathbf{x})=(1/\sqrt{2\pi})\,r^{i\lambda-1}\mathrm{Gs}_m(\theta,\tfrac14,-\lambda), \tag{4.39}$$

$x_1=r\cos\theta,\ x_2=r\sin\theta$. Here we have introduced the notation

$$\mathrm{Gc}_m(\theta,\tfrac14,-\lambda)=\exp[i\cos(2\theta)/16]\,\mathrm{gc}_m(\theta,\tfrac14,-\lambda),$$
$$\mathrm{Gs}_m(\theta,\tfrac14,-\lambda)=\exp[i\cos(2\theta)/16]\,\mathrm{gs}_m(\theta,\tfrac14,-\lambda). \tag{4.40}$$

The functions $\mathrm{gc}_m(\theta,\alpha,\beta)$ and $\mathrm{gs}_m(\theta,\alpha,\beta)$ are nonpolynomial solutions of the Whittaker–Hill equation and the subscript m (the number of zeros in the interval $[0,2\pi]$) labels the discrete eigenvalues of the operator $M^2+\frac12(B_2P_2+P_2B_2)$, i.e., $\mu=\mu_m$. This notation is due to Arscott and Urwin.[28] Each of the solutions $\mathrm{Gc}_m(\theta,\alpha,\beta)$ or $\mathrm{Gs}_m(\theta,\alpha,\beta)$ can be written as an infinite series in trignometric functions which converges for the discrete eigenvalues μ_m. For further details see Ref. 28. The three-variable basis functions are

$$\mathrm{Re}_{\lambda m}^{+}(\mathbf{x},t)=K_m^{\lambda+}v_3^{(i\lambda-1)/2}\mathrm{Gc}_m(iv_1,\tfrac14,-\lambda)\,\mathrm{Gc}_m(v_2,\tfrac14,-\lambda),$$
$$\mathrm{Re}_{\lambda m}^{-}(\mathbf{x},t)=\bar K_m^{\lambda-}v_3^{(i\lambda-1)/2}\mathrm{Gs}_m(iv_1,\tfrac14,-\lambda)\,\mathrm{Gs}_m(v_2,\tfrac14,-\lambda),$$

where

$$x_1=v_3^{1/2}\cosh v_1\cos v_2,\ x_2=v_3^{1/2}\sinh v_1\sin v_2,\ t=v_3.$$

The constants $\bar{K}_m^{\lambda\pm}$ are in principle calculable by choosing special values of the parameters v_i. In fact in the process of calculating the functions $\mathrm{Re}^{\pm}$ we get relations which to our knowledge are new, viz.,

$$K_m^{\lambda+}\,\mathrm{Gc}(iv_1,\tfrac14,-\lambda)\,\mathrm{Gc}(v_2,\tfrac14,-\lambda) = \exp[\tfrac14 i(\sinh^2 v_1+\cos^2 v_2)]\int_{-\pi}^{\pi} d\theta\,\mathrm{Gc}_m(\theta,\tfrac14,-\lambda) \times\exp[-\tfrac18 i(\cosh v_1\cos v_2\cos\theta+\sinh v_1\sin v_2\sin\theta)^2] \times D_{i\lambda-1}(-(\cosh v_1\cos v_2\cos\theta+\sinh v_1\sin v_2\sin\theta/\sqrt{2i})$$

with a similar relation holding for the functions $\mathrm{Gs}_m(\theta,\frac14,-\lambda)$. The constants $K_m^{\lambda\pm}$ can be calculated for particular values of the arguments v_i, e.g., $\mathrm{Gc}_m(\theta,\frac14,-\lambda)=\sum_{r=0}^{\infty}A_r^m\cos 2r\theta$. Then

$$K_m^{\lambda+}=\frac{2\pi D_{i\lambda-1}(0)\,A_0^m}{\mathrm{Gc}_m(\frac12\pi,\frac14,-\lambda)\,\mathrm{Gc}_m(0,\frac14,-\lambda)}.$$

Similar expressions may be obtained for the other constants. Passage to the three-variable model in this basis allows us to derive a set of orthogonal basis functions as products of two Gc or Gs functions from a knowledge of the orthogonality of single functions.

5. OVERLAP FUNCTIONS

Exactly as in Sec. 3 of 5 one can show that our results lead to a number of Hilbert space expansion theorems. Indeed if $\{f_{\lambda\mu}\}$ is an ON basis for $L_2(R_2)$, then $\{U(g)f_{\lambda\mu}\}$ for any $g\in G$ is also an ON basis. In particular, each of the three-variable models construcued in Sec. 4 provides a basis for $L_2(R_2)$. Furthermore, exactly as in (3.21) of Paper 5 we can derive discrete and continuous generating functions for each of our bases.

Now we compute overlap functions $(Aa_{\lambda\mu},Bb_{\lambda'\mu'})$ which allow us to expand eigenfunctions $Aa_{\lambda\mu}$ in terms of eigenfunctions $Bb_{\lambda'\mu'}$. The utility of these formulas is that they are invariant under the action of G so the same expressions allow us to expand $U(g)Aa_{\lambda\mu}$ in terms of $U(g)Bb_{\lambda'\mu'}$, where the results may be much less obvious. In the following we use the two-variable bases to compute some overlaps of interest. Because of G-invariance, identical results hold for the three-variable bases.

In the present paper we omit the overlaps between the three discrete bases Oc, Or, Oe, which will be treated in a forthcoming work. (However, the Oc—Or overlap is well-known.[29,30] For most of the other bases we give an overlap with either of the discrete bases Oc or Or. The principle behind these computations is obvious and the interested reader can derive for himself any of the other overlaps:

$$(\mathrm{fc}_{\gamma,\alpha}\mathrm{or}^{\pm}_{n,m})=\gamma^{1/2}\,\mathrm{or}^{\pm}_{n,m}(\gamma\cos\alpha,\gamma\sin\alpha); \tag{5.1}$$

$$(\mathrm{fr}_{\gamma,p},\mathrm{or}_{n,m})=\begin{cases}0 & \text{if } p\neq\pm m,\\ [m!\gamma/2^{m+1}(n+m)!]^{1/2}e^{-\gamma^2/4}\gamma^m L_n^m(\frac12\gamma^2) & \text{if } + \text{ and } p=\pm m\neq 0,\\ (p/m)i[m!\gamma/2^{m+1}(n+m)!]^{1/2}e^{-\gamma^2/4}\gamma^m L_n^m(\frac12\gamma^2) & \text{if } - \text{ and } p=\pm m\neq 0,\\ (4\gamma/n!)^{1/2}e^{-\gamma^2/4}L_n(\frac12\gamma^2) & \text{if } p=m=0,\end{cases} \tag{5.2}$$

$$(\mathrm{fp}^{+}_{\gamma,\mu},\mathrm{or}^{\pm}_{n,m}) =[m!\gamma/2^m\pi(n+m)!]^{1/2}\exp(-\gamma^2/4)\gamma^m L_n^m(\tfrac12\gamma^2) \times\exp[-\pi i(1\mp 1)/4]\,(a_m\pm a_{-m}), \tag{5.3}$$

$$(\mathrm{fp}^{-}_{\gamma,\mu},\mathrm{or}^{\pm}_{n,m})=(\mathrm{fp}^{+}_{\gamma,\mu},\mathrm{or}^{\pm}_{n,-m}), \tag{5.4}$$

where

$$a_m=\exp[\pi(i/2-\mu)/2]\,\Gamma(m+\tfrac12)\left[\frac{(-1)^m\Gamma(i\mu+\frac12)}{\Gamma(i\mu+m+\frac12)}\times{}_2F_1\left(\begin{matrix}i\mu+\frac12,m+\frac12\\ i\mu+m+1\end{matrix}\middle|-1\right)-\frac{i\Gamma(-i\mu+\frac12)}{\Gamma(-i\mu+m+1)}\times{}_2F_1\left(\begin{matrix}-i\mu+\frac12,m+\frac12\\ -i\mu+m+1\end{matrix}\middle|-1\right)\right.,$$

$$(\mathrm{fe}_{\gamma n'},\mathrm{or}^{+}_{mn})=\theta(n')\,(\gamma/\pi)^{1/2}\tfrac12(1+(-1)^{m-n'})A_m^{n'} \times[m!/2^m\pi(n+m)!]^{1/2}\exp(-\gamma^2/4)\,\gamma^m L_n^m(\tfrac12\gamma^2), \tag{5.5}$$

where $\theta(x)=1$ for $x\geq 0$, and zero otherwise. A similar expression for $(\mathrm{fe}_{\gamma n'},\mathrm{or}^{-}_{mn})$ can be obtained by replacing $\theta(n')$ by $\theta(-n')$ and $A_m^{n'}$ by $B_n^{n'}$ in the above equation. $A_m^{n'}$, $B_m^{n'}$ are the coefficeints in the trigonometric expansions of the even and odd Mathieu functions, respectively. All other overlaps are zero. Also,

$$(\mathrm{lc}_{\lambda,\rho},\mathrm{oc}_{n,m})=\frac{\exp(-\rho^2/4)}{(2^{m-1}\pi m!)^{1/2}}H_m(\rho/\sqrt2)\,C_n, \tag{5.6}$$

where

$$2^{2/3}\exp[-i(\tfrac16+\lambda+\rho^2/4+\sqrt{2y})]\,\mathrm{Ai}[2^{2/3}(\tfrac14-i\lambda-i\rho^2/4-i\sqrt{2y})] =\sum_{n=0}^{\infty}[(\sqrt{2i}\,y)^n/n!]\,C_n,$$

and we have normalized so that $a=-1$,

$$(\mathrm{lc}_{\lambda,\rho},\mathrm{lp}_{\mu,n})=\frac{1}{2\pi|a|}\bar{h}_n(\rho)\,\delta\left(\frac{\lambda-\mu}{a}\right), \tag{5.7}$$

$$(\mathrm{rc}^{++}_{\lambda\mu},\mathrm{oc}_{nm})=\pi^{-2}(2^{m+n+3}n!\,m!)^{-1/2}\,\mathcal{L}^{\lambda}_{m}\mathcal{L}^{\mu}_{n} \tag{5.8}$$

where

$$\mathcal{L}^{\lambda}_{m}=2^{m+i\lambda-1/2}\,\Gamma(i\lambda/2+\tfrac14)\,\Gamma((m+1)/2)\times{}_2F_1(-m/2,i\lambda/2+\tfrac14;\tfrac12;2)\quad\text{for } m \text{ even},$$
$$=2^{m+1+i\lambda}\,\Gamma(i\lambda/2+\tfrac14)\,\Gamma(m/2)\,{}_2F_1((1-m)/2,i\lambda/2+\tfrac34;\tfrac32;2)\quad\text{for } m \text{ odd}.$$

The remaining overlaps for rc^{+-}, rc^{-+}, and rc^{--} can be calculated by using relations (4.36b):

$$(\mathrm{rr}^{+}_{\lambda m},\mathrm{or}_{nm'})=\delta_{mm'}(2/n!)^{m/2-i\lambda}\times[(m+n)!/m!]^{1/2}\,\Gamma((m+1-i\lambda)/2)\times{}_2F_1(-n,(m+1-i\lambda)/2;m+1;2), \tag{5.9}$$

$$(\mathrm{rr}^{-}_{\lambda m'},\mathrm{or}_{nm'})=-i(-1)^{\mathrm{sgn}\,m}(\mathrm{rr}^{+}_{\lambda m},\mathrm{or}_{nm'}), \tag{5.10}$$

$$(\mathrm{rr}^{+}_{\lambda 0},\mathrm{or}_{nm'})=\delta_{0m'}(2^{-1/2-i\lambda}/\sqrt{n!})\,\Gamma((1-i\lambda)/2)\times{}_2F_1(-n,(1-i\lambda)/2;1;2). \tag{5.11}$$

For the basis Re we have

$$(\mathrm{re}^{+}_{\lambda m},\mathrm{or}^{+}_{nm'})=\tfrac12(1+(-1)^{m-m'})\bar{A}^m_{m'}\sqrt{2\pi}\,(\mathrm{rr}^{+}_{\lambda m'},\mathrm{or}_{nm'}) \tag{5.12}$$

$$(\mathrm{re}^-_{\lambda m}, \mathrm{or}^-_{nm'}) = \tfrac{1}{2}(1 + (-1)^{m-m'}) \times \overline{B}^m_{m'} \sqrt{2\pi}\, i(-1)^{\mathrm{sgn}\, m'} (\mathrm{rr}^-_{\lambda m'}, \mathrm{or}_{nm'}) \qquad (5.13)$$

where $\overline{A}^m_{m'}$ and $\overline{B}^m_{m'}$ are the coefficients for the expansion of the functions $\mathrm{Gc}_m(\theta, \frac{1}{4}, -\lambda)$ and $\mathrm{Gs}_m(\theta, \frac{1}{4}, -\lambda)$, respectively, in trigonometric series.[28]

*Permanent address: School of Mathematics, University of Minnesota, Minneapolis, Minn. 55455.

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