Lie theory and the wave equation in space–time. 5. R-separable solutions of the wave equation $\psi_{tt}-\Delta_3\psi=0$

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A detailed classification is made of all orthogonal coordinate systems for which the wave equation $\psi_{tt}-\Delta_3\psi=0$ admits an R-separable solution. Only those coordinate systems are given which are not conformally equivalent to coordinate systems that have already been found in previous articles. We find 106 coordinates to given a total of 368 conformally inequivalent orthogonal coordinates for which the wave equation admits an R separation of variables.

INTRODUCTION

In this article we continue our investigation of the orthogonal R-separable coordinate systems for which the wave equation in space–time,

$$\psi_{tt}-\Delta_3\psi=0, \qquad (*)$$

admits an R-separation of variables.[1–4] In a previous article[4] we have studied coordinate systems for which the Klein–Gordon equation

$$\psi_{tt}-\Delta_3\psi=\lambda\psi \qquad (**)$$

admits a separation of variables. Such coordinate systems also admit a separation of variables for the wave equation (*). In paper 4 of this series we found 262 conformally inequivalent coordinate systems of this type. It is the purpose of this article to give those coordinate systems for which (*) admits a strictly R-separable solution. By this we mean those coordinate systems for which (*) admits an R-separable solution and for which there is no conformally equivalent coordinate system such that (*) is simply separable. As with the treatment of the wave equation in two space dimensions[5] we classify all different types of orthogonal coordinate systems whose coordinate curves are cyclides or their degenerate forms.

The content of the paper is arranged as follows. In Sec. I we discuss the relevant details concerning coordinate systems whose coordinate curves are cyclides of most general type. This is a development of the methods in the fundamental book by Bôcher.[6] Also in this section we give the various differential forms corresponding to the coordinate systems of interest. In Sec. II we present the various coordinate systems together with the corresponding separation equations and triplet of mutually commuting operators $\{L_1, L_2, L_3\}$ which describe each such system.

I. R-SEPARABLE DIFFERENTIAL FORMS FOR THE WAVE EQUATION

In this section we classify all the orthogonal differential forms for which the wave equation (*) admits a strictly "R-separable" separation of variables. We recall that if ψ is a solution of (*) which is R-separable in terms of some new coordinates x_i, $(i=1, 2, 3, 4)$, then ψ can be written in the form

$$\psi=\exp[Q(x_1, x_2, x_3, x_4)]\phi, \qquad (1.1)$$

where the wave equation for the function ϕ is such that ϕ admits a separation of variables. The factor $\exp Q$ is called the modulation function and has a definite form for each R-separable coordinate system. In addition no part of the function Q should contain the sum of functions of only one of the variables x_i. For a strict R-separable system the modulation function Q should not be zero. In a previous paper[5] where we treated the wave equation in two space variables it was shown that only coordinate systems whose coordinate curves were degenerate forms of confocal cyclides of the most general type were strictly R-separable. All remaining R-separable coordinate systems could be transformed into coordinate systems for which the Klein–Gordon equation $(\partial_{tt}-\Delta_2)\psi=\lambda\psi$ also admits a separation of variables. This was done by a suitable transformation of the $O(3,2)$ conformal symmetry group of $(\partial_{tt}-\Delta_2)\psi=0$. The same situation is true in the case of three spatial dimensions and it is accordingly the purpose of this section to discuss confocal families of cyclides of general type and their degenerate forms. We now briefly outline the properties of cyclides of this type and refer the reader for details to our previous paper[5] and the book by Bôcher.[6] Families of confocal cyclides have their natural setting in pentaspherical space. This is a six-dimensional space of six homogeneous coordinates $y_1:y_2:y_3:y_4:y_5:y_6$ which are not all simultaneously zero and which are connected by the relation

$$y_1^2+y_2^2+y_3^2+y_4^2+y_5^2+y_6^2=0. \qquad (1.2)$$

The space–time coordinates are related to the homogenous coordinates via the relations

$$\begin{aligned} &y_1=i(p^2-q^2-r^2-s^2+w^2),\\ &y_2=p^2-q^2-r^2-s^2-w^2, \qquad (1.3)\\ &y_3=2pw, \quad y_4=2ipw, \quad y_5=2irw, \quad y_6=2isw, \end{aligned}$$

where $t=p/w$, $x=q/w$, $y=r/w$, $z=s/w$. A cyclide is then defined as the locus of points lying on the quadric surface

$$\Phi=\sum_{ij=1}^{6}a_{ij}y_iy_j=0,$$

with $a_{ij}=a_{ji}$ and $\det(a_{ij})\neq 0$. The classification of

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cyclides under the group of orthogonal transformations which preserves the form

$$\sum_{i=1}^{6} y_i^2$$

is then the problem of classifying the intersections of two quadratic forms in six-dimensional projective space. This is performed by the method of elementary divisors applied to the two quadratic forms.

(For the details of this classification see Ref. 5 and 6.) The equation describing the most general family of confocal cyclides in six-dimensional pentaspherical space is

$$\sum_{i=1}^{6} \frac{y_i^2}{\lambda - e_i} = 0, \quad \sum_{i=1}^{6} y_i^2 = 0. \tag{1.4}$$

Here λ is one of the new curvilinear coordinates and $e_i \neq e_j$, if $i \neq j$ $(i, j = 1, \ldots, 6)$. If we choose an orthogonal coordinate system in space–time whose coordinate curves have equations of the type (1.4), then the line element in terms of these new coordinates becomes

$$ds^2 = \frac{1}{4\sigma w^2}\left[\sum_{i=1}^{4} \frac{(x_i - x_j)(x_i - x_k)(x_i - x_l)}{f(x_i)}\, dx_i^2\right], \tag{1.5}$$

where

$$f(x_i) = \prod_{j=1}^{6} (x_i - e_j) \quad \text{and} \quad -1\sqrt{\sigma} = \sum_{i=1}^{6} e_i y_i^2.$$

The pentaspherical coordinates y_i are related to the curvilinear coordinates x_i via the equations

$$y_i = \frac{\phi(e_i)}{f'(e_i)}, \quad i = 1, \ldots, 6, \tag{1.6}$$

where $\phi(\lambda) = \prod_{j=1}^{4} (\lambda - x_j)$. If we write the solution ψ of the wave equation as

$$\psi = (\sigma^{1/2} w^2)\Phi, \tag{1.7}$$

then Φ satisfies the differential equation

$$\sum_{j=1}^{4}\left[\left(\frac{1}{\phi'(x_j)}\right)\frac{\partial^2 \Phi}{\partial v_j^2} + 3x_j \Phi\right] - 2\left(\sum_{i=1}^{6} e_i\right)\Phi = 0, \tag{1.8}$$

where $2dv_j = dx_j/\sqrt{f(x_j)}$. This equation admits separable solutions for the function Φ, i.e.,

$$\Phi = \prod_{j=1}^{4} E_j(x_j).$$

Each of the functions E_j satisfies the differential equation

$$\frac{d^2 E_j}{dv_j^2} + \left[3x_j^4 - 2\left(\sum_{i=1}^{6} e_i\right)x_j^3 + Ax_j^2 + Bx_j + C\right]E_j = 0. \tag{1.9}$$

We now proceed to classify coordinate systems of this type by considering the expression inside the square brackets in (1.5) and finding out what ranges of the coordinates x_i permit this differential form to have overall negative signature. We must also consider degenerate forms of these general coordinate systems which result when some of the e_i become equal. In addition we should mention that two confocal families of cyclides of type (1.4) are equivalent under the action of real linear transformations of the pentaspherical coordinates y_i which preserve the quantity $\sum_{i=1}^{6} y_i^2$ if their parameters e_i, e_i' and coordinates x_i, x_i' are related by the equations

$$e_i = \frac{\alpha e_i' + \beta}{\gamma e_i' + \delta}, \quad x_i = \frac{\alpha x_i' + \beta}{\gamma x_i' + \delta}, \tag{1.10}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $\alpha\delta - \beta\gamma \neq 0$. We now give the classification of the strictly R-separable coordinate systems, in particular the differential forms.

[1] The first type of differential form corresponds to R-separable coordinate systems of the type (1.6) for which all the e_i are real. In addition the relations (1.10) can be used to standardize these quantities so that $e_1 = \infty$, $e_2 = a$, $e_3 = b$, $e_4 = c$, $e_5 = 1$, $e = 0$ with $a > b > c > 1$. The differential form then becomes

$$ds^2 = \left(\frac{-y_1^2}{4w^2}\right)\left[\sum_{i=1}^{6} \frac{(x_i - x_j)(x_i - x_k)(x_i - x_l)}{h(x_i)}\, dx_i^2\right], \tag{1.11}$$

where $h(x) = (x-a)(x-b)(x-c)(x-1)x$. The ranges of variation of the variables x_i are

$$\begin{aligned}
&x_1, x_2, x_3 > a > x_4 > b,\\
&\quad x_1, x_2 > a > b > x_3 > c > x_4 > 1;\\
&x_1, x_2, x_3 > a > b > x_4 > c;\\
&x_1 > a > x_2, x_3 > b > c > x_4 > 1;\\
&x_1, x_2 > a > x_3 > b > x_4 > c,\\
&\quad x_1 > a > b > x_2 > c > x_3 > x_4 > 0.
\end{aligned} \tag{1.12}$$

[2] The differential forms of this type are as in (1.11) but with

$$b = a^* = \alpha - i\beta, \quad \alpha, \beta \in \mathbb{R}.$$

The ranges of variation of the variables x_i are

$$\begin{aligned}
&x_1, x_2, x_3 > c > x_4 > 1 > 0,\\
&x_1, x_2 > c > x_3 > 1 > x_4 > 0.
\end{aligned} \tag{1.13}$$

[3] In this case the quantities e_i can be taken to be

$$e_1 = \infty, \quad e_2^* = e_3 = \gamma + i\delta, \quad e_4^* = e_5 = \alpha + i\beta,$$

$$e_6 = 0, \quad \alpha, \beta, \gamma, \delta \in \mathbb{R}.$$

The differential form is given as in (1.11) with

$$h(x) = [(x-\gamma)^2 + \delta^2][(x-\alpha)^2 + \beta^2]x.$$

The range of variation of the variables x_i are then

$$x_1, x_2, x_3 > 0 > x_4; \quad x_1 > 0 > x_2, x_3, x_4. \tag{1.14}$$

The simplest type of degenerate differential forms corresponding to cyclides of general type (1.4) are obtained by allowing pairs of the quantities e_i to become equal. This is achieved by the prescription given by Bôcher,[6] e.g., if e_1 and e_2 become equal, then they do so according to the prescription

$$e_1 = e_2 + \epsilon, \quad x_1 = e_2 + \epsilon x_1', \tag{1.15}$$

where ϵ is a first order quantity. With this substitution and the subsequent use of the relations (1.10) to take $e_1 = \infty$ the differential form becomes

$$ds^2 = \left(-\frac{(y_1^2 + y_2^2)}{4w^2}\right)\left[\frac{dx_1'^2}{x_1'(x_1' - 1)} - \sum_{i=2}^{4} \frac{(x_i - x_j)(x_i - x_k)}{h(x_i)}\, dx_i^2\right], \tag{1.16}$$

where $h(x) = (x-a)(x-b)(x-c)(x-d)$. If we make the same substitution in (1.6) relating the pentaspherical space coordinates y_i^2, we obtain

$$y_1^2 = 1 - x_1', \quad y_2^2 = x_1',$$

$$y_3^2 = \frac{(x_2-e_3)(x_3-e_3)(x_4-e_3)}{(e_3-e_4)(e_3-e_5)(e_3-e_6)},$$

$$y_4^2 = \frac{(x_2-e_4)(x_3-e_4)(x_4-e_4)}{(e_4-e_3)(e_4-e_5)(e_4-e_6)}, \tag{1.17}$$

$$y_5^2 = \frac{(x_2-e_5)(x_3-e_5)(x_4-e_5)}{(e_5-e_3)(e_5-e_4)(e_5-e_6)},$$

$$y_6^2 = \frac{(x_2-e_6)(x_3-e_6)(x_4-e_6)}{(e_6-e_3)(e_6-e_4)(e_6-e_5)}.$$

In addition we note that the coordinate curve for the coordinate x_1' has the equation

$$\frac{y_1^2}{x_1'-1} + \frac{y_2^2}{x_1'} = 0. \tag{1.18}$$

From the form of the pentaspherical coordinates in (1.6) we see that the real linear transformations which preserve the quantity $\sum_{i=1}^6 y_i^2$ form a group isomorphic to O(4,2). In fact the representation of a point in space–time by the six pentaspherical coordinates is such that the generators $L_{ij} = y_i\partial_{y_i} - y_j\partial_{y_i}$ are directly related to the canonical generators of the conformal symmetry group of the wave equation.[3] More specifically we have the relations

$$L_{12} = \tfrac{1}{2}(K_0 - P_0), \quad L_{13} = \frac{i}{2}(K_1 - P_1), \quad L_{14} = \frac{i}{2}(K_2 - P_2),$$

$$L_{15} = \frac{i}{2}(K_2 - P_3), \quad L_{16} = iD, \quad L_{23} = iN_1, \quad L_{24} = iN_2,$$

$$L_{25} = iN_3, \quad L_{26} = \frac{i}{2}(P_0 + K_0), \; L_{34} = M_3, \quad L_{35} = M_2, \tag{1.19}$$

$$L_{36} = -\tfrac{1}{2}(P_1 + K_1), \quad L_{45} = M_1, \; L_{46} = -\tfrac{1}{2}(P_2 + K_2),$$

$$L_{56} = -\tfrac{1}{2}(P_3 + K_3).$$

Here we have used the notation of Ref. 3 for the generators of the conformal symmetry group.

Taking note of these relations we see that coordinate systems of the type given by (1.17) corresponds to the diagonalization of the generator $L_{12} = y_1\partial_{y_2} - y_2\partial_{y_1}$. This generator may correspond to a rotation or a hyperbolic rotation in pentaspherical space. If it corresponds to a hyperbolic rotation we may always use an O(4,2) group motion to ensure that $L_{12} = D$. The resulting coordinate system in space–time is then equivalent to one of the radial coordinate systems discussed in Ref. 5. Accordingly in classifying differential forms of type (1.16) we need only consider those for which $0 < x_1' < 1$.

[4] If we choose $a = b$; $c = 1$, $d = 0$, then we have the possibilities

$$a > x_2 > b > x_3 > 1 > x_4 > 0.$$

$$x_2 > a > x_3, \; x_4 > b; \; x_2 > a, \; 1 > x_3, \; x_4 > 0;$$
$$x_2 > a > x_3 > b > 1 > x_4 > 0; \; b > x_2 > 1 > x_3, \; x_4 > 0.$$

$$x_2, x_3, x_4 > a; \; b > x_2, x_3, x_4 > 1, \; 0 > x_2, x_3, x_4,$$
$$x_2, x_3 > a; \; b > x_4 > 1, \; 0 > x_4$$

$$x_2 > a; \; b > x_3, x_4 > 1, \; 0 > x_3, x_4, \; b > x_3 > 1 > 0 > x_4.$$

$$b > x_2x_3 > 1 > 0 > x_4, \; b > x_2 > 1 > 0 > x_3, x_4,$$
$$a > x_2, x_3 > b; \; b > x_4 > 1, \; 0 > x_4.$$

$$a > x_2 > b > 1 > x_3 > 0 > x_4.$$

[5] If $a = b^* = \alpha + i\beta$, $\alpha, \beta \in \mathbb{R}$ and $c = 1$, $d = 0$, then we have the possibilities

$$x_2, x_3, x_4 > 1; \; x_2, x_3 > 1 > 0 > x_4,$$
$$x_2 > 1 > 0 > x_3, x_4, \quad 0 > x_2, x_3, x_4, \tag{1.21}$$
$$x_2 > 1 > x_3, x_4 > 0 \text{ and } 1 > x_2, x_3 > 0 > x_4.$$

[6] If we have $a = b^*$ as above and $c = d^* = \gamma + i\delta$, $\gamma, \delta \in \mathbb{R}$, then the variables x_2, x_3, x_4 can be any real numbers. If in addition we allow e_3 and e_4 to become equal according to the prescription of Bôcher,[4]

$$e_3 = e_4 + \epsilon, \quad x_2 = e_4 + \epsilon x_2'. \tag{1.22}$$

The differential form is then

$$ds^2 = \left(\frac{-(y_1^2 + y_2^2)}{4w^2}\right)\left[\frac{dx_1'^2}{x_1'(x_1'-1)} + \frac{(e_4-x_3)(e_4-x_4)}{(e_4-e_5)(e_4-e_6)} \times \frac{dx_2'^2}{x_2'(1-x_2')} + (x_4 - x_3)\left(\frac{dx_3^2}{P(x_3)} - \frac{dx_4^2}{P(x_4)}\right)\right], \tag{1.23}$$

where $P(x) = (x-e_4)(x-e_5)(x-e_6)$. For all such differential forms $0 < x_2' < 1$. Differential forms of this type fall into classes in which the quantities e_4, e_5, e_6 can be chosen to be 0, 1, or a.

[7] $e_4 = 0$, $e_5 = 1$, $e_6 = a$; $a > 1$. The variables x_3, x_4 vary in the ranges

$$0 < x_3 < 1 < x_4 < a, \quad 1 < x_3 < a < x_4, \quad x_3 < 0 < 1 < x_4 < a. \tag{1.24}$$

[8] $e_4 = 1$, $e_5 = 0$, $e_6 = a$; $a > 1$;

$$1 < x_3 < a < x_4, \quad x_3 < 0 < x_4 < 1, \tag{1.25}$$
$$x_3 < 0 < 1 < x_4 < a, \quad 0 < x_3 < 1 < a < x_4.$$

If we now allow e_5 and e_6 to become equal by the usual prescription, the differential form becomes, taking $e_5 = 1$ and $e_6 = 0$,

$$ds = \left(\frac{(y_1^2 + y_2^2)}{4w^2}\right)\left[\frac{dx_1'^2}{x_1'(x_1'-1)} + (1-x_4)\frac{dx_2'^2}{x_2'(1-x_2')} + x_4\frac{dx_3'^2}{x_3'(1-x_3')} + \frac{dx_4^2}{x_4(1-x_4)}\right]. \tag{1.26}$$

There is only one differential form of this type.

[9] For this case all the variables x_i' $(i = 1, 2, 3)$, x_4 lie in the interval [0, 1].

A further class of differential forms can be obtained by taking

$$e_4 = e_6 + a\epsilon, \quad e_5 = e_6 + \epsilon, \quad x_i = e_6 + \epsilon x_i', \quad i = 3, 4. \tag{1.27}$$

If we also put $e_6 = \infty$ in the resulting differential form we obtain

$$ds = \left(\frac{-(y_4^2 + y_5^2 + y_6^2)}{4w^2}\right)\left[(x_2 - x_1)\left(\frac{dx_1^2}{P(x_1)} - \frac{dx_2^2}{P(x_2)}\right) + (x_3' - x_4')\left(\frac{dx_3'^2}{Q(x_3')} - \frac{dx_4'^2}{Q(x_4')}\right)\right], \tag{1.28}$$

where $P(x)=(x-e_1)(x-e_2)(x-e_3)$ and $Q(x)=(x-a)\times(x-1)x$. This differential form corresponds to the reductions $O(4,2)\supset O(3)\otimes O(2,1)$ and $O(4,2)\supset O(2,1)\otimes O(2,1)$ when expressed in elliptic coordinates in the case of the two reductions

$$O(3)\supset L \quad \text{and} \quad O(2,1)\supset L'. \tag{1.29}$$

With the exception of the reduction $O(2,1)\supset O(1,1)$ which can be conformally transformed into a radial system we can in principle write down all the differential forms corresponding to the reductions of the type $O(4,2)\supset O(3)\otimes O(2,1)$ and $O(4,2)\supset O(2,1)\otimes O(2,1)$ by considering degenerate forms of the differential form (1.29), but we do not do this here.

The remaining distinct type of differential form of interest in this section is obtained by taking $x_2=e_6+\epsilon' x_2'$ and $e_3=e_6+\epsilon'$ subsequent to the substitutions (1.27) and then allowing $e_6\to\infty$. We then obtain the differential form

$$ds^2=\left(\frac{(y_3^2+y_4^2+y_5^2+y_6^2)}{4w^2}\right)\left[\frac{dx_1^2}{x_1(1-x_1)} + \frac{dx_2'^2}{x_2'(x_2'-1)}+x_2'(x_3'-x_4')\left(\frac{dx_3'^2}{Q(x_3')}-\frac{dx_4'^2}{Q(x_4')}\right)\right]. \tag{1.30}$$

[10] In each class we have that $0<x_1<1$, $0<x_2'<1$. The remaining variables vary in the ranges

$$0<x_3'<1<x_4'<a,\quad 1<x_3'<a<x_4',$$

$$x_3'<0<1<a<x_4',\quad x_3'<0<x_4'<1.$$

A further differential form can be obtained from (1.29) by taking $a=1+\epsilon''$, $x_3'=1+\epsilon'' x_3''$. This gives one new differential form,

$$ds^2=\left(\frac{-(y_1^2+y_2^2+y_3^2+y_4^2)}{4w^2}\right)\left[\frac{dx_1^2}{x_1(1-x_1)}+\frac{dx_2'^2}{x_2'(x_2'-1)} + x_2'\left((1-x_4')\frac{dx_3''^2}{x_3''(x_3''-1)}+\frac{dx_4'^2}{x_4'(x_4'-1)}\right)\right], \tag{1.31}$$

where all the variables lie between 0 and 1.

[11] This gives one additional different form.

We have thus shown in this section how to get all the orthogonal coordinate systems we expect by various limiting procedures applied to coordinate systems of most general cyclidic type. We have as yet not fully understood in what sense these procedures are complete.

II. R-SEPARABLE COORDINATES FOR THE WAVE EQUATION

In this section we give the coordinate systems corresponding to the differential forms in Sec. I together with the separation equations. We also present the triplet L_1, L_2, L_3 of mutually commuting second order symmetric operators in the enveloping algebra of $O(4,2)$ whose eigenvalues are the separation constants for each coordinate system presented. We now tabulate the coordinate systems of interest starting with the most general real cyclidic type of coordinates.

A. Coordinate systems of class I

(1)–(5)

(a) A suitable choice of coordinates is

$$\begin{aligned} t&=\frac{1}{R}\left[-\frac{(x_1-a)(x_2-a)(x_3-a)(x_4-a)}{(a-b)(a-c)(a-1)a}\right]^{1/2},\\ x&=\frac{1}{R}\left[\frac{(x_1-b)(x_2-b)(x_3-b)(x_4-b)}{(b-a)(b-c)(b-1)b}\right]^{1/2},\\ y&=\frac{1}{R}\left[\frac{(x_1-c)(x_2-c)(x_3-c)(x_4-c)}{(c-a)(c-b)(c-1)c}\right]^{1/2},\\ z&=\frac{1}{R}\left[\frac{(x_1-1)(x_2-1)(x_3-1)(x_4-1)}{(1-a)(1-b)(1-c)}\right]^{1/2}, \end{aligned} \tag{2.1}$$

where

$$R=\left(1+\left[\frac{x_1x_2x_3x_4}{abc}\right]^{1/2}\right).$$

The solution of the wave equation then assumes the form $\psi=R\Phi$ where $\Phi=\Pi_{i=1}^4 E_i(x_i)$ typically. The separation equations for the functions E_i are

$$\frac{d^2E_j}{dx_j^2}+\frac{1}{2}\left(\frac{1}{x_j-a}+\frac{1}{x_j-b}+\frac{1}{x_j-c}+\frac{1}{x_j-1}+\frac{1}{x_j}\right)\frac{dE_j}{dx_j} + \frac{(-2x_j^3+l_1x_j^2+l_2x_j+l_3)}{4(x_j-a)(x_j-b)(x_j-c)(x_j-1)x_j}E_j=0. \tag{2.2}$$

The operators whose eigenvalues are the separation constants are

$$\begin{aligned} L_1&=\tfrac14(a+b+c)(P_3+K_3)^2+\tfrac14(a+b+1)(P_2+K_2)^2 \\ &\quad+\tfrac14(a+c+1)(P_1+K_1)^2-\tfrac14(b+c+1)(P_0+K_0)^2 \\ &\quad+(a+b)M_1^2+(a+c)M_2^2-(b+c)N_3^2 \\ &\quad-(c+1)N_1^2-(b+1)N_2^2+(a+1)M_3^2,\\ L_2&=\tfrac14(ac+bc+ab)(P_3+K_3)^2 \\ &\quad+\tfrac14(ab+a+b)(P_2+K_2)^2+\tfrac14(ac+a+c)(P_1+K_1)^2 \\ &\quad-\tfrac14(bc+b+c)(P_0+K_0)^2+abM_1^2 \\ &\quad+acM_2^2-bcN_3^2-cN_1^2-bN_2^2 \\ &\quad+aM_3^2,\\ L_3&=-\tfrac14abc(P_3+K_3)^2-\tfrac14ab(P_2+K_2)^2 \\ &\quad-\tfrac14ac(P_1+K_1)^2+\tfrac14bc(P_0+K_0)^2. \end{aligned} \tag{2.3}$$

The coordinates x_i vary in the ranges

$$x_1>a>b>x_2>c>x_3>1>x_4>0.$$

There are four more coordinate systems of this type. We list below the complex transformation of the space time coordinates which relates the coordinates of type (a) to the new system together with the new ranges of variation of the coordinates x_i. The separation equations for the $E_j(x_j)$ are the same in each case and the basis defining operators can be obtained by the substitution given. We now list the possibilities.

(b) $(t,x,y,z)\to(iz,x,y,it)$

$$x_1,x_2>a>x_3>b>x_4>c,$$

(c) $(t,x,y,z)\to(x,t,iy,iz)$

$$x_1>a>x_2,\ x_3>b>c>x_4>1,$$

$$x_1,x_2,x_3>a>b>c>x_4>1.$$

(d) $(t, x, z) \to (it, ix, iy, iz)$

$x_1, x_2 > a > b > x_3 > c > x_4 > 1.$

(e) $(t, x, y, z) \to (t, ix, y, iz)$

$x_1, x_2, x_3 > a > x_4 > b.$

(6)—(7)

(a) A suitable choice of coordinates is

$$t + ix = \frac{1}{R}\left[\frac{2(x_1-a)(x_2-a)(x_3-a)(x_4-a)}{(a-b)(a-c)(a-1)a}\right]^{1/2},$$
$$y = \frac{1}{R}\left[\frac{(x_1-c)(x_2-c)(x_3-c)(x_4-c)}{(c-a)(c-b)(c-1)c}\right]^{1/2}, \qquad (2.4)$$
$$z = \frac{1}{R}\left[\frac{(x_1-1)(x_2-1)(x_3-1)(x_4-1)}{(1-a)(1-b)(1-c)}\right]^{1/2}.$$

where

$$R = \left(1 + \left[\frac{x_1x_2x_3x_4}{abc}\right]^{1/2}\right)$$

and $a = b^* = \alpha + i\beta;\ \alpha, \beta \in \mathbb{R}$.

The solution of the wave equation has the form $\psi = R\Phi$ where each of the E_j satisfy Eq. (2.2). The operators whose eigenvalues are the separation constants are

$$L = \tfrac{1}{4}(2\alpha + c)(P_3+K_3)^2 + \tfrac{1}{4}(2\alpha+1)(P_2+K_2)^2 + 2\alpha M_1^2 + \tfrac{1}{4}(\alpha+c+1)[(P_1+K_1)^2 - (P_0+K_0)^2] - \frac{\beta}{4}[(P_0+K_0)(P_1+K_1) + (P_1+K_1)(P_0+K_0)] + (\alpha+c)(M_2^2-N_3^2) + \beta(N_3M_2+M_2N_3) + (\alpha+1)(M_3^2-N_2^2) + \beta(N_2M_3+M_3N_2) - (c+1)N_2^2$$

$$L_2 = -\tfrac{1}{4}(2ac+\alpha^2+\beta^2)(P_3+K_3)^2 - \tfrac{1}{4}(2\alpha+\alpha^2+\beta^2)(P_2+K_2)^2 - (\alpha^2+\beta^2)M_1^2 + \tfrac{1}{4}(\alpha c+\alpha+c)[(P_0+K_0)^2 - (P_1+K_1)^2] + \tfrac{1}{4}\beta(c+1)[(P_1+K_1)(P_0+K_0) + (P_0+K_0)(P_1+K_1)] + \alpha c(N_3^2-M_2^2) - c\beta(M_2N_3+N_3M_2) + cN_1^2 + \alpha(N_2^2-M_3^2) - \beta(M_3N_2+N_2M_3), \qquad (2.5)$$

$$L_3 = \tfrac{1}{4}(\alpha^2+\beta^2)[c(P_3+K_3)^2 + (P_2+K_2)^2] + \frac{\alpha c}{4}[(P_0+K_0)^2 - (P_1+K_1)^2] - \frac{c\beta}{4}[(P_1+K_1)(P_0+K_0) + (P_0+K_0)(P_1+K_1)].$$

The coordinates x_i can vary in the ranges

$x_1, x_2 > c > x_3 > 1 > x_4 > 0.$

(b) $(t, x, y, z) \to (it, ix, iy, iz)$

where $x_1, x_2, x_3 > c > x_4 > 1 > 0.$

(8)

A suitable choice of coordinates is

$$t + iy = \left[\frac{2(x_1-c)(x_2-c)(x_3-c)(x_4-c)}{(c-a)(c-b)(c-d)c}^{1/2}\right]R,$$
$$x = \operatorname{Im}\left[\frac{2(x_1-a)(x_2-a)(x_3-a)(x_4-a)}{(a-b)(a-c)(a-d)a}\right]^{1/2} R, \qquad (2.6)$$
$$z = [-x_1x_2x_3x_4/abcd]^{1/2}/R,$$

where

$$R = \left\{1 + \operatorname{Re}\left[-\frac{2(x_1-a)(x_2-a)(x_3-a)(x_4-a)}{(a-b)(a-c)(a-d)a}\right]^{1/2}\right\}$$

and $a = b^* = \alpha + i\beta,\ c = d^* = \gamma + i\delta,\ \alpha, \beta, \gamma, \delta \in \mathbb{R}$.

The solution of the wave equation has the form $\psi = R\Phi$ where each of the E_j satisfies the equation

$$\frac{d^2E_j}{dx_j^2} + \frac{1}{2}\left(\frac{1}{x_j-a} + \frac{1}{x_j-b} + \frac{1}{x_j-c} + \frac{1}{x_j-d} + \frac{1}{x_j}\right)\frac{dE_j}{dx_j} + \frac{(-2x_j^3 + l_1x_j^2 + l_2x_j + l_3)}{4(x_j-a)(x_j-b)(x_j-c)(x_j-d)x_j}E_j = 0. \qquad (2.7)$$

The operators whose eigenvalues are the separation constants are

$$L_1 = (2\alpha+\gamma)(M_1^2-N_3^2) + \delta(M_1N_3+N_3M_1) + (2\gamma+\alpha)[M_2^2 - \tfrac{1}{4}(P_3-K_3)^2] + \tfrac{1}{2}\beta[M_2(P_3-K_3) + (P_3-K_3)M_2] + \tfrac{1}{2}\gamma(P_0-K_0)^2 - 2\alpha N_2^2 + (\alpha+\beta)[\tfrac{1}{4}(P_0-K_0)^2 - \tfrac{1}{4}(P_2-K_2)^2 + M_3^2 - N_1^2] + \tfrac{1}{2}\beta[N_1(P_0-K_0) + (P_0-K_0)N_1] + \tfrac{1}{4}\delta[(P_0-K_0)(P_2-K_2) + (P_2-K_2)(P_0-K_0)] - \delta(N_1M_3+M_3N_1) - \frac{\beta}{2}[M_3(P_2-K_2) + (P_2-K_2)M_3],$$

$$L_2 = (\alpha^2+\beta^2+2\alpha\gamma)(N_3^2-M_1^2) - 2\alpha\delta(M_1N_3 + N_3M_1) + (\gamma^2+\delta^2+2\alpha\gamma)[\tfrac{1}{4}(P_3-K_3)^2 - M_2^2] - \gamma\beta[M_2(P_3-K_3) + (P_3-K_3)M_2] + (\alpha^2+\beta^2)N_2^2 + \tfrac{1}{4}(\gamma^2+\delta^2)(P_1-K_1)^2 + \alpha\gamma[\tfrac{1}{4}(P_2-K_2)^2 - \tfrac{1}{4}(P_0-K_0)^2 + N_1^2 - M_3^2] - \frac{\gamma\beta}{2}[(P_0-K_0)N_1 + N_1(P_0-K_0)] - \frac{\alpha\delta}{4}[(P_0-K_0)(P_2-K_2) + (P_2-K_2)(P_0-K_0)] - \frac{\beta\delta}{2}[M_3(P_0-K_0) + (P_0-K_0)M_3] - \frac{\beta\delta}{2}[N_1(P_2-K_2) + (P_2-K_2)N_1] + \alpha\delta(N_1M_3+M_3N_1) + \frac{\gamma\beta}{2}[(P_2-K_2)M_3 + M_3(P_2-K_2)], \qquad (2.8)$$

$$L_3 = (\alpha^2+\beta^2)[\gamma(N_3^2-M_1^2) - \delta(N_3M_1+M_1N_3)] + (\gamma^2+\delta^2)\{\alpha(\tfrac{1}{4}(P_3-K_3)^2 - M_2^2) - \frac{\beta}{2}[(P_3-K_3)M_2 + M_2(P_3-K_3)]\}.$$

The variables x_i can vary in the ranges $x_1 > 0 > x_2$, x_3, x_4 and x_1, x_2, $x_3 > 0 > x_4$.

B. Coordinate systems of class II

Coordinate systems this type consist of all the coordinate systems in which the operator $\frac{1}{2}(P_0 - K_0)$ is diagonal.

As has been discussed in Ref. 7 the R-separable solutions of (*) then have the form $\psi = (Y_0 - \cos\psi) \times \exp(i(2F+1)\psi)\Phi(Y_0, Y_1, Y_2, Y_3)$ where $Y_0^2 + Y_1^2 + Y_2^2 + Y_3^2 = 1$ and the space–time coordinates are given by

$$t = \frac{\sin\psi}{Y_0 - \cos\psi}, \quad x = \frac{Y_1}{Y_0 - \cos\psi}, \quad y = \frac{Y_2}{Y_0 - \cos\psi}, \quad z = \frac{Y_3}{Y_0 - \cos\psi}, \tag{2.9}$$

$i(2F+1)$ is the eigenvalue of the operator $\frac{1}{2}(P_0 - K_0)$ and F is a positive integer or half integer. The function Φ satisfies the equation

$$(\Gamma_{12}^2 + \Gamma_{13}^2 + \Gamma_{14}^2 + \Gamma_{23}^2 + \Gamma_{24}^2 + \Gamma_{34}^2)\Phi = -4F(F+1)\Phi, \tag{2.10}$$

where $\Gamma_{12} = -\frac{1}{2}(P_1 + K_1)$, $\Gamma_{13} = -\frac{1}{2}(P_2 + K_2)$, $\Gamma_{14} = -\frac{1}{2}(P_3 + K_3)$, $\Gamma_{23} = M_3$, $M_{24} = -M_2$, and $\Gamma_{34} = M_1$. Here we are using the notation of Ref. 5. The problem of separation of variables for coordinate systems in which $\frac{1}{2}(P_0 - K_0)$ is diagonal reduces to the problem of separation of variables on the three-dimensional sphere S_3 in four space. Acting on the functions Φ the operators given above have the form

$$\Gamma_{12} = Y_0\partial_1 - Y_1\partial_0, \quad \Gamma_{13} = Y_0\partial_2 - Y_2\partial_0, \quad \Gamma_{14} = Y_0\partial_3 - Y_3\partial_0, \quad \Gamma_{23} = Y_1\partial_2 - Y_2\partial_1, \quad \Gamma_{24} = Y_1\partial_3 - Y_3\partial_1, \quad \Gamma_{34} = Y_2\partial_3 - Y_3\partial_2. \tag{2.11}$$

This problem has been solved by Olevski[7,8] and the six coordinate systems on S_3 for which (2.10) admits separation of variables have recently been investigated.[7] In the interests of a complete presentation we give here the six coordinate systems mentioned, the separation equations, the operators describing the separation, and some comment on the actual solutions is also made where possible.

(9) Ellipsoidal coordinates

A suitable choice of coordinates is

$$Y_0^2 = -\frac{(x_1 - a)(x_2 - a)(x_3 - a)}{(b-a)(1-a)a}, \quad Y_1^2 = -\frac{(x_1 - b)(x_2 - b)(x_3 - b)}{(a-b)(1-b)b}, \quad Y_2^2 = -\frac{(x_1 - 1)(x_2 - 1)(x_3 - 1)}{(a-1)(b-1)}, \quad Y_3^2 = \frac{x_1 x_2 x_3}{ab}, \tag{2.12}$$

where $0 < x_3 < 1 < x_2 < b < x_1 < a$.

The separation equations for $\Phi = E_1(x_1)E_2(x_2)E_3(x_3)$ have the form

$$\frac{dE_i}{dx_i} + \frac{1}{2}\left[\frac{1}{x_i - a} + \frac{1}{x_i - b} + \frac{1}{x_i - 1} + \frac{1}{x_i}\right]\frac{dE_i}{dx_i} + \frac{[4F(F+1)x_i^2 + l_1 x_i + l_2]}{4(x_i - a)(x_i - b)(x_i - 1)x_i} E_i = 0. \tag{2.13}$$

The operators whose eigenvalues are the separation constants l_1 and l_2 are

$$L_1 = +\tfrac{1}{4}(P_1 + K_1)^2 + \frac{b}{4}(P_2 + K_2)^2 + \tfrac{1}{4}(b+1)(P_3 + K_3)^2 + aM_3^2 + (a+1)M_2^2 - (a+b)M_1^2, \quad L_2 = \tfrac{1}{4}b(P_3 + K_3)^2 - aM_2^2 - abM_1^2, \tag{2.14}$$

(10) Elliptic cylindrical coordinates of type I

A suitable choice of coordinates is

$$Y_0 = \left(\frac{x_1 x_2}{a}\right)^{1/2}\cos\phi, \quad Y_1 = \left(\frac{x_1 x_2}{a}\right)^{1/2}\sin\phi, \quad Y_2 = \left(\frac{(x_1 - a)(x_2 - a)}{a(a-1)}\right), \quad Y_3 = \left(\frac{(x_1 - 1)(x_2 - 1)}{(1-a)}\right)^{1/2}, \tag{2.15}$$

where $0 < x_1 < 1 < x_2 < a$.

The separation equations have the form for $\Phi = E_1(x_1)E_2(x_2)A(\phi)$,

$$\frac{d^2E_i}{dx_i^2} + \frac{1}{2}\left[\frac{1}{x_i - a} + \frac{1}{x_i - 1} + \frac{2}{x_i}\right]\frac{dE_i}{dx_i} + \frac{[4F(F+1)x_i^2 + l_1 x_i + l_2]}{(x_i - a)(x_1 - 1)x_i^2} E_i = 0, \tag{2.16}$$

where $i = 1, 2$,

$$a\frac{d^2A}{d\phi^2} + l_2 A = 0.$$

The operators whose eigenvalues are the separation constants l_1 and l_2 are

$$L_1 = +M_3^2 + \tfrac{1}{4}(P_2 + K_2)^2 + a[M_2^2 + \tfrac{1}{4}(P_3 + K_3)^2] + \tfrac{1}{4}(a+1)(P_1 + K_1)^2,$$

$$L_2 = -\frac{a}{4}(P_1 + K_1)^2.$$

An alternative choice of coordinates is obtained by taking $x_1 = \mathrm{sn}^2(\rho_1, k)$ and $x_2 = (1/k^2)\,\mathrm{dn}^2(\rho_2, k')$ where $a = 1/k^2$. We then have that

$$y_0 = \mathrm{sn}\rho_1\, dn\rho_2 \cos\phi, \quad y_1 = \mathrm{sn}\rho_1\, dn\rho_2 \sin\phi, \quad y_2 = dn\rho_1\, \mathrm{sn}\rho_2, \quad y_3 = \mathrm{cn}\rho_1\, \mathrm{cn}\rho_2, \tag{2.18}$$

where $0 \le \rho_1 < 2K$ and $-K' < \rho_2 < K'$. [Note: $\mathrm{sn}(z,k)$ is a Jacobi elliptic function.] In terms of these coordinates the solution for Φ has the form

$$\Phi = (\mathrm{sn}\rho_1\, \mathrm{dn}\rho_2)^m K_{Fn}^{Ps}(\mathrm{dn}\rho_2) K_{Fn}^{Ps}(k\, \mathrm{sn}\rho_1) \begin{cases} \cos m\phi, \\ \sin m\phi. \end{cases} \tag{2.19}$$

Here $K_{Fn}^{Ps}(z)$ is an associated Lamé polynomial as defined in Ref. 7.

(11) Elliptic cylindrical coordinates of type II

A suitable choice of coordinates is

$$Y_0=\left(\frac{(x_1-1)(x_2-1)}{(1-a)}\right)^{1/2}\cos\phi,$$
$$Y_1=\left(\frac{(x_1-1)(x_2-1)}{(1-a)}\right)^{1/2}\sin\phi, \tag{2.20}$$
$$Y_2=\left(\frac{x_1x_2}{a}\right)^{1/2}, \quad Y_3=\left(\frac{(x_1-a)(x_2-a)}{a(a-1)}\right)^{1/2},$$

where $0<x_1<1<x_2<a$. The separation equations have the form for $\Phi=E_1(x_1)E_2(x_2)A(\phi)$,

$$\frac{d^2E_i}{dx_i^2}+\frac{1}{2}\left[\frac{1}{x_i-a}+\frac{2}{x_i-1}+\frac{1}{x_i}\right]\frac{dE_i}{dx_i}$$
$$+\frac{[4F(F+1)x_i^2+l_1x_i+l_2]}{4(x_i-a)(x_i-1)2x_i}E_i=0, \tag{2.21}$$

where $i=1,2$,

$$(a-1)\frac{d^2A}{d\phi^2}+l_2A=0.$$

The operators whose eigenvalues are the separation constants l_1 and l_2 are

$$L_1=M_1^2+\frac{(a-1)}{4}(P_1+K_1)^2+a[M_3^2+\tfrac{1}{4}(P_2+K_2)^2],$$
$$L_2=\frac{(1-a)}{4}(P_1+K_1)^2. \tag{2.22}$$

These coordinates can also be written in terms of Jacobian elliptic functions by the same substitution as used for system (10). We then obtain

$$Y_0=\text{cn}\rho_1\,\text{cn}\rho_2\cos\phi, \quad Y_1=\text{cn}\rho_1\,\text{cn}\rho_2\sin\phi,$$
$$Y_3=\text{sn}\rho_1\,\text{dn}\rho_2, \quad Y_4=\text{dn}\rho_1\,\text{sn}\rho_2. \tag{2.23}$$

In terms of these coordinates the solution for Φ has the form

$$\Phi=(\text{cn}\rho_1\,\text{cn}\rho_2)^m K_{Fn}^{Ps}\left(-\frac{ik'}{k}\,\text{cn}\rho_2\right)K_{Fn}^{Ps}(\text{cn}\rho_1)\begin{bmatrix}\cos m\phi,\\ \sin m\phi.\end{bmatrix} \tag{2.24}$$

(12) Spheroelliptic coordinates

A suitable choice of coordinates is

$$Y_0=\sin\alpha\left(\frac{x_1x_2}{a}\right)^{1/2}, \quad Y_1=\sin\alpha\left(\frac{(x_1-1)(x_2-1)}{(1-a)}\right)^{1/2},$$
$$Y_2=\sin\alpha\left(\frac{(x_1-a)(x_2-a)}{a(a-1)}\right)^{1/2}, \quad Y_3=\cos\alpha, \tag{2.25}$$

where $0<x_1<1<x_2<a$, $0<\alpha<\pi$.

The coordinate system can also be written in terms of elliptic functions as with coordinate systems (10) and (11). This gives the parametrization,

$$Y_0=\sin\alpha\,\text{sn}\rho_1\,\text{dn}\rho_2, \quad Y_1=\sin\alpha\,\text{cn}\rho_1\,\text{cn}\rho_2,$$
$$Y_2=\sin\alpha\,\text{dn}\rho_1\,\text{sn}\rho_2, \quad Y_3=\cos\alpha. \tag{2.26}$$

A typical solution for Φ is of the form $A(\alpha)E_1(\rho_1)E_2(\rho_2)$ where

$$E_1(\rho_1)E_2(\rho_2)=F_{ln}^{pq}(-i\rho_1+iK+K',\rho_2) \tag{2.27}$$

a product of Lamé polynomials and

$$A(\alpha)=(\sin\alpha)^l C_{2F-l}^{l+1}(\cos\alpha).$$

[Here $C_\mu^\nu(z)$ is a Gegebauer polynomial.] The two operators characterizing this system are

$$L_1=\tfrac{1}{4}(P_1+K_1)^2+\tfrac{1}{4}(P_2+K_2)^2+M_3^2,$$
$$L_2=\tfrac{1}{4}(P_1+K_1)^2+\frac{a}{4}(P_2+K_2)^2, \tag{2.28}$$

with eigenvalues $-l(l+1)$ and λ_n^{pq} respectively.

(13) Spherical coordinates

A suitable choice of coordinates is

$$Y_0=\sin\alpha\sin\beta\cos\phi, \quad Y_1=\sin\alpha\sin\beta\sin\phi,$$
$$Y_2=\sin\alpha\cos\beta, \quad Y_3=\cos\alpha, \tag{2.29}$$

where $0\le\alpha,\ \beta\le\pi$, $0\le\phi<2\pi$.

A typical solution for Φ of the form $A(\alpha)B(\beta)C(\phi)$ is

$$\Phi=(\sin\alpha)^l C_{2F-l}^{l+1}(\cos\alpha)P_l^m(\cos\beta)\exp(im\phi). \tag{2.30}$$

The two operators characterizing this system are

$$L_1=\tfrac{1}{4}(P_1+K_1)^2+\tfrac{1}{4}(P_2+K_2)^2+M_3^2$$

and

$$L_2=\tfrac{1}{4}(P_2+K_1)^2, \tag{2.31}$$

with eigenvalues $-l(l+1)$ and $-m^2$ respectively.

(14) Cylindrical coordinates

A suitable choice of coordinates is

$$Y_0=\sin\alpha\cos\beta, \quad Y_2=\sin\alpha\sin\beta,$$
$$Y_3=\cos\alpha\cos\phi, \quad Y_3=\cos\alpha\sin\phi, \tag{2.32}$$

where $0<\alpha<\pi$ and $0<\beta,\ \phi<2\pi$.

A typical solution for $\Phi=A(\alpha)B(\beta)C(\phi)$ is

$$\Phi=\exp[im\phi+ip\beta](\sin\alpha)^{a+b}(\cos\alpha)^{2F-a-b}$$
$$\times{}_2F_1(b-F,\ a-F,\ a+b+1;\ -\tan^2\alpha), \tag{2.33}$$

where $m=a+b$, $p=a-b$. The two operators characterizing this system are

$$L_1=\tfrac{1}{4}(P_1+K_1)^2 \quad\text{and}\quad L_2=M_1^2 \tag{2.34}$$

with eigenvalues $-p^2$ and $-m^2$ respectively.

C. Coordinate systems of class III

These are the analogs of the elliptical coordinates of type (9). The difference in this case is that coordinate systems of this type correspond to the diagonalization of M_3^2 rather than $\frac{1}{4}(P_0-K_0)^2$. We now list the possible types of coordinates.

(15a) A suitable choice of coordinates is

$$t=\frac{1}{R}\left(\frac{(x_2-a)(x_3-a)(x_4-a)}{(b-a)(a-1)a}\right)^{1/2},$$
$$x=\frac{1}{R}\cos\phi, \quad y=\frac{1}{R}\sin\phi, \tag{2.35}$$
$$z=\frac{1}{R}\left(\frac{(x_2-b)(x_3-b)(x_4-b)}{(b-a)(b-1)b}\right)^{1/2},$$

where

$$R=\left[\left(\frac{(x_2-1)(x_3-1)(x_4-1)}{(a-1)(b-1)}\right)^{1/2}+\left(\frac{x_2x_3x_4}{ab}\right)^{1/2}\right].$$

The typical solution of the wave equation is $\psi=R\Phi$ where $\theta=E_2(x_2)E_3(x_3)E(x_4)A(\phi)$. The separation equations are the same as for system (9) with $A(\phi)=\exp(i(2F+1)\phi)$. The variables x_2, x_3, x_4 vary in the ranges $x_2, x_3 > a > b > x_4 > 1$,

$$b > x_2 > 1 > x_3, x_4 > 0, \quad b > x_2, x_3, x_4 > 1,$$

$$b > x_2 > 1 > 0 > x_3, x_4, \quad a > x_2, x_3 > b > x_4 > 1.$$

The operators whose eigenvalues are the separation constants are

$$L_1 = (a+b)D^2 - \tfrac{1}{4}(a+1)(P_3-K_3)^2 + \tfrac{1}{4}(b+1)(P_0-K_0)^2 + \frac{a}{4}(P_3+K_3)^2 - \tfrac{1}{4}b(P_0+K_0)^2 - N_3^2, \tag{2.36}$$

$$L_2 = abD^2 + \tfrac{1}{4}a(P_3-K_3)^2 + \tfrac{1}{4}b(P_0-K_0)^2,$$

and of course $L_3 = M_3^2$.

There are five further coordinate systems of this type. In each case we choose the x and y coordinates to be of the form

$$x = \frac{1}{R}\cos\phi, \quad y = \frac{1}{R}\sin\phi, \quad \text{and the operator } L_3 = M_3^2.$$

The separation equations are the same as in system (9). For each of these five further coordinate systems we give the choice of R and the coordinates t and z together with the form of the operators L_1 and L_2.

(16b) The modulation function R is

$$R = \left[\left(\frac{(x_2-1)(x_3-1)(x_4-1)}{(a-1)(b-1)}\right)^{1/2} + \left(\frac{(x_2-b)(x_3-b)(x_4-b)}{(a-b)(b-1)b}\right)\right]^{1/2} \tag{2.37}$$

and the coordinates t and z are given by

$$t = \frac{1}{R}\left(\frac{x_2x_3x_4}{ab}\right)^{1/2}, \quad z = \frac{1}{R}\left(\frac{(x_2-a)(x_3-a)(x_4-a)}{(a-b)(a-1)a}\right)^{1/2}. \tag{2.38}$$

The operators L_1 and L_2 are

$$L_1 = \tfrac{1}{4}(a+b)(P_0+K_0)^2 - \tfrac{1}{4}(a+1)(P_0-K_0)^2 + (b+1)N_3^2 + aD^2 - \tfrac{1}{4}b(P_3+K_3)^2 + \tfrac{1}{4}(P_3-K_3)^2, \tag{2.39}$$

$$L_2 = -\tfrac{1}{4}ab(P_0+K_0)^2 + \tfrac{1}{4}a(P_0-K_0)^2 - bN_3^2.$$

The ranges of variation of the coordinates x_2, x_3, and x_4 are

$$x_2 > a > x_3, x_4 > b, \quad x_2 > a > b > x_3, x_4 > 1;$$

$$x_2, x_3, x_4 > a, \quad b > x_2, x_3, x_4 > 1, \quad a > x_2, x_3 > b > x_4 > 1,$$

and

$$x_2 > a > b > 1 > 0 > x_3, x_4.$$

(17c) This coordinate system is related to (16b) via the transformation $(t, x, y, z) \to (it, ix, iy, iz)$ of the space–time coordinates. The variables x_2, x_3, x_4 vary in the ranges

$$x_2 > a > x_3 > b > 1 > x_4 > 0 \quad \text{and} \quad x_2 > a > b > 1 > x_3x_4 > 0.$$

(18d) This coordinate system is related to (16b) via the transformation $(t, x, y, z) \to (z, it, iy, t)$ of the space–time coordinates. The variables x_2, x_3, and x_4 vary in the ranges $x_2, x_3 > a > b > 1 > 0 > x_4, b > x_2, x_3 > 1 > 0 > x_4$ and $a > x_2, x_3 > b > 1 > 0 > x_4$.

(19e) This coordinate system is related the (15a) via the transformation $(t, x, y, z) \to (z, ix, iy, t)$ of the space–time coordinates. The variables x_2, x_3, and x_4 vary in the ranges $x_2 > a > b > x_3$, $x_4 > 1$.

(20f) This coordinate system is related to (16b) via the transformation $(t, x, y, z) \to (iz, x, y, it)$ of the space–time coordinates. The variables x_2, x_3, and x_4 vary in the ranges $a > x_2 > b > 1 > x_3 > 0 > x_4$.

In addition to the six types of coordinate systems we have discussed in class III we will also include coordinate systems corresponding to the differential form of type (1.16).

(21) A suitable choice of coordinates is

$$(z+it) = \frac{1}{R}\left[\frac{2(x_2-a)(x_3-a)(x_4-a)}{(a-b)(a-1)a}\right]^{1/2}, \quad x = \frac{1}{R}\cos\phi, \quad y = \frac{1}{R}\sin\phi, \tag{2.40}$$

where

$$R = \left[\left(\frac{(x_2-1)(x_3-1)(x_4-1)}{(a-1)(b-1)}\right)^{1/2} + \left(\frac{x_2x_3x_4}{ab}\right)^{1/2}\right]. \tag{2.41}$$

The separation equations are given by (2.13). The operators whose eigenvalues are l_1 and l_2 are

$$L_1 = 2\alpha D^2 + \tfrac{1}{4}(\alpha+1)[(P_3-K_3)^2 - (P_0-K_0)^2] - \frac{\beta}{2}(P_0P_3 + K_0K_3) + \tfrac{1}{4}\alpha[(P_3+K_3)^2 - (P_0+K_0)^2] - N_3^2,$$

$$L_2 = (\alpha^2+\beta^2)D^2 + \tfrac{1}{4}\alpha[(P_3-K_3)^2 - (P_0-K_0)^2] + \tfrac{1}{4}\beta[(P_3-K_3)(P_0-K_0) + (P_0-K_0)(P_3-K_3)]. \tag{2.42}$$

The variables x_2, x_3, and x_4 vary in the ranges

$$x_2, x_3, x_4 > 1, \quad x_2 > 1 > x_3, x_4 > 0, \quad x_2 > 1 > 0 > x_3, x_4.$$

(22) Coordinate systems of this type can be obtained from those of type (21) via the transformation $(t, x, y, z) \to (it, ix, iy, iz)$. The variables x_2, x_3, and x_4 lie in the ranges $x_2, x_3 > 1 > 0 > x_4$, $0 > x_2, x_3, x_4$, and $1 < x_2, x_3 > 0 > x_4$.

(23) A suitable choice of coordinates is

$$(z+it) = \frac{1}{R}\left[\frac{2(x_2-a)(x_3-a)(x_4-a)}{(a-b)(a-c)(a-d)}\right]^{1/2}, \quad x = \frac{1}{R}\cos\phi, \quad y = \frac{1}{R}\sin\phi, \tag{2.43}$$

here $R = \mathrm{Re}\,\omega - \mathrm{Im}\,\omega$

and

$$\omega = \left[\frac{2(x_2-c)(x_3-c)(x_4-c)}{(c-a)(c-b)(c-d)}\right]^{1/2}.$$

The separation equations in the variables x_2, x_3 and x_4 are

$$\frac{d^2E_i}{dx_i^2} + \frac{1}{2}\left[\frac{1}{x_i-a} + \frac{1}{x_i-b} + \frac{1}{x_i-c} + \frac{1}{x_i-d}\right]\frac{dE_i}{dx_i} + \frac{[4F(F+1)x_i^2 + l_1x_i + l_2]}{4(x_i-a)(x_i-b)(x_i-c)(x_i-d)}E_i = 0. \tag{2.44}$$

The operators whose eigenvalues are l_1 and l_2 are

$$L_1 = -2\alpha D^2 - 2\gamma N_3^2 + \tfrac{1}{2}(\alpha+\gamma)[P_3K_3 + K_3P_3 - P_0K_0 - K_0P_0] + \tfrac{1}{2}\delta[P_0^2 - P_3^2 + K_3^2 - K_0^2] - \tfrac{1}{2}\beta[P_0K_3 + K_3P_0 + P_3K_0 + K_0P_3],$$

$$L_2 = (\alpha^2+\beta^2)D^2 + (\gamma^2+\delta^2)N_3^2 + \tfrac{1}{2}\alpha\gamma[P_3K_3 + K_3P_3 - P_0K_0 - K_0P_0] + \tfrac{1}{2}\alpha\delta[P_0^2 - P_3^2 + K_3^2 - K_0^2] + \beta\delta(P_0P_3 - K_0K_3) - \tfrac{1}{2}\beta\gamma[P_3K_0 + K_0P_3 + P_0K_3 + K_3P_0];$$

the variables x_2, x_3, and x_4 can assume any real values.

(24) A suitable choice of coordinates is

$$t+z = \frac{2}{R}\,\mathrm{Im}\left[\frac{(x_1-a)(x_2-a)(x_3-a)}{(a-b)^2}\right]^{1/2},$$

$$t-z = \frac{1}{R}\,\mathrm{Im}\left[\frac{1}{(a-b)} - \frac{1}{2}\left\{\frac{1}{x_1-a} + \frac{1}{x_2-a} + \frac{1}{x_3-a}\right\}\right],$$

$$x = \frac{1}{R}\cos\phi, \quad y = \frac{1}{R}\sin\phi,$$

where

$$R = 2\,\mathrm{Re}\left[\frac{(x_1-a)(x_2-a)(x_3-a)}{(a-b)^2}\right]^{1/2}$$

The separation equations in the variables x_2, x_3 and x_4 are

$$\frac{d^2E_i}{dx_i} + \left[\frac{1}{x_i-a} + \frac{1}{x_i-b}\right]\frac{dE_i}{dx_i} + \frac{[4F(F+1)x_i^2 + l_1x_i + l_2]}{4(x_i-a)^2(x_i-b)^2}E_i = 0. \qquad (2.47)$$

The operators whose eigenvalues are l_1 and l_2 are

$$L_1 = \alpha[\tfrac{1}{4}(P_3 - P_0 - K_3 - K_0)^2 - (D+N_1)^2] + \tfrac{1}{2}\beta[(P_3 - P_0 - K_3 - K_0)(D+N_1) + (D+N_1)(P_3 - P_0 - K_3 - K_0)] + \alpha[\tfrac{1}{4}(P_0 + P_3 + K_3 - K_0)^2 - \tfrac{1}{4}(P_0 - P_3 + K_0 + K_3)^2 - (N_1 - D)^2 + \tfrac{1}{4}(P_0 + P_3 + K_0 - K_3)^2] - \tfrac{1}{4}[(P_0 + K_3)(P_3 + P_0 + K_0 - K_3) + (P_3 + P_0 + K_0 - K_3)(P_0 + K_3)], \qquad (2.48)$$

$$L_2 = -\tfrac{1}{4}(P_0+K_3)^2 + \tfrac{1}{2}(\alpha^2+\beta^2)[\tfrac{1}{4}(P_0 + P_3 + K_3 - K_0)^2 - \tfrac{1}{4}(P_0 - P_3 + K_0 + K_3)^2 - (N_1 - D)^2 - \tfrac{1}{4}(P_0 + P_3 + K_0 - K_3)^2] + \tfrac{1}{2}(\alpha^2-\beta^2)[\tfrac{1}{4}(P_3 - P_0 - K_0 - K_3)^2 - (D+N_1)^2] - \frac{\alpha\beta}{4}[(P_3 - P_0 - K_3 - K_0)(D+N_1) + (D+N_1)(P_3 - P_0 - K_3 - K_0)] + \tfrac{1}{4}(P_0+K_3)\left[\beta(D-N_1) - \frac{\alpha}{2}(P_0 + P_3 + K_0 - K_3)\right] + \tfrac{1}{4}\left[\beta(D-N_1) - \frac{\alpha}{2}(P_0 + P_3 + K_0 - K_3)\right](P_0+K_3).$$

(25) This coordinate system is of similar type to coordinate systems (10) and (11) appearing in Class II. A suitable choice of coordinates is

$$t = \frac{1}{R}\left(\frac{(x_1-a)(x_2-a)}{a(a-1)}\right)^{1/2}, \quad x = \frac{1}{R}\cos\psi\left(\frac{(x_1-1)(x_2-1)}{(a-1)}\right)^{1/2},$$

$$y = \frac{1}{R}\cos\phi, \quad z = \frac{1}{R}\sin\phi,$$

where

$$R = \left(\frac{(x_1-1)(x_2-1)}{(a-1)}\right)^{1/2}\sin\psi + \left(\frac{x_1x_2}{a}\right)^{1/2},$$

and $x_1, x_2 < 0$, $0 < x_1, x_2 < 1$.

The solution ψ of the wave equation has the form $\psi = R\Phi$. The separation equations for $\Phi = E_1(x_1)E_2(x_2) \times A(\phi)B(\psi)$ are

$$\frac{d^2E_i}{dx_i^2} + \frac{1}{2}\left[\frac{1}{x_i-a} + \frac{2}{x_i-1} + \frac{1}{x_i}\ \frac{dE_i}{dx_i}\right] \times \frac{[4F(F+1)(x_i-1)^2 + l_1(x_i-1) + l_2]}{4(x_i-a)(x_i-1)^2x_i}E_i = 0, \qquad (2.50)$$

where $i = 1, 2$,

$$\frac{d^2A}{d\phi^2} = -(2F+1)^2A, \quad (a-1)\frac{d^2B}{d\psi^2} = l_2B.$$

The operators whose eigenvalues are the separation constants are

$$L_1 = (a-1)[D^2 + \tfrac{1}{4}(P_1 - K_1)^2] - [N_1^2 + \tfrac{1}{4}(P_0 + K_0)^2] + \frac{(a-2)}{4}(P_1+K_1)^2, \qquad (2.51a)$$

$$L_2 = \frac{(a-1)}{4}(P_1+K_1)^2, \quad L_3 = M_1^2.$$

(26) A suitable choice of coordinates is

$$t = \frac{1}{R}\left(\frac{(x_1-a)(x_2-a)}{(a-1)}\right)^{1/2}, \quad x = \frac{1}{R}\cos\psi\left(\frac{-x_1x_2}{a}\right)^{1/2},$$

$$y = \frac{1}{R}\cos\phi, \quad z = \frac{1}{R}\sin\phi, \qquad (2.51b)$$

where

$$R = \left[\left(\frac{-x_1x_2}{a}\right)^{1/2}\sin\psi + \left(\frac{(x_1-1)(x_2-1)}{(1-a)}\right)\right]^{1/2}$$

and $x_1 < 0 < 1 < x_2 < a$.

The solution ψ of the wave equation has the form $\psi = R\Phi$. The separation equations for $\Phi = E_1(x_1)E_2(x_2) \times A(\phi)B(\psi)$ are

$$\frac{d^2E_i}{dx_i^2} + \frac{1}{2}\left[\frac{1}{x_i-a} + \frac{1}{x_i-1} + \frac{2}{x_i}\right]\frac{dE_i}{dx_i} + \frac{[4F(F+1)x_i^2 + l_1x_i + l_2]}{4(x_i-a)(x_i-1)x_i^2}E_i \equiv 0 \qquad (2.52)$$

where $i = 1, 2$,

$$\frac{d^2A}{d\phi^2} = -(2F+1)^2A, \quad a\frac{d^2B}{d\psi^2} = l_2B.$$

The operators whose eigenvalues are the separation constants are

$$L_1 = -a[D^2 + \tfrac{1}{4}(P_1 - K_1)^2] - M_{01}^2 + \tfrac{1}{4}(P_2+K_2)^2 + \frac{(a+1)}{4}(P_1+K_1)^2, \qquad (2.53)$$

$$L_2 = -\frac{a}{4}(P_1+K_1)^2, \quad L_3 = M_1^2.$$

This completes the list of coordinate systems of Class III.

D. Coordinate systems of class IV

Coordinate systems of this type correspond to the two direct product reductions $SO(4,2)\supset SO(2,1)\otimes SO(2,1)$ and $SO(4,2)\supset SO(3)\otimes SO(1,2)$. In each of these cases coordinates can be chosen from the nine separable classes of orthogonal coordinates on the two sheeted and one sheeted two-dimensional hyperboloids and the two separable classes of orthogonal coordinate systems on the two-dimensional sphere. The coordinate systems on these manifolds are given in the Appendix. In classifying coordinates of this type we give the general form of the space–time coordinates in terms of the above mentioned two-dimensional manifolds.

(1) Coordinate systems corresponding to the reduction

$SO(4,2)\supset SO(3)\otimes SO(1,2)$

A suitable choice of space–time coordinates is

$$t=\frac{\xi_2}{\xi_1+\xi_3},\quad x=\frac{\zeta_1}{\xi_1+\xi_3},\quad y=\frac{\zeta_2}{\xi_1+\xi_3},\quad z=\frac{\zeta_3}{\xi_1+\xi_3},\qquad (2.54)$$

where $\xi_1^2-\xi_2^2-\xi_3^2=-1$ and $\zeta_1^2+\zeta_2^2+\zeta_3^2=1$.

With the exception of coordinate systems of type (8) (which can always be chosen such that D is diagonal) there are 16 coordinate systems of this type on the single and double sheeted hyperboloids. In each case the solution of the wave equation has the form

$$\psi=(\xi_1+\xi_3)\phi(\zeta_1,\zeta_2,\zeta_3)\theta(\xi_1,\xi_2,\xi_3)$$

where the functions ϕ and θ satisfy the equations

$$(M_1^2+M_2^2+M_3^2)\phi=-l(l+1)\phi,\qquad [\{P_0,K_0\}+D^2]\theta=l(l+1)\theta,\qquad (2.55)$$

where l is a positive integer. The operators corresponding to each of the 16 possible coordinate systems can then be read off from the Appendix, if we make the identifications $N_1=\frac{1}{2}(P_0+K_0)$, $N_2=D$, and $M_3=\frac{1}{2}(P_0-K_0)$ in the case of the $SO(1,2)$ coordinates.

(2) Coordinate systems corresponding to the reduction

$SO(4,2)\supset SO(2,1)\otimes SO(2,1)$.

A suitable choice of space–time coordinates is

$$t=\frac{\xi_1}{\xi_1+\xi_3},\quad x=\frac{\xi_2}{\xi_1+\xi_3},\quad y=\frac{\zeta_2}{\xi_1+\xi_3},\quad z=\frac{\zeta_3}{\xi_1+\xi_3},\qquad (2.56)$$

where

$$\zeta_1^2-\zeta_2^2-\zeta_3^2=\epsilon,\quad \xi_1^2-\xi_2^2-\xi_3^2=-\epsilon,\quad \epsilon=\pm 1.$$

Again with the exception of coordinate systems of type (8) there are 64 coordinate systems of this type. In each case the solution of the wave equation has the form $\psi=(\xi_1+\xi_3)\phi(\xi_1,\xi_2,\xi_3)\theta(\zeta_1,\zeta_2,\zeta_3)$, where the functions ϕ and θ satisfy the equations

$$(N_2^2+N_3^2-M_1^2)\theta=j(j+1)\theta,\qquad [-\{P_1,K_1\}+D^2]\phi=j(j+1)\phi,\qquad (2.57)$$

where

$$j=-\tfrac{1}{2}+iq,\quad 0<q<\infty.$$

The operator corresponding to the $SO(2,1)$ associated with the vector (ξ_1,ξ_2,ξ_3) can be read off from the Appendix with the identifications $N_2=\frac{1}{2}(P_1-K_1)$, $N_2=D$, and $M_3=\frac{1}{2}(P_1+K_1)$.

We have looked at four classes of coordinate systems for which the wave equation (*) is strictly R-separable and found 106 distinct such coordinate systems. This gives a total of 368 inequivalent R-separable coordinate systems for the wave equation (*).

APPENDIX

In this appendix we list the orthogonal separable coordinate systems for the two-dimensional sphere, single sheeted, and double sheeted hyperboloids. In each case we list the symmetric second order operator in the enveloping algebra of the symmetry groups of these manifolds which describes each coordinate system. The coordinates (with the exception of the single sheet hyperboloid) can be found in the article by Olevski[8] and the operator characterization is due to Winternitz *et al.*[9]

A. Coordinate systems separable on the two-dimensional sphere $\zeta_1^2+\zeta_2^2+\zeta_3^2=1$

If we write the generators $M_1=\zeta_2\partial_{\zeta_3}-\zeta_3\partial_{\zeta_2}$, $M_2=\zeta_1\partial_{\zeta_3}-\zeta_3\partial_{\zeta_1}$, and $M_3=\zeta_1\partial_{\zeta_2}-\zeta_2\partial_{\zeta_1}$, the coordinate systems and operators are:

(1) $(\zeta_1^{(1)})^2=\frac{x_1x_2}{a}$, $(\zeta_2^{(1)})^2=\frac{(x_1-1)(1-x_2)}{(a-1)}$

$(\zeta_3^{(1)})^2=\frac{(x_1-a)(x_2-a)}{a(a-1)}$ $0<x_1<1<x_2<a$.

The operator is $L=aM_2^2+M_3^2$.

(2) $\boldsymbol{\zeta}^{(2)}=(\cos x_1,\ \sin x_1\cos x_2,\ \sin x_1\sin x_2)$.

The operator is $L=M_1^2$.

B. Coordinate systems on the one and two sheeted two-dimensional hyperboloids $\xi_1^2-\xi_2^2-\xi_3^2=\pm 1$

We adopt the notation $N_1=\xi_1\partial_{\xi_2}+\xi_2\partial_{\xi_1}$, $N_2=\xi_1\partial_{\xi_3}+\xi_3\partial_{\xi_1}$, and $M_3=\xi_2\partial_{\xi_3}-\xi_3\partial_{\xi_2}$.

(1) $(\xi_1^{(1)})^2=\frac{x_1x_2}{a}$, $(\xi_2^{(1)})^2=\frac{(x_1-1)(x_2-1)}{(a-1)}$

$(\xi_3^{(1)})^2=\frac{(x_1-a)(a-x_2)}{a(a-1)}$, $1<x_1<a<x_2$,

$\boldsymbol{\xi}^{(1)}\cdot\boldsymbol{\xi}^{(1)}=(\xi_1^{(1)})^2-(\xi_2^{(1)})^2-(\xi_3^{(1)})^2=1$.

The coordinates on $\boldsymbol{\xi}\cdot\boldsymbol{\xi}=-1$ are obtained by the substitution $\xi^{(1)}\to i\xi^{(1)}$ and $x_1<0<1<x_2<a$.

The operator is $L=N_1^2+aN_2^2$.

(2) $(\xi_1^{(2)})^2=\frac{(x_1-1)(1-x_2)}{(a-1)}$, $(\xi_2^{(2)})^2=-\frac{x_1x_2}{a}$,

$(\xi_3^{(2)})^2=\frac{(x_1-a)(a-x_2)}{a(a-1)}$,

$x_1<0<1<a<x_2$, $\boldsymbol{\xi}^{(2)}\cdot\boldsymbol{\xi}^{(2)}=1$.

The coordinates on the single sheeted hyperboloid $\xi = -1$ are obtained via the substitution $\xi \to i\xi$ and $1 < x_1, x_2 < a$; $x_1, x_2 > a$.

The operator is $L = N_1^2 - aM_3^2$.

(3) $(\xi_1^{(3)} + i\xi_2^{(3)})^2 = 2(x_1 - a)(x_2 - a)/a(a-b)$,

$a = b^* = \alpha + i\beta$, $(\xi_3^{(3)})^2 = -x_1x_2/ab$,

$x_1 < 0 < x_2$, $\xi^{(3)} \cdot \xi^{(3)} = 1$.

The transformation $\xi \to i\xi$ and $x_1, x_2 > 0$.

The operator is $L = \alpha(M_3^2 - N_2^2) + \beta\{M_3, N_2\}$

(4) $\xi_1^{(4)} + \xi_2^{(4)} = \sqrt{-x_1x_2}$,

$\xi_1^{(4)} - \xi_2^{(4)} = \sqrt{-x_1/x_2} + \sqrt{-x_2/x_1} + \sqrt{-x_1/x_2}$

$\xi_3^{(4)} = \sqrt{(1-x_1)(x_2-1)}$, $x_1 < 0 < 1 < x_2$,

$\xi^{(4)} \cdot \xi^{(4)} = 1$.

The coordinates on the single sheeted hyperboloid are obtained via the substitution $\xi \to i\xi$ with $x_1, x_2 > 1$, $0 < x_1, x_2 < 1$, $x_1, x_2 < 0$.

The operator is $L = N_1^2 - (N_2 + M_3)^2$.

(5) $\xi_1^{(5)} + \xi_2^{(5)} = \sqrt{x_1x_2}$,

$\xi_1^{(5)} - \xi_2^{(5)} = -(\sqrt{x_1/x_2} + \sqrt{x_2/x_1} + \sqrt{x_1x_2})$,

$\xi_3^{(5)} = \sqrt{(x_1-1)(x_2-1)}$, $0 < x_1 < 1 < x_2$,

$\xi^{(5)} \cdot \xi^{(5)} = 1$.

The coordinates on the single sheet hyperboloid are obtained via the substitution $\xi \to i\xi$ with $x_1 < 0 < x_2 < 1$.

The operator is $L = N_1^2 + (N_2 + M_3)^2$.

(6) $\xi_1^{(6)} + \xi_2^{(6)} = \sqrt{-x_1x_2}$,

$\xi_1^{(6)} - \xi_2^{(6)} = (x_1 - x_2)/[4(-x_1x_2)^{3/2}]$,

$$\xi_3^{(6)} = \frac{1}{2}\left[\left(-\frac{x_2}{x_1}\right)^{1/2} - \left(-\frac{x_1}{x_2}\right)\right]^{1/2}, \quad x_1 < 0 < x_2,$$

$\xi^{(6)} \cdot \xi^{(6)} = 1$.

The coordinates on the single sheet hyperboloid are obtained via the substitution $\xi \to i\xi$ with $x_1, x_2 > 0$.

The operator is $L = \{N_1, N_2 - M_3\}$.

(7) $\xi_1^{(7)} + \xi_2^{(7)} = \sqrt{x_1}$, $\quad \xi_1^{(7)} - \xi_2^{(7)} = \dfrac{1}{\sqrt{x_1}} + \sqrt{x_1}x_2^2$,

$\xi_3^{(7)} = x_2\sqrt{x_1}$, $\quad x_1, x_2 > 0$,

$\xi^{(7)} \cdot \xi^{(7)} = 1$.

The coordinates on the single sheet hyperboloid are obtained via the substitution $\xi \to i\xi$ with $x_1 < 0 < x_2$.

The operator is $L = (N_2 + M_3)^2$.

(8) $\xi^{(8)} = (\cosh x_1, \cosh x_2, \cosh x_1 \sinh x_2, \sinh x_1)$,

$\xi^{(8)} \cdot \xi^{(8)} = 1$,

$\xi^{(\hat{8})} = (\sinh x_1 \cosh x_2, \sinh x_1 \sinh x_2, \cosh x_1)$

$\xi^{(\hat{8})} = (\sin x_1 \sinh x_2, \sin x_1 \cosh x_2, \cos x_1)$,

$\xi^{(\hat{8})} \cdot \xi^{(\hat{8})} = -1$.

The operator is $L = N_1^2$.

(9) $\xi^{(9)} = (\cosh x_1, \sinh x_1 \cos x_2, \sinh x_1 \sin x_2)$,

$\xi^{(9)} \cdot \xi^{(9)} = 1$,

$\hat{\xi}^{(9)} = (\sinh x_1, \cosh x_1 \cos x_2, \cosh x_1 \sin x_2)$,

$\hat{\xi}^{(9)} \quad \hat{\xi}^{(9)} = -1$.

The operator is $L = M_3^2$.

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