which is identical with Eq. (4.5).

Similarly, the cylindrical limit of the field perturbation in Φ^0 is

$$\frac{2\sqrt{2}iE_0}{\pi} \lim_{\eta\to\infty} d(\cosh\eta - \cos\tau)^{1/2} \sum_{n=-\infty}^{\infty} \frac{nP_{n-1/2}(\cosh\eta)}{P_n^0} \times \left(Q_n^0 + \frac{2\cosh\eta_0}{\Pi_n^0 q_n \sinh^2\eta_0}\right) e^{in\tau} = \frac{iE_0(\epsilon-1)}{\pi^{1/2}(\epsilon+1)} r \times \lim_{\eta\to\infty} \sum_{n=-\infty}^{\infty} \frac{n\Gamma(|n|+\frac{1}{2})\cosh^{|n|+1}\eta}{|n|!2^{|n|-1}\cosh^{2|n|}\eta_0} e^{in\theta}, \qquad (6.10)$$

by Eqs. (6.1), (6.3), (6.6), and (6.7). Hence

$$\lim_{\eta\to\infty} \Phi^0 = E_0 r \sin\theta - [E_0(\epsilon-1)/(\epsilon+1)] \times (a^2/r)\sin\theta,$$

which is exactly Eq. (4.6).

It is clear from Eqs. (6.9) and (6.10) that the only terms in the n summation that contribute to the cylindrical limit are those for which $n = -1$ and $n = 1$. Since the truncation procedure of the previous section always retains these two terms, it follows that the cylindrical limit of the truncated potentials will approach the same limit as the exact solutions.

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3 *Higher Transcendental Functions* edited by A. Erdelyi (McGraw-Hill, New York, 1953), Vol. I.
4 L. M. Milne Thomsom, *The Calculus of Finite Differences* (MacMillan, London 1960), p. 531.
5 A proof of this result is given in Ref. 4, pp. 532-34.
6 The values of P_n^0 and Q_n^0 are taken from *Tables of Associated Legendre Functions*, National Bureau of Standards (Columbia U.P., New York, 1945).

Unitary Representations of the Homogeneous Lorentz Group in an $O(1,1) \otimes O(2)$ Basis and Some Applications to Relativistic Equations

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Unitary irreducible representations of the homogeneous Lorentz group $O(3,1)$ belonging to the principal series are reduced with respect to the subgroup $O(1,1) \otimes O(2)$. As an application we determine the mixed basis matrix elements between $O(3)$ and $O(1,1) \otimes O(2)$ bases and derive recurrence relations for them. This set of functions is then used to obtain invariant expansions of solutions of the Dirac and Proca free field equations. These expansions are shown to have the correct nonrelativistic limit.

INTRODUCTION

In recent years there has been considerable interest in the unitary irreducible representations (UIR's) of the homogeneous Lorentz group in various bases.[1,2] Harmonic analysis of a scalar function in terms of the four subgroup bases [i.e., $O(3)$, $O(2,1)$, $E(2)$, and $O(1,1) \otimes O(2)$] has first been given by Smorodinski and Vilenkin.[2] Since this work most of the attention has been paid to the little group bases as these also play a role in the usual Poincaré invariant partial wave analysis[3,4] of scalar functions and helicity amplitudes. The properties of the reduction of $O(3,1)$ with respect to $O(1,1) \otimes O(2)$ are, however, not so well known. It is the purpose of this paper to develop these properties and indicate some possible uses. The content of the paper is arranged as follows. In Sec. 1 we collect the pertinent facts concerning $SL(2,C)$ [the covering group of $O(3,1)$], its Lie algebra and UIR's. In Sec. 2 we carry out the reduction of the principal series of $SL(2,C)$ with respect to $D(1,1) \otimes D(2)$ (see Sec. 2) the universal covering group of $O(1,1) \otimes O(2)$. The action of the infinitesimal generators of the Lie algebra in such a basis is also determined. In Sec. 3 we develop the expansion of a single particle helicity state in terms of mixed basis matrix elements. An explicit expression for these matrix elements is obtained for the first time. In Sec. 4 we derive recurrence relations for these mixed basis matrix elements, which are used in Sec. 5 to develop invariant expansions of solutions of the free field Proca and Dirac equations. Finally in Sec. 6 the nonrelativistic limit of these solutions is obtained.

1. RESUMÉ OF $SL(2,C)$ AND ITS UIR'S

The group $SL(2,C)$[5] is the universal covering group of the homogeneous Lorentz group $O(3,1)$. The elements of $SL(2,C)$ are the unimodular complex matrices in two dimensions

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \qquad \alpha\delta - \beta\gamma = 1. \qquad (1.1)$$

The subgroup $SU(2)$ consists of all unitary unimodular matrices of the form

$$\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \qquad |\alpha|^2 + |\beta|^2 = 1. \qquad (1.2)$$

$SU(2)$ is of course the covering group of $O(3)$ the real orthogonal group in three dimensions. The covering group of $O(1,1) \otimes O(2)$ is denoted by $D(1,1) \otimes D(2)$ and consists of all diagonal unimodular matrices:

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \qquad \alpha\beta = 1. \qquad (1.3)$$

[Note: $D(2)$ is the set of all diagonal matrices of the form

$$R(\psi) = \begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{-i\psi/2} \end{pmatrix}, \qquad 0 \le \psi \le 2\pi,$$

such that to each rotation in the plane of the group $O(2)$ there corresponds the matrices $\pm R(\psi)$. This is just the usual two to one homomorphism between an orthogonal group and its spinor group. Similar remarks apply to $D(1,1)$ the set of matrices

$$\pm\begin{pmatrix} e^{a/2} & 0 \\ 0 & e^{-a/2} \end{pmatrix}, \quad -\infty < a < +\infty \Big].$$

The Lie algebra of $SL(2, C)$ is six dimensional, being spanned by the generators $M_i, N_i (i = 1, 2, 3)$ which satisfy the commutation relations

$$[M_i, M_j] = \epsilon_{ijk} M_k, \quad [M_i, N_j] = \epsilon_{ijk} N_k, \quad [N_i, N_j] = -\epsilon_{ijk} M_k. \quad (1.4)$$

There are two independent Casimir invariants of $SL(2, C)$ which label each irreducible representation. They are

$$K_1 = \mathbf{M}^2 - \mathbf{N}^2, \quad K_2 = \mathbf{M}\cdot\mathbf{N}. \quad (1.5)$$

The Casimir invariant of $SU(2)$ is well known to be $\mathbf{M}^2$. Each inequivalent UIR of $SU(2)$ is labeled by the eigenvalue j, where

$$\mathbf{M}^2 = -j(j+1), \quad j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \cdots. \quad (1.6)$$

Each UIR for given j is $(2j+1)$-dimensional and the spectrum of M_3 in it is

$$M_3 = -j, -j+1, \ldots, j-1, j. \quad (1.7)$$

A UIR of $D(1,1) \otimes D(2)$ is labeled by the two eigenvalues of M_3 and N_3, $\{m, \tau\}$ where

$$-\infty < \tau < +\infty, \quad m = 0, \pm\tfrac{1}{2}, \pm 1, \pm\tfrac{3}{2}, \cdots. \quad (1.8)$$

It is easy to see that each such UIR is one-dimensional.

We now give the spectrum of the Casimir operators K_1, K_2 corresponding to the principal series $\{j_0, \rho\}$ of $SL(2, C)$ together with the spectrum of j values of the UIR's of $SU(2)$ that appear in each such UIR of $SL(2, C)$. For the principal series

$$K_1 = 1 + \rho^2 - j_0^2, \quad K_2 = -\rho j_0, \quad j_0 = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots, \quad -\infty < \rho < +\infty, \quad (1.9)$$

and the spectrum of j values is

$$j = j_0, j_0 + 1 \cdots.$$

The other set of UIR's of $SL(2, C)$ belong to the complementary series which we write as $\{0, i\rho\}$, where

$$K_1 = 1 - \rho^2, \quad K_2 = 0, \quad 0 < \rho < 1, \quad j_0 = 0, 1, 2, \cdots.$$

This set of UIR's does not figure in the completeness relation[5] of $SL(2, C)$ and so will not be considered subsequently.

Finally in this section we give the formulas for the action of the generators M_i, N_i on an $SU(2)$ basis of the principal series

$$M_3 | j, m\rangle = m | j, m\rangle,$$
$$M_+ | j, m\rangle = -i\alpha^j_{\lambda+1} | j, m+1\rangle,$$
$$M_- | j, m\rangle = -i\alpha^j_{\lambda} | j, m-1\rangle,$$
$$N_3 | j, m\rangle = -i\sqrt{[j^2 - m^2]}\, C_j | j-1, m\rangle + i A_j m | j, m\rangle + i C_{j+1}\sqrt{[(j+1)^2 - m^2]}\, | j+1, m\rangle, \quad (1.10)$$

$$N_+ | j, m\rangle = -i C_j \sqrt{[(j-m)(j-m-1)]}\, | j-1, m+1\rangle + i A_j \sqrt{[(j-m)(j+m+1)]}\, | j, m+1\rangle - i C_{j+1}\sqrt{[(j+m+1)(j+m+2)]}\, | j+1, m+1\rangle,$$
$$N_- | j, m\rangle = i C_j \sqrt{[(j+m)(j+m-1)]}\, | j-1, m-1\rangle + i A_j \sqrt{[(j+m)(j-m+1)]}\, | j, m-1\rangle + i C_{j+1}\sqrt{[(j-m+1)(j-m+2)]}\, | j+1, m-1\rangle,$$

where

$$A_j = \frac{-j_0 \rho}{j(j+1)}, \quad C_j = \frac{i}{j}\left(\frac{(j^2 - j_0^2)(j^2 + \rho^2)}{4j^2 - 1}\right)^{1/2},$$
$$m = -j, -j+1, \ldots, j, \quad j = j_0, j_0 + 1, \ldots,$$

and $| j, m\rangle$ is an abbreviation for $| \rho j_0; jm\rangle$:

$$\alpha^j_\lambda = \sqrt{[(j(j+1) - \lambda(\lambda - 1)]}.$$

2. REDUCTION OF THE PRINCIPAL SERIES OF $SL(2, C)$ UNDER $O(1, 1) \otimes O(2)$

As is well known[5] the principal series of $SL(2, C)$ is realized via unitary transformations in a Hilbert space H of square integrable functions in a certain domain. The elements of H are specified by functions $f(z)$ of a single complex variable z varying over the entire complex plane. (This specification is only possible up to sets of measure zero.) The scalar product and norm are given by

$$(f, h) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, \overline{f(z)} h(z), \quad z = x + iy, \quad \|f\| = (f, f)^{1/2} < \infty. \quad (2.1)$$

In the UIR $\{j_0, \rho\}$ of the principal series, the unitary operator $U(g)$ representing the group element g acts on $f(z)$ in the following way:

$$[U(g)f](z) = (\delta + \beta z)^{j_0 - 1 + i\rho} (\overline{\delta} + \overline{\beta z})^{-j_0 - 1 + i\rho} \times f[(\alpha z + \gamma)/(\beta z + \delta)] \quad (2.2)$$

This realization is not the most convenient one for our purposes. In order to realize the principal series in a $D(1,1) \otimes D(2)$ basis, we make the following transformation:

$$e^a = (x^2 + y^2)^{1/2}, \quad \tan\phi = y/x, \quad -\infty \le a \le +\infty, \quad 0 \le \phi \le 2\pi. \quad (2.3)$$

Instead of specifying an element of H by $f(z)$ we specify it by the new function

$$\tilde{f}(a, \phi) = e^{-ij_0\phi} e^{a(1 - i\rho)} f(z). \quad (2.4)$$

With this indentification the scalar product can be written

$$(f, h) = \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} da\, \overline{\tilde{f}(a, \phi)} \tilde{h}(a, \phi). \quad (2.5)$$

The generators M_i, N_i acting on the $\tilde{f}(a, \phi)$ functions can be expressed as differential operators acting on a and ϕ as

$$M_1 = j_0 \cosh a \cos\phi - (\rho + i) \sinh a \sin\phi + i \left(\sinh a \cos\phi \frac{\partial}{\partial\phi} - \cosh a \sin\phi \frac{\partial}{\partial a}\right),$$

$$M_3 = -i \frac{\partial}{\partial\phi}, \tag{2.6}$$

$$N_1 = j_0 \sinh a \sin\phi + (\rho + i) \cosh a \cos\phi + i \left(\cosh a \sin\phi \frac{\partial}{\partial\phi} + \sinh a \cos\phi \frac{\partial}{\partial a}\right),$$

$$N_3 = -i \frac{\partial}{\partial a}.$$

The operators M_2, N_2 can be obtained from the expressions for M_1 and N_1, respectively, via the substitution $\phi \to -\frac{1}{2}\pi + \phi$. The principal series of $SL(2, C)$ is now realized as the set of functions $\tilde{f}(a, \phi)$ on the domain $(-\infty, +\infty) \otimes [0, 2\pi]$ which satisfy

$$(f, f) = \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} da\, |\tilde{f}(a, \phi)|^2 < \infty. \tag{2.7}$$

The two Casimir invariants of $O(1, 1) \otimes O(2)$ are N_3 and M_3, so that the simultaneous eigenfunctions of N_3 and M_3 in this realization are

$$\Psi_{\tau m} = [1/(2\pi)] e^{i\tau a} e^{im\phi}, \tag{2.8}$$

where

$$N_3 \Psi_{\tau m} = \tau \Psi_{\tau m}, \quad M_3 \Psi_{\tau m} = m \Psi_{\tau m}, \quad (\Psi_{\tau' m'}, \Psi_{\tau m}) = \delta_{m' m} \delta(\tau' - \tau); \tag{2.9}$$

so together with the completeness relations[6]

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(a'-a)\tau}\, d\tau = \delta(a' - a) \tag{2.10a}$$

$$\frac{1}{2\pi} \sum_{p=-\infty}^{\infty} e^{ip(\phi - \varphi')} = \sum_{n=-\infty}^{\infty} \delta(\phi - \phi' - 2\pi n), \tag{2.10b}$$

we get the following result.

Each UIR $\{j_0, \rho\}$ of the principal series of $SL(2, C)$ contains each UIR $\{m, \tau\}$ of $D(1, 1) \otimes D(2)$ exactly once, provided

$$m = j_0, j_0 \pm 1, j_0 \pm 2, \cdots. \tag{2.11}$$

Thus each $\tilde{f} \in H$ can be expanded in terms of the eigenfunctions $\Psi_{\tau m}$ according to

$$\tilde{f} = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} d\tau\, f_m(\tau) \Psi_{\tau m}, \quad f_m(\tau) = \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} da\, \tilde{f} \overline{\Psi_{\tau m}}. \tag{2.12}$$

Finally in this section we calculate the action of the generators $M_\pm, N_\pm$ on the $\Psi_{\tau m}$ basis

$$M_\pm \Psi_{\tau m} = \tfrac{1}{2}(j_0 \pm i\rho \mp i\tau \mp 1 - m) \Psi_{\tau - i, m \pm 1} + \tfrac{1}{2}(j_0 \mp i\rho \mp i\tau \pm 1 + m) \Psi_{\tau + i, m \pm 1},$$

$$N_\pm \Psi_{\tau m} = \tfrac{1}{2}(\mp ij_0 + \rho - \tau + i \pm im) \Psi_{\tau - i, m \pm 1} + \tfrac{1}{2}(\pm ij_0 + \rho + \tau + i \pm im) \Psi_{\tau + i, m \pm 1}, \tag{2.13}$$

$$N_\pm = N_1 \pm iN_2, \quad M_\pm = M_1 \pm iM_2;$$

The action of M_3 and N_3 already having been given in Eq. (2.9).

3. CALCULATION OF THE MIXED BASIS MATRIX ELEMENTS AND SINGLE PARTICLE HELICITY STATES

In this section we construct relativistic functions with helicity for nonvanishing mass which are at the same time basis functions of a UIR $\{j_0, \rho\}$ of the homogeneous Lorentz group $O(3, 1)$ realized on the upper sheet of a double sheeted hyperboloid. In order to do this we use the method of Integral geometry.[2,7] In this method[7,8] a one-particle state of spin s, helicity λ, and four velocity u, denoted by $|u, s, \lambda\rangle$, is expressed in terms of a function on the light cone $\Phi_{j_0 \rho}(\xi)$ via the relation

$$|u, s, \lambda\rangle = \frac{1}{2(2\pi)^3} \sum_{j_0=-s}^{s} \int_{-\infty}^{\infty} d\rho (\rho^2 + j_0^2) \times \int_\Gamma [u, \xi]^{-1-i\rho} D^s_{\lambda j_0}(R) \Phi_{j_0 \rho}(\xi) d^2\xi, \tag{3.1}$$

where Γ is the integration path on the light cone, $d^2\xi$ the invariant measure on the cone, and $[u, \xi]$ the usual Lorentz scalar product

$$[u, \xi] = u_0 \xi_0 - \mathbf{u} \cdot \boldsymbol{\xi}. \tag{3.2}$$

The rotation specified by $D_{\lambda j_0}(R)$ is the rotation necessary to account for the requantization of the helicity component from the direction ξ to that of $\mathbf{u}$. The parametrization of the four velocity u in the coordinate system of interest (the C system or cylindrical system[2]) is

$$u = (\cosh a \cosh b, \sinh a \cos\psi, \sinh a \sin\psi, \cosh a \sinh b), \tag{3.3}$$

and the 4-vector ξ is parametrized by

$$\xi = e^c (\cosh\beta, \cos\phi, \sin\phi, \sinh\beta). \tag{3.4}$$

The choice of Γ for the C system is $\xi_0^2 - \xi_3^2 = 1$, and the consequent invariant measure is $d^2\xi = d\phi d\beta$.

In the realization on the cone the generators of the Lorentz group corresponding to a "photon" of discrete helicity λ are[9]

$$M_1 = -i(\xi, \nabla)_1 + \lambda [\xi_1/(\xi_0 + \xi_3)],$$

$$M_2 = -i(\xi, \nabla)_2 + \lambda [\xi_2/(\xi_0 + \xi_3)],$$

$$M_3 = -i(\xi, \nabla)_3 + \lambda,$$

$$N_1 = -i\xi_0 \frac{\partial}{\partial \xi_1} - \lambda \frac{\xi_2}{\xi_0 + \xi_3}, \tag{3.5}$$

$$N_2 = -i\xi_0 \frac{\partial}{\partial \xi_2} + \lambda \frac{\xi_1}{\xi_0 + \xi_3},$$

$$N_3 = -i\xi_0 \frac{\partial}{\partial \xi_3}.$$

For the parametrization (3.4) of ξ, the Casimir invariants have the form

$$\mathbf{M}^2 - \mathbf{N}^2 = \frac{d^2}{dc^2} + 2\frac{d}{dc} + \lambda^2, \quad \mathbf{M} \cdot \mathbf{N} = i\lambda\left(1 + \frac{d}{dc}\right). \tag{3.6}$$

From (3.4) and (3.6) it is not hard to show that the simultaneous eigenfunctions of $\mathbf{M}^2 - \mathbf{N}^2$, $\mathbf{M} \cdot \mathbf{N}$, M_3 and N_3 have the form

$$\mathcal{C}_{p\lambda}(\tau,\rho) = e^{-(1-i\rho)c} e^{ip\phi} e^{i\tau\beta} e^{-i\lambda\phi}, \tag{3.7}$$

in particular, on the C system contour

$$\mathcal{C}_{p\lambda}(\tau,\rho) = e^{ip\phi} e^{i\tau\beta} e^{-i\lambda\phi}. \tag{3.8}$$

The function $\Phi_{j_0\rho}(\xi)$ is now expanded in terms of the $\mathcal{C}_{p\lambda}(\tau,\rho)$ functions according to

$$\Phi_{j_0\rho}(\xi) = \sum_{p,\tau} a_p^{j_0}(\tau,\rho)\mathcal{C}_{pj_0}(\tau,\rho). \tag{3.9}$$

For evaluation of the integral over $d^2\xi$ in (3.1), it is most convenient to assume u in the form

$$u = u_0 = (\cosh a, \sinh a, 0, 0); \tag{3.10}$$

the required expansion for the more general form of u can be obtained by using the simple group properties of the $O(1,1) \otimes O(2)$ matrix elements. So combining (3.9) and (3.1) requires the calculation of the following integral:

$$I = \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} d\beta\, (\cosh a \cosh\beta - \sinh a \cos\phi)^{-1-i\rho} \times D^s_{\lambda j_0}(R)\mathcal{C}_{pj_0}(\tau,\rho). \tag{3.11}$$

We now turn our attention to the explicit form of $D^s_{\lambda j_0}(R)$. For this it is convenient to write

$$\mathbf{n} = ((\cos\phi/\cosh\beta), (\sin\phi/\cosh\beta), \tanh\beta), \tag{3.12}$$

the direction vector of the photon 3-momentum. Now if $\mathbf{n}$ is rotated by $-\phi$ about the z axis, $\mathbf{n}$ becomes

$$\mathbf{n} \to \mathbf{n}_0 = ((1/\cosh\beta), 0, \tanh\beta). \tag{3.13}$$

According to the prescription of Ref. 7, the remaining rotation is a rotation in the xz plane by an amount η given by

$$\cos\eta = \frac{u_0 \cos\theta - |\mathbf{u}|}{u_0 - \mathbf{u}\cdot\mathbf{n}}, \tag{3.14}$$

where θ is the angle between $\mathbf{n}_0$ and $\mathbf{u} = (\operatorname{sh}a, 0, 0)$. In our case

$$\cos\theta = 1/\cosh\beta$$

and

$$\cos\eta = \frac{\cosh a - \cosh\beta \sinh a}{\cosh a \cosh\beta - \sinh a}, \tag{3.15}$$

so that we finally have

$$R = M_3(\tfrac{1}{2}\pi - \phi)M_1(\eta)M_3(-\tfrac{1}{2}\pi). \tag{3.16}$$

The integral I can now be evaluated. It is found to be given by

$$I = \frac{e^{i\pi(\lambda-j_0)}}{\Gamma(1+i\rho)} \sum_{r_i} \frac{\Gamma(1+2r_1+i\rho)}{\Gamma(r_1+1-\frac{1}{2}\bar{p})\Gamma(r_1+1+\frac{1}{2}\bar{p})} \times \Lambda_{sr_4,\lambda j_0} \frac{(2r_1+1+i\rho)_{r_2}(-i\tau)_{r_3}(-i\tau+\frac{1}{2})_{r_3}}{(\frac{1}{2})_{r_3} r_2!\, r_3!} \times \frac{\Gamma(s)\Gamma(c-b)}{\Gamma(c)} e^{a(\lambda-j_0+2r_4)} \times (\tfrac{1}{2}\tanh a)^{2r_1} (\cosh a)^{-1-i\rho}\, {}_2F_1(s,b;c;-e^{2a}), \tag{3.17}$$

where

$$b = \tfrac{1}{2}(\lambda - j_0) + r_2 + r_3 + r_4 + \tfrac{1}{2},$$
$$c = 2r_1 + i\rho - i\tau + \tfrac{1}{2}(\lambda - j_0) + r_2 + r_3 + r_4 + \tfrac{3}{2},$$
$$\Lambda_{sr_4,\lambda j_0} = [\Gamma(s+\lambda+1)\Gamma(s-\lambda+1)\Gamma(s+j_0+1) \times \Gamma(s-j_0+1)]^{1/2}[\Gamma(s-\lambda-j_0+1) \times \Gamma(s+j_0-r_4+1)\Gamma(r_4+\lambda-j_0+1) \times \Gamma(r^4+1)]^{-1}$$
$$(d)_n = \Gamma(d+n)/\Gamma(d), \quad \bar{p} = p - \lambda.$$

We now identify I with the mixed basis matrix element in the following way:

$$\langle \rho j_0; s\lambda | N_1(a) | \rho j_0; \tau p\rangle = C^{\rho j_0}_{s\lambda,\tau p}(a) = I. \tag{3.18}$$

The expansion of a single particle helicity state in terms of C system matrix elements is then

$$|u; s,\lambda\rangle = \frac{1}{2(2\pi)^3} \sum_{j_0=-s}^{s} \int_{-\infty}^{\infty} d\rho(\rho^2 + j_0^2) \times \sum_{p=-\infty}^{\infty} \int_{-\infty}^{\infty} d\tau a_p^{j_0}(\tau,\rho) C^{\rho j_0}_{s\lambda;\tau M}(a)\, e^{i\tau b}\, e^{ip\psi}. \tag{3.19}$$

4. RECURRENCE RELATIONS FOR THE MIXED BASIS MATRIX ELEMENTS

In this section we use the infinitesmal operator method[10,11] to establish recurrence relations and differential equations for the mixed basis matrix elements. For this method we use a fixed column of the mixed basis matrix element $\langle\rho j_0; JM | L | \rho j_0; \tau p\rangle$ (i.e., τ and p fixed) as a set of $SU(2)$ basis vectors spanning the UIR$\{j_0, \rho\}$ of $SL(2,C)$. L is a general lorentz transformation. The generators M_i, N_i are then differential operators acting on the six parameters needed to specify L. Now using Eqs. (1.10) and (1.9) and making a particular choice for L we can derive the relations we need. For the C system we parametrize L as follows

$$L = M_3(\phi)M_1(\theta)M_3(\alpha)N_1(a)N_3(b)M_3(\psi), \tag{4.1}$$

so that the mixed basis matrix element is

$$\langle\rho j_0; JM | L | \rho j_0; \tau p\rangle = \bar{C}^{\rho j_0}_{JM,\tau p} = \sum_\lambda D^J_{M\lambda}(\phi,\theta,\alpha) C^{\rho j_0}_{J\lambda,\tau p}(a) e^{i\tau b} e^{ip\psi}. \tag{4.2}$$

The generators M_i, N_i corresponding to the parametrization (4.1) are

$$M_1 = -\cot\theta \sin\phi \frac{\partial}{\partial\phi} + \cos\phi \frac{\partial}{\partial\phi} + \frac{\sin\phi}{\sin\theta}\frac{\partial}{\partial\alpha},$$
$$M_3 = \frac{\partial}{\partial\phi},$$
$$N_1 = -\sin\phi \cos\alpha \tanh a \frac{\partial}{\partial\phi} + \tanh a \sin\phi \sin\theta \sin\alpha \frac{\partial}{\partial\theta} - \frac{1}{\sinh a \cosh a}(\sin\phi \cos\theta \cos\alpha + \cos\phi \sin\alpha \cosh^2 a) \times \frac{\partial}{\partial\alpha} + (\cos\phi \cos\alpha - \sin\phi \sin\alpha \cos\theta)\frac{\partial}{\partial a} + \frac{\sin\phi \sin\theta}{\cosh a} \times \frac{\partial}{\partial b} + \frac{1}{\sinh a}(\cos\phi \sin\alpha + \sin\phi \cos\alpha \cos\theta)\frac{\partial}{\partial\psi},$$
$$N_3 = -\cot\theta \cos\alpha \tanh a \frac{\partial}{\partial\phi} + \tanh a \cos\theta \sin\alpha \frac{\partial}{\partial\theta} + \cos\alpha(\tanh a \cot\theta \cos\theta + \sin\theta \coth a)\frac{\partial}{\partial a} + \sin\theta \sin\alpha \frac{\partial}{\partial\alpha}. \tag{4.3}$$

M_2 and N_2 can be obtained from M_1 and N_1, respectively, via the transformation $\phi \to -\frac{1}{2}\pi + \phi$.

In the $SU(2)$ basis we have chosen, the Casimir invariant equations have the form

$$(\mathbf{M}^2 - \mathbf{N}^2)\bar{C}^{\rho j_0}_{JM,\tau p} = (1 + \rho^2 - j_0^2)\ \bar{C}^{\rho j_0}_{JM,\tau p},$$
$$\mathbf{M}\cdot\mathbf{N}\ \bar{C}^{\rho j_0}_{JM,\tau p} = \rho j_0 \bar{C}^{\rho j_0}_{JM,\tau p}. \quad (4.4)$$

The explicit expression of the Casimir invariants in terms of differential operators is found from (4.3) to be

$$\mathbf{N}^2 - \mathbf{M}^2 = \frac{\partial^2}{\partial a^2} + (\tanh a + \coth a)\frac{\partial}{\partial a} + \frac{1}{\cosh^2 a}\frac{\partial^2}{\partial b^2} + \frac{1}{\sinh^2 a}\frac{\partial^2}{\partial\psi^2} - 2\frac{\coth a}{\sinh a}\frac{\partial^2}{\partial\alpha\partial\psi}\tanh^2 a\ \bar{M}_2^2 - \mathbf{M}^2 + 2\frac{\tanh a}{\cosh a}\bar{M}_2\frac{\partial}{\partial b} + \coth^2 a\frac{\partial^2}{\partial\alpha^2} \quad (4.5)$$

$$\mathbf{M}\cdot\mathbf{N} = \bar{M}_1\left(\frac{\partial}{\partial a} + \tanh a\right) + \bar{M}_2\left((\tanh a - \coth a)\frac{\partial}{\partial\alpha} + \frac{1}{\sinh a}\frac{\partial}{\partial\psi}\right) + \frac{1}{\coth a}\frac{\partial^2}{\partial b\partial\alpha}, \quad (4.6)$$

where

$$\bar{M}_2 = \cot\theta\cos\alpha\frac{\partial}{\partial\alpha} + \sin\alpha\frac{\partial}{\partial\theta} - \frac{\cos\alpha}{\sin\theta}\frac{\partial}{\partial\phi},$$

$$\bar{M}_1 = -\cot\theta\sin\alpha\frac{\partial}{\partial\alpha} + \cos\alpha\frac{\partial}{\partial\theta} + \frac{\sin\alpha}{\sin\theta}\frac{\partial}{\partial\phi}.$$

So applying the Casimir invariants (4.5) to the $\bar{C}^{\rho j_0}_{JM,\tau p}$ functions and separating out all but the a dependence, using known recurrence relations of the $SU(2)$ matrix elements[12] and the orthogonality properties of the $O(1,1)\otimes O(2)$ matrix elements, we get the relations

$$\alpha^J_\lambda\left(\frac{d}{da} + \lambda\tanh a + (1-\lambda)\coth a + \frac{p}{\sinh a}\right)C^{\rho j_0}_{J,\lambda+1;\tau p} + \alpha^J_{\lambda+1}\left(\frac{d}{da} - \lambda\tanh a + (1+\lambda)\coth a - \frac{p}{\sinh a}\right) \times C^{\rho j_0}_{J,\lambda-1;\tau p} + i\left(\frac{2\lambda\tau}{\cosh a} + \rho j_0\right)C^{\rho j_0}_{J\lambda;\tau p} = 0, \quad (4.7)$$

$$\left(\frac{d^2}{da^2} + (\tanh a + \coth a)\frac{d}{da} - \frac{\tau^2}{\cosh^2 a} - \frac{p^2}{\sinh^2 a} + 2\lambda p\frac{\coth a}{\sinh a} + J(J+1) + \tfrac{1}{2}\tanh^2 a[J(J+1) - \lambda^2\coth^2 a + (1 - j_0^2 + \rho^2)\right)C^{\rho j_0}_{J\lambda;\tau p}$$
$$+ \tfrac{1}{4}\tanh^2 a[\alpha^J_{\lambda+1}\alpha^J_{\lambda+2}C^{\rho j_0}_{J,\lambda+2;\tau p} + \alpha^J_\lambda\alpha^J_{\lambda-1}C^{\rho j_0}_{J,\lambda-2;\tau p}] + i\tau(\tanh a/\cosh a)(\alpha^J_\lambda C^{\rho j_0}_{J,\lambda-1;\tau p} - \alpha^J_{\lambda+1}C^{\rho j_0}_{J,\lambda+1;\tau p}) = 0. \quad (4.8)$$

The remaining recurrence relations are determined from the known action of the generators $N_\pm$ in an $SU(2)$ basis [Eqs. (1,10)]. They are

$$[(J+\lambda)(J+\lambda+1)]^{1/2}\left(\frac{d}{da} + \frac{p}{\sinh a} + (1-\lambda)\coth a + (J-\lambda+1)\tanh a\right)C^{\rho j_0}_{J,\lambda-1;\tau p}$$
$$- [(J-\lambda)(J-\lambda+1)]^{1/2}\left(\frac{d}{da} - \frac{p}{\sinh a} + (1+\lambda)\coth a + (J+\lambda+1)\tanh a\right)C^{\rho j_0}_{J,\lambda+1;\tau p}$$
$$- (2i\tau/\cosh a)[(J+1)^2 - \lambda^2]^{1/2}C^{\rho j_0}_{J\lambda,\tau p} \quad (4.9)$$
$$= 2\{[(J+1)^2 - j_0^2][(J+1)^2 + \rho^2][(2J+1)/(2J+3)]\}^{1/2}C^{\rho j_0}_{J+1,\lambda;\tau p}$$

$$-[(J-\lambda)(J-\lambda+1)]^{1/2}\left(\frac{d}{da} + \frac{p}{\sinh a} + (1-\lambda)\coth a + (J+\lambda)\tanh a\right)C^{\rho j_0}_{J,\lambda-1;\tau p} + [(J+\lambda)(J+\lambda+1)]^{1/2}$$
$$\times\left(\frac{d}{da} - \frac{p}{\sinh a} + (1+\lambda)\coth a + (J-\lambda)\tanh a\right)C^{\rho j_0}_{J,\lambda+1;\tau p} + (J^2-\lambda^2)^{1/2}(2i\tau/\cosh a)C^{\rho j_0}_{J\lambda;\tau p}$$
$$= 2\{(J^2 - j_0^2)(J^2+\rho^2)[(2J+1)/(2J-1)]\}^{1/2}\ C^{\rho j_0}_{J-1;\lambda,\tau p}. \quad (4.10)$$

These relations we have developed here are the ones we will use in the next section in our analysis of the Proca and Dirac fields.

5. SOLUTION OF THE DIRAC AND PROCA FREE FIELD EQUATIONS IN THE C SYSTEM

As an application of the previous three sections we derive invariant expansions of solutions of the Dirac and Proca equations in terms of the functions

$$D^{\rho j_0}_{J\lambda;\tau p}(a,b,\phi) = C^{\rho j_0}_{J\lambda;\tau p}(a)\,e^{i\tau b}e^{ip\phi}. \quad (5.1)$$

This has already been done in the S system for these equations[13] and more general ones.[14,15]

An outline of the general method is as follows. In order to achieve an invariant expansion of an arbitrary field $F^{\rho j_0}_{JM}(x)$, it is convenient to go over into a coordinate system in which each component transforms independently. The components of $F^{\rho j_0}_{JM}(x)$ in this new coordinate system are

$$\bar{F}^{\rho j_0}_{JM}(g) = U(g)F^{\rho j_0}_{JM}(x) = D^{\rho j_0}_{J'M',JM}(g)F^{\rho j_0}_{J'M'}(g^{-1}x). \quad (5.2)$$

From this definition it follows that each component does indeed transform independently:

$$U(g_0)\bar{F}^{\rho j_0}_{JM}(g) = \bar{F}^{\rho j_0}_{JM}(g_0 g) \quad (5.3)$$

so that each component of $\bar{F}^{\rho j_0}_{JM}(g)$ constitutes a representation space for the Lorentz group and can, therefore, be expanded in terms of matrix elements of that group.

We now turn our attention to the Proca field $A_K(x)$ of mass μ, i.e.,

$$(\Box - \mu^2)A_K(x) = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_0^2} - \mu^2\right) \times A_K(x) = 0,\ \frac{\partial A_K}{\partial x_K} = 0. \quad (5.4)$$

We seek a solution for this equation inside the light cone, so in the C system we choose x to be parametrized by

$$x = (s\cosh a\cosh b, s\sinh a\cos\phi, s\sinh a\sin\phi, s\cosh a\sinh b). \quad (5.5)$$

The operators $\partial/\partial x_i$ have the form

$$\frac{\partial}{\partial x_0} = \cosh a\cosh b\frac{\partial}{\partial s} - \frac{\sinh a\cosh b}{s}\frac{\partial}{\partial a} - \frac{\sinh b}{s\cosh a}\frac{\partial}{\partial b},$$
$$\frac{\partial}{\partial x_1} = -\sinh a\cos\phi\frac{\partial}{\partial s} + \frac{\cosh a\cos\phi}{s}\frac{\partial}{\partial a} - \frac{\sin\phi}{s\sinh a}\frac{\partial}{\partial\phi},$$
$$\frac{\partial}{\partial x_2} = -\sinh a\sin\phi\frac{\partial}{\partial s} + \frac{\cosh a\sin\phi}{s}\frac{\partial}{\partial a} + \frac{\cos\phi}{s\cosh a}\frac{\partial}{\partial b}$$
$$\frac{\partial}{\partial x_3} = -\sinh b\cosh a\frac{\partial}{\partial s} + \frac{\sinh a\sinh b}{s}\frac{\partial}{\partial a} + \frac{\cosh b}{s\cosh a}\frac{\partial}{\partial b}. \quad (5.6)$$

The transformation to the independent variables changes the 4-vector x as if at the point (a, b, ϕ) the space has been subjected to the Lorentz transformation

$$\Omega = N_1(-a)N_3(-b)M_3(-\phi). \quad (5.7)$$

Under this transformation $\partial/\partial x_K$ and $A_K(x)$ are transformed according to

$$\frac{\partial}{\partial\bar{x}_K} = \Omega_{kn}\frac{\partial}{\partial x_n}, \quad \bar{A}_K(\bar{x}) = \Omega_{kl}A_l(x), \quad (5.8)$$

where

$$\frac{\partial}{\partial\bar{x}_0} = \frac{\partial}{\partial s}, \quad \frac{\partial}{\partial\bar{x}_1} = \frac{1}{s}\frac{\partial}{\partial a},$$
$$\frac{\partial}{\partial\bar{x}_2} = \frac{1}{s\sinh a}\frac{\partial}{\partial\phi}, \quad \frac{\partial}{\partial\bar{x}_3} = \frac{1}{s\cosh a}\frac{\partial}{\partial b}, \quad (5.9)$$

The transformed Proca equation now becomes

$$(\Box - \mu^2)\bar{A}_k(\bar{x}) - D_i\frac{\partial\bar{A}_k(\bar{x})}{\partial\bar{x}_i} - \Omega_{kl}\times\left(D_i\frac{\partial\Omega^{-1}_{lv}}{\partial\bar{x}_i} + 2\frac{\partial\Omega^{-1}_{lv}}{\partial\bar{x}_i}\frac{\partial}{\partial\bar{x}_i} + \frac{\partial^2\Omega^{-1}_{lv}}{\partial\bar{x}_i^2}\right)\bar{A}_v(\bar{x}) = 0,$$
$$\frac{\partial\bar{A}_i(\bar{x})}{\partial\bar{x}_i} + D_i\bar{A}_i(\bar{x}) = 0, \quad (5.10)$$

where

$$D_i = \frac{\partial\Omega_{iv}}{\partial\bar{x}_k}\Omega^{-1}_{vk}$$

passing to the canonical basis

$$f_0 = A_0, \quad \sqrt{2}f_\pm = iA_2 \mp A_1, \quad f_1 = A_3 \quad (5.11)$$

and expanding f_0, f_1 and $f_\pm$ according to

$$f_0 = \sum\chi_0^{(\rho,j_0)}(s)C^{\rho j_0}_{00;\tau p}e^{i\tau b}e^{ip\phi},$$
$$f_\pm = \sum\chi_\pm^{(\rho,j_0)}(s)C^{\rho j_0}_{1,\pm1;\tau p}e^{i\tau b}e^{ip\phi}, \quad (5.12)$$
$$f_1 = \sum\chi_1^{(\rho,j_0)}(s)C_{1,0;\tau p}e^{i\tau b}e^{ip\phi},$$

where the summation is over j_0, p, τ, ρ, the system of equations (5.10) becomes

$$\left(\frac{\partial\chi_0}{\partial s} + \frac{3}{s}\chi_0\right)C_0 + \frac{1}{\sqrt{2}s}\left[\left(\frac{\partial C_-}{\partial a} + (\tanh a + \coth a)C_- + \frac{p}{\sinh a}C_-\right)\chi_- - \left(\frac{\partial C_+}{\partial a} + (\tanh a + \coth a)C_+ - \frac{p}{\sinh a}C_+\right)\chi_+ + \frac{\sqrt{2}i\tau}{\cosh a}\chi_1C_1\right] = 0,$$

$$\left(\frac{\partial^2\chi_0}{\partial s} + \frac{3}{s}\frac{\partial\chi_0}{\partial s} - \frac{3\chi_0}{s^2} + u^2\chi_0\right)C_0 - \frac{1}{s^2}\left[\left(\frac{\partial^2C_0}{\partial a^2} + (\tanh a + \coth a)\frac{\partial C_0}{\partial a} - \frac{\tau^2}{\cosh^2 a}C_0 - \frac{p^2}{\sinh^2 a}C_0\right)\chi_0 + \sqrt{2}\left(\frac{\partial C_-}{\partial a} + (\tanh a + \coth a)C_- + \frac{p}{\sinh a}C_-\right)\chi_- - \sqrt{2}\left(\frac{\partial C_+}{\partial a} + (\tanh a + \coth a)C_+ - \frac{p}{\sinh a}C_+\right)\chi_+ + \frac{2i\tau}{\cosh a}\chi_1C_1\right] = 0,$$

$$\left(\frac{\partial^2\chi_1}{\partial s^2} + \frac{3}{s}\frac{\partial\chi_1}{\partial s} + \mu^2\chi_1\right)C_1 - \frac{1}{s^2}\left[\left(\frac{\partial^2C_1}{\partial a^2} + (\tanh a + \coth a)\frac{\partial C_1}{\partial a} - \frac{\tau^2}{\coth^2 a}C_1 - \frac{p^2}{\sinh^2 a}C_1 + \frac{1}{\cosh^2 a}C_1\right)\chi_1 + i\tau\sqrt{2}\frac{\tanh a}{\cosh a}(\chi_-C_- - \chi_+C_+) + \frac{2i\tau}{\cosh a}\chi_0C_0\right] = 0,$$

$$\left(\frac{\partial^2\chi_\pm}{\partial s^2} + \frac{3}{s}\frac{\partial\chi_\pm}{\partial s} + \mu^2\chi_\pm\right)C_\pm - \frac{1}{s^2}\left[\left(\frac{\partial^2C_\pm}{\partial a^2} + (\tanh a + \coth a)\frac{\partial C_\pm}{\partial a} - \frac{\tau^2}{\cosh^2 a}C_\pm - \frac{p}{\sinh^2 a}C_\pm - \frac{1}{\sinh^2 a}C_\pm - \frac{1}{2}\tanh^2 a\,C_\pm \pm 2p\frac{\coth a}{\sinh a}C_\pm\right)\chi_\pm \pm i\tau\sqrt{2}\frac{\tanh a}{\cosh a}\chi_1C_1 + \frac{1}{2}\tanh^2 a\,\chi_\mp C_\mp + 2\left(\mp\frac{\partial C_0}{\partial a} - \frac{p}{\sinh a}C_0\right)\chi_0\right] = 0, \quad (5.13)$$

where we have used the shorthand

$$C_0 = C^{\rho j_0}_{00;\tau\rho}, \quad C_\pm = C^{\rho j_0}_{1,\pm1;\tau p}, \quad C_1 = C^{\rho j_0}_{1,0;\tau p}.$$

From the recurrence relations (4.7)–(4.10) we see that the variables separate if

$$\chi_+ = \chi_- = -\chi_1. \quad (5.14)$$

We then arrive at the same system of equations as in Ref. 13 viz.

$$\left(\frac{d}{ds} + \frac{3}{s}\right)\chi_0^{(\rho,0)}(s) + \frac{[3(1+\rho^2)]^{1/2}}{s}\chi_1^{(\rho,0)}(s) = 0,$$

$$\left(\frac{d}{ds^2} + \frac{5}{s}\frac{d}{ds} + \frac{4+\rho^2}{s^2} + \mu^2\right)\chi_0^{(\rho,0)} = 0, \qquad (5.15)$$

$$\left(\frac{d^2}{ds^2} + \frac{3}{s}\frac{d}{ds} + \frac{1+\rho^2}{s^2} + \mu^2\right)\chi_1^{(\rho\pm1)}(s) = 0$$

(remember the summation on j_0 consists of $j_0 = 0$ only, for f_0). These equations have the solution

$$\chi_1^{(\rho,\pm1)}(s) = (1/\mu s)[c_1 H_{i\rho}^{(2)}(\mu s) + c_2 H_{-i\rho}^{(2)}(\mu s)],$$

$$\chi_0^{(\rho,0)}(s) = [1/(\mu s)^2][c_3 H_{i\rho}^{(2)}(\mu s) + c_4 H_{-i\rho}^{(2)}(\mu s)]. \qquad (5.16)$$

So the solutions to the Proca equation have the form

$$f_0 = \sum_{p=-\infty}^{\infty} \int_{-\infty}^{\infty} d\tau \int_0^{\infty} d\rho\, \chi_0^{(\rho,0)}(s) C_{00,\tau p}^{\rho 0} e^{i\tau b} e^{ip\phi},$$

$$h_\lambda = \sum_{j_0=-1}^{+1} \sum_{p=-\infty}^{+\infty} \int_0^{\infty} d\tau \int_0^{\infty} d\rho\, \chi_1^{(\rho,j_0)}(s) C_{1,\lambda;\tau p}^{\rho j_0} e^{i\tau b} e^{ip\phi}, \qquad (5.17)$$

where $h_{\pm1} = -f_\pm, h_0 = f_1$.

This then completes the derivation of an invariant expansion of solutions of the Proca equation inside the light cone.

We now turn our attention to the Dirac equation. In order to obtain an invariant decomposition of a solution of the Dirac equation, we write the equation in a canonical basis

$$\left(i\gamma^n \frac{\partial}{\partial x^n} - \mu\right)\psi(x) = 0, \qquad (5.18)$$

where

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^\alpha = \begin{pmatrix} 0 & -\sigma_\alpha \\ \sigma_\alpha & 0 \end{pmatrix}, \qquad (5.19)$$

($\alpha = 1, 2, 3$), where σ_α are the Pauli spin matrices and I the 2×2 identity matrix. Under the transformation Ω of (5.7), Eq. (5.18) changes to

$$i\gamma^n \frac{\partial\bar{\psi}(\bar{x})}{\partial\bar{x}^n} + i\gamma^n \Lambda \frac{\partial\Lambda^{-1}}{\partial\bar{x}^n} \bar{\psi}(\bar{x}) - \mu\bar{\psi}(\bar{x}) = 0, \qquad (5.20)$$

where $\bar{\psi}(\bar{x}) = \Lambda\psi(x)$, i.e., Λ is the 4×4 matrix according to which the spinor ψ transforms under the Lorentz transformation Ω.

In the C system we have that

$$\gamma^n \Lambda \frac{\partial\Lambda^{-1}}{\partial\bar{x}^n} = \frac{i}{2s}[(\tanh a + \coth a)\gamma^1 - 3\gamma^0]. \qquad (5.21)$$

If we now look for solutions of the form

$$\psi_i = \sum f_i(s) C_i(a) e^{i\tau b} e^{ip\phi}, \quad i = 1, 3,$$

$$\psi_j = \sum f_j(s) C_j(a) e^{i\tau b} e^{ip\phi}, \quad j = 2, 4,$$

the system of equations (5.20) becomes

$$i\left(\frac{\partial f_1}{\partial s} + \frac{3}{2s}f_1\right)C_1 + \frac{\tau}{s\cosh a} f_1 C_1 - \frac{i}{s}\left(\frac{\partial C_2}{\partial a} + \frac{p}{\sinh a}C_2 + \frac{1}{2}(\tanh a + \coth a)C_2\right)f_2 - \mu f_3 C_3 = 0,$$

$$i\left(\frac{\partial f_2}{\partial s} + \frac{3}{2s}f_2\right)C_2 - \frac{\tau}{s\cosh a} f_2 C_2 - \frac{i}{s}\left(\frac{\partial C_1}{\partial a} - \frac{p}{\sinh a}C_1 + \frac{1}{2}(\tanh a + \coth a)C_1\right)f_1 - \mu f_4 C_4 = 0,$$

$$i\left(\frac{\partial f_3}{\partial s} + \frac{3}{2s}f_3\right)C_3 - \frac{\tau}{s\cosh a} f_3 C_3 + \frac{i}{s}\left(\frac{\partial C_4}{\partial a} + \frac{p}{\sinh a}C_4 + \frac{1}{2}(\tanh a + \coth a)C_4\right)f_4 - \mu f_1 C_1 = 0,$$

$$i\left(\frac{\partial f_4}{\partial s} + \frac{3}{2s}f_4\right)C_4 + \frac{\tau}{s\cosh a} f_4 C_4 + \left(\frac{\partial C_3}{\partial a} - \frac{p}{\sinh a}C_3 + \frac{1}{2}(\tanh a + \coth a)C_3\right)f_3 - \mu f_2 C_2 = 0, \qquad (5.22)$$

from which we see that the variables separate if we take

$$f_1(s) = f_2(s), \quad f_3(s) = f_4(s),$$

$$C_i(a) = C_{1/2,1/2;\tau p}^{\rho j_0}(a), \quad i = 1, 3, \qquad (5.23)$$

$$C_j(a) = C_{1/2,-1/2;\tau p}^{\rho j_0}(a), \quad j = 2, 4.$$

The form of $f_1(s)$ and $f_3(s)$ is now determined by the pair of coupled equations

$$\left(\frac{d}{ds} + \frac{3}{2s} - 2ij_0\rho\right)f_3 + i\mu f_1 = 0, \qquad (5.24)$$

$$\left(\frac{d}{ds} + \frac{3}{2s} + 2ij_0\rho\right)f_1 + i\mu f_3 = 0,$$

which have solutions of the form[16]

$$f_1(s) = (1/\mu s)[c_1 J_\nu(\mu s) + c_2 J_\nu(\mu s)],$$

$$f_3(s) = (1/\mu s)[c_2 J_\nu(\mu s) - c_1 J_{-\nu}(\mu s)] \qquad (5.25)$$

with $\nu = \frac{1}{2} + 2ij_0\rho$.

So the solutions of the Dirac equation are

$$\psi_i = \sum_{j_0=-1/2}^{+1/2} \sum_{p=-\infty}^{\infty} \int_{-\infty}^{\infty} d\tau \int_0^{\infty} d\rho f_i(s) C_{1/2,1/2;\tau p}^{\rho j_0}(a) \times e^{i\tau b} e^{ip\phi}, \quad i = 1, 3,$$

$$\psi_j = \sum_{j_0=-1/2}^{+1/2} \sum_{p=-\infty}^{\infty} \int_{-\infty}^{\infty} d\tau \int_0^{\infty} d\rho f_j(s) C_{1/2,1/2;\tau p}^{\rho j_0}(a) \times e^{i\tau b} e^{ip\phi}, \quad j = 2, 4.$$

This then completes this section on the solution of the Proca and Dirac equation in the C system.

6. DIFFERENTIAL EQUATIONS SATISFIED BY THE EXPANSION MATRIX ELEMENTS AND THE NONRELATIVISTIC LIMIT

From the recurrence relations derived in Sec. 4 we deduce that the matrix elements used in the expansions of Sec. 5 satisfy the following differential equations:

(i) Using the shorthand

$$C_{J\lambda;\tau p}^{\rho j_0}(a) = C_{J\lambda}^{\rho j_0},$$

we have for $j_0 = J = \lambda = 0$ the differential equation

$$\left(\frac{d^2}{da^2} + (\tanh a + \coth a)\frac{d}{da} - \frac{\tau^2}{\cosh^2 a} - \frac{p^2}{\sinh^2 a}(1 + \rho^2)\right)C_{00}^0 = 0. \qquad (6.1)$$

$C^0_{1\lambda}$ may be calculated from C^0_{00} by using

$$-i\tau/\cosh a\ C^0_{00} = [\tfrac{4}{3}(1+\rho^2)]^{-1/2}C^0_{10},$$
$$\pm\sqrt{2}\left(\frac{d}{da} \pm \frac{p}{\sinh a}\right) C^0_{00} = [\tfrac{4}{3}(1+\rho^2)]^{1/2}C^0_{1,\pm 1}. \quad (6.2)$$

(ii) $j_0 = 1$; C^1_{10} satisfies the equation

$$\left[\frac{d^2}{da^2} + \left((\tanh a + \coth a) + \frac{4\tau}{\tau^2 - \rho^2\cosh^2 a}\right)\frac{d}{da} - \frac{\tau^2}{\cosh^2 a} - \frac{p^2}{\sinh^2 a} + 2 + p^2 + \tanh^2 a + \frac{4\tau \tanh a}{\tau^2 - \rho^2\cosh^2 a}(2\tau \tanh a - \rho p \coth a)\right] C^1_{10} = 0; \quad (6.3)$$

the other $j_0 = 1$ matrix elements may be deduced from the relations

$$i\left(\mp \frac{2\tau}{\cosh a} - \rho\right) C^1_{1,\pm 1} = \sqrt{2}\left(\frac{d}{da} + \tanh a \pm \frac{p}{\sinh a}\right) C^1_{10}. \quad (6.4)$$

(iii) $j_0 = \frac{1}{2}$; $C^{1/2}_{1/2,1/2}$ satisfies the equation

$$\left[\frac{d^2}{da^2} + \left((\tanh a + \coth a) + \frac{\tau \tanh a}{\tau \pm \rho \cosh a}\right)\frac{d}{da} - \frac{\tau^2}{\cosh^2 a} - \frac{p^2}{\sinh^2 a} \pm p\frac{\coth a}{\sinh a} + \tfrac{1}{4}(\tanh^2 a - \coth^2 a) + \frac{\tau}{\tau \pm \rho\cosh a}\left(\tfrac{1}{2}(\tanh a + \coth a) \mp \frac{p}{\sinh a}\right) + \rho^2\right] \times C^{1/2}_{1/2,\mp 1/2} = 0. \quad (6.5)$$

Similar equations to those of (ii) and (iii) hold for the cases $j_0 = -1, j_0 = -\frac{1}{2}$, respectively.

These equations are useful in the passage to the nonrelativistic limit.[13-17] In this limit we have

$$a \to 0, \quad s \to \infty \quad \text{s.t.} \quad sa = r, \quad (6.6)$$

where r is the polar radius in the xy plane in nonrelativistic 3-space

$$b \to 0, \quad s \to \infty \quad \text{s.t.} \quad sb = z. \quad (6.7)$$

In addition we must require that

$$\tau \to \infty \quad \text{in such a way that}$$
$$\tau/s \to \tau', \quad -\infty < \tau' < \infty; \quad (6.8)$$

finally

$$\rho \to |\mathbf{p}|s.$$

In this limit Eq. (6.1) becomes

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + |\mathbf{p}|^2 - \tau'^2 - \frac{p^2}{r^2}\right) C^0_{00}(r) = 0; \quad (6.9)$$

so taking the regular solution at $r = 0$, we have

$$C^0_{00} \to c_1 J_p(\alpha r), \quad \alpha^2 = |\mathbf{p}|^2 - \tau'^2.$$

From relations (6.2) we see that

$$C^0_{10} \to c_2 J_p(\alpha r), \quad C^0_{1,\pm 1} \to C_\pm J_{p\pm 1}(\alpha r). \quad (6.10)$$

Similar results hold in the $j_0 = 1$ case as C^1_{10} then satisfies Eq. (6.9)

This then gives the correct set of functions in 3-space corresponding to the expansion of Maxwell's equations in cylindrical coordinates,[18] viz.,

$$\bar{C}_\lambda(r, z, \phi) = J_{p+\lambda}(\alpha r)e^{i\tau' z}e^{ip\phi},$$
$$\lambda = \pm 1, 0, \quad p = 0, \pm 1, \pm 2, \ldots, \quad -\infty < \tau' < \infty. \quad (6.11)$$

We note that the solution in cylindrical coordinates is an expansion invariant with respect to the group $O(2) \otimes T_3$, the direct product of rotations about Oz, and translations along Oz. So the reduction $O(1,1) \otimes O(2) \subset O(3,1)$ becomes in the nonrelativistic limit the reduction $O(2) \otimes T_3 \subset E(3)$.

For the nonrelativistic limit of the functions used in the Dirac equation solution we have the following differential equations

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + |\mathbf{p}|^2 - \tau'^2 - \frac{(p \pm \frac{1}{2})^2}{r^2}\right) C^{1/2}_{1/2,\pm 1/2} = 0, \quad (6.12)$$

so that this corresponds to a nonrelativistic solution of the Dirac equation in terms of the complete set of functions

$$P_{\pm 1/2}(r, a, \phi) = J_{p\pm 1/2}(\alpha r)e^{i\tau' z}e^{ip\phi}. \quad (6.13)$$

This coincides with the solution in cylindrical coordinates in 3-space.

7. CONCLUSION

In this paper we have carried out the reduction of the principal series of $O(3,1)$ in an $O(1,1) \otimes O(2)$ basis and examined the properties of the $O(3) \leftrightarrow O(1,1) \otimes O(2)$ mixed basis matrix elements. It was shown that the expansion of solutions of the Proca and Dirac free fields (inside the light cone) corresponds to the relativistic generalization of cylindrical coordinates in 3-space. In future developments we propose to study the solution of other wave equations (both inside and outside the light cone) using these mixed basis matrix elements. Other related problems of interest include the reduction of the supplementary series of $O(3,1)$ with respect to $O(1,1) \otimes O(2)$[19] and a study of the matrix elements in an $O(1,1) \otimes O(2)$ basis.

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Dynamics of Harmonically Bound Semi-Infinite and Infinite Chains with Friction and Applied Forces

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The dynamics of semi-infinite and infinite linear chains of identical masses and ideal springs is studied. In addition to the harmonic coupling between nearest neighbors, each particle is harmonically bound to ts equilibrium position and is subject to friction and time-dependent applied forces. The Laplace transform method is used to express the motion of all the particles. The exact solutions are found and discussed for four different cases: (a) an infinite chain, (b) a semi-infinite chain, (c) a semi-infinite chain with the position of the end particle specified as a function of time, and (d) an infinite chain with the position of one particle specified as a function of time. By specializing some results of the present work, those of previous calculations on simpler systems by other authors are recovered.

There are two main approaches to the mathematical description of physical phenomena. One sometimes tries to study as exactly as possible a simplified model with only the main features of a real system, while some are more interested in an approximate solution of a realistic model. The one-dimensional systems have been favorite models for the first approach.[1] One such system extensively studied is the infinite chain of point masses and ideal massless springs[1,2] because it is one of the very few many-body systems in which exact calculations are possible. However, there has not been much study of an exact treatment of a semi-infinite chain. Although there have been many calculations treating semi-infinite lattices in conjunction with studies on surface phenomena,[3] most of them can be classified under the second approach above.

The present work studies the exact dynamics of semi-infinite and infinite linear chains of identical masses and ideal massless springs with identical force constants. In addition to the harmonic coupling between nearest neighbors, each mass is harmonically bound to its equilibrium position and is subject to friction and time-dependent applied forces. The motion of each of the particles is expressed exactly in terms of the given quantities and initial conditions. Four different systems are studied: (a) an infinite chain, (b) a semi-infinite chain, (c) a semi-infinite chain with the position of the end particle specified as a function of time, and (d) an infinite chain with the position of one particle specified as a function of time. By specializing some of the results, those of previous calculations on simpler systems by other authors are recovered.

Let $x_n(t)$ represent the displacement of the nth particle measured from its equilibrium position. The integer n is restricted to $n \geq 0$ for the semi-infinite systems (b) and (c). The coupled equations for the system are

$$m\ddot{x}_n = -k(x_n - x_{n+1}) - k(x_n - x_{n-1}) \begin{bmatrix} 1 \\ (1-\delta_{n0}) \end{bmatrix} - Kx_n - \beta\dot{x}_n + \phi_n, \quad \begin{matrix} (1a, 1d) \\ (1b, 1c) \end{matrix}$$

where m is the particle mass, k and K are the spring constants, β is the friction coefficient, δ is the Kronecker delta, $\phi_n(t)$ represents the external force applied to the nth particle and is assumed to be a known function of time. This system of equations is to be solved for $x_n(t)$ subject to the initial conditions

$$x_n(0) = d_n, \quad \dot{x}_n(0) = v_n. \tag{2}$$

For cases (c) and (d), in which $x_0(t)$ is specified, Eq. (1) for $n = 0$ determines the applied force $\phi_0(t)$ required to achieve such a specified motion for the particle $n = 0$.

If one assumes that x_n and ϕ_n have the Laplace transforms

$$X_n(s) = L\{x_n(t)\} = \int_0^\infty dt x_n(t) \exp(-st), \tag{3}$$

$$\Phi_n(s) = L\{\phi_n(t)/k\}, \tag{4}$$

then Eqs. (1) and (2) lead to an inhomogeneous linear difference equation of second order

$$X_{n+1} - 2\left(2\sigma^2 + 4\mu\sigma + 2\alpha^2 - (1/2)\begin{bmatrix} 0 \\ \delta_{n0} \end{bmatrix}\right)X_n + X_{n-1}\begin{bmatrix} 1 \\ (1-\delta_{n0}) \end{bmatrix} = -H_n, \quad \begin{matrix} (5a, 5d) \\ (5b, 5c) \end{matrix}$$