# Computing $\mathcal{L}$-invariants for the symmetric square of an elliptic curve

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**Abstract:** *Let $E$ be an elliptic curve over $\mathbb{Q}$, and $p \neq 2$ a prime of good ordinary reduction. The $p$-adic $L$-function for $\mathrm{Sym}^2 E$ always vanishes at $s = 1$, even though the complex $L$-function does not have a zero there. The $\mathcal{L}$-invariant itself appears on the right-hand side of the formula*

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{L}_p(\mathrm{Sym}^2 E, s)\Big|_{s=1} = \mathcal{L}_p(\mathrm{Sym}^2 E) \times (1-\alpha_p^{-2})(1-p\alpha_p^{-2}) \times \frac{L_\infty(\mathrm{Sym}^2 E, 1)}{(2\pi i)^{-1}\Omega_E^+\Omega_E^-}$$

*where $X^2 - a_p(E)X + p = (X-\alpha_p)(X-\beta_p)$ with $\alpha_p \in \mathbb{Z}_p^\times$.*

*We first devise a method to calculate $\mathcal{L}_p(\mathrm{Sym}^2 E)$ effectively, then show it is non-trivial for all elliptic curves $E$ of conductor $N_E \leq 300$ with $4|N_E$, and almost all ordinary primes $p < 17$. Hence, in these cases at least, the order of the zero in $\mathbf{L}_p(\mathrm{Sym}^2 E, s)$ at $s = 1$ is exactly one.*

## Contents



[1]To form a part of this author's PhD thesis

# 1 Introduction

Let $p$ be an odd prime. For a pure motive $M$ defined over $\mathbb{Q}$ of weight zero, there is a conjectural recipe to attach a $p$-adic $L$-function, $\mathbf{L}_p(M,s)$, provided its Galois representation is $p$-ordinary (see [6] for the precise details). The interpolated $p$-adic $L$-function should be related to its complex cousin, $L_\infty(M,s)$, at the critical point $s=0$, via the formula

$$\mathbf{L}_p(M,0) \ = \ \mathcal{E}_p(M,0) \times \frac{L_\infty(M,0)}{\Omega_\infty(M)} \ .$$

Here $\mathcal{E}_p(M,s)$ is a product of certain Euler factors at $p$, and $\Omega_\infty(M)$ denotes the Deligne period.

Curiously, sometimes $\mathcal{E}_p(M,s)$ can vanish at $s=0$ even when $L_\infty(M,0)\neq 0$, in which case we say that $M$ has an *exceptional $p$-adic zero*. Let us factorise out the trivial zero contribution into $\mathcal{E}_p(M,s) = \mathcal{E}_p^\dagger(M,s)\times\mathcal{E}_p^{\mathrm{triv}}(M,s)$, where $\mathcal{E}_p^\dagger(M,0)\neq 0$ and $\mathrm{order}_{s=0}\big(\mathcal{E}_p^{\mathrm{triv}}(M,s)\big) = \mathbf{e}_p$. Greenberg [19] has associated an explicit invariant $\mathcal{L}_p^{\mathrm{Gr}}(M)\in\mathbb{Q}_p$, and he predicts that

$$\left.\frac{\mathrm{d}^{\mathbf{e}_p}\mathbf{L}_p(M,s)}{\mathrm{d}s^{\mathbf{e}_p}}\right|_{s=0} \ = \ \mathcal{L}_p^{\mathrm{Gr}}(M)\times\mathcal{E}_p^\dagger(M,0)\times\frac{L_\infty(M,0)}{\Omega_\infty(M)} \ .$$

One is naturally left to address the following problem.

**Question.** *For a given motive $M$ as described above, and for an ordinary prime $p$ satisfying the exceptional zero condition, is Greenberg's $\mathcal{L}$-invariant term $\mathcal{L}_p^{\mathrm{Gr}}(M)$ non-zero?*

For example, let $f$ be a primitive eigenform of weight $k\geq 2$, level $N$ and trivial nebentypus. Then the symmetric square motives $M=\mathrm{Sym}^2(f)(k-1)$ and $M=\mathrm{Sym}^2(f)(k)$ both exhibit exceptional $p$-adic zero phenomena at ordinary primes $p\nmid N$.

Over two decades ago, Coates and Greenberg suggested to the first-named author to compute $\mathcal{L}_p^{\mathrm{Gr}}\big(\mathrm{Sym}^2(f)(k-1)\big)$ as part of his PhD, but he was unsuccessful and the project was shelved. Recently there has been a renewed interest in this topic [2, 22, 30], in particular with the construction of global cohomology classes (an Euler system) for the motive $M(f\otimes f)$ in [27].

Under some standard assumptions, Hida has shown [26] that in a $\Lambda$-adic family of modular forms $\{\mathcal{F}_k\}_{k\in\mathcal{W}}$, the quantity $\mathcal{L}_p^{\mathrm{Gr}}\big(\mathrm{Sym}^2(\mathcal{F}_k)(k-1)\big)$ can vanish at only finitely many points in the weight-space $\mathcal{W}$. It therefore seems an appropriate time to revisit this open problem of non-vanishing for the symmetric square $\mathcal{L}$-invariant, albeit from a computational perspective.

**Goal.** *To calculate $\mathcal{L}_p^{\mathrm{Gr}}\big(\mathrm{Sym}^2(f)(1)\big)$ mod $p^m$ numerically when $f$ arises from an elliptic curve, and then check whether the associated $\mathcal{L}$-invariant is non-vanishing in a variety of examples.*

Let $E$ be an elliptic curve over $\mathbb{Q}$, so that $E$ is necessarily modular by the work in [4, 35]. Provided $\mathrm{Re}(s)>2$, the symmetric square $L$-function for $E$ is given by an Euler product

$$L_\infty(\mathrm{Sym}^2E,s) \ = \ \prod_{\text{primes } l} \det\Big(1-\mathrm{Frob}_l^{-1}X \ \Big|\ \mathrm{Sym}^2H_{\text{ét}}\big(\overline{E},\mathbb{Q}_q(1)\big)^{I_l}\Big)^{-1}\Big|_{X=l^{-s}}$$

where $q$ is any prime different from $l$, $\mathrm{Frob}_l$ is an arithmetic Frobenius element, and $I_l\subset G_{\mathbb{Q}_l}$ denotes the inertia subgroup at $l$. If the prime number $l$ does not divide the $\mathbb{Q}$-conductor $N_E$ of the elliptic curve, then

$$\det\Big(1-\mathrm{Frob}_l^{-1}X \ \Big|\ \mathrm{Sym}^2H_{\text{ét}}\big(\overline{E},\mathbb{Q}_q(1)\big)^{I_l}\Big) \ = \ \big(1-\alpha_l^2X\big)\big(1-\beta_l^2X\big)\big(1-lX\big)$$

where $1-a_l(E)X+lX^2=\big(1-\alpha_lX\big)\big(1-\beta_lX\big)$ is the factorisation of the Hecke polynomial at $l$. Gelbart and Jacquet [17] showed that the function $L_\infty(\mathrm{Sym}^2E,s)$ has an analytic continuation to all $s\in\mathbb{C}$, and satisfies a functional equation linking the value at $s$ with the value at $3-s$.

To describe Greenberg's $\mathcal{L}$-invariant term in detail, let us first fix an ordinary prime $p$. Consider the Galois representation $V = \mathrm{Sym}^2\big(H_{\text{ét}}\big(\overline{E}, \mathbb{Q}_p(1)\big)^*\big) \cong \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{Sym}^2\big(\mathrm{Ta}_p(E)\big)$ where $\mathrm{Ta}_p(E) = \varprojlim_n E_{p^n}$ is the $p$-adic Tate module of $E$. Viewed as a $G_{\mathbb{Q}_p}$-module, there is a filtration

$$0 = \mathrm{Fil}^3 V \ \subset\ \mathrm{Fil}^2 V \ \subset\ \mathrm{Fil}^1 V \ \subset\ \mathrm{Fil}^0 V = V$$

where each quotient $\frac{\mathrm{Fil}^i V}{\mathrm{Fil}^{i+1} V}$ is isomorphic to $\mathbb{Q}_p(i)$ as an $I_p$-representation, for $i \in \{0, 1, 2\}$.

Let $\Sigma$ denote a finite set of primes containing $p$ and the primes of bad reduction for $E$. Associated to the $\mathrm{Gal}\big(\mathbb{Q}_\Sigma/\mathbb{Q}\big)$-representation $V$ in [3] are the Bloch-Kato Selmer groups

$$H^1_{f,\{p\}}(\mathbb{Q}, V) \ :=\ \mathrm{Ker}\left( H^1\big(\mathrm{Gal}(\mathbb{Q}_\Sigma/\mathbb{Q}), V\big) \overset{\oplus \mathrm{res}_l}{\longrightarrow} \bigoplus_{l \in \Sigma,\, l \neq p} H^1(I_l, V) \right)$$

and

$$H^1_f(\mathbb{Q}, V) \ :=\ \mathrm{Ker}\left( H^1_{f,\{p\}}(\mathbb{Q}, V) \overset{\mathrm{res}_p}{\longrightarrow} \frac{H^1\big(G_{\mathbb{Q}_p}, V\big)}{H^1_f\big(G_{\mathbb{Q}_p}, V\big)} \right)$$

where $H^1_f(G_{\mathbb{Q}_p}, V)$ denotes the kernel of the mapping from $H^1(G_{\mathbb{Q}_p}, V)$ to $H^1(G_{\mathbb{Q}_p}, V \otimes B_{\mathrm{cris}})$. Flach et al. [16, 35, 12] have shown $H^1_f(\mathbb{Q}, V) = \{0\}$, which implies that $\dim_{\mathbb{Q}_p} H^1_{f,\{p\}}(\mathbb{Q}, V) = 1$. Let us fix a generator $\eta$ of this line, so that $H^1_{f,\{p\}}(\mathbb{Q}, V) = \mathbb{Q}_p \cdot \eta$.

We now explain how to choose coordinates. Observe first that $H^1\big(G_{\mathbb{Q}_p}, V\big) = H^1\big(G_{\mathbb{Q}_p}, \mathrm{Fil}^1 V\big)$, an assertion that can be checked from the local formula

$$\dim_{\mathbb{Q}_p} H^1\big(G_{\mathbb{Q}_p}, W\big) \ =\ \dim_{\mathbb{Q}_p}\big(W \otimes B_{\mathrm{cris}}\big)^{G_{\mathbb{Q}_p}} + \dim_{\mathbb{Q}_p} H^0\big(G_{\mathbb{Q}_p}, W\big) + \dim_{\mathbb{Q}_p} H^0\big(G_{\mathbb{Q}_p}, W^*(1)\big)$$

which yields the value $3+0+0$ if $W = V$, and $2+0+1$ if $W = \mathrm{Fil}^1 V$. Applying Kummer theory, there is a natural identification $H^1\big(G_{\mathbb{Q}_p}, \mathbb{Q}_p(1)\big) \cong \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Big(\varprojlim_n \mathbb{Q}_p^\times / \mathbb{Q}_p^{\times\, p^n}\Big)$, from which one obtains the homomorphism

$$\mathfrak{q} : H^1\big(G_{\mathbb{Q}_p}, V\big) = H^1\big(G_{\mathbb{Q}_p}, \mathrm{Fil}^1 V\big) \overset{\text{mod } \mathrm{Fil}^2}{\longrightarrow} H^1\big(G_{\mathbb{Q}_p}, \mathrm{Fil}^1 V/\mathrm{Fil}^2 V\big) \overset{\sim}{\longrightarrow} \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Big(\varprojlim_n \mathbb{Q}_p^\times / \mathbb{Q}_p^{\times\, p^n}\Big).$$

Furthermore, on the right-hand target space there is an isomorphism

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Big(\varprojlim_n \mathbb{Q}_p^\times / \mathbb{Q}_p^{\times\, p^n}\Big) \overset{\sim}{\longrightarrow} \mathbb{Q}_p \times \mathbb{Q}_p \quad \text{sending } \ q \mapsto \big(\log_p(q), \mathrm{ord}_p(q)\big),$$

where $\log_p : \mathbb{Q}_p^\times \to \mathbb{Q}_p$ denotes Iwasawa's logarithm map, normalised so that $\log_p(p) = 0$.

**Definition 1.1.** *The arithmetic $\mathcal{L}$-invariant is defined to be the slope of $\mathfrak{q} \circ \mathrm{res}_p(\eta)$ inside the vector space $H^1\big(G_{\mathbb{Q}_p}, \mathrm{Fil}^1 V/\mathrm{Fil}^2 V\big) \cong \mathbb{Q}_p \times \mathbb{Q}_p$, i.e.*

$$\mathcal{L}_p^{\mathrm{Gr}}(\mathrm{Sym}^2 E) \ :=\ \frac{\log_p\big(\mathfrak{q}\big(\mathrm{res}_p(\eta)\big)\big)}{\mathrm{ord}_p\big(\mathfrak{q}\big(\mathrm{res}_p(\eta)\big)\big)}$$

*which is clearly independent of the choice of generator $\eta$ for the $\mathbb{Q}_p$-line $H^1_{f,\{p\}}(\mathbb{Q}, V)$.*

In fact, there is a more analytic way to introduce the $\mathcal{L}$-invariant if we work with the $p$-adic $L$-function directly. Depending on the reduction type of the curve, $p$-adic $L$-functions which interpolate Dirichlet twists of $\mathrm{Sym}^2\big(h^1(E)\big)(1)$ have been constructed in [7, 9, 11, 21, 24, 31]. Now if $E$ has good ordinary reduction at $p$, there exists $\mathcal{F}(X) \in X \cdot \mathbb{Z}_p[\![X]\!] \otimes \mathbb{Q}$ such that

$$\mathcal{F}\big(\chi(1+p) - 1\big) \ =\ \frac{\tau(\overline{\chi})}{\alpha_p^{2m_\chi}} \times \frac{L_\infty(\mathrm{Sym}^2 E \otimes \chi, 1)}{(2\pi i)^{-1}\Omega_E^+ \Omega_E^-}$$

at all non-trivial characters $\chi$ of conductor $\mathfrak{f}_\chi = p^{m_\chi} > 1$ satisfying $\chi\big|_{\mathbb{F}_p^\times} = \mathbf{1}$, while $\mathcal{F}(0) = 0$. Here $\alpha_p$ is the $p$-adic unit root of $X^2 - a_p(E)X + p$, secondly $\tau(\overline{\chi})$ denotes a Gauss sum for $\chi^{-1}$, and lastly $\Omega_E^\pm$ are real/imaginary periods associated to a minimal Weierstrass equation for $E_{/\mathbb{Z}}$.

**Definition 1.2.** *We write* $\mathbf{L}_p(\mathrm{Sym}^2 E, -) : \mathbb{Z}_p \to \mathbb{Q}_p$ *for the Mazur-Mellin transform*

$$\mathbf{L}_p(\mathrm{Sym}^2 E, s) \;:=\; \mathcal{F}\big((1+p)^{s-1} - 1\big),$$

*so that* $\mathbf{L}_p(\mathrm{Sym}^2 E, s)$ *has an exceptional zero at* $s = 1$.

In the late 1980s, Coates and Greenberg made the following prediction about its first derivative.

**Conjecture 1.3.** *If $E$ has good ordinary reduction at $p$, the $\mathcal{L}$-invariant given by the ratio*

$$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E) \;:=\; \frac{\mathrm{d}}{\mathrm{d}s}\mathbf{L}_p(\mathrm{Sym}^2 E, s)\Big|_{s=1} \;\times\; \left((1-\alpha_p^{-2})(1-p\alpha_p^{-2}) \times \frac{L_\infty(\mathrm{Sym}^2 E, 1)}{(2\pi i)^{-1}\Omega_E^+\Omega_E^-}\right)^{-1}$$

*is a* **non-zero** *$p$-adic number, so in particular*

$$\mathrm{order}_{s=1}\big(\mathbf{L}_p(\mathrm{Sym}^2 E, s)\big) \;=\; 1.$$

As will be discussed at length in §3.3, in most situations the work of Citro, Dasgupta and Hida [5, 10, 26] implies that $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E) = \mathcal{L}_p^{\mathrm{Gr}}(\mathrm{Sym}^2 E)$, so we may shift between these two definitions as appropriate. In particular, the non-vanishing of the analytic $\mathcal{L}$-invariant means that the line $H^1_{f,\{p\}}(\mathbb{Q}, V)$ has a non-trivial slope inside $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \left(\varprojlim_n \mathbb{Q}_p^\times/\mathbb{Q}_p^{\times\, p^n}\right) \cong \mathbb{Q}_p \times \mathbb{Q}_p$.

*Remarks:* (a) If $E$ has complex multiplication, then a result of Ferrero and Greenberg [14] implies that $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E) = \log_p(\alpha_p^{-2})$; therefore in the CM case, Conjecture 1.3 is at least known to be true.

(b) If $E$ has split multiplicative reduction at $p$, under certain restrictions Rosso [30] recently proved $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E) = \log_p(q_E)/\mathrm{ord}_p(q_E)$ where $q_E$ is the Tate period of the rigid analytic curve; moreover $\log_p(q_E) \neq 0$ by [1, Theorem 3], so Conjecture 1.3 holds in this situation too.

(c) We should also point out that in the case where $E$ has split multiplicative reduction at $p$, the Tate period $q_E$ is a universal norm for the $\mathbb{Z}_p$-extension $F_\infty/\mathbb{Q}_p$ cut out by

$$\mathrm{Im}\Big(H^1\big(G_{\mathbb{Q}_p}, \mathrm{Sym}^2\mathrm{Ta}_p(E)\big) \overset{\mathrm{mod\ Fil}^1}{\longrightarrow} H^1\big(G_{\mathbb{Q}_p}, \mathbb{Z}_p\big)\Big)$$

inside $H^1\big(G_{\mathbb{Q}_p}, \mathbb{Z}_p\big) = \mathrm{Hom}\big(G_{\mathbb{Q}_p}, \mathbb{Z}_p\big) \cong \mathbb{Z}_p^2$. Under the Tate local pairing

$$H^1\big(G_{\mathbb{Q}_p}, \mathbb{Q}_p(1)\big) \times H^1\big(G_{\mathbb{Q}_p}, \mathbb{Q}_p\big) \to \mathbb{Q}_p,$$

the line $\mathfrak{q} \circ \mathrm{res}_p\left(H^1_{f,\{p\}}(\mathbb{Q}, V)\right)$ will then be orthogonal to the subspace $\mathrm{Hom}\big(\mathrm{Gal}(F_\infty/\mathbb{Q}_p), \mathbb{Q}_p\big)$. Applying exactly the same reasoning as [19, p154], it follows that the slopes $\log_p(q_E)/\mathrm{ord}_p(q_E)$ and $\log_p\big(\mathfrak{q}\big(\mathrm{res}_p(\eta)\big)\big)/\mathrm{ord}_p\big(\mathfrak{q}\big(\mathrm{res}_p(\eta)\big)\big)$ are actually equal.[2]

(d) Using efficient methods to compute overconvergent modular symbols, Dummit et al [13, §7.2] have computed $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ for $(E, p) = \big(X_0(11), 11\big)$, and for $p = 5$ and $E \in \{15a1, 19a1, 95a1\}$ (here we employ Cremona's elliptic curve labelling from [8]).

By devising algorithms to compute $\mathbf{L}_p(\mathrm{Sym}^2 E, 1)'$ and $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ numerically to a reasonable accuracy, and then implementing them into SAGE, we have established the following result.

[2] In the case of split multiplicative reduction the $\mathcal{L}$-invariant for $\mathrm{Sym}^2 E$ is the same as the $\mathcal{L}$-invariant for $E$, and it is further conjectured (by Greenberg) that the $\mathcal{L}$-invariants for $\mathrm{Sym}^m E$ should be independent of $m > 0$.

**Theorem 1.4.** *Let $E$ be an elliptic curve over $\mathbb{Q}$ of conductor $N_E \leq 300$, with 4 dividing $N_E$.*
*(i) If $p \in \{3,5,7\}$ is a prime of good ordinary reduction for $E$ then Conjecture 1.3 is true,*

$$\text{i.e.}\quad \mathcal{L}_p^{\text{an}}(\text{Sym}^2 E) \neq 0 \quad \textit{and} \quad \text{order}_{s=1}\big(\mathbf{L}_p(\text{Sym}^2 E, s)\big) = 1.$$

*(ii) If $p = 11$ is a prime of good ordinary reduction for $E$ then Conjecture 1.3 is true, with the possible exception of the following four elliptic curves:*

$$116a1,\ \ 124b1,\ \ 200a1,\ \ 296a1.$$

*(iii) If $p = 13$ is a prime of good ordinary reduction for $E$ then Conjecture 1.3 is true, with the possible exception of the following six elliptic curves:*

$$140a1,\ \ 200b1,\ \ 232b1,\ \ 244a1,\ \ 272b1,\ \ 280a1.$$

Here is a brief plan of the paper.

Sections 2.1–2.5 explain the theory behind our method. We first derive a technical result about the Petersson inner product in §2.1. In the next two sections, we relate the moments of the $p$-adic measure interpolating $\text{Sym}^2 E(1)$ to a specific inner product involving the weight two newform $f_E$ obtained from $E$ by modularity, and an auxiliary weight two form "$R|U_p^{2m-1}$" which is independent of the elliptic curve. Then in §2.4, we use our identity from §2.1 to obtain an expression for $\mathbf{L}_p(\text{Sym}^2 E, 1)' \bmod p^m$, thereby yielding an approximation to $\mathcal{L}_p^{\text{an}}(\text{Sym}^2 E)$. Lastly in §2.5, we discuss how one might increase both the speed and the accuracy of our computations by instead evaluating Tate-twists for the $p$-adic measure, although there remain significant technical hurdles to overcome if one adopts this approach.

Sections 3.1–3.2 contain an implementation of our algorithms. We chose $N_E \leq 300$ and $p \leq 13$ as our bounds to determine whether or not $\mathbf{L}_p(\text{Sym}^2 E, 1)'$ was non-zero, and thence to tabulate the $\mathcal{L}$-invariants to a decent accuracy – see Appendix B for the full numerical results. It took ten months to run our programs within these limited ranges, on a single core of an Intel i5-2400. The $4 + 6 = 10$ missing pairs $(E, p)$ in Theorem 1.4(ii)-(iii) occurred because the run-time required to show that $\mathbf{L}_p(\text{Sym}^2 E, 1)' \neq 0$ for these specimens was too slow.

Finally in §3.3, we discuss how the work of Hida, Citro and Dasgupta [26, 5, 10] combined with Theorem 1.4 implies the non-vanishing for the derivative of the Hecke eigenvalue $a_p(\mathcal{F}_k)$ at weight $k = 2$, where $\mathcal{F}$ denotes the $p$-ordinary family lifting $f_E$.

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## 2 The Analytic Theory

For an integer $N > 1$, we write $\mathcal{M}_2\big(\Gamma_0(N)\big)$ for the $\mathbb{C}$-vector space of modular forms of weight two, level $N$ and trivial nebentypus. We also denote by $\mathcal{S}_2\big(\Gamma_0(N)\big)$ the subspace of cusp forms. Throughout one normalises the Petersson inner product by

$$\langle g, h\rangle_N \ := \int_{\Gamma_0(N)\backslash\mathfrak{H}} \overline{g(z)}h(z)\cdot \mathrm{d}x\mathrm{d}y \qquad \text{for all } g \in \mathcal{S}_2\big(\Gamma_0(N)\big) \text{ and } h \in \mathcal{M}_2\big(\Gamma_0(N)\big).$$

Let $f_E \in \mathcal{S}_2^{\text{new}}\big(\Gamma_0(N_E)\big)$ denote the primitive form associated to the modular elliptic curve $E$. Without loss of generality, we assume that

- the conductor $N_E$ of the newform $f_E$ is divisible by 4.

Because $L_\infty(\text{Sym}^2 E, s)$ is invariant under taking quadratic twists, one can always ensure that the above holds by replacing $E$ with its twist by the unique character of conductor 4 (if necessary). We also modify the quantities in Conjecture 1.3, as follows:

- we swap the motivic period $(2\pi i)^{-1}\Omega_E^+\Omega_E^-$ with the automorphic period $\pi\langle f_E, f_E\rangle_{N_E}$;
- we shall exchange the primitive symmetric square $L$-function $L_\infty(\text{Sym}^2 E \otimes \chi, s)$ with its imprimitive version

$$D(E,\chi,s) \ := \ L_{\mathfrak{f}_\chi N_E}\big(\chi^2, 2s-2\big) \times \sum_{n=1}^{\infty} \frac{\chi(n) a_{n^2}(E)}{n^s} \ ;$$

- we replace the $p$-adic $L$-function with $\mathbf{L}_p^{\text{imp}}(\text{Sym}^2 E, s) := \mathcal{F}^{\text{imp}}\big((1+p)^{s-1}-1\big)$, where

$$\mathcal{F}^{\text{imp}}\big(\chi(1+p)-1\big) \ = \ \frac{\tau(\overline{\chi})}{\alpha_p^{2m_\chi}} \times \frac{D(E,\chi,1)}{\pi\langle f_E, f_E\rangle_{N_E}} \ \text{ if } \chi \neq \mathbf{1}, \quad \text{with } \mathcal{F}^{\text{imp}}(0) = 0.$$

**Providing the imprimitive $L$-function is non-vanishing at $s=1$ so that** $D(E,1) \neq 0$, the $\mathcal{L}$-invariant may be equivalently rewritten as

$$\mathcal{L}_p^{\text{an}}(\text{Sym}^2 E) \ := \ \frac{\mathrm{d}}{\mathrm{d}s}\mathbf{L}_p^{\text{imp}}(\text{Sym}^2 E, s)\Big|_{s=1} \times \left((1-\alpha_p^{-2})(1-p\alpha_p^{-2}) \times \frac{D(E,1)}{\pi\langle f_E, f_E\rangle_{N_E}}\right)^{-1}. \quad (1)$$

The right-hand bracketed term in Equation (1) is reasonably straightforward to evaluate.

**Lemma 2.1.** *Assume $E$ has minimal conductor amongst its quadratic twists. If the geometric conductor of* $\text{Sym}^2\big(h^1(E)\big)$ *is denoted by $C_{\text{Sym}^2 E} \in \mathbb{N}^2$, then one has the formula*

$$\frac{D(E,1)}{\pi\langle f_E, f_E\rangle_{N_E}} \ = \ \frac{4\cdot\sqrt{C_{\text{Sym}^2 E}}}{N_E} \times \prod_{l \in S_1} \frac{H_l(l^{-1})}{H_l(l^{-2})}$$

*where $L_\infty(\text{Sym}^2 E \otimes \chi, s) = D(E,\chi,s) \times \prod_{l\in S_1} H_l\big(\chi(l) l^{-s}\big)^{-1}$ for a finite set of bad primes $S_1$.*

*Proof.* If we define $\Lambda_\infty(\text{Sym}^2 E, s) := \big(C_{\text{Sym}^2 E}\big)^{s/2} \cdot \pi^{-s/2}\Gamma(s/2)(2\pi)^{-s}\Gamma(s) \times L_\infty(\text{Sym}^2 E, s)$, then the functional equation [7, Thm 2.2] for this completed $L$-function states that

$$\Lambda_\infty(\text{Sym}^2 E, s) \ = \ \Lambda_\infty(\text{Sym}^2 E, 3-s).$$

Combining the above equation at $s=2$ with the formula $D(E,2) = \frac{8\pi^3}{N_E} \times \langle f_E, f_E\rangle_{N_E}$ for the imprimitive symmetric square $L$-function in [15, Equation (5)], the result follows easily. $\square$

To calculate $\mathcal{L}_p^{\text{an}}(\text{Sym}^2 E)$ numerically, we must therefore evaluate $\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{L}_p^{\text{imp}}(\text{Sym}^2 E, s)$ at $s=1$ to a reasonable accuracy. If $\mu_E^{\text{imp}} \in \text{Meas}(\mathbb{Z}_p^\times, \mathbb{Q}_p)$ is the $p$-bounded measure corresponding to the power series $\mathcal{F}^{\text{imp}}(X) \in X \cdot \mathbb{Z}_p[\![X]\!][1/p]$, then

$$\mathbf{L}_p^{\text{imp}}(\text{Sym}^2 E, s) \ = \ \int_{x\in\mathbb{Z}_p^\times} <x>_p^{s-1} \cdot \mathrm{d}\mu_E^{\text{imp}}(x) \quad \text{for every } s \in \mathbb{Z}_p,$$

where $<->_p\colon \mathbb{Z}_p^\times \twoheadrightarrow 1+p\mathbb{Z}_p$ denotes the projection to the principal local units. It follows that

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, s)\Big|_{s=1} = \int_{x\in\mathbb{Z}_p^\times} \log_p <x>_p \cdot \mathrm{d}\mu_E^{\mathrm{imp}}(x) \;\approx \sum_{e\in(\mathbb{Z}/p^m\mathbb{Z})^\times} \log_p <e>_p \times \mu_E^{\mathrm{imp}}(e+p^m\mathbb{Z}_p)$$

upon using a Riemann sum approximation for the covering $\mathbb{Z}_p^\times = \bigsqcup_e \left(e+p^m\mathbb{Z}_p\right)$.

**Question.** *Given a class $e\in(\mathbb{Z}/p^m\mathbb{Z})^\times$, how do we calculate each $\mu_E^{\mathrm{imp}}(e+p^m\mathbb{Z}_p)$ efficiently?*

It is well known [7, 9, 31] the moments $\mu_E^{\mathrm{imp}}(e+p^m\mathbb{Z}_p)$ can be written as an inner product of

$$f^0(z) \;:=\; \Big(f_E(z)-\beta_p f_E(pz)\Big)\Big|\begin{pmatrix} 0 & -1 \\ pN_E & 0 \end{pmatrix} \;\in\; \mathcal{S}_2\big(\Gamma_0(pN_E)\big)$$

with a certain modular form $R_{m,e}\in\mathcal{M}_2\big(\Gamma_0(pN_E)\big)$, whose Fourier coefficients are $p$-integral. The integrality of $\mu_E^{\mathrm{imp}}(-)$ is then controlled by that of $\frac{\langle f^0,R_{m,e}\rangle_{pN_E}}{\langle f_E,f_E\rangle}$ for varying $m$ and $e$.

## 2.1 Petersson inner product identities for $f^0$

Recall that the functional equation for the completed Hasse-Weil $L$-function, $\Lambda_\infty(E,s)$, has the form $\Lambda_\infty(E,2-s)=w_E\times\Lambda_\infty(E,s)$ where $w_E\in\{\pm1\}$ denotes the root number for $E$ over $\mathbb{Q}$. In terms of the associated newform,

$$f_E\big|W(N_E) \;=\; -w_E\cdot f_E \qquad \text{under the action of } W(N_E)=\begin{pmatrix} 0 & -1 \\ N_E & 0 \end{pmatrix}.$$

Let $h(z)$ denote a weight 2 holomorphic modular form of level $pN_E$, and with trivial character. Our goal here is to derive the following technical result, which we repeatedly make use of later.

**Lemma 2.2.** *(i) If $\mathbb{C}\cdot h(z)\cap\big(\mathbb{C}\cdot f_E(z)\oplus\mathbb{C}\cdot f_E(pz)\big)=\{0\}$, then $\langle f^0,h\rangle_{pN_E}=0$;*

*(ii) If $w_E\in\{\pm1\}$ is the root number for $E_{/\mathbb{Q}}$, then $\langle f^0,f_E(z)\rangle_{pN_E}=-w_E\cdot\frac{\alpha_p^2-1}{\alpha_p}\times\langle f_E,f_E\rangle_{N_E}$;*

*(iii) Replacing $f_E(z)$ with $f_E(pz)$, one instead has $\langle f^0,f_E(pz)\rangle_{pN_E}=-w_E\cdot\frac{\alpha_p^2-1}{\alpha_p^2}\times\langle f_E,f_E\rangle_{N_E}$.*

*Proof.* The $f_E$-isotypic part of $\mathcal{M}_2\big(\Gamma_0(pN_E)\big)$ consists of the subspace $\mathbb{C}\cdot f_E(z)\oplus\mathbb{C}\cdot f_E(pz)$. Without loss of generality, assume $h(z)$ is an eigenform for the Hecke algebra at level $pN_E$. Then by multiplicity one, we can pick a prime $l\nmid pN_E$ such that $a_l(f_E)\neq a_l(h)$; consequently

$$a_l(f_E)\times\langle f^0,h\rangle_{pN_E} \;=\; \langle f^0|T_l^*,h\rangle_{pN_E} \;=\; \langle f^0,h|T_l\rangle_{pN_E} \;=\; a_l(h)\times\langle f^0,h\rangle_{pN_E}$$

in which case $\langle f^0,h\rangle_{pN_E}=0$, so part (i) is true.

To establish statement (ii), let us first introduce the $p$-stabilisation

$$f_0(z) \;:=\; f_E(z)-\beta_p f_E(pz) \;=\; \alpha_p^{-1}\cdot f_E\Big|\big(U_p-\beta_p I_2\big) \;\in\; \mathcal{S}_2\big(\Gamma_0(pN_E)\big). \tag{2}$$

This cusp form $f_0$ is related to $f^0$ through the formula

$$f^0(z) \;=\; f_E^\rho\big|W(pN_E)-\alpha_p p^{-1} f_E^\rho\big|W(N_E) \;=\; f_0^\rho\big|W(pN_E), \tag{3}$$

where the involution $(-)^\rho$ above sends each $h(z)=\sum_{n\geq 1} h_n e^{2\pi i n z}$ to $h^\rho(z)=\sum_{n\geq 1}\overline{h_n}e^{2\pi i n z}$. Now using Equation (3) and observing that $f_E^\rho = f_E$, one obtains the equalities

$$\begin{aligned}\left\langle f^0, f_E\right\rangle_{pN_E} &= \left\langle f_E^\rho \big| W(pN_E), f_E\right\rangle_{pN_E} - \overline{\alpha_p p^{-1}}\cdot \left\langle f_E^\rho\big|W(N_E), f_E\right\rangle_{pN_E} \\ &= \left\langle f_E^\rho \Big| W(N_E)\Big| \begin{pmatrix} p & 0 \\ 0 & 1\end{pmatrix}, f_E\right\rangle_{pN_E} - \beta_p p^{-1}\cdot\left\langle -w_E\cdot f_E, f_E\right\rangle_{pN_E} \\ = \left\langle -w_E\cdot f_E\Big|\begin{pmatrix} p & 0 \\ 0 & 1\end{pmatrix}\Big|\mathrm{Tr}^{\Gamma_0(pN_E)}_{\Gamma_0(N_E)}, f_E\right\rangle_{N_E} &+ w_E\beta_p p^{-1}\big[\Gamma_0(N_E):\Gamma_0(pN_E)\big]\cdot\left\langle f_E, f_E\right\rangle_{N_E}.\end{aligned}$$

Note from the trace map identity $f_E\Big|\begin{pmatrix} p & 0 \\ 0 & 1\end{pmatrix}\Big|\mathrm{Tr}^{\Gamma_0(pN_E)}_{\Gamma_0(N_E)} = f_E\big|T_p^* = \overline{a_p(E)}f_E$ together with the index formula $\big[\Gamma_0(N_E):\Gamma_0(pN_E)\big]=p+1$, the above becomes

$$\left\langle f^0, f_E\right\rangle_{pN_E} = w_E\cdot\left(-a_p(E)+\beta_p\cdot\frac{p+1}{p}\right)\times\left\langle f_E, f_E\right\rangle_{N_E} = -w_E\cdot\left(\frac{\alpha_p^2-1}{\alpha_p}\right)\times\left\langle f_E, f_E\right\rangle_{N_E}.$$

Lastly to prove that (iii) is true, one knows from Equation (2) that

$$\left\langle f^0, f_E(pz)\right\rangle_{pN_E} = \left\langle f^0,\ \beta_p^{-1}\big(f_E(z)-f_0(z)\big)\right\rangle_{pN_E} = \beta_p^{-1}\times\left(\left\langle f^0, f_E\right\rangle_{pN_E} - \left\langle f^0, f_0\right\rangle_{pN_E}\right).$$

The first term $\langle f^0, f_E\rangle_{pN_E}$ is already determined from (ii) above. To compute the second term,

$$\left\langle f^0, f_0\right\rangle_{pN_E} = (p\alpha_p)^{-1}(\alpha_p-\beta_p)(p\alpha_p-\beta_p)\cdot\left\langle f_E^\rho\big|W(N_E), f_E\right\rangle_{N_E}$$

upon applying [18, §C.5, Lemma 1], and clearly one has $\left\langle f_E^\rho\big|W(N_E), f_E\right\rangle_{N_E} = -w_E\left\langle f_E, f_E\right\rangle_{N_E}$. Combining these strands together:

$$\begin{aligned}\left\langle f^0, f_E(pz)\right\rangle_{pN_E} &= \beta_p^{-1}\times\left(-w_E\cdot\frac{\alpha_p^2-1}{\alpha_p} + w_E\cdot\frac{(\alpha_p-\beta_p)(p\alpha_p-\beta_p)}{p\alpha_p}\right)\times\left\langle f_E, f_E\right\rangle_{N_E} \\ = -w_E\cdot\left(\frac{\alpha_p^2-1}{p}-\frac{(\alpha_p-\beta_p)(p\alpha_p-\beta_p)}{p^2}\right)&\times\left\langle f_E, f_E\right\rangle_{N_E} = -w_E\cdot\left(\frac{\alpha_p^2-1}{\alpha_p^2}\right)\times\left\langle f_E, f_E\right\rangle_{N_E}\end{aligned}$$

which completes the demonstration of (iii), and thereby the lemma. □

## 2.2 The $q$-expansion of the modular form $R_{m,e}$

The key ingredient in calculating the first derivative of $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, s)$ at $s=1$, is that the moments of the measure $\mathrm{d}\mu_E^{\mathrm{imp}}(-)$ can be written in terms of the $f^0$-isotypic projection of a holomorphic modular form. More precisely, let us recall from [7, Eqs (3.22)-(3.23)] that

$$\mu_E^{\mathrm{imp}}(e+p^m\mathbb{Z}_p) = 2\alpha_p^{-2m}\times\frac{\left\langle f^0, R_{m,e}\big|U_p^{2m-1}\right\rangle_{pN_E}}{\left\langle f_E, f_E\right\rangle_{N_E}} \tag{4}$$

where $R_{m,e}\in\mathcal{M}_2\big(\Gamma_0(p^{2m}N_E)\big)$ is obtained by summing up products of certain theta-functions of weight $1/2$ with Eisenstein series of weight $3/2$ (the precise definitions will not be needed). Note also from [7, Lemma 3.10(ii)], the classical trace map identity

$$h\big|U_p^{2m-1} = h\Big|W(p^{2m}N_E)\Big|\mathrm{Tr}^{\Gamma_0(p^{2m}N_E)}_{\Gamma_0(pN_E)}\Big|W(pN_E)$$

implies that $R_{m,e}\big|U_p^{2m-1}$ actually has level $pN_E$, so the inner product above is well-defined.

*Remarks:* (i) If $R_{m,e} = \sum_{n=0}^{\infty} r_n(m,e)q^n$ then clearly $R_{m,e}\big|U_p^{2m-1} = \sum_{n=0}^{\infty} r_{np^{2m-1}}(m,e)q^n$; furthermore, $r_0(m,e) = 0$ since the theta-functions of weight $1/2$ vanish at the cusp $\infty$.

(ii) Applying [7, Theorem 3.11] each coefficient $r_n(m,e) \in \mathbb{Q}$, in fact $r_n(m,e) \in \mathbb{Z}_p$ if $p^{2m-1}\big|n$; it therefore follows that $R_{m,e}\big|U_p^{2m-1} \in q \cdot \mathbb{Z}_{(p)}\big[\big[q\big]\big]$.

(iii) Assuming $p^{2m-1}$ divides $n$, from [7, p133] the $q^n$-coefficient of $R_{m,e}$ is given by

$$r_n(m,e) \;=\; \frac{-2}{\phi(p^m)} \times \sum_{\chi\in\Delta_m}\ \sum_{(n_1,n_2)\in\mathcal{W}_n}\ \sum_{(a,b)\in\mathcal{V}_{n_2}} \mu(a)b\cdot\varepsilon_{n_2}(a)\chi(b^2a)\chi^{-1}(n_1e)\cdot L_{N_E}(\chi\varepsilon_{n_2},0). \tag{5}$$

Here we have employed the notation:

- $\Delta_m$ denotes the set of *non-trivial* Dirichlet characters of conductor dividing $p^m$;
- $\mathcal{W}_n$ is the set of pairs $(n_1,n_2) \in \mathbb{N}\times\mathbb{N}$ coprime to $p$, and satisfying $n_1^2 \times \frac{N_E}{4} + n_2 = n$;
- $\mathcal{V}_{n_2}$ consists of pairs $(a,b) \in \mathbb{N}\times\mathbb{N}$ that are coprime to $pN_E$, such that $(ab)^2$ divides $n_2$;
- $\varepsilon_{n_2}$ refers to the character of the imaginary quadratic field $\mathbb{Q}\big(\sqrt{-n_2N_E}\big)$.

As usual, $L_{N_E}(\chi\varepsilon_{n_2},s)$ indicates the $\chi\varepsilon_{n_2}$-twisted zeta-function with its Euler factors at the primes dividing $N_E$ removed.

**Definition 2.3.** *(a) For an integer $t \geq 1$ and $y \in \mathbb{Z}$ with $p \nmid y$, one defines*

$$\vartheta_t(y) \;=\; \begin{cases} (p-1)^2/p^2 & \text{if } t\geq 2 \text{ and } y\equiv 1 \pmod{p^t} \\ -(p-1)/p^2 & \text{if } t\geq 2,\ y\not\equiv 1 \pmod{p^t} \text{ but } y\equiv 1 \pmod{p^{t-1}} \\ 0 & \text{if } t\geq 2 \text{ and } y\not\equiv 1 \pmod{p^{t-1}} \\ (p-2)/p & \text{if } t=1 \text{ and } y\equiv 1 \pmod{p} \\ -1/p & \text{if } t=1 \text{ and } y\not\equiv 1 \pmod{p}. \end{cases}$$

*(b) For any $m \in \mathbb{N}$ and integers $x, n_2$ both coprime to $p$, we set*

$$M_m^{(n_2)}(x) \;:=\; \sum_{t=1}^{m}\ \sum_{\substack{j=1,\\ p\nmid j}}^{p^t} p^t\cdot\vartheta_t(xj) \;\times\; \frac{-1}{\mathfrak{f}_{\varepsilon_{n_2}}}\cdot \sum_{i=0}^{\mathfrak{f}_{\varepsilon_{n_2}}-1} \varepsilon_{n_2}(i)\cdot\big((i-j)p^{-t}\big)^{\sharp}$$

*where $\big((i-j)p^{-t}\big)^{\sharp} \in \big\{0,\dots,\mathfrak{f}_{\varepsilon_{n_2}}-1\big\}$ is the unique integer congruent to $(i-j)p^{-t}$ mod $\mathfrak{f}_{\varepsilon_{n_2}}$.*

The following yields an alternate expression for $r_n(m,e)$, designed for use in our programs.

**Proposition 2.4.** *If $p^{2m-1}$ divides $n$, then the $q^n$-coefficient of $R_{m,e}$ is given by*

$$r_n(m,e) \;=\; -2\cdot \sum_{(n_1,n_2)\in\mathcal{W}_n}\ \sum_{(a,b)\in\mathcal{V}_{n_2}}\ \sum_{d|N_E} \mu(ad)b\varepsilon_{n_2}(ad) \times \phi(p^m)^{-1}\cdot M_m^{(n_2)}\big(ab^2d(n_1e)^*\big)$$

*where $(n_1e)^* \in \{1,\dots,p^m-1\}$ denotes the multiplicative inverse of $n_1e$ modulo $p^m$.*

Before we give the demonstration, we make a couple of observations.

Firstly, the main expense in computing $r_n(m,e)$ is in tabulating the values of $\varepsilon_{n_2}$ necessary to compute $M_m^{(n_2)}(-)$. The length of time required to compute $r_n(m,e)$ is roughly proportional to the sum $\sum_{(n_1,n_2)\in\mathcal{W}_n}\mathfrak{f}_{\varepsilon_{n_2}}$, which has order $O(p^{3m})$ as a function of $m$.

Secondly, the quantity $\phi(p^m)^{-1}\cdot M_m^{(n_2)}\big(ab^2d(n_1e)^*\big)$ occurring above is actually $p$-integral. The reason is that $M_m^{(n_2)}(-)$ coincides with '$M_m(-)$' defined in [7, Eq (3.30)], and then by Lemma 3.12 of *op. cit.*, the latter is congruent to zero modulo $p^{m-1}$. However, once one has programmed in the function $\vartheta_t$, our version $M_m^{(n_2)}(-)$ is the quicker to calculate numerically.

*Proof.* If one recalls the standard identity $L_{N_E}(\chi\varepsilon_{n_2},s)=\sum_{d|N_E}\mu(d)\chi(d)\varepsilon_{n_2}(d)d^{-s}\cdot L(\chi\varepsilon_{n_2},s)$, then Equation (5) can be rewritten as

$$r_n(m,e) \;=\; \frac{-2}{\phi(p^m)}\times\sum_{(n_1,n_2)\in\mathcal{W}_n}\;\sum_{(a,b)\in\mathcal{V}_{n_2}}\;\sum_{d|N_E}\mu(ad)b\cdot\varepsilon_{n_2}(ad)\cdot\sum_{\chi\in\Delta_m}\chi\left(\frac{ab^2d}{n_1e}\right)L(\chi\varepsilon_{n_2},0).$$

Therefore, it is enough to show that $\sum_{\chi\in\Delta_m}\chi(x)L(\chi\varepsilon_{n_2},0)$ is equal to the quantity $M_m^{(n_2)}(x)$. Now as each $L(\chi\varepsilon_{n_2},0)=-B_{1,\chi\varepsilon_{n_2}}$ with $B_{1,\chi\varepsilon_{n_2}}$ denoting a $\chi\varepsilon_{n_2}$-twisted Bernoulli number,

$$L(\chi\varepsilon_{n_2},0) \;=\; \frac{-1}{\mathfrak{f}_\chi\mathfrak{f}_{\varepsilon_{n_2}}}\cdot\sum_{a=1}^{\mathfrak{f}_\chi\mathfrak{f}_{\varepsilon_{n_2}}}\chi\varepsilon_{n_2}(a)\cdot a \;=\; \frac{-1}{\mathfrak{f}_{\varepsilon_{n_2}}}\cdot p^{-t}\times\sum_{i=1}^{\mathfrak{f}_{\varepsilon_{n_2}}}\sum_{j=1}^{p^t}\chi(a_{i,j})\varepsilon_{n_2}(a_{i,j})\cdot a_{i,j}$$

where $\mathfrak{f}_\chi=p^t>1$ say, and the integers $a_{i,j}:=(i-1)p^t+j$. Moreover $\chi(a_{i,j})=\chi(j)$, so decomposing $\Delta_m$ into a disjoint union of $\Delta_t-\Delta_{t-1}$'s yields

$$\begin{aligned}\sum_{\chi\in\Delta_m}\chi(x)L(\chi\varepsilon_{n_2},0) \;&=\; \sum_{t=1}^{m}\sum_{\chi\in\Delta_t-\Delta_{t-1}}\chi(x)L(\chi\varepsilon_{n_2},0)\\ &=\; \sum_{t=1}^{m}\sum_{\substack{j=1,\\ p\nmid j}}^{p^t}\left(p^{-t}\cdot\sum_{\chi\in\Delta_t-\Delta_{t-1}}\chi(xj)\right)\times\frac{-1}{\mathfrak{f}_{\varepsilon_{n_2}}}\cdot\sum_{i=1}^{\mathfrak{f}_{\varepsilon_{n_2}}}\varepsilon_{n_2}(a_{i,j})\cdot a_{i,j}.\end{aligned}$$

The lemma will now follow, provided one can verify that:

(i) $p^{-t}\cdot\sum_{\chi\in\Delta_t-\Delta_{t-1}}\chi(xj)$ equals $\vartheta_t(xj)$;

(ii) $\sum_{i=1}^{\mathfrak{f}_{\varepsilon_{n_2}}}\varepsilon_{n_2}(a_{i,j})\cdot a_{i,j}$ coincides with $p^t\cdot\sum_{i=0}^{\mathfrak{f}_{\varepsilon_{n_2}}-1}\varepsilon_{n_2}(i)\cdot\big((i-j)p^{-t}\big)^\sharp$.

To establish statement (i), if $\overline{\Delta}_t=\Delta_t\cup\{\mathbf{1}\}$ for $t>0$ with $\overline{\Delta}_0=\{\mathbf{1}\}$ then

$$\begin{aligned}p^{-t}\cdot\sum_{\chi\in\Delta_t-\Delta_{t-1}}\chi(xj) \;&=\; p^{-t}\cdot\left(\sum_{\chi\in\overline{\Delta}_t}\chi(xj)\;-\;\sum_{\chi\in\overline{\Delta}_{t-1}}\chi(xj)\right)\\ &=\; p^{-t}\cdot\Big(\phi(p^t)\times\mathrm{char}_{1\bmod p^t}(xj)\;-\;\phi(p^{t-1})\times\mathrm{char}_{1\bmod p^{t-1}}(xj)\Big)\end{aligned}$$

where $\mathrm{char}_{1\bmod p^t}(y)$ returns 1 if $p^t$ divides $y-1$, and returns 0 otherwise. It is then routine to check that the above formula agrees with $\vartheta_t(xj)$ from Definition 2.3.

To prove that (ii) is true, we first observe that

$$\sum_{i=1}^{\mathfrak{f}_{\varepsilon_{n_2}}}\varepsilon_{n_2}(a_{i,j})\cdot a_{i,j} \;=\; p^t\sum_{i=0}^{\mathfrak{f}_{\varepsilon_{n_2}}-1}\varepsilon_{n_2}\big(ip^t+j\big)\cdot i\;+\;j\sum_{i=0}^{\mathfrak{f}_{\varepsilon_{n_2}}-1}\varepsilon_{n_2}\big(ip^t+j\big)$$

and the right-most summation is identically zero. Furthermore

$$p^t\cdot\sum_{i=0}^{\mathfrak{f}_{\varepsilon_{n_2}}-1}\varepsilon_{n_2}\big(ip^t+j\big)\cdot i \;=\; p^t\cdot\sum_{i=0}^{\mathfrak{f}_{\varepsilon_{n_2}}-1}\varepsilon_{n_2}(i)\cdot\big((i-j)p^{-t}\big)^\sharp$$

so statement (ii) is also verified. □

### 2.3 Expressing $R_{m,e}\big|U_p^{2m-1}$ in terms of a rational basis

The next stage is to write $R_{m,e}\big|U_p^{2m-1}$ in terms of an explicit rational basis of $\mathcal{M}_2\big(\Gamma_0(pN_E)\big)$. One first uses the decomposition

$$\mathcal{M}_2\big(\Gamma_0(pN_E)\big) \;=\; \mathcal{S}_2\big(\Gamma_0(pN_E)\big) \oplus \mathrm{Eis}_2\big(\Gamma_0(pN_E)\big)$$

where the second summand denotes the space of generalised Eisenstein series of weight two, level $pN_E$ and trivial nebentypus.

**Definition 2.5.** *For two Dirichlet characters $\chi$ and $\psi$, we define the q-expansion*

$$\mathcal{E}_2(\chi,\psi)(q) \;=\; c_0(\chi,\psi)q^0 \;+\; \sum_{m=1}^{\infty}\left(\sum_{d|m}\psi(d)\chi\big(m/d\big)\cdot d\right) q^m \;\in\; \overline{\mathbb{Q}}[[q]]$$

*where the constant term* $c_0(\chi,\psi) = \begin{cases} 0 & \text{if } \mathfrak{f}_\chi > 1 \\ -B_{2,\psi}/4 & \text{if } \mathfrak{f}_\chi = 1. \end{cases}$

Let $\mathfrak{X}_{pN_E}$ be the group of Dirichlet characters $\chi : (\mathbb{Z}/pN_E\mathbb{Z})^\times \to \mathbb{C}^\times$. One can then form sets

$$\Sigma_1 \;:=\; \Big\{\mathcal{E}_2(\mathbf{1},\mathbf{1})(q) - c\cdot\mathcal{E}_2(\mathbf{1},\mathbf{1})(q^c) \text{ where } c\in\mathbb{N},\, c\neq 1 \text{ and } c\big|pN_E\Big\}, \qquad \text{and}$$

$$\Sigma_2 \;:=\; \Big\{\mathcal{E}_2(\chi,\chi^{-1})(q^c) \text{ where } c\in\mathbb{N},\, \chi\in\mathfrak{X}_{pN_E}-\{\mathbf{1}\} \text{ and } c\mathfrak{f}_\chi^2\big|pN_E\Big\}.$$

Remember throughout we have assumed the conductor $N_E$ is divisible by 4, and that $p\nmid N_E$. The following is a special case of a very well known result – the full details can be found in Miyake's book [28, Thm 4.7.1 and §7.2].

**Proposition 2.6.** *(i)* $\dim_{\mathbb{C}}\Big(\mathrm{Eis}_2\big(\Gamma_0(pN_E)\big)\Big) \;=\; 2\cdot\sum_{d|N_E}\phi\big(\gcd(d,N_E/d)\big) - 1$*;*

*(ii) The union $\Sigma_1\cup\Sigma_2$ consists of Hecke eigenforms, and yields a $\mathbb{C}$-basis for* $\mathrm{Eis}_2\big(\Gamma_0(pN_E)\big)$.

Turning our attention to the space of cusp forms,

$$\mathcal{S}_2\big(\Gamma_0(pN_E)\big) \;\cong\; \bigoplus_{M|pN_E}\;\bigoplus_{c|pN_E/M} \mathcal{S}_2^{\mathrm{new}}\big(\Gamma_0(M)\big)\Big|V_c$$

and one can express an arbitrary cusp form as a linear combination of Hecke eigenforms.

**Proposition 2.7.** *Since 4 divides $N_E$, the size of the space of cusp forms is given by*

$$\dim_{\mathbb{C}}\Big(\mathcal{S}_2\big(\Gamma_0(pN_E)\big)\Big) \;=\; 1+\left(\frac{p+1}{12}\right)\cdot\prod_{l|N_E}(l+1)\cdot l^{\mathrm{ord}_l(N_E)-1} - \sum_{d|N_E}\phi\big(\gcd(d,N_E/d)\big).$$

*Proof.* This follows immediately from a classical formula of Shimura (e.g. see [34, Prop 6.1]), upon noting that $4|pN_E$ and $p$ exactly divides the level $pN_E$. □

Write $\mathbf{d}_{\mathcal{S}}$ for the dimension of $\mathcal{S}_2\big(\Gamma_0(pN_E)\big)$, and let $\mathbf{d}_{\mathrm{Eis}}$ be the dimension of $\mathrm{Eis}_2\big(\Gamma_0(pN_E)\big)$. Then there exist coefficients $\delta_\bullet(m,e)\in\mathbb{Q}$ such that

$$R_{m,e}\big|U_p^{2m-1} = \delta_1(m,e)\cdot f_E(z) + \delta_2(m,e)\cdot f_E(pz) + \sum_{i=3}^{\mathbf{d}_{\mathcal{S}}}\delta_i(m,e)\cdot g_i + \sum_{j=1}^{\mathbf{d}_{\mathrm{Eis}}}\delta_{j+\mathbf{d}_{\mathcal{S}}}(m,e)\cdot h_j \qquad (6)$$

where $\{f_E(z), f_E(pz), g_3(z), g_4(z), \ \dots \ , g_{\mathbf{d}_{\mathcal{S}}}(z)\}$ is a basis of cuspidal eigenforms at level $pN_E$, and $\{h_1, \dots, h_{\mathbf{d}_{\mathrm{Eis}}}\}$ denotes an arbitrary $\mathbb{Q}$-basis for the Eisenstein component.

*Remarks:* (a) We have adopted the labelling convention that $g_1(z) = f_E(z)$ and $g_2(z) = f_E(pz)$.
(b) To find the basis elements $g_1, \dots g_{\mathbf{d}_{\mathcal{S}}}$ generating the space of cusp forms, at every $M \mid pN_E$ we compute a basis of $q$-expansions for $\mathcal{S}_2^{\mathrm{new}}(\Gamma_0(M))$, by using the SAGE command

```
CuspForms(Gamma0(N),k).new_subspace().q_expansion_basis(prec)
```

with level $N = M$ and for weight $k = 2$. We then compute $g(q^c)$ for each $g \in \mathcal{S}_2^{\mathrm{new}}(\Gamma_0(M))$ and $c$ dividing $pN_E/M$ (note the precision of the $q$-expansions is determined by the value of `prec`).
(c) To find the elements $h_1, \dots, h_{\mathbf{d}_{\mathrm{Eis}}}$ one can code up an implementation of Proposition 2.6(ii), or instead produce a basis of $q$-expansions (in echelon form) via the SAGE command

```
EisensteinForms(Gamma0(N),k).q_expansion_basis(prec)
```

with level $N = pN_E$ and for weight $k = 2$, once again.

We are then left with the task of determining the $\delta_\bullet(m, e)$'s, especially $\delta_1(m, e)$ and $\delta_2(m, e)$. To accomplish this we select an ordered tuple $\mathfrak{N} = [n_1, n_2, \ \dots \ , n_{\mathbf{d}_{\mathcal{S}}+\mathbf{d}_{\mathrm{Eis}}}] \in \mathbb{N}^{\mathbf{d}_{\mathcal{S}}+\mathbf{d}_{\mathrm{Eis}}}$ of distinct positive integers, then consider the $(\mathbf{d}_{\mathcal{S}} + \mathbf{d}_{\mathrm{Eis}}) \times (\mathbf{d}_{\mathcal{S}} + \mathbf{d}_{\mathrm{Eis}})$-linear system of equations

$$r_{np^{2m-1}}(m, e) \ = \ \sum_{i=1}^{\mathbf{d}_{\mathcal{S}}} a_n(g_i) \cdot \delta_i(m, e) \ + \ \sum_{j=1}^{\mathbf{d}_{\mathrm{Eis}}} a_n(h_j) \cdot \delta_{j+\mathbf{d}_{\mathcal{S}}}(m, e) \quad \text{for each } n \in \mathfrak{N},$$

arising from Equation (6). The corresponding $q$-coefficient matrix is given by

$$M \ = \ \begin{pmatrix} a_{n_1}(g_1) & \dots & a_{n_1}(g_{\mathbf{d}_{\mathcal{S}}}) & a_{n_1}(h_1) & \dots & a_{n_1}(h_{\mathbf{d}_{\mathrm{Eis}}}) \\ a_{n_2}(g_1) & \dots & a_{n_2}(g_{\mathbf{d}_{\mathcal{S}}}) & a_{n_2}(h_1) & \dots & a_{n_2}(h_{\mathbf{d}_{\mathrm{Eis}}}) \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n_{\mathbf{d}_{\mathcal{S}}+\mathbf{d}_{\mathrm{Eis}}}}(g_1) & \dots & a_{n_{\mathbf{d}_{\mathcal{S}}+\mathbf{d}_{\mathrm{Eis}}}}(g_{\mathbf{d}_{\mathcal{S}}}) & a_{n_{\mathbf{d}_{\mathcal{S}}+\mathbf{d}_{\mathrm{Eis}}}}(h_1) & \dots & a_{n_{\mathbf{d}_{\mathcal{S}}+\mathbf{d}_{\mathrm{Eis}}}}(h_{\mathbf{d}_{\mathrm{Eis}}}) \end{pmatrix}$$

so we can write the system as $\underline{r(m,e)}^T = M \times \underline{\delta(m,e)}^T$, where $\underline{r(m,e)} = (r_{np^{2m-1}}(m,e))_{n\in\mathfrak{N}}$ and $\underline{\delta(m,e)} = (\delta_i(m,e))_{i=1,\dots,\#\mathfrak{N}}$.

**Hypothesis** ($\det M \neq 0$)**.** *The matrix $M = M(\mathfrak{N})$ is invertible for the choice of tuple $\mathfrak{N}$.*

Clearly one can always find an $\mathfrak{N}$ for which the above holds, otherwise $\{g_1, \dots, g_{\mathbf{d}_{\mathcal{S}}}, h_1, \dots, h_{\mathbf{d}_{\mathrm{Eis}}}\}$ would not be a basis for $\mathcal{M}_2(\Gamma_0(pN_E))$. In practice, we choose a tuple $\mathfrak{N}$ that will minimise $\sum_{n\in\mathfrak{N}} \sum_{(n_1,n_2)\in\mathcal{W}_n} \mathfrak{f}_{\varepsilon_{n_2}}$, and hence the time needed to compute the vector $\underline{r(m,e)}$.

**Corollary 2.8.** *If Hypothesis (*$\det M \neq 0$*) is satisfied for a tuple $\mathfrak{N}$, and $W = (w_{i,j})_{1\leq i,j\leq\#\mathfrak{N}}$ denotes the inverse matrix to $M = M(m, e, \mathfrak{N})$, then $\underline{\delta(m,e)}^T = W \times \underline{r(m,e)}^T$; in particular*

$$\delta_1(m, e) \ = \ \sum_{j=1}^{\#\mathfrak{N}} w_{1,j} \cdot \underline{r(m,e)}_j \quad \textit{and} \quad \delta_2(m, e) \ = \ \sum_{j=1}^{\#\mathfrak{N}} w_{2,j} \cdot \underline{r(m,e)}_j \ .$$

Therefore, to obtain these first two components of $\underline{\delta(m,e)}$, we must:

- compute the dimensions $\mathbf{d}_{\mathcal{S}}$ and $\mathbf{d}_{\mathrm{Eis}}$ by using Propositions 2.7 and 2.6(i), respectively;
- calculate $g_1, \dots g_{\mathbf{d}_{\mathcal{S}}}$ and $h_1, \dots, h_{\mathbf{d}_{\mathrm{Eis}}}$ using the SAGE commands in (b) and (c) above;

- find an optimal choice of $\mathfrak{N} \in \mathbb{N}^{\mathbf{d}_{\mathcal{S}}+\mathbf{d}_{\mathrm{Eis}}}$ such that Hypothesis ($\det M \neq 0$) holds;
- produce the vector of $q$-coefficients $\underline{r(m,e)} = \big(r_{np^{2m-1}}(m,e)\big)_{n\in\mathfrak{N}}$ from Propositon 2.4;
- evaluate the first two basis coefficients, i.e. $\delta_1(m,e)$ and $\delta_2(m,e)$, using Corollary 2.8.

The slowest part of the algorithm is the penultimate line, and as we need $\#\mathfrak{N} = \mathbf{d}_{\mathcal{S}} + \mathbf{d}_{\mathrm{Eis}}$ of these $r_{np^{2m-1}}(m,e)$'s, the time required for this step has order $O\big((\mathbf{d}_{\mathcal{S}} + \mathbf{d}_{\mathrm{Eis}}) \times p^{3m}\big)$.

## 2.4 An explicit formula for $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ modulo $p^m$, when $D(E,1) \neq 0$

We shall begin by expressing the moments of the measure $\mathrm{d}\mu_E^{\mathrm{imp}}$ in terms of the vector $\underline{\delta(m,e)}$. For each $m \geq 1$, define an integer $\nu_{m,p} = \nu_{m,p}\big(\mathcal{F}^{\mathrm{imp}}\big)$ by the rule

$$\nu_{m,p}\big(\mathcal{F}^{\mathrm{imp}}\big) \ := \ \min\Big\{\mathrm{ord}_p\big(\delta_1(m,e)\big), \mathrm{ord}_p\big(\delta_2(m,e)\big) \text{ where } e \in \big(\mathbb{Z}/p^m\mathbb{Z}\big)^{\times}\Big\}.$$

Therefore to compute $\nu_{m,p}$ we must calculate the $2(p-1)p^{m-1}$ coefficients $\delta_\bullet(m,e)$, $\bullet = 1,2$. The $\mathbb{Z}_p$-module $\mathfrak{L}_{m,p} \subset \mathbb{Q}_p$ generated by the $\delta_1(m,e)$'s and the $\delta_2(m,e)$'s evidently satisfies

$$\mathfrak{L}_{m,p} \ := \ \mathbb{Z}_p \cdot \Big\langle \delta_1(m,e), \delta_2(m,e) \ \Big| \ e \in (\mathbb{Z}/p^m\mathbb{Z})^{\times} \Big\rangle \ = \ p^{\nu_{m,p}} \cdot \mathbb{Z}_p.$$

In particular, if the values $\delta_\bullet(m,e)$'s are $p$-integral then $\mathfrak{L}_{m,p} \subset \mathbb{Z}_p$, whence $\nu_{m,p}\big(\mathcal{F}^{\mathrm{imp}}\big) \geq 0$.

**Lemma 2.9.** *For each integer $m \geq 1$ and congruence class $e \in \big(\mathbb{Z}/p^m\mathbb{Z}\big)^{\times}$,*

$$\mu_E^{\mathrm{imp}}\big(e + p^m\mathbb{Z}_p\big) \ = \ \frac{-2\, w_E}{\alpha_p^{2m}} \cdot (1 - \alpha_p^{-2}) \times \Big(\alpha_p \cdot \delta_1(m,e) + \delta_2(m,e)\Big)$$

*and these moments lie inside $p^{\nu_{m,p}}(1 - \alpha_p^{-2}) \cdot \mathbb{Z}_p$.*

*Proof.* Considering Equations (4) and (6) in turn, one deduces that

$$\begin{aligned}\mu_E^{\mathrm{imp}}(e + p^m\mathbb{Z}_p) \ &= \ 2\alpha_p^{-2m} \times \frac{\big\langle f^0 \ , \ R_{m,e}\big|U_p^{2m-1}\big\rangle_{pN_E}}{\big\langle f_E, f_E\big\rangle_{N_E}} \\ &= \ 2\alpha_p^{-2m} \times \frac{\big\langle f^0 \ , \ \delta_1(m,e) f_E(z) + \delta_2(m,e) f_E(pz) + \widetilde{R}(z)\big\rangle_{pN_E}}{\big\langle f_E, f_E\big\rangle_{N_E}}\end{aligned}$$

where $\widetilde{R}(z) \in \mathcal{M}_2\big(\Gamma_0(pN_E)\big)$ intersects trivially with the isotypic subspace $\big(\mathbb{C}\cdot f_E \oplus \mathbb{C}\cdot f_E(pz)\big)$. If we make full use of Lemma 2.2, the three Petersson inner product identities imply

$$\mu_E^{\mathrm{imp}}(e + p^m\mathbb{Z}_p) \ = \ 2\alpha_p^{-2m} \times \left(\delta_1(m,e) \cdot \left(-w_E \cdot \frac{\alpha_p^2 - 1}{\alpha_p}\right) + \ \delta_2(m,e) \cdot \left(-w_E \cdot \frac{\alpha_p^2 - 1}{\alpha_p^2}\right) + \ 0\right)$$

which is equivalent to the stated formula.

Note the integrality statement for $\mu_E^{\mathrm{imp}}(-)$ follows as $\delta_\bullet(m,e) \in p^{\nu_{m,p}}\cdot\mathbb{Z}_p$ and $\frac{-2w_E}{\alpha_p^{2m}} \in \mathbb{Z}_p^{\times}$. $\square$

An important corollary of this result is that the power series $\mathcal{F}^{\mathrm{imp}}(X)$ belongs to $p^{\nu_{m,p}} \cdot \mathbb{Z}_p[\![X]\!]$, hence the imprimitive $p$-adic $L$-function is $p$-integral if $\big|\delta_1(m,e)\big|_p, \big|\delta_2(m,e)\big|_p \leq 1$ for all $e$. Furthermore, if $S_{\mathrm{ord}}$ denotes the set of primes where $E$ has good ordinary reduction over $\mathbb{Q}_p$, and $S_{\rm denom}$ consists of those primes dividing $6 \times \prod_{l \in S_1} H_l(l^{-1}) \times \frac{(2\pi i)^{-1}\Omega_E^+\Omega_E^-}{\pi\langle f_E, f_E\rangle_{N_E}}$ (cf. Lemma 2.1), an easy exercise verifies that

$$\mathcal{F}(X) \in p^{\nu_{m,p}} \cdot \mathbb{Z}_p[\![X]\!] \quad \text{at every prime } p \in S_{\rm ord} - S_{\rm denom}.$$

**Consequently, the primitive $p$-adic $L$-function $\mathbf{L}_p(\mathrm{Sym}^2 E, s)$ is a $p$-integral Iwasawa function at good ordinary primes $p \notin S_{\rm denom}$ for which $\sup\{\nu_{m,p}(\mathcal{F}^{\rm imp}) \mid m \in \mathbb{N}\} \geq 0$.**

For each $m$, the quantities $\nu_{m,p}$ give a lower bound on the $\mu$-invariant of $\mathcal{F}(X)$ when $p \notin S_{\rm denom}$. In all of our numerical calculations, we found that the exponent $\nu_{m,p}(\mathcal{F}^{\rm imp})$ stabilized as a function of $m \geq 3$, and was only once smaller than $-2$ in value. In fact, this was the single instance where $\mathbf{L}_p'(\mathrm{Sym}^2 E, 1) \notin \mathbb{Z}_p$, occurring at the prime $p = 3$ for the curve $E = 268\mathrm{a}1$.

**Theorem 2.10.** *Provided that $D(E,1) \neq 0$, if one defines $\xi_{\mathrm{Sym}^2 E} := \frac{D(E,1)}{\pi\langle f_E, f_E\rangle_{N_E}}$ and sets $\epsilon_p = \mathrm{ord}_p\big((1-\alpha_p^{-2}) \cdot \xi_{\mathrm{Sym}^2 E}\big)$, then the $\mathcal{L}$-invariant will satisfy the congruences*

$$\mathcal{L}_p^{\rm an}(\mathrm{Sym}^2 E) \equiv \frac{-2\, w_E \cdot \xi_{\mathrm{Sym}^2 E}^{-1}}{\alpha_p^{2m}(1 - p\alpha_p^{-2})} \times \sum_{e \in (\mathbb{Z}/p^m\mathbb{Z})^\times} \log_p <e>_p \cdot \Big(\alpha_p \cdot \delta_1(m,e) + \delta_2(m,e)\Big) \mod p^{m+\nu_{m,p}-\epsilon_p}$$

*for every integer $m \geq 1$.*

*Proof.* From Lemma 2.9, our formulae for the moments of the measure $\mathrm{d}\mu_E$ imply that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{L}_p^{\rm imp}(\mathrm{Sym}^2 E, s)\Big|_{s=1} &= \int_{x \in \mathbb{Z}_p^\times} \log_p <x>_p \cdot \mathrm{d}\mu_E^{\rm imp}(x) \\
&\equiv \sum_{e \in (\mathbb{Z}/p^m\mathbb{Z})^\times} \log_p <e>_p \cdot \mu_E^{\rm imp}(e + p^m\mathbb{Z}_p) \mod p^{m+\nu_{m,p}} \\
&\equiv \frac{-2\, w_E}{\alpha_p^{2m}} \cdot (1-\alpha_p^{-2}) \times \sum_{e \in (\mathbb{Z}/p^m\mathbb{Z})^\times} \log_p <e>_p \cdot \Big(\alpha_p \cdot \delta_1(m,e) + \delta_2(m,e)\Big) \mod p^{m+\nu_{m,p}}.
\end{aligned}$$

Now using Equation (1) which is valid as $D(E,1) \neq 0$, the $\mathcal{L}$-invariant can be expressed as

$$\mathcal{L}_p^{\rm an}(\mathrm{Sym}^2 E) = \Big((1-\alpha_p^{-2})(1-p\alpha_p^{-2})\, \xi_{\mathrm{Sym}^2 E}\Big)^{-1} \times \frac{\mathrm{d}}{\mathrm{d}s}\mathbf{L}_p^{\rm imp}(\mathrm{Sym}^2 E, s)\Big|_{s=1}$$

and since $(1-\alpha_p^{-2})(1-p\alpha_p^{-2})\, \xi_{\mathrm{Sym}^2 E} \cdot \mathbb{Z}_p = p^{\epsilon_p} \cdot \mathbb{Z}_p$, the result follows directly. □

## 2.5 Attempts at evaluating the moments $\int x^j \cdot \mathrm{d}\mu_E^{\rm imp}$ for $j \neq 0$?

Theoretically at least, there should be a more efficient way to compute the derivative of the imprimitive $p$-adic $L$-function at $s = 1$, which we now outline. Keeping our previous notation,

$$\begin{aligned}
\frac{\mathrm{d}\, \mathbf{L}_p^{\rm imp}(\mathrm{Sym}^2 E, s)}{\mathrm{d}s} &= \int_{\mathbb{Z}_p^\times} <x>_p^{s-1} \log_p <x>_p \cdot \mathrm{d}\mu_E^{\rm imp}(x) \\
&= \sum_{e=1}^{p-1} \sum_{j=0}^{\infty} \mathcal{A}_{e,j}(s) \cdot \int_{e+p\mathbb{Z}_p} x^j \cdot \mathrm{d}\mu_E^{\rm imp}(x)
\end{aligned}$$

where $\sum_{j=0}^\infty \mathcal{A}_{e,j}(s) x^j$ is the power series development for $<x>_p^{s-1} \log_p <x>_p$ along $e + p\mathbb{Z}_p$.

**Question.** *Is there an efficient algorithm to determine $\int_{e+p\mathbb{Z}_p} x^j \cdot \mathrm{d}\mu_E^{\rm imp}(x)$ when $j \neq 0$?*

If there is a positive answer, then one simply needs to evaluate $\sum_{j=0}^{\infty} \mathcal{A}_{e,j}(1) \cdot \int_{e+p\mathbb{Z}_p} x^j \cdot \mathrm{d}\mu_E^{\mathrm{imp}}$ to some prescribed $p$-adic precision, and next sum the values over the range $e = 1, \ldots, p-1$. In theory this should yield a far quicker and more accurate method than using Riemann sums, but in practice there are a number of difficulties that arise.

To better illustrate these difficulties, let us assume that $\mathcal{F}_k$ is a $p$-stabilised ordinary Hecke eigenform of weight $k \geq 2$ and level $Np$. The critical points for the $L$-function of the symmetric square of $\mathcal{F}_k$ are $\{1, \ldots, 2k-2\}$ which, after $p$-adically interpolating $L^{\mathrm{imp}}\big(\mathrm{Sym}^2\mathcal{F}_k, s\big)$ at positive integer values, naturally subdivide into the disjoint subsets $\{1, \ldots, k-1\}$ and $\{k, \ldots, 2k-2\}$. If $\mathrm{d}\mu^{\mathrm{imp},-}_{\mathrm{Sym}^2\mathcal{F}_k(j)}$ is the measure interpolating $\chi$-twists of $\mathrm{Sym}^2\mathcal{F}_k(j)$ at each $j \in \{1, \ldots, k-1\}$, then the analytic methods in [9, 31] imply for some non-zero constant $c_k \in \overline{\mathbb{Q}}^{\times}$:

$$\int_{e+p^m\mathbb{Z}_p} x^{j-1} \cdot \mathrm{d}\mu^{\mathrm{imp},-}_{\mathrm{Sym}^2\mathcal{F}_k(1)}(x) \;=\; \mu^{\mathrm{imp},-}_{\mathrm{Sym}^2\mathcal{F}_k(j)}\big(e + p^m\mathbb{Z}_p\big) \;=\; c_k \times \frac{\big\langle \mathcal{F}_k^0 \,,\, \mathrm{Hol}\big(\tilde{R}^{(k,j)}_{m,e}\big)\big|U_p^{2m-1}\big\rangle_{pN}}{\big\langle \mathcal{F}_k, \mathcal{F}_k\big\rangle_{pN}}$$

where $\tilde{R}^{(k,j)}_{m,e}$ are certain $\mathcal{C}^{\infty}$-modular forms exhibiting moderate growth at the cusps of $X_1(p^{2m}N)$, and 'Hol' denotes the operator of holomorphic projection, in the terminology of [20].

*Remarks:* (i) If $j = k-1$ then the modular forms $\tilde{R}^{(k,k-1)}_{m,e}$ are already holomorphic, and there is no need to hit them with the operator 'Hol' (e.g. for weight $k = 2$, one has $R_{m,e} = \tilde{R}^{(2,1)}_{m,e}$).

(ii) However if $j \in \{1, \ldots, k-1\}$ and $j \neq k-1$, then $\tilde{R}^{(k,j)}_{m,e}$ is *not* a holomorphic modular form.

(iii) More alarmingly, if $j \in \mathbb{Z} - \{1, \ldots, k-1\}$ then $\tilde{R}^{(k,j)}_{m,e}$ no longer has moderate growth at the cusps of $X_1(p^{2m}N)$, so attempting to evaluate $\mathrm{Hol}\big(\tilde{R}^{(k,j)}_{m,e}\big)$ does not even make sense.

For each critical value $j \in \{1, \ldots, k-1\}$, the Fourier expansion of $\mathrm{Hol}\big(\tilde{R}^{(k,j)}_{m,e}\big)\big|U_p^{2m-1}$ can be readily computed [9, pp.592-594], and is of the form

$$\mathrm{Hol}\big(\tilde{R}^{(k,j)}_{m,e}\big)\Big|U_p^{2m-1} \;=\; \sum_{n=1}^{\infty} \left( \sum_{p^{2m-1}n = Nn_1^2 + n_2} C^{(k,j)}_{n_2,m,n} \times \int_{x \in e+p^m\mathbb{Z}_p} x^{j-k+1} \cdot \mathrm{d}\mu^-(x, \varepsilon_{n_2}) \right) \cdot q^n \tag{7}$$

where the scalars $C^{(k,j)}_{n_2,m,n} \in \overline{\mathbb{Q}}$, and $\mathrm{d}\mu^-(x, \varepsilon_{n_2})$ is the twisted Kubota-Leopoldt pseudo-measure interpolating $\int_{\mathbb{Z}_p^\times} \chi x^s \cdot \mathrm{d}\mu^-(x, \varepsilon_{n_2}) = \zeta_p(s, \chi^{-1}\varepsilon_{n_2})$ at finite order characters $\chi$, with $1 - s \in \mathbb{N}$.

In order to evaluate $\int x^j \cdot \mathrm{d}\mu^{\mathrm{imp},-}_{\mathrm{Sym}^2\mathcal{F}_k(k-1)}$, one could naively try to Tate twist the $q$-expansions in Equation (7) at integer values $j \notin [1, k-1]$, and then compute the $\mathcal{F}_k^0$-isotypic component. We attempted this for both the ranges $j > k-1$ and $j < 1$ (which lie outside the region of $p$-adic interpolation), but found that the corresponding $q$-expansions could not possibly come from modular forms of level $Np$. Essentially these methods fail because the operator 'Hol' cannot be extended to real analytic forms that do not exhibit moderate growth.

A possible salvage is to allow the $p$-stabilised eigenform $\mathcal{F}_k$ to vary in an ordinary family. For example, one could pick another weight $k' = k + t(p-1)p^r$ for some $t, r \in \mathbb{N}$, and a Hecke eigenform $\mathcal{F}_{k'} \in \mathcal{S}_{k'}\big(\Gamma_1(Np)\big)$ such that $\mathcal{F}_{k'} \equiv \mathcal{F}_k \mod p^r$. Morally it should be the case that $\mathcal{L}_p^{\mathrm{an}}\big(\mathrm{Sym}^2(\mathcal{F}_{k'})(k'-1)\big)$ and $\mathcal{L}_p^{\mathrm{an}}\big(\mathrm{Sym}^2(\mathcal{F}_k)(k-1)\big)$ are also congruent, albeit modulo a lesser power of $p$. Suppose that we want to compute the moments $\int x^j \cdot \mathrm{d}\mu^{\mathrm{imp},-}_{\mathrm{Sym}^2\mathcal{F}_{k'}(k'-1)}$ instead. Because $k' = k + t(p-1)p^r$ with the chosen $r > 1$, the strip $\{1, \ldots, k'-1\}$ is considerable larger than the strip $\{1, \ldots, k-1\}$, so the range of $j$'s for which $\mathrm{Hol}\big(\tilde{R}^{(k',j)}_{m,e}\big)\Big|U_p^{2m-1}$ is a classical weight $k'$ modular form is now bigger. There are also more moments $\int x^j \cdot \mathrm{d}\mu^{\mathrm{imp},-}_{\mathrm{Sym}^2\mathcal{F}_{k'}(k'-1)}$ available.

The main hindrance is that expressing $\mathrm{Hol}\big(\tilde{R}_{m,e}^{(k',j)}\big)\Big|U_p^{2m-1}$ in terms of a basis of weight $k'$ modular forms is computationally far slower than before, as the dimension of $\mathcal{M}_{k'}\big(\Gamma_0(Np)\big)$ grows rapidly with $k'$. Therefore any advantage gained by calculating this larger set of moments is immediately offset by the slowness in writing each $\mathrm{Hol}\big(\tilde{R}_{m,e}^{(k',j)}\big)\Big|U_p^{2m-1}$ in terms of a $\mathbb{C}$-basis. For example, if $p = 5$, $N = 11$, $k = 2$ and $k' = 2 + (5-1)5^{10}$ then a simple SAGE calculation reveals $\dim_{\mathbb{C}}\Big(\mathcal{M}_{k'}\big(\Gamma_0(Np)\big)\Big) = 234,375,008$, which is crippling from a numerical standpoint. Nevertheless, because the subspace of $p$-ordinary modular forms has fixed dimension by Hida's control theory, any theoretical result which could bypass the slowness in computing a full basis for $\mathcal{M}_{k'}\big(\Gamma_0(Np)\big)$ would make the algorithm far more efficient.

# 3 The Basic Method

Using the SAGE computer package, we implemented the method outlined in §2.1- §2.4 to compile tables of $\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2E, s)\big|_{s=1}$ for all curves $E$ of conductor $N_E \leq 300$ such that $4\big|N_E$, as well as their symmetric square $\mathcal{L}$-invariants. These numerical values are tabulated in Appendix B. Here we were mainly interested in verifying that $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2E)$ was non-zero, rather than in computing it to a high $p$-adic accuracy.

## 3.1 An algorithm to compute the $\mathcal{L}$-invariant numerically

We begin with some general observations. Assume we are given an elliptic curve $E_{/\mathbb{Q}}$ with no restriction on its conductor $N_E$. Then $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2E)$ depends only on the $\mathbb{Q}$-isogeny class of $E$. Indeed Nastasescu [29] has shown that the $p$-adic $L$-function for $\mathrm{Sym}^2E$ uniquely determines the $\mathbb{Q}$-isogeny class of the elliptic curve $E$, up to a twist by a quadratic character.

Let $l \neq 2$ be a prime. We write $\omega_l : \mathbb{F}_l^{\times} \to \mu_{l-1}$ for the Teichmüller character modulo $l$, which associates to each $x \in \mathbb{F}_l^{\times}$ the unique $(l-1)$-st root of unity congruent to $x$ modulo $l$. One can then define a quadratic character $\varpi_l : \mathbb{F}_l^{\times} \to \{\pm 1\}$ by the rule $\varpi_l(x) = \omega_l^{(l-1)/2}(x)$. However if $l = 2$, then $\varpi_2 : (\mathbb{Z}/4\mathbb{Z})^{\times} \to \{\pm 1\}$ denotes the quadratic character of conductor 4.

*Step 1:* If $E$ has conductor $N_E$ divisible by 4 and $2 \leq \mathrm{ord}_2(N_{E\otimes\theta}) < \mathrm{ord}_2(N_E)$ where $\theta$ is one of $\varpi_2$, $\big(\frac{-}{2}\big)$, and $\big(\frac{-}{-2}\big)$, then replace $E$ with its twist $E \otimes \theta$; alternatively, if $E$ has conductor $N_E$ such that $\mathrm{ord}_2(N_E) \leq 1$, then replace $E$ with its twist $E \otimes \varpi_2$ to ensure that $4|N_E$ holds.

*Step 2:* For our (possibly new) choice of $E$, let us define the set

$$S_1 = S_1(E) \;:=\; \big\{\text{primes } l\big|N_E \text{ such that } \mathrm{ord}_l(j_E) \geq 0\big\}.$$

Note if the $j$-invariant of $E$ satisfies $\mathrm{ord}_l(j_E) < 0$, then the Euler factor at $l$ for $D(E, s)$ equals $1 - (\pm 1)^2 l^{-s} = 1 - l^{-s}$ which agrees with that of $L_\infty\big(\mathrm{Sym}^2E, s\big)$, hence there is no discrepancy; the contribution to the $l$-part of the conductor is then given by $\mathrm{ord}_l\big(C_{\mathrm{Sym}^2E}\big) = 2$ if $l\big\|N_E$.

We shall now compute the bad Euler factors, $H_l(X)$, at each prime number $l \in S_1$, and also the $l$-part of $C_{\mathrm{Sym}^2E}$ (we should point out that $E$ has potential good reduction at all $l \in S_1$). If $l \neq 2$, we write $\Phi_l \subset \mathrm{Gal}\big(\mathbb{Q}_l(E_4)/\mathbb{Q}_l\big)$ for the inertia subgroup; alternatively, if $l = 2$ then $\Phi_2 \subset \mathrm{Gal}\big(\mathbb{Q}_2(E_3)/\mathbb{Q}_2\big)$ will denote the inertia subgroup.

*Step 3a:* Suppose $E \otimes \theta$ has good reduction at $l$ where $\theta = \varpi_l$ if $l > 2$, or $\theta \in \Big\{\varpi_2, \big(\frac{-}{2}\big), \big(\frac{-}{-2}\big)\Big\}$ if $l = 2$; then $H_l(X) = (1 - \hat{\alpha}_l{}^2X)(1 - \hat{\beta}_l{}^2X)(1 - lX)$ with $\Phi_l \cong C_2$ and $\mathrm{ord}_l\big(C_{\mathrm{Sym}^2E}\big) = 0$, where the Hecke polynomial $1 - a_l\big(E \otimes \theta\big)X + lX^2 = (1 - \hat{\alpha}_lX)(1 - \hat{\beta}_lX)$.

*Step 3b:* If $\#\Phi_l > 2$, then each factor $H_l(X)$ is determined by Proposition A.1 in the Appendix, in which case $\mathrm{ord}_l\big(C_{\mathrm{Sym}^2 E}\big)$ can be read off from the results in [7, pp120-121].

*Step 4:* Evaluate the imprimitive $L$-value $\xi_{\mathrm{Sym}^2 E} = \frac{D(E,1)}{\pi\langle f_E, f_E\rangle_{N_E}}$ using the formula in Lemma 2.1, which requires both $\prod_{l\in S_1} H_l(X)$ and $C_{\mathrm{Sym}^2 E} = \prod_{l|N_E} l^{\mathrm{ord}_l(C_{\mathrm{Sym}^2 E})}$ from the previous steps. **However, if $D(E,1)=0$ then STOP!**

*Step 5:* Compute $\mathbf{d}_{\mathcal{S}}$ and $\mathbf{d}_{\mathrm{Eis}}$, then find a $q$-coefficient matrix $M = M(\mathfrak{N})$ with $\det(M) \neq 0$.

*Step 6:* Fix the desired accuracy $m \geq 1$; then for each $e \in (\mathbb{Z}/p^m\mathbb{Z})^\times$, compute both of the terms $\delta_1(m,e)$ and $\delta_2(m,e)$ by following the method described at the end of Corollary 2.8.

*Step 7:* Calculate $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E) \bmod p^{m+\nu_{m,p}-\epsilon_p}$ via the numerical congruences in Theorem 2.10.

The structure of these inertia subgroups $\Phi_l$ was worked out completely by Serre in [32, §5.6]. To summarise, if $\mathrm{ord}_l(j_E) \geq 0$ and $l\big|N_E$ then $\Phi_l \in \big\{C_2, C_3, C_4, C_6\big\}$ provided that $l \neq 2,3$. If $l=3$ then the semi-direct product $C_4 \rtimes C_3$ is also a possibility, while if $l=2$ then both $\mathrm{SL}_2(\mathbb{F}_3)$ and $Q_8$ (the quaternion group of size 8) can also occur as $\Phi_l$.

Fortunately, there is an extensive table given in [7, p121] which contains the information required to pin down the structure of $\Phi_2$ and $\Phi_3$, as well as the 2- and 3-parts of $C_{\mathrm{Sym}^2 E}$. Therefore Step 3 can be fully automated.

We should also point out that the matrix $M(\mathfrak{N})$ in Step 5 need only be determined once, which is lucky because $\mathbf{d}_{\mathcal{S}} + \mathbf{d}_{\mathrm{Eis}}$ can typically be greater than $10^4$ even if $N_E$ is relatively small.

### 3.2 A general formula for $\mathbf{L}_p(\mathrm{Sym}^2 E, 1)'$ modulo $p^m$, even when $D(E,1)=0$

It is important to mention for the six elliptic curves 176$b$1, 196$a$1, 200$b$1, 240$d$1, 272$b$1, 300$c$1, the value of $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E,1)'$ is zero at all primes $p$ simply because $D(E,s)$ vanishes at $s=1$. **One should note that the triviality of $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E,1)'$ does not imply either the triviality of $\mathbf{L}_p(\mathrm{Sym}^2 E,1)'$, nor the triviality of $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$.**

In order that our study of Conjecture 1.3 is not missing out any curves of conductor $\leq 300$, for those six elliptic curves listed above with $D(E,1)=0$, we shall now describe a general method to approximate $\mathbf{L}_p(\mathrm{Sym}^2 E,1)'$ that will work irrespective of whether $D(E,1)$ is zero. Let us begin by partitioning the set $S_1 = S_1(E)$ into a disjoint union of

$$S'_{1,-} := \big\{l \in S_1 - \{2\} \text{ such that } \#\Phi_l > 2 \text{ and } \mathrm{Gal}\big(\mathbb{Q}_l(E_4)/\mathbb{Q}_l\big) \text{ is abelian}\big\},$$
$$S''_{1,-} := \big\{l \in S_1 \text{ such that } \#\Phi_l = 2\big\}, \quad \text{and} \quad S_{1,+} := S_1 - S'_{1,-} - S''_{1,-}.$$

A careful reading of the argument in [7, pp119-121] indicates that for each prime $l \in S'_{1,-} \cup S''_{1,-}$, one has $H_l(X) = (1-lX)\cdot \Upsilon_l(X)$ where

$$\Upsilon_l(X) := \begin{cases} 1 & \text{if } l \in S'_{1,-} \\ \big(1-\hat{\alpha}_l^2 X\big)\big(1-\hat{\beta}_l^2 X\big) & \text{if } l \in S''_{1,-}. \end{cases} \tag{8}$$

Alternatively, if a prime $l \in S_{1,+}$ then $H_l(X) = (1+lX)$ unless either $l=3$ and $\Phi_3 \cong C_4 \rtimes C_3$, or instead $l=2$ and $\Phi_2 \in \big\{\mathrm{SL}_2(\mathbb{F}_3), Q_8\big\}$, in which case $H_l(X)=1$.
It follows that for $s \in \mathbb{C}$, there is a natural separation of Euler factors given by

$$\prod_{l\in S_1} H_l(l^{-s}) = \left(\prod_{l\in S_{1,+}} H_l(l^{-s}) \times \prod_{l\in S''_{1,-}} \Upsilon_l(l^{-s})\right) \times \prod_{l\in S'_{1,-}\cup S''_{1,-}} \big(1-l^{1-s}\big)$$

with the bracketted term non-zero at $s=1$, while the other term has order $\#S'_{1,-} + \#S''_{1,-}$.

**Definition 3.1.** *For any $s \in \mathbb{Z}_p$, let us define the (period modified) p-adic L-function by*

$$\mathbf{L}_p^{\rm aut}(\mathrm{Sym}^2 E, s) \;:=\; \frac{(2\pi i)^{-1}\Omega_E^+\Omega_E^-}{\pi\big\langle f_E, f_E\big\rangle_{N_E}} \times \mathbf{L}_p(\mathrm{Sym}^2 E, s).$$

Comparing the above with the imprimitive $p$-adic $L$-function, one can factorise the latter into

$$\mathbf{L}_p^{\rm imp}(\mathrm{Sym}^2 E, s) \;=\; \mathfrak{I}_p(s)\cdot \prod_{l\in S'_{1,-}\cup S''_{1,-}} \big(1-<l>_p^{s-1}\big) \times \mathbf{L}_p^{\rm aut}(\mathrm{Sym}^2 E, s) \tag{9}$$

where $\mathfrak{I}_p(s)\in\mathbb{Z}_p\langle\!\langle s\rangle\!\rangle$ is an Iwasawa function satisfying

$$\mathfrak{I}_p(1) \;=\; \prod_{l\in S_{1,+}} H_l(l^{-1}) \times \prod_{l\in S''_{1,-}} \Upsilon_l(l^{-1}).$$

It follows directly from this factorisation that

$$\mathrm{order}_{s=1}\big(\mathbf{L}_p^{\rm imp}(\mathrm{Sym}^2 E, s)\big) \;=\; \#S'_{1,-} + \#S''_{1,-} + \mathrm{order}_{s=1}\big(\mathbf{L}_p^{\rm aut}(\mathrm{Sym}^2 E, s)\big),$$

hence Conjecture 1.3 is equivalent to $\mathbf{L}_p^{\rm imp}(\mathrm{Sym}^2 E, s)$ vanishing with order $1+\#S'_{1,-}+\#S''_{1,-}$ at the critical point $s=1$.

To verify Coates and Greenberg's conjecture for a given elliptic curve $E$ when $S'_{1,-}\cup S''_{1,-}\neq\emptyset$, we must therefore supply a method to calculate $\mathbf{L}_p^{\rm aut}(\mathrm{Sym}^2 E, 1)'$, then check it is non-zero.

**Theorem 3.2.** *For all integers $m\geq 1$, there are congruences*

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{L}_p^{\rm aut}(\mathrm{Sym}^2 E, s)\Big|_{s=1} \;\equiv\; \frac{\sum_{e\in(\mathbb{Z}/p^m\mathbb{Z})^\times}\big(\log_p<e>_p\big)^{1+\#S'_{1,-}+\#S''_{1,-}}\cdot \mu_E^{\rm imp}(e+p^m\mathbb{Z}_p)}{(1+\#S'_{1,-}+\#S''_{1,-})!\times\mathfrak{I}_p(1)\times\prod_{l\in S'_{1,-}\cup S''_{1,-}}\log_p(1/l)}$$

*modulo* $p^{m+\nu_{m,p}-\mathrm{ord}_p\left(\mathfrak{I}_p(1)\times\prod_{l\in S'_{1,-}\cup S''_{1,-}}\log_p(1/l)\right)}$.

*Proof.* Let us first set $\kappa_p := \#S'_{1,-}+\#S''_{1,-}\geq 0$. We have the following Taylor series at $s=1$:

- $\mathbf{L}_p^{\rm imp}(\mathrm{Sym}^2 E, s) \;=\; \frac{\mathrm{d}^{\kappa_p+1}\mathbf{L}_p^{\rm imp}(\mathrm{Sym}^2 E, s)}{\mathrm{d}s^{\kappa_p+1}}\Big|_{s=1}\cdot\frac{(s-1)^{\kappa_p+1}}{(\kappa_p+1)!} \;+\; O\big((s-1)^{\kappa_p+2}\big)$
- $\mathbf{L}_p^{\rm aut}(\mathrm{Sym}^2 E, s) \;=\; \frac{\mathrm{d}\mathbf{L}_p^{\rm aut}(\mathrm{Sym}^2 E, s)}{\mathrm{d}s}\Big|_{s=1}\cdot(s-1) \;+\; O\big((s-1)^2\big)$
- $\mathfrak{I}_p(s) \;=\; \mathfrak{I}_p(1)\cdot(s-1)^0 \;+\; O\big((s-1)^1\big)$
- $\big(1-<l>_p^{s-1}\big) \;=\; \log_p(1/l)\cdot(s-1)^1 \;+\; O\big((s-1)^2\big)$ for each prime $l\neq p$.

Plugging these directly into Equation (9), one reads off from the $(s-1)^{\kappa_p+1}$-term that

$$\frac{\mathrm{d}\mathbf{L}_p^{\rm aut}(\mathrm{Sym}^2 E, s)}{\mathrm{d}s}\Big|_{s=1} \;=\; \frac{\frac{\mathrm{d}^{\kappa_p+1}\mathbf{L}_p^{\rm imp}(\mathrm{Sym}^2 E, s)}{\mathrm{d}s^{\kappa_p+1}}\Big|_{s=1}}{(\kappa_p+1)!\cdot\mathfrak{I}_p(1)\times\prod_{l\in S'_{1,-}\cup S''_{1,-}}\log_p(1/l)}\,.$$

Further, upon differentiating the Mazur-Mellin transform $(\kappa_p+1)$-times, one easily deduces

$$\begin{aligned}\frac{\mathrm{d}^{\kappa_p+1}\mathbf{L}_p^{\rm imp}(\mathrm{Sym}^2 E, s)}{\mathrm{d}s^{\kappa_p+1}}\Big|_{s=1} \;&=\; \int_{x\in\mathbb{Z}_p^\times}\big(\log_p<x>_p\big)^{\kappa_p+1}\cdot\;\mathrm{d}\mu_E^{\rm imp}(x)\\ &\equiv\; \sum_{e\in(\mathbb{Z}/p^m\mathbb{Z})^\times}\big(\log_p<e>_p\big)^{\kappa_p+1}\times\mu_E^{\rm imp}(e+p^m\mathbb{Z}_p)\quad\mathrm{mod}\;p^{m+\nu_p}.\end{aligned}$$

The approximation now follows after dividing by $(\kappa_p+1)!\cdot\mathfrak{I}_p(1)\times\prod_{l\in S'_{1,-}\cup S''_{1,-}}\log_p(1/l)$. □

*Remarks:* (a) The preceding theorem yields an effective method to calculate $\mathbf{L}_p^{\rm aut}({\rm Sym}^2 E, 1)'$, as a formula for the moments of the measure ${\rm d}\mu_E^{\rm imp}$ has already been given in Lemma 2.9.
(b) The $\mathcal{L}$-invariant itself is then obtained simply by working out the ratio

$$\mathcal{L}_p^{\rm an}({\rm Sym}^2 E) \;=\; \frac{\rm d}{{\rm d}s}\mathbf{L}_p^{\rm aut}({\rm Sym}^2 E, s)\Big|_{s=1} \times \left((1-\alpha_p^{-2})(1-p\alpha_p^{-2}) \times \frac{L_\infty({\rm Sym}^2 E, 1)}{\pi\langle f_E, f_E\rangle_{N_E}}\right)^{-1}.$$

(c) In the Appendix A, we give a method to determine $S_{1,+}$, $S'_{1,-}$, $S''_{1,-}$ and also the $H_l(X)$'s.

(d) If $S'_{1,-} = S''_{1,-} = \emptyset$ so that $D(E,1) \neq 0$, then Theorem 3.2 and the $\mathcal{L}$-invariant equation specialise to the situation covered in §2.4 – here $\mathbf{L}_p^{\rm aut}$ and $\mathbf{L}_p^{\rm imp}$ have the same order at $s = 1$.

**A worked example at level** 176**.** Consider the elliptic curve

$$E = 176b1 \;:\; y^2 \;=\; x^3 + x^2 - 5x - 13$$

of conductor $N_E = 2^4 \cdot 11$. Its first few good ordinary primes are $p = 3, 5, 7, 13, \ldots$ with corresponding Hecke eigenvalues $a_3(E) = 1$, $a_5(E) = 1$, $a_7(E) = 2$, $a_{13}(E) = 4, \ldots$ respectively. The quadratic twist $E \otimes \varpi_2$ has conductor 11, and therefore will be $\mathbb{Q}$-isogenous to $X_0(11)$. One determines that $S_{1,+}(E) = S'_{1,-}(E) = \emptyset$ and $S''_{1,-}(E) = \{2\}$, with

$$\Upsilon_2(X) \;=\; \big(1 - \hat{\alpha}_2^2 X\big)\big(1 - \hat{\beta}_2^2 X\big) \;=\; 1 + 4X^2$$

where $\hat{\alpha}_2 = -1 + i$ and $\hat{\beta}_2 = -1 - i$ are the roots of $X^2 - a_2\big(X_0(11)\big)X + 2 = X^2 + 2X + 2$. Furthermore at $s = 1$, the primitive complex $L$-function satisfies

$$\frac{L_\infty({\rm Sym}^2 E, 1)}{\pi\langle f_{E\otimes\varpi_2}, f_{E\otimes\varpi_2}\rangle_{N_{E\otimes\varpi_2}}} \;=\; \frac{4 \cdot \sqrt{C_{{\rm Sym}^2 E}}}{N_{E\otimes\varpi_2}} \times \prod_{l \in S_1(E\otimes\varpi_2)} \frac{1}{H_l(l^{-2})} \;=\; \frac{4 \cdot \sqrt{121}}{11} \times 1 \;=\; 4.$$

The period ratio is given by

$$\frac{\langle f_{E\otimes\varpi_2}, f_{E\otimes\varpi_2}\rangle_{N_{E\otimes\varpi_2}}}{\langle f_E, f_E\rangle_{N_E}} \;=\; \big[\Gamma_0(N_{E\otimes\varpi_2}) : \Gamma_0(N_E)\big]^{-1} \times {\rm Res}_{s=2}\left(\frac{\sum_{n=1}^\infty a_n(E\otimes\varpi_2)\cdot n^{-s}}{\sum_{n=1}^\infty a_n(E)\cdot n^{-s}}\right)$$
$$= \frac{N_{E\otimes\varpi_2} \cdot \prod_{l | N_{E\otimes\varpi_2}}(1+1/l)}{N_E \cdot \prod_{l|N_E}(1+1/l)} \times \frac{1+1/2}{D_2(E\otimes\varpi_2, 2)} \;=\; \frac{1}{16 \times \big((1-2^{1-2})\cdot\Upsilon_2(2^{-2})\big)} \;=\; \frac{1}{10},$$

in which case

$$\frac{L_\infty({\rm Sym}^2 E, 1)}{\pi\langle f_E, f_E\rangle_{N_E}} \;=\; \frac{L_\infty({\rm Sym}^2 E, 1)}{\pi\langle f_{E\otimes\varpi_2}, f_{E\otimes\varpi_2}\rangle_{N_{E\otimes\varpi_2}}} \times \frac{\langle f_{E\otimes\varpi_2}, f_{E\otimes\varpi_2}\rangle_{N_{E\otimes\varpi_2}}}{\langle f_E, f_E\rangle_{N_E}} \;=\; 4 \times \frac{1}{10} \;=\; \frac{2}{5}.$$

Now for each choice of prime $p \in \{3, 5, 7, 13\}$, applying Theorem 3.2 yields the congruences

$$\mathbf{L}_p^{\rm aut}({\rm Sym}^2 E, 1)' \;\equiv\; \frac{\sum_{e\in(\mathbb{Z}/p^m\mathbb{Z})^\times}\big(\log_p {<}e{>}_p\big)^2 \cdot \mu_E^{\rm imp}(e + p^m\mathbb{Z}_p)}{2! \times \Upsilon_2(2^{-1}) \times \log_p(1/2)} \quad {\rm mod}\; p^{m+\nu_{m,p}-1}.$$

Evaluating the moments of the measure ${\rm d}\mu_E^{\rm imp}$ (via Lemma 2.9) for varying $m \geq 2$, we obtain

$$\mathbf{L}_3^{\rm aut}({\rm Sym}^2 E, 1)' \;=\; p + O(p^4), \qquad \mathbf{L}_5^{\rm aut}({\rm Sym}^2 E, 1)' \;=\; p + O(p^2),$$
$$\mathbf{L}_7^{\rm aut}({\rm Sym}^2 E, 1)' \;=\; 2p + O(p^2), \qquad \mathbf{L}_{13}^{\rm aut}({\rm Sym}^2 E, 1)' \;=\; 4p + O(p^2).$$

Finally, dividing the above derivatives by $(1-\alpha_p^{-2})(1-p\alpha_p^{-2}) \times \frac{L_\infty(\mathrm{Sym}^2 E,1)}{\pi\langle f_E,f_E\rangle_{N_E}}$, we conclude that

$$\begin{aligned}
\mathcal{L}_3^{\mathrm{an}}\big(\mathrm{Sym}^2 X_0(11)\big) &= \mathcal{L}_3^{\mathrm{an}}(\mathrm{Sym}^2 E) = 1+2p^2+O(p^3)\\
\mathcal{L}_5^{\mathrm{an}}\big(\mathrm{Sym}^2 X_0(11)\big) &= \mathcal{L}_5^{\mathrm{an}}(\mathrm{Sym}^2 E) = p+O(p^2)\\
\mathcal{L}_7^{\mathrm{an}}\big(\mathrm{Sym}^2 X_0(11)\big) &= \mathcal{L}_7^{\mathrm{an}}(\mathrm{Sym}^2 E) = 2p+O(p^2)\\
\mathcal{L}_{13}^{\mathrm{an}}\big(\mathrm{Sym}^2 X_0(11)\big) &= \mathcal{L}_{13}^{\mathrm{an}}(\mathrm{Sym}^2 E) = 2p+O(p^2)
\end{aligned}$$

which are all non-zero elements of $\mathbb{Z}_p$.

*Remark:* In fact, if one chooses $p=11$ so that $X_0(11)$ has split multiplicative reduction at $p$, then it is established in [13, p51] that $\mathcal{L}_{11}^{\mathrm{an}}\big(\mathrm{Sym}^2 X_0(11)\big) = 6p+5p^2+7p^3+7p^4+O(p^5) \neq 0$ by using an approach based on overconvergent modular symbols[3]. It follows immediately that Conjecture 1.3 must hold for the modular elliptic curve $X_0(11)$, at **all odd primes** $p<17$.

### 3.3 The connection with deformation theory

We conclude by interpreting these numerical calculations in the context of $\Lambda$-adic cusp forms. For a given elliptic curve $E_{/\mathbb{Q}}$ and a good ordinary prime $p\geq 3$, one can lift the $p$-stabilisation $f_0 \in \mathcal{S}_2\big(\Gamma_0(pN_E)\big)$ to an $\mathbb{I}$-adic eigenform, $\mathcal{F}$, where $\mathbb{I}$ denotes a suitable finite, flat extension of $\mathbb{Z}_p[\![X]\!]$, isomorphic to the irreducible component of the universal ordinary Hecke algebra carrying the form $f_0$.

For a sufficiently small choice of $p$-adic disk $\mathcal{W}\subset\mathbb{Z}_p$ centred on $k=2$, each specialisation

$$\mathcal{F}_k := \mathcal{F}\big|_{X=(1+p)^{k-2}-1} \in \mathcal{S}_k\big(\Gamma_0(N_E p^\infty),\omega_p^{2-k}\big) \quad \text{for } k\in\mathcal{W}\cap\mathbb{Z}_{\geq 2}$$

yields a classical cuspidal Hecke eigenform, with the $q$-expansion $\mathcal{F}_k(q) = \sum_{n=1}^\infty a\big(\mathcal{F}_k,n\big)q^n$. One can then interpolate each $q^n$-coefficient to yield a function, $a\big(\mathcal{F}(X),n\big)$, on the disk $\mathcal{W}$.

If $n=p$, then the derivative of $a\big(\mathcal{F}(X),p\big)$ with respect to $X$ is rigid meromorphic on $\mathcal{W}$. Hida established in [25, Prop 7.1] under suitable hypotheses (which are true, for instance, if the versal deformation ring $\mathcal{R}_E$ is Gorenstein) that $\frac{\mathrm{d}a(\mathcal{F},p)}{\mathrm{d}X}$ is non-zero, and can thus vanish at only finitely many unspecified bad weights. Furthermore, the main formula in [26, Thm 1.1] yields

$$\mathcal{L}_p^{\mathrm{Gr}}\big(\mathrm{Sym}^2(\mathcal{F}_k)(k)\big) = -2\log_p(1+p)\cdot a\big(\mathcal{F}_k,p\big)^{-1} \times \frac{\mathrm{d}a(\mathcal{F},p)}{\mathrm{d}X}\bigg|_{X=(1+p)^{k-2}-1} \tag{10}$$

for every weight $k\in\mathcal{W}\cap\mathbb{Z}_{\geq 2}$, where $\mathcal{L}_p^{\mathrm{Gr}}\big(-\big)$ again denotes Greenberg's algebraic $\mathcal{L}$-invariant.

Note that the Gorenstein property of the versal deformation ring $\mathcal{R}_E$ above has been verified for numerous elliptic curves $E$, and ordinary primes $p\geq 3$ (see [4, 24, 35]). For example, it is known to hold if the conductor $N_E$ of the elliptic curve is a square-free integer.

*Remarks:* (a) Let $\mathbf{L}_p\big(\mathcal{F}_k\otimes\mathcal{F}_k,s\big)$ denote the analytic $p$-adic $L$-function constructed in [23], which interpolates the special values $L\big(\mathcal{F}_k\otimes\mathcal{F}_k\otimes\chi,k\big)$. From Dasgupta's result in [10, Thm 1], one has a factorisation

$$\mathbf{L}_p\big(\mathcal{F}_k\otimes\mathcal{F}_k,s\big) = \star \times \zeta_p\big(s-k+1,\omega_p^0\big)\times\mathbf{L}_p\big(\mathrm{Sym}^2(\mathcal{F}_k),s\big).$$

Here $\star$ consists of some Euler factors which are non-zero at classical weights, and so $\star\big|_{s=k}\neq 0$.

[3] We also computed $\mathcal{L}_p(\mathrm{Sym}^2 E)$ for $E=304e1$ at the good ordinary prime $p=5$, using an identical method. In fact $\mathcal{L}_5(\mathrm{Sym}^2(304e1)) = \mathcal{L}_5(\mathrm{Sym}^2(19a1))$ because $E\otimes\varpi_2$ is $\mathbb{Q}$-isogenous to $19a1$; thankfully, the value we obtained numerically agreed with the 5-adic expansion for $\mathcal{L}_5(19a1)$ given in [13, p52], at the weight $k+2=2$.

(b) Allowing $s \to k$ and observing that $\text{Res}_{s=k}\Big(\zeta_p\big(s-k+1,\omega_p^0\big)\Big) = 1-p^{-1}$, the above implies

$$\mathbf{L}_p\big(\mathcal{F}_k \otimes \mathcal{F}_k, k\big) \;=\; \mathcal{L}_p^{\text{an}}\big(\text{Sym}^2(\mathcal{F}_k)(k)\big) \times \mathcal{E}_p(\mathcal{F}_k) \cdot \frac{L_\infty(\text{Sym}^2\mathcal{F}_k, k)}{\Omega_{\infty,\text{Sym}^2(\mathcal{F}_k)}} \tag{11}$$

with $\mathcal{E}_p(\mathcal{F}_k) = \star\big|_{s=k} \cdot \big(1-p^{-1}\big)\big(1-\alpha_p(\mathcal{F}_k)^{-2}p^{k-1}\big)\big(1-\beta(\mathcal{F}_k)_p^{-2}p^{-k}\big) \neq 0$.

(c) Under the same assumptions as [26, Thm 1.1], Citro proves in [5, Thm 1] that

$$\mathbf{L}_p\big(\mathcal{F}_k \otimes \mathcal{F}_k, k\big) \;=\; \mathcal{L}_p^{\text{Gr}}\big(\text{Sym}^2(\mathcal{F}_k)(k)\big) \times \mathcal{E}_p(\mathcal{F}_k) \cdot \frac{L_\infty(\text{Sym}^2\mathcal{F}_k, k)}{\Omega_{\infty,\text{Sym}^2(\mathcal{F}_k)}}. \tag{12}$$

Using Equations (11) and (12), Dasgupta [10, Thm 4] then reads off Greenberg's prediction that

$$\mathcal{L}_p^{\text{Gr}}\big(\text{Sym}^2(\mathcal{F}_k)(k)\big) \;=\; \mathcal{L}_p^{\text{an}}\big(\text{Sym}^2(\mathcal{F}_k)(k)\big) \;=\; \mathcal{L}_p^{\text{an}}\big(\text{Sym}^2(\mathcal{F}_k)(k-1)\big)$$

(note the second equality is a consequence of the $p$-adic functional equation for $\text{Sym}^2(\mathcal{F}_k)$).

A corollary of these remarks is that we can replace the algebraic $\mathcal{L}$-invariant in Equation (10) with either analytic version. In particular, at weight two Hida's formula now becomes

$$\frac{\mathrm{d}a(\mathcal{F},p)}{\mathrm{d}X}\bigg|_{X=0} \;=\; -\frac{\alpha_p}{2\log_p(1+p)} \times \mathcal{L}_p^{\text{an}}\big(\text{Sym}^2 E\big) \tag{13}$$

hence the derivative of $a(\mathcal{F},p)$ at zero coincides with $\frac{\mathcal{L}_p^{\text{an}}(\text{Sym}^2E)}{p}$, up to an explicit $p$-adic unit. Of course, this could just end up being be the equation "$0=0$" in disguise!

Nevertheless, combining our numerical calculations from Appendix B with Equation (13):

**Corollary 3.3.** *Suppose that $E$ is an elliptic curve over $\mathbb{Q}$ of conductor $N_E \le 300$ with $4|N_E$, and let $p \le 13$ be a prime of good ordinary reduction for $E$. Provided that $(E,p)$ is not one of the ten missing pairs listed in Theorem 1.4(ii)-(iii), it immediately follows that*

$$\frac{\mathrm{d}a(\mathcal{F},p)}{\mathrm{d}X}\bigg|_{X=0} \;=\; \delta_p(E) \cdot \frac{\mathcal{L}_p^{\text{an}}(\text{Sym}^2E)}{p} \;\neq\; 0$$

*where $\mathcal{F}$ denotes the Hida family lifting $f_E \in \mathcal{S}_2^{\text{new}}\big(\Gamma_0(N_E)\big)$, and $\delta_p(E) := -\frac{p\alpha_p}{2\log_p(1+p)} \in \mathbb{Z}_p^\times$.*

Dummit, Hablicsek, Harron, Jain, Pollack and Ross [13] have a direct method to calculate $a(\mathcal{F},p)'(0)$ through the use of overconvergent modular symbols, and they have computed four examples in *op. cit.*, thereby establishing the non-triviality of $\mathcal{L}_p^{\text{an}}(\text{Sym}^2E)$ in these cases. Their results further determine power series expansions for $a(\mathcal{F}_k,p)$, as a function of $k$, over the weight-space $\mathcal{W}$.

The non-triviality of this $\mathcal{L}$-invariant has a key consequence for the Iwasawa Main Conjecture for $\text{Sym}^2E$ over the cyclotomic $\mathbb{Z}_p$-extension $\mathbb{Q}^{\text{cyc}}$ of $\mathbb{Q}$. The property that $\mathcal{L}_p^{\text{an}}(\text{Sym}^2E) \neq 0$ allows one to deduce that the order of the *algebraic $p$-adic $L$-function* at $s=1$ is exactly one. Here the algebraic $p$-adic $L$-function denotes the Mazur-Mellin transform of a generator, for the characteristic ideal of $\text{Hom}_{\text{cont}}\big(\text{Sel}_{p^\infty}\big(\text{Sym}^2E(1)/\mathbb{Q}^{\text{cyc}}\big), \mathbb{Q}/\mathbb{Z}\big)$ over the cyclotomic Iwasawa algebra $\mathbb{Z}_p\big[\big[\text{Gal}(\mathbb{Q}^{\text{cyc}}/\mathbb{Q})\big]\big]$ – we refer the reader to [30, Sect 10] for a fuller discussion.

# A Determining the set $S_1$, and the bad factors $H_l(X)$ with $l \in S_1$

The purpose of this Appendix is to compute the decomposition $S_1 = S_{1,+} \cup S'_{1,-} \cup S''_{1,-}$, and the corresponding Euler factors $H_l(X)$. We retain the same notation and assumptions as §3.2. Let $\Delta_E$ denote the discriminant associated to a minimal Weierstrass equation for $E$ over $\mathbb{Z}$.

**Proposition A.1.** *(a) A prime $l \in S_1$ belongs to the subset $S''_{1,-}$ if and only if $\mathrm{ord}_l\big(N_{E\otimes\theta}\big) = 0$ at the character $\theta = \varpi_l$ if $l > 2$, or instead at $\theta \in \Big\{\varpi_2, \big(\frac{-}{2}\big), \big(\frac{-}{-2}\big)\Big\}$ if $l = 2$, in which case*

$$H_l(X) \;=\; (1 - \hat{\alpha}_l^2 X)(1 - \hat{\beta}_l^2 X)(1 - lX)$$

*where $1 - a_l(E\otimes\theta)X + lX^2 = (1 - \hat{\alpha}_l X)(1 - \hat{\beta}_l X)$.*

*(b) A prime $l \in S_1 - S''_{1,-} - \{2,3\}$ belongs to $S'_{1,-}$ if and only if either*

- $\mathrm{ord}_l\big(\Delta_E\big) = 2, 4, 8, 10$ *and* $l \equiv 1 \pmod 3$, *or*
- $\mathrm{ord}_l\big(\Delta_E\big) = 3, 9$ *and* $l \equiv 1 \pmod 4$

*in which case $H_l(X) = 1 - lX$.*

*(c) A prime $l \in S_1 - S''_{1,-} - \{2,3\}$ belongs to $S_{1,+}$ if and only if either*

- $\mathrm{ord}_l\big(\Delta_E\big) = 2, 4, 8, 10$ *and* $l \equiv 2 \pmod 3$, *or*
- $\mathrm{ord}_l\big(\Delta_E\big) = 3, 9$ *and* $l \equiv 3 \pmod 4$

*in which case $H_l(X) = 1 + lX$.*

*(d) For a prime $l \in \big(S_1 \cap \{2,3\}\big) - S''_{1,-}$, one determines whether it belongs to $S'_{1,-}$ or to $S_{1,+}$, and also its Euler factor $H_l(X)$, by using the tables in [7, p121] and Lemma 2.13 of op. cit.*

*Proof.* Most of these statements follow from the description in [32] of the Galois representation $\rho_{E,p^\infty} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Z}_p)$ associated to the $p$-adic Tate module $\mathrm{Ta}_p(E) := \varprojlim_n E_{p^n}$.

Firstly (a) is true because $\rho_{E,p^\infty} \otimes \theta$ will be unramified at $l$, and corresponds to the Tate module of the quadratic twist $E \otimes \theta$, which has good reduction at $l$ by the criterion of Néron, Ogg and Shafarevich; consequently $\mathrm{Sym}^2(\rho_{E,p^\infty}) \cong \mathrm{Sym}^2(\rho_{E,p^\infty} \otimes \theta)$ is also unramified at $l$.

To establish (b) and (c), let us now assume (i) the prime $l \geq 5$, and also (ii) $d_l := \#\Phi_l > 2$ so that $\Phi_l \in \{C_3, C_4, C_6\}$ here. Then using [7, Lemma 1.4],

$$H_l(X) \;=\; \begin{cases} 1 - lX & \text{if } \mathbb{Q}_l(E_p)/\mathbb{Q}_l \text{ is abelian} \\ 1 + lX & \text{if } \mathbb{Q}_l(E_p)/\mathbb{Q}_l \text{ is non-abelian.} \end{cases}$$

Since $\mathbb{Q}_l(E_{p^\infty})/\mathbb{Q}_l(E_p)$ is unramified, we observe that $\mathbb{Q}_l(E_p)/\mathbb{Q}_l$ is abelian if and only if $\mathbb{Q}_l(E_{p^\infty})/\mathbb{Q}_l$ is abelian.

If $\mathbb{Q}_l(E_p)/\mathbb{Q}_l$ is abelian, then $\Phi_l$ factors through the inertia subgroup inside $\mathrm{Gal}\big(\mathbb{Q}_l^{\mathrm{ab}}/\mathbb{Q}_l\big)$, and hence through $\mathrm{Gal}\big(\mathbb{Q}_l(\mu_{l^\infty})/\mathbb{Q}_l\big)$. Because $l \nmid d_l$ clearly $\mathrm{Gal}\big(\mathbb{Q}_l(\mu_{l^\infty})/\mathbb{Q}_l(\mu_l)\big)$ acts trivially on $\mathrm{Ta}_p(E)$, in which case $\Phi_l$ factors through $\mathrm{Gal}\big(\mathbb{Q}_l(\mu_l)/\mathbb{Q}_l\big)$, whence $l \equiv 1 \pmod{d_l}$.

Conversely, there exists a unique tamely ramified extension $H_d$ of $\mathbb{Q}_p^{\mathrm{nr}}$ with degree $d > 0$. If $l \equiv 1 \pmod{d_l}$ then the action of $\Phi_l \cong \rho_{E,p^\infty}(I_l)$ on $\mathrm{Ta}_p(E)$ factors through the algebraic extension $H_{d_l} = \mathbb{Q}_l^{\mathrm{nr}}(E_{p^\infty}) \subset \mathbb{Q}_l^{\mathrm{nr}}(\mu_l)$, which is certainly an abelian extension of $\mathbb{Q}_l$.

*Conclusion:* The extension $\mathbb{Q}_l(E_p)/\mathbb{Q}_l$ is abelian if and only if $l \equiv 1 \pmod{d_l}$.

To complete the proof, we note that $d_l = \#\Phi_l$ can be read off from [32, p312] as follows:

- $\#\Phi_l = 3$ if and only if $\mathrm{ord}_l(\Delta_E) \equiv 4$ or $8 \bmod 12$;

- $\#\Phi_l = 4$ if and only if $\text{ord}_l(\Delta_E) \equiv 3$ or $9 \bmod 12$;
- $\#\Phi_l = 6$ if and only if $\text{ord}_l(\Delta_E) \equiv 2$ or $10 \bmod 12$.

It is then a tedious but straightforward exercise to verify that the conditions stated in (b) correspond to $l \equiv 1 \pmod{d_l}$, while the conditions in (c) correspond to $l \not\equiv 1 \pmod{d_l}$. $\square$

# B Results for odd primes $p \leq 13$, in the range $11 \leq N_E \leq 300$

Throughout we have only considered elliptic curves $E_{/\mathbb{Q}}$ whose conductors are divisible by 4. We first treat the curves with $D(E,1) \neq 0$, and then the six exceptional curves with $D(E,1) = 0$.

## B.1 Tables of $\mathcal{L}$-invariants for elliptic curves $E$ with $D(E,1) \neq 0$

Tabulated below are the values we computed for both the derivative of $\mathbf{L}_p^{\text{imp}}(\text{Sym}^2 E, s)$ at $s = 1$ together with the corresponding $\mathcal{L}$-invariant term, for the elliptic curves $E$ with $D(E,1) \neq 0$. If the elliptic curve $E$ is already a quadratic twist of another (earlier) elliptic curve listed in our tables, then we omit the $\mathcal{L}$-invariant data for $E$ completely.

The reader will notice for the elliptic curves of conductor 32 and 36, which have complex multiplication by $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$ respectively, that $\mathcal{L}_p(\text{Sym}^2 E)$ coincides with $\log_p(\alpha_p^{-2})$ in agreement with the Ferrero-Greenberg formula. However if $E$ has no complex multiplication, this identity no longer appears to hold in general.

$E = 20\text{a}1$, $C_{\text{Sym}^2 E} = 10^2$, $S_1 = \{2\}$, $\xi_{\text{Sym}^2 E} = \frac{8}{3}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\text{imp}}(\text{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\text{an}}(\text{Sym}^2 E)$ |
|---|---|---|---|
| 3 | $-2$ | $p^2 + p^3 + 2p^4 + 2p^5 + 2p^6 + O(p^7)$ | $p^2 + 2p^4 + O(p^7)$ |
| 7 | 2 | $2p + 2p^2 + p^3 + O(p^4)$ | $p + 2p^2 + p^3 + O(p^4)$ |
| 13 | 2 | $11p + 2p^2 + O(p^3)$ | $12p + 12p^2 + O(p^3)$ |

$E = 24\text{a}1$, $C_{\text{Sym}^2 E} = 12^2$, $S_1 = \{2\}$, $\xi_{\text{Sym}^2 E} = 2$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\text{imp}}(\text{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\text{an}}(\text{Sym}^2 E)$ |
|---|---|---|---|
| 5 | $-2$ | $2p^2 + p^3 + p^4 + O(p^5)$ | $3p^2 + 2p^3 + O(p^5)$ |
| 11 | 4 | $6p + 8p^2 + O(p^3)$ | $p + O(p^3)$ |
| 13 | $-2$ | $7p + 9p^2 + O(p^3)$ | $9p + 4p^2 + O(p^3)$ |

$E = 32\text{a}1$, $C_{\text{Sym}^2 E} = 8^2$, $S_1 = \{2\}$, $\xi_{\text{Sym}^2 E} = 1$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\text{imp}}(\text{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\text{an}}(\text{Sym}^2 E)$ |
|---|---|---|---|
| 5 | $-2$ | $3p + p^2 + 2p^3 + 4p^4 + O(p^5)$ | $4p + 3p^2 + 3p^3 + 3p^4 + O(p^5)$ |
| 13 | 6 | $p + 9p^2 + O(p^3)$ | $4p + 7p^2 + O(p^3)$ |

$E = 36\text{a}1$, $C_{\text{Sym}^2 E} = 6^2$, $S_1 = \{2, 3\}$, $\xi_{\text{Sym}^2 E} = \frac{4}{3}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\text{imp}}(\text{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\text{an}}(\text{Sym}^2 E)$ |
|---|---|---|---|
| 7 | $-4$ | $6p + 3p^2 + 2p^3 + O(p^4)$ | $2p + 3p^2 + 2p^3 + O(p^4)$ |
| 13 | 2 | $p^2 + O(p^3)$ | $p^2 + O(p^3)$ |

$E = 40\text{a}1$, $C_{\mathrm{Sym}^2 E} = 20^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 2$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 7 | $-4$ | $p + 6p^3 + O(p^4)$ | $p + 4p^2 + 6p^3 + O(p^4)$ |
| 11 | 4 | $5p + 4p^2 + O(p^3)$ | $10p + 3p^2 + O(p^3)$ |
| 13 | $-2$ | $10p + 2p^2 + O(p^3)$ | $11p + 11p^2 + O(p^3)$ |

$E = 44\text{a}1$, $C_{\mathrm{Sym}^2 E} = 22^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 3 | 1 | $2p^3 + O(p^6)$ | $p^3 + p^5 + O(p^6)$ |
| 5 | $-3$ | $4p + 2p^2 + p^4 + O(p^5)$ | $2p + 3p^3 + 3p^4 + O(p^5)$ |
| 7 | 2 | $3p + 5p^3 + O(p^4)$ | $5p + p^2 + O(p^4)$ |
| 13 | $-4$ | $3p + p^2 + O(p^3)$ | $9p + 11p^2 + O(p^3)$ |

$E = 52\text{a}1$, $C_{\mathrm{Sym}^2 E} = 26^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 5 | 2 | $4p + 2p^4 + O(p^5)$ | $2p + 2p^3 + 3p^4 + O(p^5)$ |
| 7 | $-2$ | $p^3 + O(p^4)$ | $4p^3 + O(p^4)$ |
| 11 | $-2$ | $10p + p^2 + O(p^3)$ | $5p + 9p^2 + O(p^3)$ |

$E = 56\text{a}1$, $C_{\mathrm{Sym}^2 E} = 28^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 2$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 5 | 2 | $p + p^2 + 2p^3 + O(p^4)$ | $4p + 2p^3 + O(p^4)$ |
| 11 | $-4$ | $p + O(p^3)$ | $2p + 10p^2 + O(p^3)$ |
| 13 | 2 | $9p + O(p^2)$ | $6p + O(p^2)$ |

$E = 56\text{b}1$, $C_{\mathrm{Sym}^2 E} = 28^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 2$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 3 | 2 | $2p + p^4 + O(p^5)$ | $2 + p^3 + O(p^4)$ |
| 5 | $-4$ | $3p + p^2 + O(p^3)$ | $4 + 2p + O(p^2)$ |

$E = 76\text{a}1$, $C_{\mathrm{Sym}^2 E} = 38^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 3 | 2 | $1 + 2p + O(p^4)$ | $1 + p + 2p^2 + O(p^4)$ |
| 5 | $-1$ | $4p + 3p^2 + O(p^3)$ | $3 + 2p + O(p^2)$ |
| 7 | $-3$ | $p^2 + O(p^3)$ | $6p^2 + O(p^3)$ |
| 11 | 5 | $5p + 4p^2 + O(p^3)$ | $9p + 4p^2 + O(p^3)$ |
| 13 | $-4$ | $4p + 9p^2 + O(p^3)$ | $12p + 3p^2 + O(p^3)$ |

$E = 84\text{a}1$, $C_{\mathrm{Sym}^2 E} = 42^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 11 | $-6$ | $5p + 3p^2 + O(p^3)$ | $9p + 4p^2 + O(p^3)$ |
| 13 | 2 | $10p + 10p^2 + O(p^3)$ | $5p + 6p^2 + O(p^3)$ |

$E = 84\text{b}1$, $C_{\mathrm{Sym}^2 E} = 42^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 5 | 4 | $p + 2p^2 + O(p^3)$ | $1 + 3p + O(p^2)$ |
| 11 | 2 | $5p + 4p^2 + O(p^3)$ | $8p + 3p^2 + O(p^3)$ |
| 13 | $-6$ | $7p + 3p^2 + O(p^3)$ | $4p + 2p^2 + O(p^3)$ |

$E = 88\text{a}1$, $C_{\mathrm{Sym}^2 E} = 44^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 2$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 5 | $-3$ | $4p + p^3 + O(p^4)$ | $p + 4p^2 + p^3 + O(p^4)$ |
| 7 | $-2$ | $6p + 5p^2 + O(p^3)$ | $4p + 5p^2 + O(p^3)$ |

$E = 92\text{a}1$, $C_{\mathrm{Sym}^2 E} = 46^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 3 | 1 | $p^4 + 2p^5 + O(p^6)$ | $2p^4 + 2p^5 + O(p^6)$ |
| 7 | 2 | $6p + 2p^2 + O(p^3)$ | $3p + 4p^2 + O(p^3)$ |
| 13 | $-1$ | $11p + O(p^2)$ | $2 + O(p)$ |

$E = 92\text{b}1$, $C_{\mathrm{Sym}^2 E} = 46^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 5 | $-2$ | $4p + p^2 + 2p^3 + O(p^4)$ | $2p + 3p^2 + O(p^4)$ |
| 7 | $-4$ | $3p + 3p^2 + O(p^3)$ | $4p + 4p^2 + O(p^3)$ |
| 11 | 2 | $10p + O(p^2)$ | $5p + O(p^2)$ |
| 13 | $-5$ | $12p + O(p^2)$ | $12p + O(p^2)$ |

$E = 96\text{a}1$, $C_{\mathrm{Sym}^2 E} = 24^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 1$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 5 | 2 | $p^2 + 2p^3 + O(p^4)$ | $3p^2 + 4p^3 + O(p^4)$ |
| 7 | $-4$ | $4p + 2p^2 + O(p^3)$ | $p + 2p^2 + O(p^3)$ |
| 11 | 4 | $9p + O(p^2)$ | $3p + O(p^2)$ |
| 13 | $-2$ | $3p + O(p^2)$ | $4p + O(p^2)$ |

$E = 104\text{a}1$, $C_{\mathrm{Sym}^2 E} = 52^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 2$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 3 | 1 | $p + p^2 + 2p^3 + p^4 + O(p^5)$ | $2 + 2p + 2p^3 + O(p^4)$ |
| 5 | $-1$ | $2p^2 + O(p^3)$ | $2p + O(p^2)$ |
| 7 | 5 | $p + O(p^3)$ | $3p + 6p^2 + O(p^3)$ |
| 11 | $-2$ | $9p + O(p^2)$ | $6p + O(p^2)$ |

$E = 108\text{a}1$, $C_{\mathrm{Sym}^2 E} = 18^2$, $S_1 = \{2, 3\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{9}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 7 | 5 | $6p + 5p^2 + O(p^3)$ | $2p + 3p^2 + O(p^3)$ |
| 13 | $-7$ | $6p^2 + O(p^3)$ | $p^2 + O(p^3)$ |

$E = 112\text{c}1$, $C_{\mathrm{Sym}^2 E} = 14^2$, $S_1 = \{\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{1}{2}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 3 | 2 | $p + 2p^2 + O(p^5)$ | $1 + p^3 + O(p^4)$ |
| 13 | $-4$ | $9p + O(p^2)$ | $p + O(p^2)$ |

$E = 116\text{a}1$, $C_{\mathrm{Sym}^2 E} = 58^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 5 | 3 | $p + 3p^2 + O(p^3)$ | $3p + 2p^2 + O(p^3)$ |
| 7 | 4 | $4p + 3p^2 + O(p^3)$ | $3p + 2p^2 + O(p^3)$ |
| 11 | $-1$ | $O(p^2)$ | $O(p)$ |
| 13 | $-3$ | $12p + O(p^2)$ | $p + O(p^2)$ |

$E = 116\text{b}1$, $C_{\mathrm{Sym}^2 E} = 58^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 3 | 1 | $2p^2 + 2p^3 + O(p^5)$ | $p^2 + p^3 + p^4 + O(p^5)$ |
| 5 | 3 | $3p + 4p^2 + 4p^3 + O(p^4)$ | $4p + 2p^2 + 4p^3 + O(p^4)$ |
| 7 | $-4$ | $6p + 4p^2 + O(p^3)$ | $p + 4p^2 + O(p^3)$ |
| 11 | 3 | $5p + O(p^2)$ | $4p + O(p^2)$ |
| 13 | 5 | $9p + O(p^2)$ | $9p + O(p^2)$ |

$E = 116\text{c}1$, $C_{\mathrm{Sym}^2 E} = 58^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 3 | 2 | $1 + 2p + p^2 + 2p^3 + O(p^4)$ | $1 + p + 2p^3 + O(p^4)$ |
| 5 | $-2$ | $p + 4p^2 + O(p^3)$ | $3p + 4p^2 + O(p^3)$ |
| 7 | 4 | $3p + 4p^2 + O(p^3)$ | $4p + 3p^2 + O(p^3)$ |
| 11 | $-6$ | $2p + O(p^2)$ | $8p + O(p^2)$ |
| 13 | 2 | $8p + O(p^2)$ | $4p + O(p^2)$ |

$E = 120\text{a}1$, $C_{\mathrm{Sym}^2 E} = 60^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 2$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 11 | $-4$ | $6p + O(p^2)$ | $p + O(p^2)$ |
| 13 | 6 | $11p + O(p^2)$ | $9p + O(p^2)$ |

$E = 120\text{b}1$, $C_{\mathrm{Sym}^2 E} = 60^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 2$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 7 | 4 | $5p + 3p^2 + O(p^3)$ | $5p + 2p^2 + O(p^3)$ |
| 13 | $-6$ | $6p + O(p^2)$ | $12p + O(p^2)$ |

$E = 124\text{a}1$, $C_{\mathrm{Sym}^2E} = 62^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2E} = \frac{8}{3}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2E,1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2E)$ |
|---|---|---|---|
| 3 | $-2$ | $2+2p+p^2+O(p^4)$ | $2+p^2+2p^3+O(p^4)$ |
| 5 | $-3$ | $2p+3p^2+2p^3+O(p^4)$ | $p+p^2+2p^3+O(p^4)$ |
| 7 | $-1$ | $2p^2+O(p^3)$ | $4p+O(p^2)$ |
| 11 | $-6$ | $7p^2+O(p^3)$ | $6p^2+O(p^3)$ |
| 13 | 2 | $4p+O(p^2)$ | $2p+O(p^2)$ |

$E = 124\text{b}1$, $C_{\mathrm{Sym}^2E} = 62^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2E} = \frac{8}{3}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2E,1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2E)$ |
|---|---|---|---|
| 5 | 1 | $p+3p^2+O(p^3)$ | $2+p+O(p^2)$ |
| 7 | 3 | $5p+6p^2+O(p^3)$ | $2p+5p^2+O(p^3)$ |
| 11 | 6 | $O(p^3)$ | $O(p^3)$ |
| 13 | $-4$ | $11p+O(p^2)$ | $7p+O(p^2)$ |

$E = 128\text{a}1$, $C_{\mathrm{Sym}^2E} = 16^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2E} = \frac{1}{2}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2E,1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2E)$ |
|---|---|---|---|
| 3 | $-2$ | $2p+2p^2+p^3+O(p^5)$ | $2+p+p^2+2p^3+O(p^4)$ |
| 5 | $-2$ | $4p+2p^2+p^3+O(p^4)$ | $4p+3p^3+O(p^4)$ |
| 7 | $-4$ | $3p+5p^2+O(p^3)$ | $5p+6p^2+O(p^3)$ |
| 11 | 2 | $7p+O(p^2)$ | $4p+O(p^2)$ |
| 13 | $-2$ | $5p+O(p^2)$ | $9p+O(p^2)$ |

$E = 132\text{a}1$, $C_{\mathrm{Sym}^2E} = 66^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2E} = \frac{8}{3}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2E,1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2E)$ |
|---|---|---|---|
| 5 | 2 | $4p+p^2+p^3+O(p^4)$ | $2p+3p^2+2p^3+O(p^4)$ |
| 7 | $-2$ | $p+O(p^3)$ | $4p+O(p^3)$ |
| 13 | $-2$ | $12p+O(p^2)$ | $6p+O(p^2)$ |

$E = 132\text{b}1$, $C_{\mathrm{Sym}^2E} = 66^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2E} = \frac{8}{3}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2E,1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2E)$ |
|---|---|---|---|
| 5 | 2 | $2p+3p^2+O(p^3)$ | $p+4p^2+O(p^3)$ |
| 7 | 2 | $3p+p^2+O(p^3)$ | $5p+5p^2+O(p^3)$ |
| 13 | 6 | $3p+O(p^2)$ | $11p+O(p^2)$ |

$E = 136\text{a}1$, $C_{\mathrm{Sym}^2E} = 68^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2E} = 2$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2E,1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2E)$ |
|---|---|---|---|
| 3 | $-2$ | $2p^2+2p^3+p^4+O(p^5)$ | $2p+2p^2+p^3+O(p^4)$ |
| 5 | $-2$ | $3p+3p^2+3p^3+O(p^4)$ | $2p+2p^2+4p^3+O(p^4)$ |
| 7 | $-2$ | $2p+2p^2+O(p^3)$ | $6p+2p^2+O(p^3)$ |
| 11 | $-6$ | $10p+O(p^2)$ | $2p+O(p^2)$ |
| 13 | 2 | $9p+O(p^2)$ | $6p+O(p^2)$ |

$E = 136\text{b}1$, $C_{\mathrm{Sym}^2 E} = 68^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 2$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 3 | 2 | $2p + 2p^2 + 2p^3 + 2p^4 + O(p^5)$ | $2 + 2p + 2p^2 + 2p^3 + O(p^4)$ |
| 11 | 2 | $3p + O(p^2)$ | $2p + O(p^2)$ |
| 13 | $-6$ | $p + O(p^2)$ | $2p + O(p^2)$ |

$E = 140\text{a}1$, $C_{\mathrm{Sym}^2 E} = 70^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 3 | 1 | $2p + p^2 + O(p^4)$ | $p + 2p^2 + 2p^3 + O(p^4)$ |
| 11 | 3 | $p + O(p^2)$ | $3p + O(p^2)$ |
| 13 | $-1$ | $O(p^2)$ | $O(p)$ |

$E = 140\text{b}1$, $C_{\mathrm{Sym}^2 E} = 70^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 11 | $-5$ | $2p + O(p^2)$ | $8p + O(p^2)$ |
| 13 | $-3$ | $8p + O(p^2)$ | $5p + O(p^2)$ |

$E = 148\text{a}1$, $C_{\mathrm{Sym}^2 E} = 74^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 3 | $-1$ | $2p + p^2 + O(p^4)$ | $p + 2p^2 + 2p^3 + O(p^4)$ |
| 5 | $-4$ | $3p + 4p^2 + O(p^3)$ | $3 + 2p + O(p^2)$ |
| 7 | $-3$ | $p + 3p^2 + O(p^3)$ | $6p + 4p^2 + O(p^3)$ |
| 11 | 5 | $8p + O(p^2)$ | $10p + O(p^2)$ |

$E = 152\text{a}1$, $C_{\mathrm{Sym}^2 E} = 76^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 2$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 3 | $-2$ | $2p + O(p^5)$ | $2 + O(p^4)$ |
| 5 | $-1$ | $p + 2p^2 + O(p^3)$ | $1 + 4p + O(p^2)$ |
| 7 | $-3$ | $p + 6p^2 + O(p^3)$ | $p + O(p^3)$ |
| 11 | $-3$ | $3p + O(p^2)$ | $p + O(p^2)$ |
| 13 | $-4$ | $4p + O(p^2)$ | $3p + O(p^2)$ |

$E = 152\text{b}1$, $C_{\mathrm{Sym}^2 E} = 76^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 2$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 3 | 1 | $2p + 2p^2 + 2p^3 + 2p^4 + O(p^5)$ | $1 + 2p + O(p^4)$ |
| 7 | 3 | $p + O(p^3)$ | $p + p^2 + O(p^3)$ |
| 11 | 2 | $8p + O(p^2)$ | $9p + O(p^2)$ |
| 13 | 1 | $7p + O(p^2)$ | $8 + O(p)$ |

$E = 156\text{a}1$, $C_{\mathrm{Sym}^2 E} = 78^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 5 | $-4$ | $2p + p^2 + O(p^3)$ | $2 + 3p + O(p^2)$ |
| 7 | $-2$ | $4p + O(p^3)$ | $2p + 2p^2 + O(p^3)$ |
| 11 | $-4$ | $7p + O(p^2)$ | $5p + O(p^2)$ |

$E = 156\text{b}1$, $C_{\mathrm{Sym}^2 E} = 78^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 7 | 2 | $6p + O(p^3)$ | $3p + 3p^2 + O(p^3)$ |

$E = 160\text{a}1$, $C_{\mathrm{Sym}^2 E} = 40^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 1$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 3 | $-2$ | $2p + 2p^2 + p^3 + p^4 + O(p^5)$ | $1 + 2p + O(p^4)$ |
| 7 | $-2$ | $p + 3p^2 + O(p^3)$ | $6p + O(p^3)$ |
| 11 | $-4$ | $2p + O(p^2)$ | $8p + O(p^2)$ |
| 13 | $-6$ | $10p + O(p^2)$ | $p + O(p^2)$ |

$E = 168\text{a}1$, $C_{\mathrm{Sym}^2 E} = 84^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 2$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 5 | 2 | $p + p^2 + 3p^3 + O(p^4)$ | $4p + p^3 + O(p^4)$ |
| 13 | $-2$ | $3p + O(p^2)$ | $2p + O(p^2)$ |

$E = 168\text{b}1$, $C_{\mathrm{Sym}^2 E} = 84^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 2$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 5 | 2 | $4p^2 + 2p^3 + O(p^4)$ | $p^2 + O(p^4)$ |
| 13 | 6 | $10p + O(p^2)$ | $7p + O(p^2)$ |

$E = 172\text{a}1$, $C_{\mathrm{Sym}^2 E} = 86^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 3 | $-2$ | $2 + p + O(p^4)$ | $2 + 2p + 2p^3 + O(p^4)$ |
| 7 | $-4$ | $4p + 2p^2 + O(p^3)$ | $3p + 3p^2 + O(p^3)$ |
| 11 | $-3$ | $3p + O(p^2)$ | $9p + O(p^2)$ |
| 13 | $-1$ | $9p + O(p^2)$ | $4 + O(p)$ |

$E = 184\text{a}1$, $C_{\mathrm{Sym}^2 E} = 92^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 2$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 3 | $-1$ | $2p^2 + p^3 + p^4 + O(p^5)$ | $p + p^3 + O(p^4)$ |
| 5 | $-4$ | $2p^2 + O(p^3)$ | $p + O(p^2)$ |
| 7 | 2 | $p + 6p^2 + O(p^3)$ | $3p + 2p^2 + O(p^3)$ |
| 11 | $-4$ | $6p + O(p^2)$ | $p + O(p^2)$ |
| 13 | $-5$ | $3p + O(p^2)$ | $4p + O(p^2)$ |

$E = 184\text{b}1$, $C_{\mathrm{Sym}^2 E} = 92^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 2$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 3 | $-1$ | $2p^2 + 2p^3 + O(p^5)$ | $p + 2p^2 + 2p^3 + O(p^4)$ |
| 5 | $-2$ | $p + 3p^2 + O(p^4)$ | $4p + 3p^2 + 2p^3 + O(p^4)$ |
| 7 | $-4$ | $3p + p^2 + O(p^3)$ | $3p + 6p^2 + O(p^3)$ |
| 11 | $-2$ | $4p + O(p^2)$ | $10p + O(p^2)$ |
| 13 | 7 | $3p + O(p^2)$ | $6p + O(p^2)$ |

$E = 184\text{c}1$, $C_{\mathrm{Sym}^2 E} = 92^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 2$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 7 | 4 | $5p + 5p^2 + O(p^3)$ | $5p + 4p^2 + O(p^3)$ |
| 11 | 6 | $8p + O(p^2)$ | $6p + O(p^2)$ |
| 13 | $-2$ | $7p + O(p^2)$ | $9p + O(p^2)$ |

$E = 184\text{d}1$, $C_{\mathrm{Sym}^2 E} = 92^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 2$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 7 | $-2$ | $p + O(p^3)$ | $3p + 5p^2 + O(p^3)$ |
| 13 | $-5$ | $p + O(p^2)$ | $10p + O(p^2)$ |

$E = 200\text{a}1$, $C_{\mathrm{Sym}^2 E} = 20^2$, $S_1 = \{2, 5\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{2}{3}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 7 | 2 | $5p + 6p^2 + O(p^3)$ | $3p + 2p^2 + O(p^3)$ |
| 11 | 1 | $O(p^2)$ | $O(p)$ |
| 13 | 4 | $6p + O(p^2)$ | $7p + O(p^2)$ |

$E = 204\text{a}1$, $C_{\mathrm{Sym}^2 E} = 102^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 5 | $-1$ | $2p + p^2 + O(p^3)$ | $4 + 2p + O(p^2)$ |
| 7 | 4 | $6p^3 + O(p^4)$ | $p^3 + O(p^4)$ |
| 11 | 3 | $8p + O(p^2)$ | $2p + O(p^2)$ |
| 13 | 3 | $3p + O(p^2)$ | $10p + O(p^2)$ |

$E = 204\text{b}1$, $C_{\mathrm{Sym}^2 E} = 102^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 5 | 1 | $2p^2 + O(p^3)$ | $4p + O(p^2)$ |
| 11 | 5 | $10p + O(p^2)$ | $7p + O(p^2)$ |
| 13 | $-5$ | $8p + O(p^2)$ | $8p + O(p^2)$ |

$E = 208\text{a}1$, $C_{\mathrm{Sym}^2 E} = 26^2$, $S_1 = \{\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{1}{2}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 3 | $-1$ | $p + 2p^2 + 2p^4 + O(p^5)$ | $2 + 2p^2 + 2p^3 + O(p^4)$ |
| 5 | $-3$ | $2p + 3p^2 + 4p^3 + O(p^4)$ | $2p + p^2 + 2p^3 + O(p^4)$ |
| 7 | 1 | $6p + O(p^2)$ | $1 + O(p)$ |
| 11 | $-6$ | $7p + O(p^2)$ | $10p + O(p^2)$ |

$E = 208\text{d}1$, $C_{\mathrm{Sym}^2 E} = 26^2$, $S_1 = \{\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{1}{2}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 5 | $-1$ | $4p + 4p^2 + O(p^3)$ | $1 + p + O(p^2)$ |
| 7 | $-1$ | $3p^2 + O(p^3)$ | $4p + O(p^2)$ |
| 11 | 2 | $6p + O(p^2)$ | $5p + O(p^2)$ |

$E = 212\text{a}1$, $C_{\mathrm{Sym}^2 E} = 106^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 3 | $-1$ | $2 + 2p + p^3 + O(p^4)$ | $1 + p + p^2 + p^3 + O(p^4)$ |
| 5 | $-2$ | $p^2 + 2p^3 + O(p^4)$ | $3p^2 + 3p^3 + O(p^4)$ |
| 7 | $-2$ | $3p + 4p^2 + O(p^3)$ | $5p + 3p^2 + O(p^3)$ |
| 11 | 2 | $8p + O(p^2)$ | $4p + O(p^2)$ |
| 13 | $-7$ | $11p + O(p^2)$ | $10p + O(p^2)$ |

$E = 212\text{b}1$, $C_{\mathrm{Sym}^2 E} = 106^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 3 | 2 | $1 + p + p^2 + 2p^3 + O(p^4)$ | $1 + p^2 + p^3 + O(p^4)$ |
| 5 | 2 | $4p + p^2 + 3p^3 + O(p^4)$ | $2p + 3p^2 + 3p^3 + O(p^4)$ |
| 11 | $-4$ | $10p + O(p^2)$ | $4p + O(p^2)$ |
| 13 | $-2$ | $3p + O(p^2)$ | $8p + O(p^2)$ |

$E = 216\text{a}1$, $C_{\mathrm{Sym}^2 E} = 36^2$, $S_1 = \{2, 3\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{2}{3}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 5 | $-4$ | $2p + 3p^2 + O(p^3)$ | $3 + p + O(p^2)$ |
| 7 | $-3$ | $p + 4p^2 + O(p^3)$ | $3p + p^2 + O(p^3)$ |
| 11 | $-4$ | $6p + O(p^2)$ | $3p + O(p^2)$ |
| 13 | 1 | $4p + O(p^2)$ | $10 + O(p)$ |

$E = 216\text{c}1$, $C_{\mathrm{Sym}^2 E} = 36^2$, $S_1 = \{2, 3\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{2}{3}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 5 | 1 | $2p + O(p^2)$ | $1 + O(p)$ |
| 7 | 3 | $4p + 2p^2 + O(p^3)$ | $5p + 5p^2 + O(p^3)$ |
| 11 | $-5$ | $10p + O(p^2)$ | $6p + O(p^2)$ |
| 13 | 4 | $2p + O(p^2)$ | $11p + O(p^2)$ |

$E = 220\text{a}1$, $C_{\mathrm{Sym}^2 E} = 110^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 3 | $-2$ | $2 + O(p^2)$ | $2 + p + O(p^2)$ |
| 7 | $-4$ | $p + O(p^3)$ | $6p + 4p^2 + O(p^3)$ |
| 13 | $-4$ | $2p + O(p^2)$ | $6p + O(p^2)$ |

$E = 220\text{b}1$, $C_{\mathrm{Sym}^2 E} = 110^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 3 | 2 | $1 + p^2 + O(p^3)$ | $1 + 2p + p^2 + O(p^3)$ |

$E = 224\text{a}1$, $C_{\mathrm{Sym}^2 E} = 56^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 1$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 3 | $-2$ | $p^3 + O(p^4)$ | $2p^2 + O(p^3)$ |
| 11 | $-4$ | $10p + O(p^2)$ | $7p + O(p^2)$ |
| 13 | $-4$ | $8p + O(p^2)$ | $12p + O(p^2)$ |

$E = 228\text{a}1$, $C_{\mathrm{Sym}^2 E} = 114^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 5 | 2 | $p + O(p^3)$ | $3p + 2p^2 + O(p^3)$ |
| 11 | 2 | $9p + O(p^2)$ | $10p + O(p^2)$ |
| 13 | 2 | $11p + O(p^2)$ | $12p + O(p^2)$ |

$E = 228\text{b}1$, $C_{\mathrm{Sym}^2 E} = 114^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 5 | $-3$ | $2p + 4p^2 + O(p^3)$ | $p + 4p^2 + O(p^3)$ |
| 7 | 1 | $6p + O(p^2)$ | $5 + O(p)$ |
| 11 | $-5$ | $2p + O(p^2)$ | $8p + O(p^2)$ |
| 13 | $-6$ | $p + O(p^2)$ | $8p + O(p^2)$ |

$E = 232\text{a}1$, $C_{\mathrm{Sym}^2 E} = 116^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 2$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 3 | $-1$ | $p + O(p^4)$ | $2 + 2p^2 + O(p^3)$ |
| 5 | $-3$ | $p + 3p^2 + O(p^3)$ | $4p + p^2 + O(p^3)$ |
| 7 | 2 | $p + O(p^3)$ | $3p + 5p^2 + O(p^3)$ |
| 11 | $-3$ | $3p + O(p^2)$ | $p + O(p^2)$ |
| 13 | $-5$ | $2p + O(p^2)$ | $7p + O(p^2)$ |

$E = 232\text{b}1$, $C_{\mathrm{Sym}^2 E} = 116^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 2$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 3 | 1 | $p + 2p^3 + O(p^4)$ | $2 + O(p^3)$ |
| 5 | 1 | $3p + O(p^2)$ | $3 + O(p)$ |
| 7 | 2 | $2p + O(p^3)$ | $6p + 3p^2 + O(p^3)$ |
| 11 | 3 | $6p + O(p^2)$ | $2p + O(p^2)$ |
| 13 | $-1$ | $O(p)$ | $O(1)$ |

$E = 236\text{a}1$, $C_{\mathrm{Sym}^2 E} = 118^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 3 | $-1$ | $2p^2 + O(p^3)$ | $p^2 + O(p^3)$ |
| 5 | $-1$ | $4p + O(p^2)$ | $3 + O(p)$ |
| 7 | $-3$ | $6p + O(p^3)$ | $p + 5p^2 + O(p^3)$ |
| 11 | $-2$ | $3p + O(p^2)$ | $7p + O(p^2)$ |

$E = 236\text{b}1$, $C_{\mathrm{Sym}^2 E} = 118^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 3 | 1 | $2p^2 + O(p^4)$ | $p^2 + O(p^4)$ |
| 5 | 3 | $2p^3 + O(p^4)$ | $p^3 + O(p^4)$ |
| 7 | $-1$ | $6 + O(p)$ | $5p^{-1} + O(1)$ |
| 11 | 6 | $5p + O(p^2)$ | $9p + O(p^2)$ |
| 13 | $-4$ | $7p + O(p^2)$ | $8p + O(p^2)$ |

$E = 240\text{b}1$, $C_{\mathrm{Sym}^2 E} = 30^2$, $S_1 = \{\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{1}{2}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 7 | 4 | $p + 2p^2 + O(p^3)$ | $4p + 3p^2 + O(p^3)$ |
| 13 | 2 | $10p + O(p^2)$ | $5p + O(p^2)$ |

$E = 244\text{a}1$, $C_{\mathrm{Sym}^2 E} = 122^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 5 | $-3$ | $p + 4p^2 + O(p^3)$ | $3p + O(p^3)$ |
| 7 | $-3$ | $3p + 2p^2 + O(p^3)$ | $4p + O(p^3)$ |
| 11 | $-1$ | $4p + O(p^2)$ | $2 + O(p)$ |
| 13 | 1 | $O(p^2)$ | $O(p)$ |

$E = 248\text{a}1$, $C_{\mathrm{Sym}^2 E} = 124^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 2$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 3 | $-2$ | $p^2 + O(p^4)$ | $p + O(p^3)$ |
| 5 | 1 | $3p + O(p^2)$ | $3 + O(p)$ |
| 7 | $-3$ | $2p + 2p^2 + O(p^3)$ | $2p + 4p^2 + O(p^3)$ |
| 11 | $-2$ | $7p^2 + O(p^3)$ | $p^2 + O(p^3)$ |
| 13 | $-2$ | $6p + O(p^2)$ | $4p + O(p^2)$ |

$E = 248\text{b}1$, $C_{\mathrm{Sym}^2E} = 124^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2E} = 2$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2E)$ |
|---|---|---|---|
| 3 | $-2$ | $2p^3 + O(p^4)$ | $2p^2 + O(p^3)$ |
| 5 | 2 | $4p + O(p^3)$ | $p + 2p^2 + O(p^3)$ |
| 11 | 2 | $5p + 5p^2 + O(p^3)$ | $7p + 5p^2 + O(p^3)$ |
| 13 | 4 | $12p + O(p^2)$ | $9p + O(p^2)$ |

$E = 248\text{c}1$, $C_{\mathrm{Sym}^2E} = 124^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2E} = 2$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2E)$ |
|---|---|---|---|
| 5 | $-3$ | $p^3 + O(p^4)$ | $4p^3 + O(p^4)$ |
| 7 | $-3$ | $6p + O(p^3)$ | $6p + 6p^2 + O(p^3)$ |
| 11 | 2 | $3p^2 + O(p^3)$ | $2p^2 + O(p^3)$ |
| 13 | $-4$ | $7p + O(p^2)$ | $2p + O(p^2)$ |

$E = 256\text{a}1$, $C_{\mathrm{Sym}^2E} = 8^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2E} = \frac{1}{6}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2E)$ |
|---|---|---|---|
| 3 | $-2$ | $p + p^3 + O(p^4)$ | $p + p^2 + p^3 + O(p^4)$ |
| 11 | $-6$ | $2p + O(p^2)$ | $7p + O(p^2)$ |

$E = 256\text{b}1$, $C_{\mathrm{Sym}^2E} = 8^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2E} = \frac{1}{6}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2E)$ |
|---|---|---|---|
| 5 | $-4$ | $p^2 + O(p^3)$ | $p + O(p^2)$ |
| 13 | $-4$ | $5p + O(p^2)$ | $6p + O(p^2)$ |

$E = 260\text{a}1$, $C_{\mathrm{Sym}^2E} = 130^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2E} = \frac{8}{3}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2E)$ |
|---|---|---|---|
| 3 | 2 | $2 + p^2 + O(p^3)$ | $2 + p + 2p^2 + O(p^3)$ |
| 7 | 2 | $p^2 + O(p^3)$ | $4p^2 + O(p^3)$ |
| 11 | 4 | $4p + O(p^2)$ | $6p + O(p^2)$ |

$E = 264\text{a}1$, $C_{\mathrm{Sym}^2E} = 132^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2E} = 2$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2E)$ |
|---|---|---|---|
| 7 | 2 | $3p + p^2 + O(p^3)$ | $2p + 5p^2 + O(p^3)$ |

$E = 264\text{b}1$, $C_{\mathrm{Sym}^2E} = 132^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2E} = 2$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2E)$ |
|---|---|---|---|
| 5 | 2 | $p + 2p^2 + O(p^3)$ | $4p + 4p^2 + O(p^3)$ |
| 13 | 2 | $12p + O(p^2)$ | $8p + O(p^2)$ |

$E = 264\text{c}1$, $C_{\mathrm{Sym}^2 E} = 132^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 2$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 5 | $-2$ | $p^2 + O(p^3)$ | $4p^2 + O(p^3)$ |
| 7 | 4 | $5p + 6p^2 + O(p^3)$ | $5p + 5p^2 + O(p^3)$ |
| 13 | 6 | $5p + O(p^2)$ | $10p + O(p^2)$ |

$E = 264\text{d}1$, $C_{\mathrm{Sym}^2 E} = 132^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 2$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 5 | 4 | $3p + O(p^2)$ | $4 + O(p)$ |
| 7 | $-2$ | $6p + O(p^2)$ | $4p + O(p^2)$ |

$E = 268\text{a}1$, $C_{\mathrm{Sym}^2 E} = 134^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 3 | 2 | $p^{-1} + 2 + p + O(p^2)$ | $p^{-1} + 1 + O(p^2)$ |
| 5 | 2 | $4p + 3p^2 + O(p^3)$ | $2p + 4p^2 + O(p^3)$ |
| 7 | 2 | $p + 6p^2 + O(p^3)$ | $4p + 3p^2 + O(p^3)$ |
| 11 | $-4$ | $3p + O(p^2)$ | $10p + O(p^2)$ |
| 13 | $-6$ | $6p + O(p^2)$ | $9p + O(p^2)$ |

$E = 272\text{d}1$, $C_{\mathrm{Sym}^2 E} = 34^2$, $S_1 = \{\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{1}{2}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 3 | 2 | $p + p^2 + p^3 + O(p^4)$ | $1 + 2p + 2p^2 + O(p^3)$ |
| 7 | 4 | $3p + 2p^2 + O(p^3)$ | $5p + p^2 + O(p^3)$ |
| 11 | $-6$ | $p + O(p^2)$ | $3p + O(p^2)$ |
| 13 | 2 | $7p + O(p^2)$ | $10p + O(p^2)$ |

$E = 280\text{a}1$, $C_{\mathrm{Sym}^2 E} = 140^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 2$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 3 | $-1$ | $p + 2p^2 + 2p^3 + O(p^4)$ | $2 + p + p^2 + O(p^3)$ |
| 11 | $-5$ | $7p + O(p^2)$ | $8p + O(p^2)$ |
| 13 | 1 | $O(p)$ | $O(1)$ |

$E = 280\text{b}1$, $C_{\mathrm{Sym}^2 E} = 140^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 2$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 11 | $-5$ | $4p + O(p^2)$ | $3p + O(p^2)$ |
| 13 | $-5$ | $10p + O(p^2)$ | $9p + O(p^2)$ |

$E = 288\text{a}1$, $C_{\mathrm{Sym}^2 E} = 24^2$, $S_1 = \{2, 3\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{1}{2}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 5 | $-4$ | $2p^2 + O(p^3)$ | $4p + O(p^2)$ |
| 13 | $-6$ | $7p + O(p^2)$ | $4p + O(p^2)$ |

$E = 296\text{a}1$, $C_{\mathrm{Sym}^2E} = 148^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2E} = 2$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2E)$ |
|---|---|---|---|
| 3 | $-1$ | $p^2 + O(p^4)$ | $2p + O(p^3)$ |
| 5 | $-2$ | $4p + p^2 + O(p^3)$ | $p + p^2 + O(p^3)$ |
| 7 | 1 | $4p + O(p^2)$ | $6 + O(p)$ |
| 11 | 1 | $O(p)$ | $O(1)$ |
| 13 | $-6$ | $2p + O(p^2)$ | $4p + O(p^2)$ |

$E = 296\text{b}1$, $C_{\mathrm{Sym}^2E} = 148^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2E} = 2$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2E)$ |
|---|---|---|---|
| 3 | $-1$ | $2p + 2p^2 + p^3 + O(p^4)$ | $1 + 2p + p^2 + O(p^3)$ |
| 7 | $-3$ | $2p + 3p^2 + O(p^3)$ | $2p + 5p^2 + O(p^3)$ |
| 11 | $-3$ | $8p + O(p^2)$ | $10p + O(p^2)$ |

$E = 300\text{a}1$, $C_{\mathrm{Sym}^2E} = 30^2$, $S_1 = \{2, 5\}$, $\xi_{\mathrm{Sym}^2E} = \frac{8}{9}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2E)$ |
|---|---|---|---|
| 7 | 1 | $p + O(p^2)$ | $6 + O(p)$ |
| 11 | 6 | $5p^2 + O(p^3)$ | $5p^2 + O(p^3)$ |
| 13 | $-5$ | $3p + O(p^2)$ | $9p + O(p^2)$ |

## B.2 Tables of $\mathcal{L}$-invariants for elliptic curves $E$ with $D(E, 1) = 0$

Included below are the values we computed for both the derivative of $\mathbf{L}_p^{\mathrm{aut}}(\mathrm{Sym}^2E, s)$ at $s = 1$ and the corresponding $\mathcal{L}$-invariant term, for the six exceptional elliptic curves with $D(E, 1) = 0$ (we omitted these specimens from §B.1 as $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2E, 1)' = 0$ for each of these six curves). To calculate these $p$-adic numbers, we used the generalised congruences given in Theorem 3.2.

$E = 176\text{b}1$, $C_{\mathrm{Sym}^2E} = 11^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2E} = \frac{2}{5}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2E)$ |
|---|---|---|---|
| 3 | 1 | $p + O(p^4)$ | $1 + 2p^2 + O(p^3)$ |
| 5 | 1 | $p + O(p^2)$ | $p + O(p^2)$ |
| 7 | 2 | $2p + O(p^2)$ | $2p + O(p^2)$ |
| 13 | 4 | $4p + O(p^2)$ | $2p + O(p^2)$ |

$E = 196\text{a}1$, $C_{\mathrm{Sym}^2E} = 14^2$, $S_1 = \{2, 7\}$, $\xi_{\mathrm{Sym}^2E} = \frac{2}{9}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2E)$ |
|---|---|---|---|
| 3 | $-1$ | $1 + 2p + p^2 + O(p^3)$ | $2p + p^2 + 2p^3 + O(p^4)$ |
| 5 | $-3$ | $3p + O(p^2)$ | $3p + O(p^2)$ |
| 11 | $-3$ | $p + O(p^2)$ | $3p + O(p^2)$ |
| 13 | $-2$ | $10p + O(p^2)$ | $8p + O(p^2)$ |

$E = 200\text{b}1$, $C_{\mathrm{Sym}^2 E} = 20^2$, $S_1 = \{2, 5\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{1}{2}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 3 | $-2$ | $p + p^3 + O(p^4)$ | $1 + p + p^2 + O(p^3)$ |
| 7 | $-2$ | $4p + O(p^2)$ | $6p + O(p^2)$ |
| 11 | $-4$ | $7p + O(p^2)$ | $p + O(p^2)$ |
| 13 | $-4$ | $O(p^2)$ | $O(p^2)$ |

$E = 240\text{d}1$, $C_{\mathrm{Sym}^2 E} = 15^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{1}{4}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 11 | 4 | $10p + O(p^2)$ | $6p + O(p^2)$ |
| 13 | $-2$ | $3p + O(p^2)$ | $3p + O(p^2)$ |

$E = 272\text{b}1$, $C_{\mathrm{Sym}^2 E} = 17^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{1}{4}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 5 | $-2$ | $p + 4p^2 + O(p^3)$ | $2p + 2p^2 + O(p^3)$ |
| 7 | $-4$ | $2p + O(p^2)$ | $2p + O(p^2)$ |
| 13 | $-2$ | $O(p)$ | $O(p)$ |

$E = 300\text{c}1$, $C_{\mathrm{Sym}^2 E} = 30^2$, $S_1 = \{2, 5\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{1}{3}$

| $p$ | $a_p(E)$ | $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ | $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ |
|---|---|---|---|
| 7 | 4 | $2p + O(p^2)$ | $5p + O(p^2)$ |
| 11 | $-4$ | $4p + O(p^2)$ | $4p + O(p^2)$ |

# References


[1] K. Barré-Sireix, G. Diaz, F. Gramain and G. Philibert, *Une preuve de la conjecture Mahler-Manin*, Invent. Math. **124** (1996), 1-9.

[2] D. Benois, *A generalization of Greenberg's $\mathcal{L}$-invariant*, American Journal of Mathematics **133** (2011), 1573-1632.

[3] S. Bloch and K. Kato, *L-functions and Tamagawa numbers of motives*, in the Grothendieck Festschrift I, Progress in Math., Birkäuser, Boston (1990), 333-400.

[4] C. Breuil, B. Conrad, F. Diamond and R. Taylor, *On the modularity of elliptic curves over $\mathbb{Q}$: wild 3-adic exercises*, Journal of the American Math. Soc. **14** (2001), 843-939.

[5] C. Citro, *$\mathcal{L}$-invariants of adjoint square Galois representations coming from modular forms*, Int. Math. Res. Notices no. 14 (2008), Art. ID rnn048.

[6] J. Coates and B. Perrin-Riou, *On p-adic L-functions attached to motives over $\mathbb{Q}$*, Advanced Studies in Pure Mathematics **17**, Academic Press, Boston (1989), 23-54.

[7] J. Coates and C.-G. Schmidt, *Iwasawa theory for the symmetric square of an elliptic curve*, Journal Reine Angew. Math. **375/376** (1987), 104-156.

[8] J. Cremona, *Elliptic curve data*, available at **http://johncremona.github.io/ecdata/**

[9] A. Dabrowski and D. Delbourgo, *$S$-adic $L$-functions attached to the symmetric square of a newform*, Proc. of the London Math. Soc. **74** (1997), 559-611.

[10] S. Dasgupta, *Factorization of $p$-adic Rankin series*, Invent. Math. **205** (2016), 221-268.

[11] D. Delbourgo, *$\mathcal{L}$-invariants arising from $p$-adic measures of $Sym^2 E$*, Glasgow Mathematical Journal **44** (2001), 45-64.

[12] F. Diamond, *On deformation rings and Hecke rings*, Annals of Mathematics **144** (1996), 137-166.

[13] E. Dummit, M. Hablicsek, R. Harron, L. Jain, R. Pollack and D. Ross, *Explicit computations of Hida families via overconvergent modular symbols*, Research in Number Theory **2:25** (2016), DOI: 10.1007/s40993-016-0052-8

[14] B. Ferrero and R. Greenberg, *On the behaviour of $p$-adic $L$-functions at $s = 0$*, Invent. Math. **50** (1978), 91-102.

[15] M. Flach, *On the degree of modular parametrizations*, Séminaire de Théorie des Nombres, Paris 1991/92, Progr. Math. **116**, Birkhuser Boston, 23-36.

[16] M. Flach, *A finiteness theorem for the symmetric square of an elliptic curve*, Invent. Math. **109** (1992), 307-327.

[17] S. Gelbart and H. Jacquet, *A relation between automorphic $L$-functions of $GL(2)$ and $GL(3)$*, Ann. Sci. École Norm. Sup. **11** (1978), 471-542.

[18] B. Gorsse and G. Robert, Appendix to *Mesures $p$-adiques associées aux carrés symétriques*, Thèse de Doctorat, Université de Grenoble I (2006).

[19] R. Greenberg, *Trivial zeros of $p$-adic $L$-functions*, in "$p$-adic monodromy and the Birch and Swinnerton-Dyer conjecture", Contemp. Math. **165** (1991), 149-181.

[20] B. Gross and D. Zagier, *Heegner points and derivatives of $L$-series*, Invent. Math. **84** (1984), 225-320.

[21] P. Guerzhoy, *Jacobi-Eisenstein series and $p$-adic interpolation of symmetric squares of cusp forms*, Annales Inst. Fourier **45** (1995), 605-625.

[22] R. Harron and A. Lei, *Iwasawa theory for symmetric powers of CM modular forms at non-ordinary primes*, Journal de Théorie des Nombres de Bordeaux **26** (2014), 673-707.

[23] H. Hida, *A $p$-adic measure attached to the zeta functions associated with two elliptic modular forms II*, Ann. Inst. Fourier **38** (1988), 1-83.

[24] H. Hida, *$p$-adic $L$-functions for base change lifts of $GL_2$ to $GL_3$*, in Automorphic Forms, Shimura Varieties and $L$-functions, Academic Press, San Diego (1990), 93-142.

[25] H. Hida, *Adjoint Selmer groups as Iwasawa modules*, Israel Journal of Math. **120** (2000), 361-427.

[26] H. Hida, *$\mathcal{L}$-invariants of adjoint square Galois representations*, Int. Math. Res. Notices **59** (2004), 3177-3189.

[27] D. Loeffler and S. Zerbes, *Iwasawa theory for the symmetric square of a modular form*, to appear in Journal Reine Angew. Math.

[28] T. Miyake, *Modular forms*, Springer Monographs in Mathematics, Springer-Verlag, Berlin (1989), 338 pages.

[29] M. Nastasescu, *Determination of elliptic curves by their adjoint $p$-adic $L$-functions*, Journal of Number Theory **152** (2015), 156-181.

[30] G. Rosso, *A formula for the derivative of the $p$-adic $L$-function of the symmetric square of a finite slope modular form*, Amer. J. Math. **138** (2016), 821-878.

[31] C.-G. Schmidt, *$p$-adic measures attached to automorphic representations of $GL_3$*, Invent. Math. **92** (1988), 597-631.

[32] J.-P. Serre, *Propriétés galoisiennes des points d'ordre finis des courbes elliptiques*, Invent. Math. **15** (1972), 259-331.

[33] G. Shimura, *The special values of the zeta functions associated with cusp forms*, Com. Pure and Appl. Math. **29** (1976), 783-804.

[34] W. Stein, *Modular forms, a computational approach*, Graduate Studies in Mathematics, **79** (2007), 268 pages.

[35] A. Wiles, *Modular elliptic curves and Fermat's last theorem*, Annals of Mathematics **141** (1995), 443-551.