Biembeddings of cycle systems using integer Heffter arrays

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Abstract

In this paper we use constructions of Heffter arrays to verify the existence of face 2-colourable embeddings of cycle decompositions of the complete graph. Specifically, for $n \equiv 1 \pmod 4$ and $k \equiv 3 \pmod 4$, $n \gg k \geqslant 7$ and when $n \equiv 0 \pmod 3$ then $k \equiv 7 \pmod{12}$, there exist face 2-colourable embeddings of the complete graph K_{2nk+1} onto an orientable surface where each face is a cycle of a fixed length k. In these embeddings the vertices of K_{2nk+1} will be labelled with the elements of $\mathbb{Z}_{2nk+1}$ in such a way that the group, $(\mathbb{Z}_{2nk+1}, +)$ acts sharply transitively on the vertices of the embedding. This result is achieved by verifying the existence of non-equivalent Heffter arrays, $H(n; k)$, which satisfy the conditions: (1) for each row and each column the sequential partial sums determined by the natural ordering must be distinct modulo $2nk + 1$; (2) the composition of the natural orderings of the rows and columns is equivalent to a single cycle permutation on the entries in the array. The existence of Heffter arrays $H(n; k)$ that satisfy condition (1) was established earlier in [5] and in this current paper we vary this construction and show, for $k \geqslant 11$, that there are at least $(n-2)[((k-11)/4)!/e]^2$ such non-equivalent $H(n; k)$ that satisfy both conditions (1) and (2).

1 Introduction

Throughout this paper the set of integers $\{0, 1, \ldots, n-1\}$ is denoted by $[n]$ and the rows and columns of an $m \times n$ array will be indexed by $[m]$ and $[n]$, respectively.

A k-cycle system of order v is an edge disjoint decomposition of the complete graph K_v into cycles of length k. Cycle systems can be represented as embeddings of the underlying graph on a surface (or pseudosurface), where the cycles correspond to faces in the embedding.

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Researchers have exploited this connection in the study of the "Heawood Map Colouring Conjecture", see [16, 26] and related problems. In this paper we focus on decompositions of K_v into cycles of constant length, however in general the underlying graph need not be the complete graph and the cycles need not be of constant length. When the embedding is a proper face 2-colourable embedding of the complete graph K_v, in which each face corresponds to a cycle of length k and each colour class corresponds to a k-cycle system, we say the pair of k-cycle systems *biembeds* in the surface.

It is clear that not all pairs of k-cycle systems biembed on a surface, however it not obvious which pairs of such systems biembed. Further, to date, comparatively little is known about the spectrum of values of k for which there exists a pair of biembeddable k-cycle systems.

When $k = 3$, a k-cycle system is commonly referred to as a Steiner triple systems of order v or STS(v). For these systems, it is known that necessary and sufficient conditions for the existence of biembeddings of pairs of STS(v) are: (a) $v \equiv 1, 3 \pmod 6$ and $v \geqslant 9$ for non-orientable surfaces; and (b) $v \equiv 3, 7 \pmod{12}$ for orientable surfaces, [17, 26]. These systems have been studied extensively and the early survey by Grannell and Griggs [16] is an excellent starting point for further information. The reader may also refer to the recent work by Korzhik [22].

Ellingham and Stephens [15] show that for odd $v \geqslant 7$ there exists a pair of biembeddable Hamilton cycle systems (i.e. $k = v$) of order v. In 2016 Griggs and McCourt [19] developed new constructions for biembeddings of symmetric ($k = (v-1)/2$) k-cycle decompositions of K_v and established necessary and sufficient conditions for the existence of a biembedding of symmetric k-cycle systems on a non-orientable surface if $k \geqslant 4$ and on an orientable surface if k is odd and $k \geqslant 3$. Other studies connecting cycle decompositions and embeddings on orientable and non-orientable surfaces include [1, 8, 11, 13, 16, 18, 19, 23, 24] and [10].

In 2015 Archdeacon [1] studied biembeddings of cycle systems of the complete graph on a surface and formalized the connection between biembeddings and Heffter arrays. Heffter arrays arise as an extension to Heffter's [21] famous first difference problem: partition the set $\{1, \ldots, 3m\}$ into m triples $\{a, b, c\}$ such that either $a + b = c$ or $a + b + c$ is divisible by $6m + 1$. This problem was solved by Peltesohn, [25], some 40 years later, for all $m \neq 3$, a result that also implies the existence of cyclic Steiner triple systems on the given order; see [7]. A natural extension to Heffter's first difference problem is: can we identify a set of m subsets $\{x_1, \ldots, x_s\} \subset \{-ms, \ldots, -1, 1, \ldots, ms\}$ such that the sum of the entries in each subset is divisible by $2ms + 1$ and further if x occurs in one of the subsets, $-x$ does not occur in any of the subsets? We call the set of m such subsets a *Heffter system*. Two Heffter systems, $H_0 = \{H_{0,0}, \ldots, H_{0,m-1}\}$, $|H_{0,i}| = s$ for $i \in [m]$, and $H_1 = \{H_{1,0}, \ldots, H_{1,n-1}\}$, $|H_{1,j}| = t$ for $j \in [n]$, where $sm = nt$, are said to be *orthogonal* if for all i, j, $|H_{0,i} \cap H_{1,j}| \leqslant 1$.

Given the connection between Heffter's first difference problem and biembeddings of pairs of 3-cycle systems (STS(v)), one may ask which Heffter systems yield biembeddings of cycle decompositions, where the length of the cycles may vary.

To study this problem we follow the work of Archdeacon, [1], and Dinitz and Mattern, [13], who observed that an orthogonal Heffter system is equivalent to a *Heffter array* $H(m, n; s, t)$ which is an $m \times n$ array of integers such that:

- each row contains s filled cells and each column contains t filled cells;
- the elements in every row and column sum to $0 \pmod{2ms+1}$; and
- for each integer $1 \leqslant x \leqslant ms$, either x or $-x$ appears in the array.

The *support* of an array is taken to be the set of absolute values of the entries occurring in the array. In [2] it was shown that a $H(m, n; n, m)$ exists if and only if $m, n \geqslant 3$.

If $m = n$ and necessarily $s = t = k$, we say the Heffter array is *square*, usually denoted by $H(n; k)$. If the elements in every row and column sum to 0 in $\mathbb{Z}$, we refer to the array as an *integer* Heffter array. The spectrum for square Heffter arrays has been completely determined as stated in the following theorem, see [3, 6, 14].

Theorem 1.1. [3, 6, 14] *There exists an $H(n; k)$ if and only if $3 \leqslant k \leqslant n$. Also there exists an integer $H(n; k)$ if and only if $3 \leqslant k \leqslant n$ and $nk \equiv 0, 3 \pmod 4$.*

Archdeacon [1] went on to prove that a Heffter array $H(m, n; s, t)$ that admits a simple and compatible ordering of the rows and columns, can be used to construct a face 2-colourable embedding of the complete graph K_{2ms+1} on an orientable surface. The definitions of simple and compatible are as follows.

Given a row r of a Heffter array $H(m, n; s, t)$, if there exists a cyclic ordering $\phi_r = (a_0, a_1, \dots, a_{s-1})$ of the entries of row r such that, for $i \in [s]$, the partial sums

$$\alpha_i = \sum_{j=0}^{i} a_j \pmod{2ms+1} \tag{1}$$

are all distinct, we say that ϕ_r is *simple*. A simple ordering of the entries of a column may be defined similarly. If every row and column of a Heffter array $H(m, n; s, t)$ has a simple ordering, we say that the array is *simple* [11].

Suppose that a simple cyclic ordering $\phi_r = (a_0, a_1, \dots, a_{s-1})$ of a row r of a Heffter array has the property that whenever entry a_i lies in cell (r, c) and entry a_{i+1} lies in cell (r, c'), then $c < c'$. That is, the ordering for the row r is taken from left to right across the array. Observe that if this ordering is simple then the ordering from right to left is also simple and vice versa. We say that ϕ_r is the *natural* ordering for the rows and define a natural column ordering in a similar way with the ordering going from top to bottom. If the natural ordering for every row and column is also a simple ordering, we say that the Heffter array is *globally simple*.

The composition of the cycles ϕ_r, for each row $r \in [m]$, is the permutation ω_r on the entries of the Heffter array. Similarly we may define the permutation ω_c as the composition of the cycles ϕ_c, for the columns $c \in [n]$. If, the permutation $\omega_r \circ \omega_c$ can be written as a single cycle of length $ms = nt$, we say that ω_r and ω_c are *compatible* orderings for the Heffter array.

For $\mathcal{G}$ an embedding of a graph G on an orientable surface, we define the *rotation scheme* for a vertex of G to be the clockwise cyclic ordering of the neighbors of that vertex on the surface. Collectively, the rotation schemes for the vertices of G form the *rotation scheme* of G. Two embeddings of graphs G_1 and G_2 are homeomorphic if there is a graph isomorphism from G_1 to G_2 which preserves the rotation schemes (see, for example, [20]). This allows for a completely combinatorial description of the embedding. In particular, if the rotation schemes of two embeddings are distinct we say that the embeddings are distinct.

The cyclic group $(\mathbb{Z}_{|G|}, +)$ has a *sharply vertex-transitive* action on an embedded graph G if the vertices of G are labelled with the elements of $\mathbb{Z}_{|G|}$ and the permutation $x \to x+1$, when applied to the vertices, is a homeomorphism of the embedding of G onto itself (that is, the rotation scheme is invariant under this action).

Archdeacon's result is as follows:

Theorem 1.2. [1] *Suppose there exists a Heffter array $H(m, n; s, t)$ with orderings ω_r on the entries in the rows of the array and ω_c on the entries in the columns of the array, where ω_r and ω_c are both simple and compatible. Then there exists a face 2-colourable embedding $\mathcal{G}$ of K_{2ms+1} on an orientable surface such that the faces of one colour are cycles of length s and the faces of the other colour are cycles of length t. Moreover, $(\mathbb{Z}_{2ms+1}, +)$ has a sharply vertex-transitive action on $\mathcal{G}$.*

We next describe the rotation scheme in a proof of the previous theorem. Let ω_r and ω_c be simple and compatible orderings of the entries of a Heffter array. We extend these permutations to permutations of $\mathbb{Z}_{2ms+1} \setminus \{0\}$ by taking $\omega_r(-x) = -\omega_r(x)$ and $\omega_c(-x) = -\omega_c(x)$ for each $x \in \mathbb{Z}_{2ms+1} \setminus \{0\}$. We may now use these permutations to write down a rotation scheme for the embedding $\mathcal{G}$ arising from the Heffter array. We begin by defining the rotation scheme for vertex 0 in this embedding to be the permutation:

$$\rho_0 = (x_1, x_2, \ldots, x_{2ms}),$$

where $x_1 = 1$, $x_{2i} = -\omega_r(x_{2i-1})$ for $1 \leqslant i \leqslant ms$ and $x_{2i+1} = \omega_c(-x_{2i})$ for $1 \leqslant i < ms$. The rotation scheme for each vertex $j \in \mathbb{Z}_{2ms+1}$ is then given by:

$$\rho_j = (x_1 + j, x_2 + j, \ldots, x_{2ms} + j),$$

where arithmetic is calculated modulo $2ms + 1$.

A corollary of the previous theorem is that the decompositions $\mathcal{C}$ and $\mathcal{C}'$ of the graph K_{2ms+1} into s-cycles and t-cycles (respectively) are orthogonal. If we relax the condition of simplicity in the above theorem, we still have a embedding on an orientable surface but the faces collapse into smaller ones (and the cycles become circuits). On the other hand if we relax only the condition of compatibility, we have an embedding onto a pseudosurface rather than a surface, but $\mathcal{C}$ and $\mathcal{C}'$ remain orthogonal.

Example 1.3. Here we give an example of the previous theorem. Consider the following globally simple Heffter array:

1	−2	−10	11
−8	6	−3	5
7	−4	−12	9

.

Define

$$\omega_c = (1, 7, -8)(-2, -4, 6)(-10, -12, -3)(11, 9, 5),$$
$$\omega_r = (11, -10, -2, 1)(5, -3, 6, -8)(7, -4, -12, 9).$$

Observe that ω_r and ω_c are simple (see Equation (1)). Composing left to right,

$$\omega_r \circ \omega_c = (1, 9, -8, 11, -12, 5, -10, -4, -3, -2, 7, 6),$$

so ω_r and ω_c are also compatible.

Thus there exists a face 2-colourable embedding of K_{25} onto an orientable surface, with the faces of one colour each bounded by a 3-cycle and the faces of the other colour each bounded by a 4-cycle. The rotation scheme for vertex 0 is given by:

$$(1, -11, 9, -7, -8, -5, 11, 10, -12, -9, 5, 3, -10, 2, -4, 12, -3, -6, -2, -1, 7, 4, 6, 8).$$

The rotation scheme for vertex i is to be found by adding $i \pmod{25}$ to each integer in the above rotation scheme.

□

In [13] it was verified that there exists an $H(m, n; n, m)$ which admits both simple and compatible orderings, for all $n \geqslant 3$ when $m = 3$, and for all $3 \leqslant n \leqslant 100$ when $m = 5$. In [3, 11, 14] it was verified that there exists an integer $H(n; k)$, $n \geqslant k$ and $nk \equiv 3 \pmod 4$, that admit both simple and compatible orderings, when $k \in \{3, 5, 7, 9\}$ and simple orderings when $k \in \{4, 6, 8, 10\}$. Focusing on simple orderings only, the authors of [5] constructed simple Heffter arrays, $H(n; k)$, satisfying the conditions: (a) $k \equiv 0 \pmod 4$; or (b) $k \equiv 3 \pmod 4$ and $n \equiv 1 \pmod 4$; or (c) $k \equiv 3 \pmod 4$, $n \equiv 0 \pmod 4$, and $n \gg k$. A new type of Heffter array, called a *relative Heffter array*, in which certain entries are excluded with the aim of embedding complete multipartite graphs rather than complete graphs, is studied in [12]. In the current paper, we establish existence results for Heffter arrays $H(n; k)$ with simple and compatible orderings where $n \equiv 1 \pmod 4$, $k \equiv 3 \pmod 4$ and n is prime or; $n \gg k$ and either $n \not\equiv 0 \pmod 3$ or $p \equiv 1 \pmod 3$.

The starting point for our study is the following result providing necessary conditions for the existence of compatible Heffter arrays. It is a generalization of results given in [8, 11, 13].

Theorem 1.4. *If there exist compatible orderings ω_r and ω_c for a Heffter array $H(m, n; s, t)$, then either:*

- *m, n, s and t are all odd;*
- *m is odd, n is even and s is even; or*
- *m is even, n is odd and t is even.*

Proof. Let ω_r and ω_c be compatible orderings for a Heffter array $H(m, n; s, t)$. A permutation is *odd* (parity 1) or *even* (parity 0) if it can be written as a product of an *odd* or *even* number of transpositions, respectively. To be clear we say this is the *parity* of the permutation. If a permutation is a cycle of even length it has odd parity, and vice versa.

It follows that the parity of ω_r is equal to $m(s-1) \pmod 2$ and the parity of ω_c is equal to $n(t-1) \pmod 2$. Thus the parity of $\omega_r \circ \omega_c$ is equal to $m(s-1) + n(t-1) \pmod 2$. But the parity of a cycle of length ms is equal to $ms - 1 \pmod 2$. So if the orderings are compatible, $m(s-1) + n(t-1) - ms + 1 = n(t-1) - (m-1)$ is even.

Hence if $n(t-1)$ is odd, then n is odd, t is even and m is even. Otherwise $n(t-1)$ is even and m is odd. If n is odd, t is odd, thus since $ms = nt$, s is also odd. Otherwise n is even and m is odd. Since $ms = nt$, s is even. □

Thus by Theorem 1.1 and 1.4, if there exists an integer $H(n; k)$ with both compatible and simple orderings, then $nk \equiv 3 \pmod 4$. In other words either $n \equiv 1 \pmod 4$ and $k \equiv 3 \pmod 4$; or $n \equiv 3 \pmod 4$ and $k \equiv 1 \pmod 4$. In this context, we will verify the following theorem and show existence of Heffter arrays $H(n; k)$ with simple and compatible orderings for $n \equiv 1 \pmod 4$ and $k \equiv 3 \pmod 4$, with infinite sporadic exceptions. The case $n \equiv 3 \pmod 4$ and $k \equiv 1 \pmod 4$ remains unsolved in general.

The next four theorems are the main results of this paper. Theorem 1.5 is proven in Section 2.

Theorem 1.5. *Let $n \equiv 1 \pmod 4$, $p > 0$ and $n > 4p + 3$. If there exists α such that $2p+2 \leqslant \alpha \leqslant n-2-2p$, $\gcd(n, \alpha) = 1$, $\gcd(n, \alpha - 2p - 1) = 1$ and $\gcd(n, n-1-\alpha-2p) = 1$, then there exists a globally simple integer Heffter array $H(n; 4p+3)$ with an ordering that is both simple and compatible.*

We then show that under certain conditions a suitable α exists and prove the following theorem in Subsection 2.1.

Theorem 1.6. *Let $n \equiv 1 \pmod 4$, $p > 0$, $n > k = 4p + 3$ and either: (a) n is prime; (b) $n = k + 2$; or (c) $n \geqslant 7(k+1)/3$ and if $n \equiv 3 \pmod 6$ then $p \equiv 1 \pmod 3$. Then there exists a globally simple integer Heffter array $H(n; k)$ with an ordering that is both simple and compatible. Furthermore [by Theorem 1.2], there exists a face 2-colourable embedding $\mathcal{G}$ of K_{2nk+1} on an orientable surface such that the faces of each colour are cycles of length k. Moreover, $\mathbb{Z}_{2nk+1}$ has a sharply vertex-transitive action on $\mathcal{G}$.*

Let $\mathcal{G}$ be a biembedding of two cycle decompositions of the complete graph on an orientable surface corresponding to a Heffter array H. Rearranging the rows and columns of H, and adjusting the orderings ω_r and ω_c accordingly, has no effect on $\mathcal{G}$. Replacing every entry x in a Heffter array by $-x$, replacing ω_r and ω_c by ω_r^{-1} and ω_c^{-1}, respectively, and reversing the rotation scheme at each vertex also preserves the embedding $\mathcal{G}$. We say that two Heffter arrays H and H' are *equivalent* if one can be obtained from the other by (i) rearranging rows or columns; or (ii) replacing each entry x of the array with $-x$.

Conversely, we next show how two non-equivalent Heffter arrays can give rise to non-equal embeddings, that is, distinct rotation schemes. To this end define mappings, Ω_r and Ω_c, on the non-empty cells of a Heffter array $A = A[(i,j)]$ as follows:

$$\begin{aligned}\Omega_r((i,j)) &= (i,j') \text{ iff } \omega_r(A(i,j)) = A(i,j'), \text{ and} \\ \Omega_c((i,j)) &= (i',j) \text{ iff } \omega_c(A(i,j)) = A(i',j).\end{aligned}$$

Essentially Ω_r and Ω_c act on the non-empty cells of a Heffter array rather than its entries.

Lemma 1.7. *Let H_1 and H_2 be Heffter arrays of the same dimensions with the same set of filled cells, each with entry 1 in the same cell. Suppose that H_1 and H_2 have identical mappings Ω_r and Ω_c corresponding to their respective simple and compatible orderings, but $H_1 \neq H_2$. Then the embeddings $\mathcal{G}_1$ and $\mathcal{G}_2$ corresponding to H_1 and H_2 are distinct.*

Proof. Suppose, for the sake of contradiction that the embeddings $\mathcal{G}_1$ and $\mathcal{G}_2$ are equal (i.e. have the same rotation schemes). Let $(1, x_2, \ldots, x_{2ms})$ be the rotation schemes for vertex 0 in $\mathcal{G}_1$ and $\mathcal{G}_2$. Let ω_r and ω_c be the row and columns orderings, respectively, for the entries of H_1 and let ω_r' and ω_c' be the row and columns orderings, respectively, for the entries of H_2. Let k be the least integer such that x_k lies in different cells of H_1 and H_2. Then x_{k-1} lies in the same cell (i,j) of H_1 and H_2.

If k is even, by the definition of the rotation scheme, $x_k = -\omega_c(x_{k-1}) = -\omega_c'(x_{k-1})$. Thus by the definition of Ω_r and Ω_c, x_k lies in the same cell of H_1 and H_2, a contradiction. Otherwise, k is odd and $\omega_r(-x_{k-1}) = \omega_r'(-x_{k-1})$. We obtain a similar contradiction. □

Theorem 1.8 gives a lower bound on the number of non-equivalent Heffter arrays $H(n; 4p+3)$ that satisfy Theorem 1.5, with a proof provided in Section 3.3. Let $\mathcal{H}(n)$ represent the number of derangements on $[n]$. It is a well-known asymptotic result that $\mathcal{H}(n) \sim n!/e$.

Theorem 1.8. *Let $n \equiv 1 \pmod 4$, $p \geqslant 2$ and $n > 4p+3$. If there exists α such that $2p+2 \leqslant \alpha \leqslant n-2-2p$, $\gcd(n,\alpha) = 1$, $\gcd(n, \alpha-2p-1) = 1$ and $\gcd(n, n-1-\alpha-2p) = 1$, then there exist at least $(n-2)(\mathcal{H}(p-2))^2 \sim (n-2)[(p-2)!/e]^2$ non-equivalent globally simple integer Heffter arrays $H(n; 4p+3)$, each with an ordering that is both simple and compatible.*

Each of these Heffter arrays have orderings corresponding to the same Ω_r and Ω_c as described in Section 2. Thus from Lemmata 2.4, 2.5 and 1.7 and Theorem 1.8 we have:

Corollary 1.9. *Let $n \equiv 1 \pmod 4$ and $k \equiv 3 \pmod 4$, for $k \geqslant 11$ and either n is prime or $n \geqslant (7k+1)/3$. Further if $n \equiv 0 \pmod 3$ assume $k \equiv 7 \pmod{12}$. Then there exists at least $(n-2)[((k-11)/4)!/e]^2$ distinct face 2-colourable embeddings of the complete graph K_{2nk+1} onto an orientable surface where each face is a cycle of fixed length k, and the vertices can be labelled with the elements of $\mathbb{Z}_{2nk+1}$ in such a way that this group $(\mathbb{Z}_{2nk+1}, +)$ has a sharply vertex-transitive action on the embedding.*

2 $H(n;4p+3)$ with simple and compatible orderings

First we work towards a proof of Theorem 1.5. In [5], constructions were provided that verify the existence of the globally simple Heffter arrays $H(n;4p+3)$. Thus, it suffices to show that the particular Heffter arrays have orderings which are compatible and simple. To obtain this result we will apply Lemma 2.1 which a generalization of results given in [8, 11, 13].

In what follows, for a partially filled array $A = [A(i,j)]$ we use $A(i,j)$ to denote the entry in cell (i,j) of array A. The cells of an $n \times n$ array can be partitioned into n disjoint *diagonals* D_d, $d \in [n]$, where

$$D_d := \{(i + d \,(\mathrm{mod}\ n), i) \mid i \in [n]\}.$$

We use the convention that if α and β are two permutations acting on a set X, then $(\alpha \circ \beta)(x)$ is defined to be $\beta(\alpha(x))$, for each $x \in X$.

We define an ordering for a Heffter array in terms of the natural orderings of each row (left to right) and column (top to bottom). Let $\alpha_r = \phi_r$ for each row r, $r \in [n-1]$ and let $\alpha_{n-1} = \phi_{n-1}^{-1}$, where ϕ_r is the natural ordering for each row $r \in [n]$. For each column c, $c \in [n]$, let $\alpha_c = \phi_c$, where ϕ_c is the natural ordering for column c. Next, define ω_r and ω_c to be compositions of the orderings α_r, $r \in [n]$ and α_c, $c \in [n]$, respectively. This is the ordering used for every Heffter array construction in this paper.

Lemma 2.1. *Assume that k is odd and that the non-empty cells of a Heffter array $H(n;k)$, $A = [A(i,j)]$, can be partitioned into diagonals $D_{g(1)}, \dots, D_{g(k)}$, where $g(1) < g(2) < \dots < g(k)$. For $h = 2, \dots, k$ define gaps of the diagonals as $s_h = g(h) - g(h-1) \,(\mathrm{mod}\ n)$ and $s_1 = g(1) - g(k) \,(\mathrm{mod}\ n)$. Suppose that for all $h = 1, \dots, k$, $\gcd(n, s_h) = 1$. Then if A is globally simple, the orderings ω_r and ω_c defined above are both simple and compatible.*

Proof. Observe that ω_r and ω_c are simple because A is globally simple. It remains to show that ω_r and ω_c are compatible orderings, that is $w_r \circ w_c$ can be written as a single permutation of length nk. While we have defined compatible orderings based on entries above, such orderings can also be defined on the cells of an array. Now, $\omega_r \circ \omega_c$ can be written as a single cycle if and only if $\Omega_r \circ \Omega_c$ can be written as a single cycle. For simplicity, we will abuse notation and remove brackets writing $\Omega_r(i,j)$ instead of $\Omega_r((i,j))$; similarly for $\Omega_c(i,j)$.

For fixed h, consider the diagonals $D_{g(h)}, D_{g(h+1)}, D_{g(h+2)}$ and cell $(n-1, n-1-g(h)) \in D_{g(h)}$. Then working modulo n on the row and column indices, with residues in $[n]$,

$$\begin{aligned}
(\Omega_r \circ \Omega_c)(n-1, n-1-g(h)) &= \Omega_c(n-1, n-1-g(h+1)) \\
&= (s_{h+2} - 1, n-1-g(h+1)), \\
\therefore (\Omega_r \circ \Omega_c)^2(n-1, n-1-g(h)) &= (2s_{h+2} - 1, n - g(h+1) + s_{h+2} - 1), \\
\therefore (\Omega_r \circ \Omega_c)^i(n-1, n-1-g(h)) &= (is_{h+2} - 1, n - g(h+1) + (i-1)is_{h+2} - 1), 1 \leqslant i \leqslant n, \\
\therefore (\Omega_r \circ \Omega_c)^n(n-1, n-1-g(h)) &= (n-1, n-1-g(h+2)).
\end{aligned}$$

Now since k is odd and each s_h is coprime to n, we see that mapping $\Omega_r \circ \Omega_c$ is a full cycle of length nk. □

Theorem 2.2. [5] *Let $n \equiv 1 \pmod 4$, $p > 0$ and $n > 4p+3$. If there exists α such that $\gcd(\alpha, n) = 1$ and $2p+2 \leqslant \alpha \leqslant n-2-2p$, then there exists a globally simple Heffter array $H(n; 4p+3)$, denoted by B, with occupied cells on the set of diagonals*

$$D_0, D_1, \ldots, D_{4p-2}, D_{2p+\alpha-3}, D_{2p+\alpha-1}, D_{2p+\alpha}, D_{2p+\alpha+1}.$$

Now observe that the gaps between the diagonals for the Heffter array in Theorem 2.2 are of size either 1, $2p+\alpha-3-(4p-2) = \alpha-2p-1$, 2 or $n-(2p+\alpha+1)$. So Lemma 2.1 together with Theorem 2.2 then imply Theorem 1.5.

2.1 Existence of a suitable α

In this section we will give some lemmata using number theory to determine when a suitable α exists for use in Theorem 1.5 to prove Theorem 1.6.

Lemma 2.3. *If $n \equiv 0 \pmod 3$ and $p \not\equiv 1 \pmod 3$, there does not exist $\alpha < n$ such that $\gcd(n, \alpha) = 1$, $\gcd(n, \alpha-2p-1) = 1$ and $\gcd(n, n-(\alpha+2p+1)) = 1$.*

Proof. Let $n \equiv 0 \pmod 3$ and $p \not\equiv 1 \pmod 3$. Suppose there exists an α that satisfies all three of the gcd conditions. Then $\gcd(n, \alpha) = 1$ implies $\alpha \not\equiv 0 \pmod 3$, hence we have two options $\alpha \equiv 1$ or $2 \pmod 3$. Since $p \not\equiv 1 \pmod 3$, there are further two options to consider: $p \equiv 0$ or $2 \pmod 3$. Thus there are four cases to consider in all. However each of these cases leads to a contradiction. □

Lemma 2.4. *If $n = 4p+5$, $\alpha = 2p+2$ satisfies the conditions $2p+2 \leqslant \alpha \leqslant n-2-2p$, $\gcd(n, \alpha) = 1$, $\gcd(n, \alpha-2p-1) = 1$ and $\gcd(n, n-1-\alpha-2p) = 1$.*

Lemma 2.5. *Let $p > 0$, and let $n \geqslant 4p+7$ be an odd integer. Further, suppose that $n \geqslant 28(p+1)/3$ if n is not a prime; and if $n \equiv 3 \pmod 6$ then $p \equiv 1 \pmod 3$. Then there exists α satisfying $2p+2 \leqslant \alpha \leqslant n-2-2p$, $\gcd(n, \alpha) = 1$, $\gcd(n, \alpha-2p-1) = 1$ and $\gcd(n, n-1-\alpha-2p) = 1$.*

Proof. The proof is trivial if n is prime. Otherwise let $q_1 < q_2 < \cdots < q_h$ be the prime factors of n where $h \geqslant 2$. For each i, there exists $0 < b_i < q_i$ such that $b_i - 2p - 1 \not\equiv 0 \pmod{q_i}$ and $-1 - b_i - 2p \not\equiv 0 \pmod{q_i}$. (Note that if $q_1 = 3$, we need $p \equiv 1 \pmod 3$ here for b_1 to exist.) By the Chinese remainder theorem, there is a unique x satisfying $x \equiv b_i \pmod{q_i}$ for each $1 \leqslant i \leqslant h$ and $0 < x < q_1 q_2 \ldots q_h$. By construction x satisfies $\gcd(n, x) = 1$, $\gcd(n, x-2p-1) = 1$ and $\gcd(n, n-1-x-2p) = 1$.

Thus if $2p+2 \leqslant x \leqslant n-2-2p$, then set $\alpha = x$ and we are done. Otherwise we need to make some adjustments to x and we proceed as follows. Let $Q = q_1 q_2 \ldots q_h$ and consider two cases either there is a prime q such that q^2 divides n or $n = Q$. In the former case, since $n \geqslant 28(p+1)/3 > 6(p+1)$, we have that $n - 4p - 4 > n/3$. Thus, $(n-2-2p) - (2p+2) > n/3 \geqslant n/q \geqslant Q$. Thus there exists $\alpha \equiv x \pmod Q$ such that $2p+2 \leqslant \alpha \leqslant n-2-2p$ and we are done.

Otherwise, n is a product of at least two distinct primes. If $n = 15$ then since $n \equiv 3 \pmod 6$ and $n \geqslant 4p + 7$, we must have $p = 1$. Observe that $\alpha = 4$ satisfies the conditions of the lemma. Otherwise, there exists a prime $q \geqslant 7$ which divides n. Since $n \geqslant 28(p+1)/3$,

$$(n - 2 - 2p) - (2p + 2) = n - 4p - 4 \geqslant 4n/7 \geqslant 4n/q.$$

Next, let x' be the least integer such that $x' \equiv x \pmod{n/q}$ and $2p + 2 \leqslant x'$. Since $(n - 2 - 2p) - (2p + 2) \geqslant 4n/q$,

$$2p + 2 \leqslant x' < x' + 3n/q \leqslant n - 2 - 2p.$$

Let $\alpha_i = x' + in/q$ for each $0 \leqslant i \leqslant 3$. Since $q \geqslant 7$ and q is prime, q does not divides in/q for $1 \leqslant i \leqslant 3$. Thus $\alpha_j - \alpha_i$ is not divisible by q for any $0 \leqslant i < j \leqslant 3$. In particular, α_0, α_1, α_2 and α_3 are distinct modulo q. Therefore there exists $\alpha \in \{\alpha_0, \alpha_1, \alpha_2, \alpha_3\}$ such that α, $\alpha - 2p - 1$ and $-1 - \alpha - 2p$ are each coprime to q. Furthermore, since $\alpha \equiv x \pmod{n/q}$, α, $\alpha - 2p - 1$ and $-1 - \alpha - 2p$ are each coprime to every prime factor of n/q and hence coprime to n itself. □

Then Theorem 1.6 follows directly from Theorem 1.5 and previous lemmata.

3 Non-equivalent globally simple integer Heffter arrays, $H(n; 4p + 3)$

In this section we work towards proving Theorem 1.8.

We start with a generalization of Heffter arrays. An array A is defined to be a *support shifted simple integer Heffter array* $H(n; 4p, \gamma)$, where $\gamma \geqslant 0$, if it satisfies the following properties:

P1. Every row and every column of A has $4p$ filled cells.

P2. The support of A is $\{\gamma n + 1, \ldots, (4p + \gamma)n\}$.

P3. Elements in every row and every column sum to 0.

P4. Partial sums are distinct in each row and each column of A modulo $2(4p + \gamma)n + 1$.

A related generalization of Heffter arrays is studied in [9]. Note that a support shifted integer Heffter array $H(n; 4p, 0)$ is in fact an integer Heffter array $H(n; 4p)$. Support shifted simple integer Heffter arrays were constructed for all $n \geqslant 4p$ and $\gamma \geqslant 1$ in [5]. Then these arrays for $\gamma = 3$ were merged with a Heffter array $H(n; 3)$ to obtain simple Heffter arrays $H(n; 4p + 3)$. In this section we first document the existing constructions from [5] then we will generalize these constructions to obtain $(p - 1)!(p - 2)!$ non-equivalent support shifted simple $H(n; 4p, \gamma)$. Then as in [5] we will merge each of these arrays with Heffter arrays $H(n; 3)$ to prove Theorem 1.8.

3.1 Existing results on support shifted simple integer Heffter arrays, $H(n;4p,\gamma)$

First, we outline the precise results needed from [5].

For an $n\times n$ array let the entries in row a and column a of diagonal D_i be denoted by $d_i(r_a)$ and $d_i(c_a)$ respectively, with these values defined to be 0 when there is no entry. For a given row a we define $\Sigma(x)=\sum_{i=0}^{x} d_i(r_a)$ and for a given column a we define $\overline{\Sigma}(x)=\sum_{i=0}^{x} d_i(c_a)$. For a given row a, the values of $\Sigma(x)$ such that $d_x(r_a)$ is non-zero are called the *row partial sums* for a. For a given column a, the values of $\overline{\Sigma}(x)$ such that $d_x(c_a)$ is non-zero are called the *column partial sums* for a. Thus to show that a Heffter array $H(n;k)$ is globally simple, it suffices to show that the row partial sums are distinct (modulo $2nk+1$) for each row a and that the column partial sums are distinct (modulo $2nk+1$) for each column c. It is important to be aware that throughout this section, row and column indices are *always* calculated modulo n, while entries of arrays are *always* evaluated as integers.

Remark 3.1. It will be useful to refer to the following basic observations. Let m, x_1, x_2, α_1, α_2, β_1, β_2 be integers and $m>0$. Then for:

$$-m \leqslant x_1, x_2 \leqslant m, \qquad x_1 \equiv x_2 \pmod{2m+1} \Rightarrow x_1 = x_2; \tag{2}$$

$$0 \leqslant x_1, x_2 < m, \qquad x_1 \equiv x_2 \pmod{m} \Rightarrow x_1 = x_2; \tag{3}$$

$$-\frac{m}{2} < \alpha_1, \alpha_2 < \frac{m}{2}, \qquad \beta_1 m + \alpha_1 = \beta_2 m + \alpha_2 \Rightarrow \beta_1 = \beta_2 \textit{ and } \alpha_1 = \alpha_2; \tag{4}$$

$$-m < x_1 < 0 < x_2 < m, \qquad x_1 \equiv x_2 \pmod{m} \Rightarrow x_2 = m + x_1. \tag{5}$$

In [5] a globally simple array A was constructed as follows. Let $\gamma>0$, $n\geqslant 4p$, $2p-1\leqslant \alpha \leqslant n-2p-1$, and $\gcd(\alpha,n)=1$. Define $I=[p]$, $J=[p-1]$ and $A=[A(i,j)]$ to be the $n\times n$ array with filled cells defined by the $4p$ diagonals

$$D_{2i}, D_{2i+1}, D_{2p}, D_{2p+1+2j}, D_{2p+2+2j}, D_{2p+\alpha},$$

where $i\in I$ and $j\in J$, and with entries for each $x\in[n]$:

$$\begin{array}{rll} (\gamma+2)n+4in-2x & \text{in cell} & (2i-x,-x)\in D_{2i}, \\ -\gamma n-4in-1-2x & \text{in cell} & (2i+1+x,x)\in D_{2i+1}, \\ -(4p+\gamma)n+2x & \text{in cell} & (2p-\alpha x,-\alpha x)\in D_{2p}, \\ (4p+\gamma-6)n-4jn+1+2x & \text{in cell} & (2p+1+2j-x,-x)\in D_{2p+1+2j}, \\ -(4p+\gamma-4)n+4jn+2x & \text{in cell} & (2p+2+2j+x,x)\in D_{2p+2+2j}, \\ (4p+\gamma-2)n+1+2x & \text{in cell} & (2p+\alpha+\alpha x,\alpha x)\in D_{2p+\alpha}. \end{array} \tag{6}$$

Theorem 3.2. (Theorem 3.1 of [5]) *Let $\gamma>0$, $n\geqslant 4p$, $2p-1\leqslant\alpha\leqslant n-2p-1$, and $\gcd(\alpha,n)=1$. Then the array A constructed above is a support shifted simple integer Heffter array $H(n;4p,\gamma)$.*

Remark 3.3. If $n\equiv 2 \pmod 4$, let $\alpha=n/2-2$. Otherwise, let $\alpha=\lfloor(n-1)/2\rfloor$. Observe that $2p-1\leqslant\alpha\leqslant n-2p-1$ and $\gcd(\alpha,n)=1$. Hence, if $\gamma>0$ and $n\geqslant 4p$, then there exists a support shifted simple integer Heffter array $H(n;4p,\gamma)$.

From Equations (12) and (13) in [5], we have the following lemma.

Lemma 3.4. [5] *The row partial sums and the column partial sums of A satisfy the following inequalities.*

$$\begin{aligned}\Sigma(4p-2) < \Sigma(4p-4) < \cdots < \Sigma(2p+2) < \Sigma(2p) < -(4p+\gamma-3)n < 0 \\ 0 < \Sigma(1) < \Sigma(3) < \cdots < \Sigma(2p-1) < \gamma n.\end{aligned} \tag{7}$$

$$\begin{aligned}-(4p+\gamma+1)n < \overline{\Sigma}(2p) < \overline{\Sigma}(2p+2) < \cdots < \overline{\Sigma}(4p-2) < \overline{\Sigma}(2p) + p < -n \\ -n < \overline{\Sigma}(2p-1) < \cdots < \overline{\Sigma}(3) < \overline{\Sigma}(1) < 0.\end{aligned} \tag{8}$$

3.2 The existence of support shifted simple integer Heffter arrays, $H(n;4p,\gamma)$

In this section we reorder the entries in each column of the array A given in the previous section to get a new array A', obtained by applying a bijection $f_I : I \to I$ to the entries in the coupled diagonals D_{2i} and D_{2i+1} of A and a bijection $f_J : J \to J$ to the entries in the coupled diagonals $D_{2p+2j+1}$ and $D_{2p+2j+2}$ of A.

Let $\gamma > 0$, $n \geqslant 4p$, $2p-1 \leqslant \alpha \leqslant n-2p-1$, and $\gcd(\alpha, n) = 1$. For each pair of functions (f_I, f_J), we construct an $n \times n$ array A' with support $s(A') = \{\gamma n+1, \ldots, (4p+\gamma)n\}$ as follows: for all $i \in I$, $j \in J$, and $x \in [n]$ in A' place entry

$$\begin{aligned}(\gamma+2)n + 4f_I(i)n - 2x \quad &\text{in cell} \quad (2i-x, -x) \in D_{2i}, \\ -\gamma n - 4f_I(i)n - 1 - 2x \quad &\text{in cell} \quad (2i+1+x, x) \in D_{2i+1}, \\ -(4p+\gamma)n + 2x \quad &\text{in cell} \quad (2p-\alpha x, -\alpha x) \in D_{2p}, \\ (4p+\gamma-6)n - 4f_J(j)n + 1 + 2x \quad &\text{in cell} \quad (2p+1+2j-x, -x) \in D_{2p+1+2j}, \\ -(4p+\gamma-4)n + 4f_J(j)n + 2x \quad &\text{in cell} \quad (2p+2+2j+x, x) \in D_{2p+2+2j}, \\ (4p+\gamma-2)n + 1 + 2x \quad &\text{in cell} \quad (2p+\alpha+\alpha x, \alpha x) \in D_{2p+\alpha}.\end{aligned} \tag{9}$$

We illustrate this new construction with an example.

Example 3.5. Here $n = 17$, $p = 3$, $\alpha = 2p = (2 \times 3) = 6$, $f_I(0) = 0$, $f_I(1) = 2$, $f_I(2) = 1$, $f_J(0) = 1$ and $f_J(1) = 0$.

85					252		-173	172	-101	100	-253	-144	145	-216	217	-84
-52	53					224		-171	170	-99	98	-225	-146	147	-218	219
221	-54	55					230		-169	168	-97	96	-231	-148	149	-220
-188	189	-56	57					236		-167	166	-95	94	-237	-150	151
153	-190	191	-58	59					242		-165	164	-93	92	-243	-152
-120	121	-192	193	-60	61					248		-163	162	-91	90	-249
-255	-122	123	-194	195	-62	63					254		-161	160	-89	88
86	-227	-124	125	-196	197	-64	65					226		-159	158	-87
-119	118	-233	-126	127	-198	199	-66	67					232		-157	156
154	-153	116	-239	-128	129	-200	201	-68	69					238		-155
-187	186	-115	114	-245	-130	131	-202	203	-70	71					244	
	-185	184	-113	112	-251	-132	133	-204	205	-72	73					250
222		-183	182	-111	110	-223	-134	135	-206	207	-74	75				
	228		-181	180	-109	108	-229	-136	137	-208	209	-76	77			
		234		-179	178	-107	106	-235	-138	139	-210	211	-78	79		
			240		-177	176	-105	104	-241	-140	141	-212	213	-80	81	
				246		-175	174	-103	102	-247	-142	143	-214	215	-82	83

Theorem 3.6. *Let $p > 0$, $n \geqslant 4p$, $\gcd(n, \alpha) = 1$, $2p - 1 \leqslant \alpha \leqslant n - 2p - 1$ and $\gamma > 0$. Then each choice of the pair of functions (f_I, f_J) gives rise to a support shifted simple integer Heffter array $H(n; 4p, \gamma)$ where filled cells are precisely the set of diagonals $\{D_0, D_1, D_2, \dots, D_{4p-2}, D_{2p+\alpha}\}$.*

We prove this theorem through a series of lemmata and corollaries, namely results 3.7 to 3.13, where we show that for each pair of functions (f_I, f_J), the array A' constructed above carries the Properties P1, P2, P3, P4. We will use the notation $\Sigma_A(x)$ and $\overline{\Sigma}_A(x)$ to denote the row partial sums and column partial sums in the array A as given in [5] and $\Sigma_{A'}(x)$ and $\overline{\Sigma}_{A'}(x)$ to denote the row partial sums and column partial sums respectively in the array A' as constructed here. Observe that A is a special form of the array A' where both f_I and f_J are identity mappings.

Now since A' is obtained by permuting the entries in columns of A, $s(A') = \{\gamma n + 1, \dots, (4p+\gamma)n\}$ and all columns sums are equal to 0. Next the equations in Lemma 3.7 can be used to verify that the row sums are 0.

Lemma 3.7. *The rows and columns A' satisfy the following equations for each $i \in I$ and $j \in J$. (Note, that since the context is clear here we have reduced notation and represented $d_x(r_a)$ and $d_x(c_a)$ by d_x.)*

For rows a:	*For columns $a \neq 0$:*	*For columns $a = 0$:*
$d_{2i} + d_{2i+1} = 1,$	$d_{2i} + d_{2i+1} = -1$	$d_{2i} + d_{2i+1} = 2n - 1,$
$d_{2p+2j+1} + d_{2p+2j+2} = -1,$	$d_{2p+2j+1} + d_{2p+2j+2} = 1,$	$d_{2p+2j+1} + d_{2p+2j+2} = -2n + 1,$
$d_{2p} + d_{2p+\alpha} = -1,$	$d_{2p} + d_{2p+\alpha} = 1,$	$d_{2p} + d_{2p+\alpha} = -2n + 1.$

Proof. In [5] it was shown the above statements are true for the array A and the result follows directly by definition for the columns of A'.

For the rest of this subsection we use the notation ($a \bmod n$) to denote the element of $[n]$ equivalent to $a \pmod n$. For a given row $a \in [n]$ and for all $i \in I$, let $x_1 = ((2i - a) \bmod n)$ and $x_2 = ((a - 2i - 1) \bmod n)$. Thus $x_1 + x_2 + 1 = 0 \pmod n$ and so $x_1 + x_2 = n - 1$. Consequently for all $i \in I$,

$$\begin{aligned} d_{2i}(r_a) + d_{2i+1}(r_a) &= (\gamma+2)n + 4f_I(i)n - 2x_1 - \gamma n - 4f_I(i)n - 1 - 2x_2 \\ &= 2n - 1 - 2(n-1) = 1. \end{aligned} \tag{10}$$

The remaining observations for the rows hold similarly. □

Recall that with respect to an array B, $\Sigma_B(x)$ is defined to be $\sum_{i=0}^{x} d_i(r_a)$ for a given row a and $\overline{\Sigma}_B(x)$ is defined to be $\sum_{i=0}^{x} d_i(c_a)$ for a given column a. Thus we have the following corollary.

Corollary 3.8. *For any choice of f_I and f_J and for all rows and columns of A' we have:*

$$\begin{aligned} \Sigma_{A'}(2i+1) &= \Sigma_A(2i+1), & \overline{\Sigma}_{A'}(2i+1) &= \overline{\Sigma}_A(2i+1), \\ \Sigma_{A'}(2p+2j+2) &= \Sigma_A(2p+2j+2), & \overline{\Sigma}_{A'}(2p+2j+2) &= \overline{\Sigma}_A(2p+2j+2), \\ \Sigma_{A'}(2p) &= \Sigma_A(2p), & \overline{\Sigma}_{A'}(2p) &= \overline{\Sigma}_A(2p), \\ \Sigma_{A'}(2p+\alpha) &= \Sigma_A(2p+\alpha), & \overline{\Sigma}_{A'}(2p+\alpha) &= \overline{\Sigma}_A(2p+\alpha). \end{aligned}$$

We will also need the following lemma to bound certain partial sums.

Lemma 3.9. *The following bounds hold for partial sums on rows and non-zero columns.*

$$\begin{aligned}-(4p+\gamma)n &< \Sigma_{A'}(2p) = & d_{2p}(r_a)+p < -(4p+\gamma-2)n+p-1,\\ & \Sigma_{A'}(4p-2) = & d_{2p}(r_a)+1,\\ -(4p+\gamma)n-p &\leqslant \overline{\Sigma}_{A'}(2p) = & d_{2p}(c_a)-p \leqslant -(4p+\gamma-2)n-p-2.\end{aligned}$$

Proof. From Lemma 3.7, for each row a, $\Sigma_{A'}(2p) = d_{2p}(r_a)+p$ and for each non-zero column a, $\overline{\Sigma}_{A'}(2p) = d_{2p}(c_a) - p$. Also, $\Sigma_{A'}(4p-2) = \Sigma_{A'}(2p) - (p-1)$.

The result then follows from the definition of A'. □

By Theorem 3.2, A has distinct row and column partial sums; hence by Corollary 3.8 it is only necessary to show that the row partial sums $\Sigma_{A'}(2i)$ and $\Sigma_{A'}(2p+2j+1)$ and column partial sums $\overline{\Sigma}_{A'}(2i)$ and $\overline{\Sigma}_{A'}(2p+2j+1)$ are distinct from each other and from the other partial sums.

Lemma 3.10. *For all $i \in I$ and $j \in J$,*

$$\begin{aligned}(4p+\gamma)n > && \Sigma_{A'}(2i) && \geqslant \gamma n+2,\\ 0 > -n > \Sigma_{A'}(2p+2j+1) > && \Sigma_{A'}(2p)+(\gamma+1)n && = d_{2p}(r_a)+p+(\gamma+1)n.\end{aligned}$$

Proof. By Lemma 3.7 and the definition of A', for all rows a

$$\begin{aligned}\Sigma_{A'}(2i) &= d_{2i}(r_a)+i = (\gamma+2)n+4f_I(i)n-2x+i\\ &\qquad \text{where } x = ((2i-a) \bmod n),\\ \Sigma_{A'}(2p+2j+1) &= \Sigma_{A'}(2p)+d_{2p+2j+1}(r_a)-j\\ &= \Sigma_{A'}(2p)+(4p+\gamma-6)n-4f_J(j)n+1+2x-j,\\ &\qquad \text{where } x = ((2p+2j+1-a) \bmod n).\end{aligned}$$

Since $0 \leqslant f_I(i) \leqslant p-1$ and $0 \leqslant f_J(j) \leqslant p-2$ for $i \in I$ and $j \in J$, the result follows. □

We next show that all row partial sums are distinct.

Lemma 3.11. *For all $i \in I$ and $j \in J$, the row partial sums*

$$\Sigma_{A'}(2i) \text{ and } \Sigma_{A'}(2p+2j+1)$$

are distinct modulo $2(4p+\gamma)n+1$.

Proof. Consider any $i_1, i_2 \in I$, where without loss of generality $f_I(i_1) \geqslant f_I(i_2)$, and suppose that

$$\Sigma_{A'}(2i_1) \equiv \Sigma_{A'}(2i_2) \pmod{2(4p+\gamma)n+1}.$$

Then

$$4f_I(i_1)n - 2((2i_1 - a) \bmod n) + i_1 = 4f_I(i_2)n - 2((2i_2 - a) \bmod n) + i_2$$

by Lemma 3.10 and (2), which implies $4(f_I(i_1) - f_I(i_2))n \leqslant (i_2 - i_1) + 2(n-1) \leqslant 3n$. Consequently $f_I(i_1) - f_I(i_2) = 0$ and so $i_1 = i_2$. Similarly, suppose

$$\Sigma_{A'}(2p + 2j_1 + 1) \equiv \Sigma_{A'}(2p + 2j_2 + 1) \pmod{2(4p+\gamma)n+1}$$

for some $j_1, j_2 \in J$, where without loss of generality we assume $f_J(j_1) \geqslant f_J(j_2)$. Then

$$4f_J(j_1)n - 2((2p + 2j_1 + 1 - a) \bmod n) + j_1 = 4f_J(j_2)n - 2((2p + 2j_2 + 1 - a) \bmod n) + j_2$$

by Lemma 3.10 and (2). It now follows that $4(f_J(j_1) - f_J(j_2))n \leqslant (j_2 - j_1) + 2(n-1) \leqslant 3n$ and hence $j_1 = j_2$ as before. Also, by inequality (7),

$$\Sigma_{A'}(2i) \neq \Sigma_{A'}(2p + 2j + 1).$$

Thus all row partial sums are distinct. □

Lemma 3.12. *Let $a \neq 0$ be a column. Then:*

$$\begin{array}{rcl} (4p+\gamma)n > & \overline{\Sigma}_{A'}(2i) & \geqslant \gamma n + 2 - i > 0, \\ \overline{\Sigma}_{A'}(2p-1) > & \overline{\Sigma}_{A'}(2p+2j+1) & > \overline{\Sigma}_{A'}(2p) + (\gamma+2)n > \overline{\Sigma}_{A'}(2p) + p. \end{array}$$

Proof. By Lemma 3.7 and the definition of A' we have:

$$\begin{array}{rcl} \overline{\Sigma}_{A'}(2i) & = & d_{2i}(c_a) - i = (\gamma+2)n + 4f_I(i)n - 2((n-a) \bmod n) - i, \\ \overline{\Sigma}_{A'}(2p+2j+1) & = & \overline{\Sigma}_{A'}(2p) + d_{2p+2j+1}(c_a) + j = d_{2p}(c_a) - p + d_{2p+2j+1}(c_a) + j \\ & < & -(4p+\gamma)n + 2n - 2 - p + (4p+\gamma-6)n - 4f_J(j)n + 1 + 2n - 2 + j \\ & = & -2n - 3 - p - 4f_J(j)n + j < -n, \\ \overline{\Sigma}_{A'}(2p+2j+1) & \geqslant & d_{2p}(c_a) - p + (4p+\gamma-6)n - 4f_J(j)n + 1 + j \\ & > & \overline{\Sigma}_{A'}(2p) + (\gamma+2)n. \end{array}$$

From Corollary 3.8 and (8), $-n < \overline{\Sigma}_{A'}(2p-1)$. The result follows. □

We now use this results to show that column partial sums are distinct.

Lemma 3.13. *For all $i \in I$ and $j \in J$, the column partial sums*

$$\overline{\Sigma}_{A'}(2i) \text{ and } \overline{\Sigma}_{A'}(2p+2j+1)$$

are distinct modulo $2(4p+\gamma)n+1$.

Proof. Let $a \neq 0$ be a column. Suppose

$$\overline{\Sigma}_{A'}(2i_1) \equiv \overline{\Sigma}_{A'}(2i_2) \pmod{2(4p+\gamma)n+1}.$$

Now Lemma 3.12 implies $4f_I(i_1)n - i_1 = 4f_I(i_2)n - i_2$ by (2). Then $4(f_I(i_1) - 4f_I(i_2))n = i_1 - i_2$, and so $i_1 = i_2$. Similarly, suppose

$$\overline{\Sigma}_{A'}(2p+2j_1+1) = \overline{\Sigma}_{A'}(2p+2j_2+1) \mod (2(4p+\gamma)n+1)$$

implying $-4f_I(j_1)n + j_1 = -4f_I(j_2)n + j_2$ so $j_1 = j_2$ as before. Therefore by inequality (8) all column partial sums are distinct.

This leaves column 0, where

$$\begin{aligned} \overline{\Sigma}_{A'}(2i) &= (2n-1)i + (\gamma+2)n + 4f_I(i)n \leqslant 2(4p+\gamma)n+1, & (11)\\ \overline{\Sigma}_{A'}(2p+2j+1) &= (p-j)(2n-1) - (4p+\gamma)n + (4p+\gamma-6)n - 4f_J(j)n + 1 & (12)\\ &= (p-j)(2n-1) - 6n - 4f_J(j)n + 1 > -(4p+\gamma)n. \end{aligned}$$

By Lemma 3.7 and Corollary 3.8, related partial sums for column 0 can be calculated as:

$$\begin{aligned} \overline{\Sigma}_{A'}(2i+1) &= 2(i+1)n - (i+1) > 0,\\ \overline{\Sigma}_{A'}(2p) &= -(2p+\gamma)n - p < 0,\\ \overline{\Sigma}_{A'}(2p+2j+2) &= -(2p+2j+\gamma+2)n - (p-j-1) < 0,\\ \overline{\Sigma}_{A'}(2p+\alpha) &= 0. \end{aligned}$$

Observe that for column 0 and for all non-empty diagonals x,

$$|\overline{\Sigma}_{A'}(x)| \leqslant 2(4p+\gamma)n+1. \tag{13}$$

We will show that for each $i \in I$ and $j \in J$, $\overline{\Sigma}_{A'}(2i)$ and $\overline{\Sigma}_{A'}(2p+2j+1)$ are distinct $\pmod{2(4p+\gamma)n+1}$ from each other and each of the other partial sums in column 0. In what follows we will make extensive use of (13) together with (4).

1(i) Suppose that $\overline{\Sigma}(2i_1) = \overline{\Sigma}(2i_2) \pmod{2(4p+\gamma)n+1}$ for some $i_1, i_2 \in I$. Then $2(i_1 - i_2)n - (i_1 - i_2) = 4(f_I(i_2) - f(i_1))n$ by (3) but then $i_1 - i_2 = 0$.

1(ii) Suppose that $\overline{\Sigma}(2i_1) = \overline{\Sigma}(2i_2+1) \pmod{2(4p+\gamma)n+1}$ for some $i_1, i_2 \in I$. Then (3) implies

$$(2n-1)i_1 + (\gamma+2)n + 4f_I(i_1)n = (2i_2+2)n - (i_2+1).$$

Hence $i_1 = i_2 + 1$ and $2i_1 + \gamma + 2 + 4f_I(i_1) = 2i_2 + 2$. But then $4f_I(i_1) = -\gamma - 2$. This is a contradiction since $\gamma > 0$.

1(iii) Suppose that $\overline{\Sigma}(2i) = \overline{\Sigma}(2p) \pmod{2(4p+\gamma)n+1}$, for some $i \in I$. Then by (5)

$$\begin{aligned} (2n-1)i + (\gamma+2)n + 4f_I(i)n &= 2(4p+\gamma)n + 1 - (2p+\gamma)n - p,\\ (\gamma+2+4f_I(i)+2i)n - i &= (6p+\gamma)n + 1 - p. \end{aligned}$$

This implies $i = p-1$ and $\gamma + 2 + 4f_I(i) + 2i = 6p + \gamma$ leading to the contradiction $f_I(i) = p$.

1(iv) Suppose that $\overline{\Sigma}(2i) = \overline{\Sigma}(2p+2j+1) \pmod{2(4p+\gamma)n+1}$, for some $i \in I$ and $j \in J$ then by (5)

$$\begin{aligned}(2n-1)i + (\gamma+2)n + 4f_I(i)n &= (p-j)(2n-1) - 6n - 4f_J(j)n + 1 \quad \text{or} \\ (2n-1)i + (\gamma+2)n + 4f_I(i)n &= (p-j)(2n-1) - 6n - 4f_J(j)n + 2(4p+\gamma)n + 2.\end{aligned}$$

The former case implies $i = p-j-1$ and so $2(p-j-1)+\gamma+2+4f_I(i) = 2p-2j-6-4f_J(j)$, or equivalently $4(f_I(i)+f_J(j)) = -6-\gamma < 0$ which is a contradiction. For the second case we have $(2i+2j+8-\gamma+4f_I(i)+4f_J(j))n - i = 10pn-(p-j)+2$, which implies $p-2 = i+j$ and so $4(f_I(i)+f_J(j)) = 10p-2(i+j)-8+\gamma = 10p-2p-4+\gamma = 8p-4+\gamma > 8p-4$. This is a contradiction since $f_I(i)+f_J(j) \leqslant 2p-3$ and so $4(f_I(i)+f_J(j)) \leqslant 8p-12$.

1(v) Suppose that $\overline{\Sigma}(2i) = \overline{\Sigma}(2p+2j+2) \pmod{2(4p+\gamma)n+1}$ for some $i \in I$ and $j \in J$. Then by (5)

$$\begin{aligned}(2n-1)i + (\gamma+2)n + 4f_I(i)n &= 2(4p+\gamma)n + 1 - (2p+2j+\gamma+2)n - (p-j-1) \\ &= (6p-2j+\gamma-2)n - (p-j-2).\end{aligned}$$

Thus $i = p-j-2$ and $2i+\gamma+2+4f_I(i) = 6p-2j+\gamma-2$ or equivalently $2(p-j-2)+4f_I(i)+2 = 6p-2j-2$ and so $f_I(i) = p$, a contradiction.

2(i) Assume $\overline{\Sigma}(2p+2j+1) \equiv \overline{\Sigma}(2i+1) \pmod{2(4p+\gamma)n+1}$ for some $i \in I$ and $j \in J$. Then by (2) we have

$$(p-j)(2n-1) - 6n - 4f_J(j)n + 1 = (2i+2)n - (i+1).$$

Then $p = j+i+2$ and $2p-2j-6-4f_J(j)-2i-2 = 0$ which implies $2(i+j+2)-2i-2j-8-4f_J(j) = 0$ and so $f_J(j) = -1$, a contradiction.

2(ii) Assume $\overline{\Sigma}(2p+2j+1) \equiv \overline{\Sigma}(2p) \pmod{2(4p+\gamma)n+1}$ for some $j \in J$. Then by (2)

$$(p-j)(2n-1) - 6n - 4f_J(j)n + 1 = -(2p+\gamma)n - p.$$

This implies $-p = -p+j+1$ which is a contradiction as $j \neq -1$.

2(iii) Assume $\overline{\Sigma}(2p+2j_1+1) \equiv \overline{\Sigma}(2p+2j_2+2) \pmod{2(4p+\gamma)n+1}$ for some $j_1, j_2 \in J$. Then by (2)

$$(p-j_1)(2n-1) - 6n - 4f_J(j_1)n + 1 = -(2p+2j_2+\gamma+2)n - (p-j_2-1).$$

Then we have $-(p-j_1)+1 = -p+j_2+1$ and $4p-2j_1-6-4f_J(j_1)+2j_2+\gamma+2 = 0$ which implies $j_1 = j_2$ and $4f_J(j_1) = 4p-4+\gamma \geqslant 4(p-1)$. This is a contradiction since $f_J(j_1) \leqslant p-2$.

2(iv) Assume $\overline{\Sigma}(2p+2j+1) \equiv 0 \pmod{2(4p+\gamma)n+1}$ for some $j \in J$. Then by (2) we have $-p+j+1 = 0$ which implies $j = p-1 > p-2$, a contradiction.

2(v) Assume $\overline{\Sigma}(2p+2j_1+1) \equiv \overline{\Sigma}(2p+2j_2+1) \pmod{2(4p+\gamma)n+1}$. Then by (2) we have $-p+j_1+1 = -p+j_2+1$ which implies $j_1 = j_2$.

Thus it can be concluded that all column partial sums are distinct.

□

The above results and commentary validate Theorem 3.6.

3.3 Non-equivalent globally simple Heffter arrays $H(n; 4p+3)$

In this section we prove Theorem 1.8. First we need the following theorems from [5].

Theorem 3.14. [5] *For each $n \equiv 1 \pmod 4$ and $0 \leqslant \beta \leqslant n-5$, a Heffter array $H(n;3)$, L, exists that satisfies the following conditions:*

- *The non-empty cells are exactly on the diagonals D_β, $D_{\beta+2}$ and $D_{\beta+4}$,*
- *$s(D_{\beta+2}) = \{1, \ldots, n\}$,*
- *$s(D_\beta \cup D_{\beta+4}) = \{n+1, \ldots, 3n\}$,*
- *entries on D_β are all positive,*
- *entries on $D_{\beta+4}$ are all negative,*
- *the array defined by $M = [M(i,j)]$ where $M(i,j) = L(i+1, j+1)$, $i, j \in [n]$ retains the above properties.*

Theorem 3.15. [5] *Let $n \equiv 1 \pmod 4$, $p > 0$ and $n > 4p+3$. Let α be an integer such that $2p+2 \leqslant \alpha \leqslant n-2-2p$ and $\gcd(n, \alpha) = 1$. Let $\beta = 2p+\alpha-3$ and let L be a Heffter array $H(n;3)$ based on β satisfying the properties of Theorem 3.14 where $\{L(\beta, 0), -L(\beta+4, 0)\} \cap \{2n-1, 2n-(2p+1)/3\} = \emptyset$. Then the union of arrays L and A (defined in (6), with $\gamma = 3$) is a globally simple Heffter array $H(n; 4p+3)$ where entries are on the set of diagonals D_i where i is in $\mathbb{D} = \{0, 1, \ldots, 4p-2, 2p+\alpha\} \cup \{2p+\alpha-3, 2p+\alpha-1, 2p+\alpha+1\}$.*

As A' satisfies properties P1, P2, P3 and P4, and is indeed on the same set of cells as A, a similar theorem can be proven if we replace A with A' under certain conditions on f_I and f_J in A': $f_I(0) = 0$, $f_I(i) \neq (2p-i+1)/2$ for $i \in I$ and $f_J(j) \neq (p-j-4)/4$ for $j \in J$. Furthermore assume $\{L(\beta, 0), -L(\beta+4, 0)\} \cap \{2n-1\} = \emptyset$.

Let B' be the merged array constructed as below.

$$B'(i,j) = \begin{cases} A'(i,j) \text{ if } i-j \notin \{2p+\alpha-3, 2p+\alpha-1, 2p+\alpha+1\}, \\ L(i,j) \text{ if } i-j \in \{2p+\alpha-3, 2p+\alpha-1, 2p+\alpha+1\}. \end{cases}$$

Hence we are positioning $D_{2p+\alpha-3}$, $D_{2p+\alpha-1}$ and $D_{2p+\alpha+1}$ of L to match the diagonals $D_{2p+\alpha-3}$, $D_{2p+\alpha-1}$ and $D_{2p+\alpha+1}$ of B'.

Theorem 3.16. *Let $n \equiv 1 \pmod 4$, $p > 0$, $n > 4p+3$ and α be an integer such that $(n,\alpha) = 1$ and $2p+2 \leqslant \alpha \leqslant n-2-2p$. Now assume $f_I(0) = 0$, $f_I(i) \neq (2p-i+1)/2$ for $i \in I$ and $f_J(j) \neq (p-j-4)/4$ for $j \in J$ and $\{L(\beta,0), -L(\beta+4,0)\} \cap \{2n-1\} = \emptyset$. Then the array B' constructed as above by merging arrays A' and L is a globally simple Heftter array $H(n; 4p+3)$.*

Proof. Now, $s(A') = \{3n+1, \dots, (4p+3)n\}$ and $s(L) = \{1, \dots, 3n\}$ so it is obvious that $s(B) = \{1, \dots, (4p+3)n\}$. Also since all row and column sums of both A' and L are 0, all rows and columns sum to 0 in B'.

For rows and columns of B', $\Sigma_{B'}(x) = \Sigma_{A'}(x)$ and $\overline{\Sigma}_{B'}(x) = \overline{\Sigma}_{A'}(x)$ for $0 \leqslant x \leqslant 4p-2$. Hence by the previous section the row partial sums and column partial sums are distinct for $0 \leqslant x \leqslant 4p-2$.

Also $\Sigma_{B'}(x) = \Sigma_B(x)$ and $\overline{\Sigma}_{B'}(x) = \overline{\Sigma}_B(x)$ for all $x \in \{2i+1, 2p+2j+2, 2p, 2p+\alpha-3, 2p+\alpha-1, 2p+\alpha, 2p+\alpha+1 | i \in I, j \in J\}$ where B is the array given in Theorem 2.2, in effect when f_I and f_J are identity mappings. Hence to prove that row and non-zero columns have distinct partial sums in B', we only need to show that when $i \in I$ and $j \in J$

$$\{\Sigma_{B'}(2i), \Sigma_{B'}(2p+2j+1)\} \cap \{\Sigma_{B'}(2p+\alpha-3), \Sigma_{B'}(2p+\alpha-1), \Sigma_{B'}(2p+\alpha)\} = \emptyset \text{ and}$$
$$\{\overline{\Sigma}_{B'}(2i), \overline{\Sigma}_{B'}(2p+2j+1)\} \cap \{\overline{\Sigma}_{B'}(2p+\alpha-3), \overline{\Sigma}_{B'}(2p+\alpha-1), \overline{\Sigma}_{B'}(2p+\alpha)\} = \emptyset,$$

where the above elements are calculated as residues modulo $2(4p+3)n+1$.

First note that

$$n+1 \leqslant \Sigma_{B'}(2p+\alpha), \overline{\Sigma}_{B'}(2p+\alpha) \leqslant 3n. \tag{14}$$

Now consider row a, since $\Sigma_{B'}(4p-2) = \Sigma_{A'}(4p-2) = d_{2p}(r_a) + 1$ (from Lemma 3.9), the row partial sums for array B' are as follows:

$$\begin{aligned}
\Sigma_{B'}(2p+\alpha-3) &= d_{2p}(r_a) + 1 + L(a, a-2p-\alpha+3) < 0, \\
\Sigma_{B'}(2p+\alpha-1) &= d_{2p}(r_a) + 1 - L(a, a-2p-\alpha-1) < 0, \\
\Sigma_{B'}(2p+\alpha) &= -L(a, a-2p-\alpha-1) > 0, \\
\Sigma_{B'}(2p+\alpha+1) &= 0.
\end{aligned}$$

Then by Lemma 3.10 we have: $(4p+3)n > \Sigma_{B'}(2i) \geqslant 3n+2 > \Sigma_{B'}(2p+\alpha) > n > 0 > \Sigma_{B'}(2p+2j+1) > d_{2p}(r_a)+3n+2 > \Sigma_{B'}(2p+\alpha-3), \Sigma_{B'}(2p+\alpha-1) > d_{2p}(r_a) \geqslant -(4p+3)n$. The final inequality can be inferred from the definition of A'.

Next, consider column $a \neq 0$. Since $f_I(0) = 0$, from Lemma 3.12, we may deduce that $\overline{\Sigma}_{B'}(2i) \geqslant 3n+1$. Furthermore, by Lemma 3.12 we have:

$(4p+3)n > \overline{\Sigma}_{B'}(2i) \geqslant 3n+1 > \overline{\Sigma}_{B'}(2p+\alpha) > n > 0 > \overline{\Sigma}_{B'}(2p+2j+1) > d_{2p}(c_a)+3n+2 > \overline{\Sigma}_{B'}(2p+\alpha-3), \overline{\Sigma}_{B'}(2p+\alpha-1) > -(4p+3)n$.

Now consider column $a = 0$. By (11) and (12) we have:

$$\begin{aligned}
\overline{\Sigma}_{B'}(2i) &= (2n-1)i + 5n + 4f_I(i)n \leqslant 2(4p+3)n+1, \\
\overline{\Sigma}_{B'}(2p+2j+1) &= (p-j)(2n-1) - 6n - 4f_J(j)n + 1 > -(4p+3)n.
\end{aligned}$$

Since $\overline{\Sigma}_{B'}(4p-2) = -(4p+1)n-1$, we also have:

$$\begin{aligned} \overline{\Sigma}_{B'}(2p+\alpha-3) &= \overline{\Sigma}_{B}(2p+\alpha-3) = -(4p+1)n-1+L(2p+\alpha-3,0) < 0, \quad (15)\\ \overline{\Sigma}_{B'}(2p+\alpha-1) &= \overline{\Sigma}_{B}(2p+\alpha-1) = -(4p+1)n-1-L(2p+\alpha+1,0) < 0, \quad (16)\\ \overline{\Sigma}_{B'}(2p+\alpha) &= \overline{\Sigma}_{B}(2p+\alpha) = -L(2p+\alpha+1,0) > 0,\\ \overline{\Sigma}_{B'}(2p+\alpha+1) &= 0. \end{aligned}$$

Assuming $\{L(\beta,0), -L(\beta+4,0)\} \cap \{2n-1\} = \emptyset$, it was shown in [5] (Proof of Theorem 3.10) that partial sums $\overline{\Sigma}_B(2p+\alpha-3)$, $\overline{\Sigma}_B(2p+\alpha-1)$ and $\overline{\Sigma}_B(2p+\alpha)$ are distinct from partial sums $\overline{\Sigma}_A(2i+1)$, $\overline{\Sigma}_A(2p)$ and $\overline{\Sigma}_A(2p+2j+2)$ for all $i \in I$ and $j \in J$ and; 0. So by Corollary 3.8 we only need to show that partial sums $\overline{\Sigma}_{B'}(2p+\alpha-3)$, $\overline{\Sigma}_{B'}(2p+\alpha-1)$ and $\overline{\Sigma}_{B'}(2p+\alpha)$ are also distinct from each of the other partial sums $\overline{\Sigma}_{B'}(2i)$ and $\overline{\Sigma}_{B'}(2p+2j+1)$ for all $i \in I$ and $j \in J$.

In what follows we will make extensive use of (13) together with (4).

1(i) $\overline{\Sigma}_{B'}(2i) = (2n-1)i + 5n + 4f_I(i)n \geqslant 5n$ so by (14) and (3),

$$\overline{\Sigma}_{B'}(2p+\alpha) \not\equiv \overline{\Sigma}_{B'}(2i) \pmod{2(4p+3)n+1}$$

for all $i \in I$.

1(ii) Suppose that $\overline{\Sigma}_{B'}(2i) \equiv \overline{\Sigma}_{B'}(2p+\alpha-3) \pmod{2(4p+3)n+1}$ for some $i \in I$. Then by (5), $\overline{\Sigma}_{B'}(2i) = (2n-1)i+5n+4f_I(i)n = 2(4p+3)n+1-(4p+1)n-1+L(2p+\alpha-3,0)$. Hence $(4f_I(i)+2i-4p)n - i = L(2p+\alpha-3,0)$. But $3n \geqslant L(2p+\alpha-3,0) > n$ which implies that $4f_I(i)+2i-4p = 2$ and so $f_I(i) = (2p-i+1)/2$, contradicting the definition of f_I.

2(i) Suppose that $\overline{\Sigma}_{B'}(2p+2j+1) \equiv \overline{\Sigma}_{B'}(2p+\alpha) \pmod{2(4p+3)n+1}$ for some $j \in J$. Then

$$(p-j)(2n-1) - 6n - 4f_J(j)n + 1 = -L(2p+\alpha+1,0)$$

which implies $2p-2j-6-4f_J(j) = 2$ by (14). Then $4f_J(j) = 2p-2j-8$ so $f_J(j) = (p-j-4)/2$, contradicting the definition of f_J.

2(ii) Suppose that $\overline{\Sigma}_{B'}(2p+2j+1) \equiv \Sigma(2p+\alpha-3) \pmod{2(4p+3)n+1}$ for some $j \in J$. Then

$$(p-j)(2n-1) - 6n - 4f_J(j)n + 1 = -(4p+1)n - 1 + L(2p+\alpha-3,0)$$

which implies $L(2p+\alpha-3,0) = (6p-5-2j-4f_J(j))n - p + j + 2$. Now by (14) $6p-5-2j-4f_J(j) \in \{1,2,3\}$. Thus $f_J(j) \geqslant p + (p-j-3)/2 \geqslant p-1$, a contradiction.

Similar arguments to above can be used to verify the result for $\overline{\Sigma}_{B'}(2p+\alpha-1)$ since (15) and (16) imply this term is bounded by the same range of values as $\overline{\Sigma}_{B'}(2p+\alpha-3)$.

This completes the proof of Theorem 3.16. □

It remains to invoke the above results and complete the proof of Theorem 1.8.

Assume $n \equiv 1 \pmod 4$, $p > 0$, $n > 4p+3$ and α be an integer such that $2p+2 \leqslant \alpha \leqslant n-2-2p$, $\gcd(n,\alpha) = 1$, $\gcd(n, \alpha - 2p - 1) = 1$ and $\gcd(n, n-1-\alpha-2p) = 1$. Let B' and B'' be two Heffter arrays constructed as in the the proof of Theorem 3.16 for distinct valid choices of f_I and f_J. Firstly, from the definition of A', the diagonals D_{2p} and $D_{2p+\alpha}$ do not depend on the choice of f_I and f_J. Then since these two diagonals of B' and B'' are identical yet other entries are different, it is impossible to obtain B'' from B' by rearranging rows and/or columns, or replacing each entry x with $-x$. Therefore, B' and B'' are non-equivalent Heffter arrays. Also by Lemma 2.1 they have simple and compatible orderings.

Next, recall the restrictions $f_I(0) = 0$, $f_I(i) \neq (2p-i+1)/2$ for $i \in I$ and $f_J(j) \neq (p-j-4)/4$ for $j \in J$. Thus, if $\mathcal{H}(\epsilon)$ represents the number of derangements on a set of size ϵ then there are more than $\mathcal{H}(p-2)$ choices for each of f_I and f_J. Moreover, from the final condition of Theorem 3.14, the subarray L of B' may be adjusted provided $\{L(\beta,0), -L(\beta+4,0)\} \cap \{2n-1\} = \emptyset$. This yields an extra factor of $n-2$ in the number of Heffter arrays. Theorem 1.8 then follows by Theorem 3.6.

Consequently, Theorem 1.8 verifies the existence of many non-equivalent globally simple integer Heffter arrays $H(n; 4p+3)$ for an infinite class of n and p. Further, Corollary 1.9 draws the connection with the existence of many distinct face 2-colourable embedding of the complete graph K_{2nk+1} onto an orientable surface where each face is a cycle of fixed length k.

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