A NOTE ON FAMILIES OVER A P-ADIC DISK, CIRCA 2012

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§1 – Introduction and Main Results

The following notes discuss how the local conditions H^1_e and H^1_g of Bloch-Kato vary in an analytic family, culminating in the proof of a control theorem (Theorem 1.1). However this document is not intended for publication for the following reasons:

(i) The principal results are not surprisingly out of date after a full decade;

(ii) The use of continuous cohomology over affinoid rings is not explained here;

(iii) Some of the arguments duplicate those already given in Appendix C of [De];

(iv) Many of the proofs can be shortened, and are in truth shoddily written up;

(v) The effort required to make these notes publishable exceeds their utility.

Fix a prime number $p \neq 2$, and consider a family of p-adic Galois representations $\{V_x\}_{x\in\mathfrak{X}}$ interpolated over a rigid-analytic space $\mathfrak{X}$, say. To study the arithmetic of such a family, one often tries to attach a Selmer group to it (which amounts to imposing local conditions at the places of the number field you are considering). At places not lying above p, it is perfectly natural to choose unramified cocycles. However, if the deformation is non-ordinary, there is a significant dilemma in how to make a 'good choice' of local condition above p.

Much recent progress was made by Bellaiche, Chenevier, Pottharst and others, who focussed on defining Selmer families by using the theory of (φ,Γ)-modules, combined with the powerful methods associated to Nekovář's Selmer complexes. The purpose of these notes is to outline an alternative approach to interpolating such families, again in terms of subgroups "H^1_g" introduced by Bloch and Kato. The big Selmer groups we study are cut out by a single local condition $H^1_{\mathcal{G},\dagger}(-,\mathbb{V})$ at primes above p, and do not utilise the theory of Selmer complexes in any way. Certainly our proofs are rather more low-brow in nature (and less sophisticated) than the other approaches, but are arguably more accessible for the non-expert.

Unfortunately the $\mathbb{Q}_p$-dimension of each $H^1_g(-,V_x)$ jumps around alarmingly with the variation in $x \in \mathfrak{X}$ (as has already been remarked upon in [Be,Po,Ki]). To overcome this instability we will impose conditions only at pseudo-geometric points x, contained in some prescribed region $\mathfrak{X}^+_{\mathrm{dR}} \subset \mathfrak{X} \cap \mathbb{N}$ where the dimension of $H^1_g(-,V_x)$ is stable. Here one identifies an integer k with an element of weight space, by sending it to the character which maps each $a \in \mathbb{Z}_p^\times$ to the value a^k.

The Galois representations we shall consider include the p-ordinary families of Greenberg [Gr]. However they might well differ, in general, from the trianguline examples which were initially studied by Pottharst, and subsequently Bellaiche; the local conditions at primes above p used to define our Selmer groups have a different formulation from those occurring in the construction of $H^1_{\mathrm{pot},\underline{a}}$ in [Po,Be]. Needless to say if the family is p-ordinary, all these definitions collapse down to give Greenberg's original version, though the proof of this statement is non-trivial.

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Let F be an algebraic number field, and $\mathbb{L}$ a free $\mathbb{Z}_p\langle\langle s\rangle\rangle$-module of finite rank equipped with an action of $G_F = \text{Gal}(\overline{F}/F)$. We assume that $\mathbb{L}$ is *pseudo-geometric* in a suitable sense – see page 15 for the precise definition; in particular, the lattice $\mathbb{L}$ should be unramified outside a finite set of places Σ, and de Rham (as a local Galois representation, at every place $\nu|p$) for infinitely many positive weights k.

We will now write λ_k for the multiplication by $(s-k+1)$ map operating on $\mathbb{L}$. If χ_{cy} is the p-cyclotomic character, one may define the χ_{cy}-twisted discrete dual by $\mathcal{A}_{\mathbb{L}} := \text{Hom}_{\text{cont}}\big(\mathbb{L}, \mu_{p^\infty}\big)$ where μ_{p^∞} denotes the group of p-power roots of unity. Moreover for a p-adic weight k, one can make a similar definition if $\mathbb{L}$ is replaced instead by the free $\mathbb{Z}_p$-module $\mathbb{L}/\lambda_k$.

Recall that Bloch and Kato [BK] attach a discrete Selmer group $H^1_{e,\Sigma}\big(F, \mathcal{A}_{\mathbb{L}/\lambda_k}\big)$ for de Rham specialisations k, defined using the local conditions $H^1_e\big(F_\nu, -\big)$ at $\nu|p$. The following is a simplification of the main result of this paper (Theorem 3.3.2).

Theorem 1.1. *There exists a big Selmer group* $\text{Sel}_{F,\Sigma}\big(\mathcal{A}_{\mathbb{L}}\big) \subset H^1_{\text{cont}}\big(\overline{F}/F, \mathcal{A}_{\mathbb{L}}\big)$ *cofinitely-generated over* $\mathbb{Z}_p\langle\langle s\rangle\rangle$*, with the following properties:*

(i) For almost all de Rham weights $k \in \mathbb{N}$*, the transition maps*

$$H^1_{e,\Sigma}\big(F, \mathcal{A}_{\mathbb{L}/\lambda_k}\big) \longrightarrow \text{Sel}_{F,\Sigma}\big(\mathcal{A}_{\mathbb{L}}\big)[\lambda_k]$$

have finite kernel bounded independently of λ_k*, and finite cokernel;*

(ii) The cokernel of these maps in (i) will also be bounded independently of λ_k *provided one assumes* $\mathbb{L}$ *satisfies an additional Hypothesis*(Bnd)*, given on p17.*

This extra condition (Bnd) is equivalent to controlling the p-valuation for each of the Tamagawa factors, associated to the specialisations $\mathbb{L}/\lambda_k$ at the primes over p. For instance if $\mathbb{L}$ is a lattice obtained from a slope zero family and the prime $p \geq 5$, we shall prove in §§3.3 that (Bnd) always holds for such $\mathbb{L}$.

Corollary 1.2. *For almost all de Rham weights* $k \in \mathbb{N}$*,*

$$\text{co-rk}_{\mathbb{Z}_p}\Big(H^1_{e,\Sigma}\big(F, \mathcal{A}_{\mathbb{L}/\lambda_k}\big)\Big) \;=\; \text{co-rk}_{\mathbb{Z}_p\langle\langle s\rangle\rangle}\Big(\text{Sel}_{F,\Sigma}\big(\mathcal{A}_{\mathbb{L}}\big)\Big).$$

One can easily see that, in general, the corollary does not work at weights k which are either non-positive or non de Rham (in the latter scenario, the Bloch-Kato Selmer group is **not** the correct object to interpolate anyway).

As an illustration of this theorem, we'll give an application to families on $\text{GL}_{2/\mathbb{Q}}$. Let f be a modular eigenform on $\Gamma_1(N) \cap \Gamma_0(p)$ of weight $k_0 \geq 2$, and assume that E is a finite extension of $\mathbb{Q}_p$ containing values of the Fourier coefficients $a_n(f)$. Providing $\text{val}_p\big(a_p(f)\big) < k_0 - 1$ and $a_p(f)^2 \neq p^{k_0-1}$, Hida [Hi] and Coleman [Co] have shown existence of an affinoid subdomain $U(f)$ in weight-space containing k_0, and a power series $\mathcal{F}(s) = \sum_{n=1}^{\infty} A_n(s)q^n \in E_U\langle\langle s\rangle\rangle\big[\big[q\big]\big]$ satisfying:

- For all $k \in U(f)$, each $\mathcal{F}(k)$ is an overconvergent modular form of weight k;
- If $k \in U(f) \cap \mathbb{Z}_{\geq k_0}$, then $\mathcal{F}(k)$ is a classical eigenform on $\Gamma_1(N)$ of weight k;
- Lastly $\mathcal{F}(k_0)$ yields the original Hecke eigenform f we started off with.

Here $E_U\langle\langle s\rangle\rangle$ denotes the affinoid E-algebra associated to $U(f)$. By taking a smaller p-adic neighborhood if necessary, one may assume that $U = \mathbb{U}_{k_0}$ is a closed disk centred on k_0 (we shall now drop the U from the notation entirely).

We also suppose $\rho_{\mathcal{F}} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2\big(E\langle\langle s\rangle\rangle\big)$ denotes a big Galois representation with the property that for all $k \in \mathbb{U}_{k_0} \cap \mathbb{Z}_{\geq k_0}$, each specialisation $\rho_{\mathcal{F}}\big|_{s=k-1}$ coincides with the contragredient $\rho^{\vee}_{\mathcal{F}(k)} : G_{\mathbb{Q}} \longrightarrow \mathrm{Aut}\big(V(\mathcal{F}(k))^*\big)$ of Deligne's representation attached to $\mathcal{F}(k)$. Additionally if the weight $k \in \mathbb{U}_{k_0}$ is non-classical, then the restriction of $\rho_{\mathcal{F}}\big|_{s=k-1}$ to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ will be overconvergent but not de Rham.

Proposition 1.3. *For all but finitely many even classical weights $k \in \mathbb{U}_{k_0} \cap 2\cdot\mathbb{Z}$, the co-rank of the Selmer group $H^1_{f,\Sigma}\Big(F, \rho_{\mathcal{F}(k)} \otimes \omega^{k-1}\langle\chi_{\mathrm{cy}}\rangle^{k/2}\Big)$ is a fixed constant.*

A version of this statement was proved by Bellaiche-Chenevier via another method. Moreover if $F = \mathbb{Q}$, then Kato has already shown

$$\text{co-rk}_{\mathcal{O}_E}\Big(H^1_{f,\Sigma}\big(\mathbb{Q}, \rho_{\mathcal{F}(k)} \otimes \omega^{k-1}\langle\chi_{\mathrm{cy}}\rangle^{k/2}\big)\Big) \;\leq\; \mathrm{order}_{s=k/2}\Big(L^{\mathrm{MTT}}_{p\text{-adic}}\big(\mathcal{F}(k), s\big)\Big)$$

and one expects an equality, certainly if there is no trivial zero condition at p. The above proposition suggests the order of the Mazur-Tate-Teitelbaum L-function $L^{\mathrm{MTT}}_{p\text{-adic}}\big(\mathcal{F}(k), s\big)$ along the central critical line $s = k/2$ should then also be constant, at all but finitely many exceptional weights.

Sketch Proof of 1.3: To obtain this result from Corollary 1.2 is straightforward. Let $\mathbb{V}\big(\rho_{\mathcal{F}}\big)$ denote the representation space associated to $\rho_{\mathcal{F}}$ over the algebra $E\langle\langle s\rangle\rangle$, and fix an $\mathcal{O}_E\langle\langle s\rangle\rangle$-lattice $\mathbb{L}_{\mathcal{F}} \subset \mathbb{V}\big(\rho_{\mathcal{F}}\big)$ which is Galois stable (and clearly rank two). If ξ_2 denotes the doubling transformation on weight-space, then the specialisations $(\xi_2)_*\mathbb{L}_{\mathcal{F}}\Big|_{s=k'-1}$ will be equivalent to $\rho^{\vee}_{\mathcal{F}(2k')} : G_{\mathbb{Q}} \to \mathrm{GL}_2(E)$ at each $2k' \in \mathbb{U}_{k_0} \cap \mathbb{N}$.

Remark: The universal character $\Psi : G_{\mathbb{Q}} \to \mathbb{Z}_p\langle\langle s\rangle\rangle^{\times}$ is determined by the expansion

$$\Psi(\sigma) \;:=\; \sum_{k=0}^{\infty} \frac{\log^k\big(\chi_{\mathrm{cy}}(\sigma)\big)}{\Gamma(k+1)} \times s^k.$$

Therefore for the twisted lattice $(\xi_2)_*\mathbb{L}_{\mathcal{F}}\otimes\Psi^{-1}$, its specialisations at integer classical weights $k' \geq k_0/2$ yield the p-adic representations

$$(\xi_2)_*\mathbb{L}_{\mathcal{F}} \otimes \Psi^{-1}\Big|_{s=k'-1} \;\sim\; \rho^{\vee}_{\mathcal{F}(2k')} \otimes \langle\chi_{\mathrm{cy}}\rangle^{1-k'} \;\cong\; \rho_{\mathcal{F}(2k')^*} \otimes \omega^{2k'-1}\langle\chi_{\mathrm{cy}}\rangle^{k'}.$$

One must then check:

(I) The lattice $(\xi_2)_*\mathbb{L}_{\mathcal{F}} \otimes \Psi^{-1}$ is pseudo-geometric over its affinoid E-algebra;

(II) The Selmer groups $H^1_{e,\Sigma}$ and $H^1_{f,\Sigma}$ coincide on each $\rho_{\mathcal{F}(2k')} \otimes \omega^{2k'-1}\langle\chi_{\mathrm{cy}}\rangle^{k'}$.

The second assertion is well known for the classical modular forms $\mathcal{F}(2k') \otimes \omega^m$ with $0 \leq m \leq p-2$, at their central critical values $s+1 = k'$.

To establish (I), we firstly show that $\mathbb{L}_{\mathcal{F}}$ is pseudo-geometric (Proposition 2.3.1). Crucially this property is well behaved under the affine transformation ξ_2 on $\mathbb{U}_{k_0}$, and is also invariant under twisting by integer powers of the universal character Ψ. This means we can directly apply our Corollary 1.2 to the lattice $(\xi_2)_*\mathbb{L}_{\mathcal{F}} \otimes \Psi^{-1}$, and thereby conclude

$$\text{co-rk}_{\mathcal{O}_E}\Big(H^1_{f,\Sigma}\big(F, \rho_{\mathcal{F}(2k')} \otimes \omega^{2k'-1}\langle\chi_{\mathrm{cy}}\rangle^{k'}\big)\Big) \;=\; \text{co-rk}_{\mathcal{O}_E\langle\langle s\rangle\rangle}\Big(\mathrm{Sel}_{F,\Sigma}\big(\mathcal{A}_{(\xi_2)_*\mathbb{L}_{\mathcal{F}}\otimes\Psi^{-1}}\big)\Big)$$

at almost all classical weights $k = 2k' \geq k_0$, which gives the required constant.

□

Notation and Conventions.
(a) Though we won't need to mention it again, we always fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. The Tate field $\mathbb{C}_p := \widehat{\overline{\mathbb{Q}}_p}$ is known to be algebraically closed, and as usual one normalises the p-adic valuation $\left|x\right|_p = p^{-\mathrm{val}_p(x)}$ on it to ensure that $\left|p\right|_p = 1/p$.

(b) Let K be a finite extension of $\mathbb{Q}_p$. We write $K_\infty = \bigcup_{n\geq 0} K_n$ to indicate the cyclotomic $\mathbb{Z}_p$-extension of K, so that $\left[K_n : K\right] = p^n$ and $\Gamma_K = \mathrm{Gal}\big(K_\infty/K\big) \cong \mathbb{Z}_p$. The universal character $\Psi : G_{\mathbb{Q}} \twoheadrightarrow \mathrm{Gal}\big(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}\big) \hookrightarrow \mathbb{Z}_p\langle\langle s\rangle\rangle^\times$ above will then interpolate the cyclotomic character via the rule $\Psi\big|_{s=k-1} = \chi_{\mathrm{cy}}^{k-1} \cdot \omega^{1-k}$ for $k \in \mathbb{Z}$.

(c) We assume the reader is familiar with the p-adic period rings $B_{\max}$, B_{cris}, B_{dR}. In this paper we consider their affinoid analogues $\mathbb{B}_{\max}$, $\mathbb{B}_{\mathrm{cris}}$, $\mathbb{B}_{\mathrm{dR}}$ consisting of power series convergent on $\mathbb{U}_{k_0}$, whose coefficients lie in the corresponding B-ring. For example, if $\mathbb{U}_{k_0} = \mathbb{Z}_p$ then the deformed ring

$$\mathbb{B}_{\mathrm{dR}} \;=\; \left\{ \sum_{n=0}^{\infty} b_n s^n \;\text{ where } b_n \in B_{\mathrm{dR}} \text{ and } b_n \to 0 \text{ in the } B_{\mathrm{dR}}\text{-topology} \right\}$$

has a natural $G_{\mathbb{Q}_p}$-action via the coefficients, and inherits its filtration from B_{dR}. In addition both $\mathbb{B}_{\max}$ and $\mathbb{B}_{\mathrm{cris}}$ are endowed with a Frobenius operator φ induced from their p-adic versions $B_{\max}$ and B_{cris}, respectively (for details see [De, §§5.2]).

(d) For a module M equipped with an action of a topological group G, throughout we'll write $H^j(G,M) = H^j_{\mathrm{cont}}(G,M)$ to denote its continuous cohomology groups. If G satisfies the p-finiteness condition and M is finite and free as a $\mathbb{Q}_p\langle\langle s\rangle\rangle$-module, then it's well known $H^j(G,M)$ is of finite-type over $\mathbb{Q}_p\langle\langle s\rangle\rangle$ (e.g. see [Be, Prop 3]). However if one considers instead the G_K-cohomology of finitely-generated modules over the rings $\mathbb{B}_{\max}$, $\mathbb{B}_{\mathrm{cris}}$, $\mathbb{B}_{\mathrm{dR}}$ then this ceases to be true, so one must be careful.

(e) If M is a module over an integral domain R and $M' \subset M$ is an R-submodule, then the R-saturation of M' inside M is defined to be the intersection $\bigcap M''$ of all R-saturated modules M'' satisfying $M' \subset M'' \subset M$. For instance the quotient module $M\big/M'_{R\text{-sat}}$ is automatically R-torsion free, whilst $M'_{R\text{-sat}}\big/M'$ is R-torsion.

(f) For an R-module M, we use the notation $\mathrm{Tors}_R(M)$ for its maximal R-torsion submodule. For example, if $R = \mathbb{Z}_p$ then $\mathrm{Tors}_{\mathbb{Z}_p}(M)$ coincides with the p-primary torsion submodule $M[p^\infty]$ of M. On the other hand, if $R = E\langle\langle s\rangle\rangle$ for some finite normal extension $E/\mathbb{Q}_p$ one has $\mathrm{Tors}_{E\langle\langle s\rangle\rangle}(M) = \mathrm{Tors}_{\mathbb{Q}_p\langle\langle s\rangle\rangle}(M)$, as any element killed by $g \in E\langle\langle s\rangle\rangle$ is also killed by $\mathrm{Norm}_{E/\mathbb{Q}_p}(g) := \prod_{\sigma\in\mathrm{Gal}(E/\mathbb{Q}_p)} g^\sigma(s) \in \mathbb{Q}_p\langle\langle s\rangle\rangle$.

(g) Finally, if $\mathcal{R}$ is a finite and flat extension of the weight algebra $\Lambda^{\mathrm{wt}} = \mathbb{Z}_p[\![1+p\mathbb{Z}_p]\!]$ then we write $\widetilde{\mathcal{R}}^{\mathrm{norm}}$ to denote the normal closure of $\mathcal{R}$ inside $\mathrm{Frac}\big(\Lambda^{\mathrm{wt}}\big) \otimes_{\Lambda^{\mathrm{wt}}} \mathcal{R}$. A point $\widetilde{P} \in \mathrm{Spec}\big(\widetilde{\mathcal{R}}^{\mathrm{norm}}\big)$ is called 'arithmetic of type (k,ϵ)' if its restriction $\widetilde{P}\big|_{\Lambda^{\mathrm{wt}}}$ is the power series representing the map $[\tau] \mapsto \epsilon(\tau)\cdot\omega^{k_0-k}(\tau)\cdot\tau^k$.

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§2 – The Local Theory

Let K be a finite extension of $\mathbb{Q}_p$. Henceforth $\mathbb{W}$ denotes a free $\mathbb{Q}_p\langle\langle s\rangle\rangle$-module equipped with a continuous action of the absolute Galois group $G_K = \mathrm{Gal}(\overline{K}/K)$.

Hypothesis(dR). *(i) There exists an integer $n(\mathbb{W}) \geq 0$ such that*

$$\mathbb{W} \otimes_{\mathbb{Q}_p\langle\langle s\rangle\rangle} \mathbb{C}_p\langle\langle s\rangle\rangle[s^{-1}] \;\cong\; \bigoplus_{i,j\in\mathbb{Z}} \mathbb{C}_p\langle\langle s\rangle\rangle[s^{-1}] \otimes \left(\Psi^i \chi_{\mathrm{cy}}^j\right)^{\oplus e_{i,j}(\mathbb{W})}$$

as $\mathrm{Gal}(\overline{K}/K_{n(\mathbb{W})})$-modules, where almost all of the integers $e_{i,j}(\mathbb{W})$ are zero;

(ii) The specialisations $\mathbb{W}_k := \mathbb{W}/\big(s-k+1\big)\cdot\mathbb{W}$ are de Rham representations at all but finitely many distinct weights $k \in \mathbb{U}_{k_0} \cap \mathbb{Z}_{\geq 2}$.

We denote this set of de Rham specialisations by $\mathfrak{X}^+_{\mathrm{dR}} = \mathfrak{X}^+_{\mathrm{dR}}(\mathbb{W})$, and endow it with a topology induced from the rule $\mathcal{U} \subset \mathfrak{X}^+_{\mathrm{dR}}$ is open whenever $\#\big(\mathfrak{X}^+_{\mathrm{dR}} - \mathcal{U}\big) < \infty$.

§§2.1 – Determining the Image of the Big Exponential

Let's begin by deforming the local Bloch-Kato homomorphisms [BK, §3] over $\mathbb{U}_{k_0}$ (unless otherwise indicated, the default tensor products will be taken over $\mathbb{Q}_p\langle\langle s\rangle\rangle$).

Definition 2.1.1. *The exponential part $H^1_{\mathcal{E}}\big(K, \mathbb{W}^*(1)\big)$ is the $\mathbb{Q}_p\langle\langle s\rangle\rangle$-saturation of*

$$H^1_{\mathcal{E},0}\big(K, \mathbb{W}^*(1)\big) \;:=\; \mathrm{Ker}\Big(H^1\big(K, \mathbb{W}^*(1)\big) \longrightarrow H^1\big(K, \mathbb{W}^*(1) \otimes \mathbb{B}^{\varphi=1}_{\mathrm{cris}}\big)\Big)\,.$$

The reason we call it the 'exponential subgroup' is because it is intimately related to the boundary map in the G_K-cohomology of the following sequence.

Proposition 2.1.2. *There exists a short exact sequence of $\mathrm{Gal}\big(\overline{\mathbb{Q}}_p/\mathbb{Q}_p\big)$-modules,*

$$0 \longrightarrow \mathbb{Q}_p\langle\langle s\rangle\rangle \longrightarrow \mathbb{B}^{\varphi=1}_{\mathrm{cris}} \overset{\mathrm{mod\ Fil}^0}{\longrightarrow} \mathbb{B}_{\mathrm{dR}}/\mathrm{Fil}^0\mathbb{B}_{\mathrm{dR}} \longrightarrow 0.$$

Proof: With many thanks to Adrian Iovita, we have included (verbatim) his demonstration of how the Fontaine-Messing sequence [FM] can be stretched to fit a closed domain $\mathbb{U}_{k_0} \subset \mathbb{Z}_p$. This sequence was quoted in [DS, p208] but the proof was omitted due to its length.

Note first that $B^{\varphi=1}_{\mathrm{cris}} = B^{\varphi=1}_{\mathrm{max}}$, so we can replace $\mathbb{B}^{\varphi=1}_{\mathrm{cris}}$ with $\mathbb{B}^{\varphi=1}_{\mathrm{max}}$ throughout. One may assume that $\mathbb{U}_{k_0} = \mathbb{Z}_p$ (if not, do a translation, then a scaling). It is clear the first map is injective, and also that this sequence forms a complex. To show that we have exactness in the middle, let $\sum_{n=0}^{\infty} a_n s^n \in \mathbb{B}^{\varphi=1}_{\mathrm{max}} \cap \mathrm{Fil}^0\mathbb{B}_{\mathrm{dR}}$. Then $\big\|a_n\big\|_{\mathrm{max}} \longrightarrow 0$ as the index $n \to \infty$, and each $a_n \in B^{\varphi=1}_{\mathrm{max}} \cap \mathrm{Fil}^0 B_{\mathrm{dR}} = \mathbb{Q}_p$. However the norm on B_{max} yields the usual p-adic absolute value on $\mathbb{Q}_p$, hence $\sum_{n=0}^{\infty} a_n s^n \in \mathbb{Q}_p\langle\langle s\rangle\rangle$ which gives us exactness in the middle.

Remark: It remains to prove surjectivity for the natural map $\mathbb{B}^{\varphi=1}_{\mathrm{max}} \longrightarrow \mathbb{B}_{\mathrm{dR}}/\mathrm{Fil}^0$. In fact, we shall establish the surjectivity of

$$\big(t^{-j}\cdot\mathbb{B}^+_{\mathrm{max}}\big)^{\varphi=1} \longrightarrow t^{-j}\cdot\mathbb{B}^+_{\mathrm{dR}}/\mathbb{B}^+_{\mathrm{dR}} \qquad \text{at every integer } j \geq 0.$$

As usual $t = \log[\varepsilon]_R$ for a compatible system $\varepsilon = (\zeta_{p^n})_n$ of $p^{n\text{-th}}$-roots of unity where $R := \varprojlim_{(-)^p} \mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}$, and $[-]_R$ is the Teichmüller lift to $\mathbf{A}_{\mathrm{inf}} = \mathrm{Witt}(R)$.

Let $\theta : \mathbf{A}_{\text{inf}} \twoheadrightarrow \mathcal{O}_{\mathbb{C}_p}$ be the map sending a vector $\sum_{n=0}^{\infty} p^n [c_n]_R \mapsto \sum_{n=0}^{\infty} p^n c_n^{(0)}$; in other words, the pair $\big(\mathbf{A}_{\text{inf}}, \theta\big)$ denotes the "universal p-adic thickening" of $\mathcal{O}_{\mathbb{C}_p}$. For any generator π of the ideal $\text{Ker}(\theta)$, the ring $\mathbf{A}_{\max}$ denotes the subring of B_{dR}^+ consisting of expansions

$$\left\{ x = \sum_{n=0}^{\infty} c_n \big(\pi/p\big)^n \quad \text{with} \;\; c_n \in \mathbf{A}_{\text{inf}} \;\; \text{satisfying} \; \lim_{n\to\infty} c_n = 0 \right\}.$$

(One then defines $B_{\max}^+ := \mathbf{A}_{\max}[1/p]$, and $B_{\max} := B_{\max}^+[1/t]$ where t is as above.)

To establish that the map $\big(t^{-j} \cdot \mathbb{B}_{\max}^+\big)^{\varphi=1} \longrightarrow t^{-j} \cdot \mathbb{B}_{\text{dR}}^+/\mathbb{B}_{\text{dR}}^+$ is a surjection, we shall proceed by induction on $j \geq 0$.

If $j = 0$ there is nothing to prove. Let's therefore take $j = 1$, and consider $y \in t^{-1} \cdot \mathbb{B}_{\text{dR}}^+/\mathbb{B}_{\text{dR}}^+ = t^{-1}\mathbb{C}_p\langle\langle s\rangle\rangle$. In particular, one may expand $ty = \sum_{n=0}^{\infty} a_n s^n$ where each coefficient $a_n \in \mathbb{C}_p$, with the a_n's tending to zero p-adically as $n \to +\infty$.

Remark: Contained in the proof of Proposition III.3.1 of [Cz], there is a recipe for constructing sequences of elements $b_n \in B_{\max}^{\varphi=p}$ mapping down to a_n modulo $t \cdot B_{\text{dR}}^+$. The delicate part is to show that one can choose the b_n's precisely, so that they tend to zero under the $B_{\max}$-topology (i.e. under its associated norm $\| - \|_{\max}$).

Fix some $\alpha \in \mathbb{C}_p$ such that $m_\alpha := \text{val}_p(\alpha - 1) \geq 2$, and let $\widetilde{\alpha} = (\alpha_i)_i \in R$ be a sequence whose initial terms satisfy $\alpha_j = \exp\big(p^{-j}\log(\alpha)\big)$ for $0 \leq j \leq m_\alpha - 1$. Clearly $\log\big[\widetilde{\alpha}\big]_R$ belongs to $\mathbf{A}_{\max}$, but we also claim the

Upper Bound Estimate: $\quad \Big\| \log\big[\widetilde{\alpha}\big]_R \Big\|_{\max} \leq p^{-n_\alpha} \quad$ where $n_\alpha = \lfloor m_\alpha - 1 \rfloor$.

To derive this estimate, consider $x = \exp\big(p^{-n_\alpha}\log(\alpha)\big) \in \mathcal{O}_{\mathbb{C}_p}$ and $\widetilde{x} = \widetilde{\alpha}^{p^{-n_\alpha}} \in R$. Then, working out the p-adic valuation

$$\text{val}_p(x-1) \;=\; -n_\alpha + \text{val}_p\big(\log(\alpha)\big) \;=\; m_\alpha - n_\alpha \;\geq\; 1, \quad \text{hence } x \in 1 + p\mathcal{O}_{\mathbb{C}_p}.$$

Given $z \in \mathcal{O}_{\mathbb{C}_p}$ satisfies $x - 1 = pz$, one sees immediately $\theta\big([\widetilde{x}]_R - 1\big) = x - 1 = pz$. As a corollary, $[\widetilde{x}]_R - 1 - pu \in \text{Ker}\Big(\theta : \mathbf{A}_{\text{inf}} \twoheadrightarrow \mathcal{O}_{\mathbb{C}_p}\Big)$ where $u \in \mathbf{A}_{\text{inf}}$ with $\theta(u) = z$. If π generates $\text{Ker}(\theta)$ principally, then

$$[\widetilde{x}]_R - 1 \;=\; p \times u + \pi \times \delta \;=\; p \times u + p \times \left(\frac{\pi}{p}\right) \times \delta \;\in\; p\mathbf{A}_{\max} \;\; \text{for some } \delta \in \mathbf{A}_{\text{inf}}.$$

Consequently $\log[\widetilde{x}]_R \in \mathbf{A}_{\max}$ and $\log[\widetilde{\alpha}]_R = p^{n_\alpha}\log[\widetilde{x}]_R \in p^{n_\alpha}\mathbf{A}_{\max}$, so we're done.

Now we finish off the $j = 1$ induction step. Recall that $ty = \sum_{n=0}^{\infty} a_n s^n \in \mathbb{C}_p\langle\langle s\rangle\rangle$: for every positive integer n, there exist elements $\alpha^{(n)} \in \mathbb{C}_p$ with $\text{val}_p\big(\alpha^{(n)} - 1\big) > 0$ such that $\log\big(\alpha^{(n)}\big) = a_n$. Indeed, providing that $n \gg 0$ one must have

$$\text{val}_p(a_n) \;=\; \text{val}_p\big(\log(\alpha^{(n)})\big) \;=\; \text{val}_p\big(\alpha^{(n)} - 1\big) \;\geq\; 2$$

in which case our 'upper bound estimate' ensures that $b_n := \log\big[\widetilde{\alpha^{(n)}}\big]_R \in B_{\max}^{\varphi=p}$. Moreover $\lim_{n\to\infty} \big\|t^{-1}b_n\big\|_{\max} = 0$, and it follows that $\beta := t^{-1}\sum_{n=0}^{\infty} b_n s^n \in \mathbb{B}_{\max}^{\varphi=1}$ maps down to $y = t^{-1}\sum_{n=0}^{\infty} a_n s^n$ modulo $\mathbb{B}_{\text{dR}}^+$.

The induction for $j > 1$ is standard, using the 'upper bound estimate' repeatedly. $\square$

From now on we suppose that $\mathbb{W}$ is a G_K-representation satisfying Hypothesis(dR). Tensoring the sequence in Proposition 2.1.2 by $\mathbb{W}^*(1)$ and taking G_K-cohomology, one obtains the long exact sequence

$$0 \longrightarrow \mathbb{W}^*(1)^{G_K} \longrightarrow \left(\mathbb{W}^* \otimes \mathbb{B}_{\mathrm{cris}}^{\varphi=p}\right)^{G_K}$$
$$\longrightarrow \left(\mathbb{W}^* \otimes \mathbb{B}_{\mathrm{dR}}/\mathrm{Fil}^1\mathbb{B}_{\mathrm{dR}}\right)^{G_K} \overset{\mathbf{EXP}_{\mathbb{W}^*(1)}}{\longrightarrow} H^1_{\mathcal{E},0}\left(K, \mathbb{W}^*(1)\right) \longrightarrow 0.$$

It is natural to ask, how much does the image of $\mathbf{EXP}_{\mathbb{W}^*(1)}$ differ from the full $H^1_{\mathcal{E}}$? Below is a partial answer that works over the finite extension $L = K_{n(\mathbb{W})}$.

Lemma 2.1.3. *There exists an $f(s) \in s \cdot \mathbb{Q}_p[s]$ such that after localisation at f, the composition*

$$\widetilde{\mathbf{EXP}} : \left(\mathbb{W}^*(1) \otimes \mathbb{B}_{\mathrm{dR}}\right)^{G_L} \overset{(-\times t) \bmod \mathrm{Fil}^1}{\longrightarrow} \left(\mathbb{W}^* \otimes \mathbb{B}_{\mathrm{dR}}/\mathrm{Fil}^1\right)^{G_L} \overset{\mathbf{EXP}_{\mathbb{W}^*(1)}}{\longrightarrow} H^1_{\mathcal{E}}\left(L, \mathbb{W}^*(1)\right)$$

becomes surjective.

This lemma is a straightforward consequence of the *Key Claim* made below, whose proof will occupy the remainder of this section. Consider the tautological exact sequence $0 \longrightarrow \mathrm{Fil}^1\mathbb{B}_{\mathrm{dR}} \overset{\alpha}{\longrightarrow} \mathbb{B}_{\mathrm{dR}} \overset{\beta}{\longrightarrow} \mathbb{B}_{\mathrm{dR}}/\mathrm{Fil}^1\mathbb{B}_{\mathrm{dR}} \longrightarrow 0$ over the closed disk $\mathbb{U}_{k_0}$.

Key Claim: For any integer $m \geq 0$, the cokernel of the induced map

$$\beta_*^{(0)} : \left(\mathbb{W}^*(m) \otimes \mathbb{B}_{\mathrm{dR}}\right)^{G_L} \longrightarrow \left(\mathbb{W}^*(m) \otimes \mathbb{B}_{\mathrm{dR}}/\mathrm{Fil}^1\right)^{G_L}$$

is a $\mathbb{Q}_p\langle\langle s\rangle\rangle$-torsion module; moreover there exists a polynomial $g_m(s) \in s \cdot \mathbb{Q}_p[s]$ which kills off the cokernel of $\beta_*^{(0)}$ after localisation.

Deferring its demonstration for the moment, let's see how to obtain 2.1.3 from it. Inverting by the polynomial $g_0(s)$, the localised sequence

$$\left(\left(\mathbb{W}^*(1)\otimes\mathbb{B}_{\mathrm{dR}}\right)^{G_L}\right)_{(g_0)} \overset{\sim}{\longrightarrow} \left(\left(\mathbb{W}^*\otimes\mathbb{B}_{\mathrm{dR}}\right)^{G_L}\right)_{(g_0)} \overset{\bmod \mathrm{Fil}^1}{\longrightarrow} \left(\left(\mathbb{W}^*\otimes\mathbb{B}_{\mathrm{dR}}/\mathrm{Fil}^1\right)^{G_L}\right)_{(g_0)}$$

is surjective on the right. Now $H^1_{\mathcal{E}}\left(L, \mathbb{W}^*(1)\right)/H^1_{\mathcal{E},0}\left(L, \mathbb{W}^*(1)\right)$ is a finitely-generated $\mathbb{Q}_p\langle\langle s\rangle\rangle$-torsion module, hence there exists $a(s) \in \mathbb{Q}_p[s]$ which kills off this quotient and thus also $\mathrm{Coker}\left(\widetilde{\mathbf{EXP}}_{(g_0)}\right)$; choosing $f(s) = g_0(s) \times a(s)$, clearly 2.1.3 follows.

N.B. Establishing that $\beta_*^{(0)}$ is locally surjective involves quite a tricky calculation. It's enough to prove that the kernel of the homomorphism

$$\alpha_*^{(1)} : H^1\left(L, \mathbb{W}^*(m) \otimes \mathrm{Fil}^1\mathbb{B}_{\mathrm{dR}}\right) \longrightarrow H^1\left(L, \mathbb{W}^*(m) \otimes \mathbb{B}_{\mathrm{dR}}\right)$$

is killed off after localising at some $g_m(s) \in s \cdot \mathbb{Q}_p[s]$.

(Were such a polynomial $g_m(s)$ to exist, then under the boundary mapping $\left(\mathbb{W}^*(m)\otimes\mathbb{B}_{\mathrm{dR}}/\mathrm{Fil}^1\right)^{G_L} \overset{\partial}{\longrightarrow} H^1\left(L, \mathbb{W}^*(m)\otimes\mathrm{Fil}^1\mathbb{B}_{\mathrm{dR}}\right)$ one knows $\mathrm{Im}(\partial) = \mathrm{Ker}\left(\alpha_*^{(1)}\right)$, therefore the image of ∂ would also be killed by g_m. After localisation at g_m, the kernel of ∂ would then consist of the whole of $\left(\mathbb{W}^*(m)\otimes\mathbb{B}_{\mathrm{dR}}/\mathrm{Fil}^1\right)^{G_L}\otimes \mathbb{Q}_p\langle\langle s\rangle\rangle[g_m^{-1}]$; since $\mathrm{Im}\left(\beta_*^{(0)}\right) = \mathrm{Ker}(\partial)$, it follows that $\mathrm{Coker}\left(\beta_*^{(0)}\right)$ is killed by g_m as required.)

A proof that $\mathrm{Ker}\big(\alpha_*^{(1)}\big) \otimes \mathbb{Q}_p\langle\langle s\rangle\rangle\big[g_m^{-1}\big]$ vanishes at some $g_m(s)$:
Let $\mathbf{x} \in \mathrm{Ker}\big(\alpha_*^{(1)}\big)$ – our goal is to show $\mathbf{x}$ trivialises after localisation at a $g_m(s)$. For each integer $n \geq 1$, there exists an open set $\mathcal{U}_n \subset \mathfrak{X}_{\mathrm{dR}}^+$ such that

$$\begin{array}{ccc} H^1\left(L, \mathbb{W}^*(m) \otimes \dfrac{\mathrm{Fil}^1\mathbb{B}_{\mathrm{dR}}}{\mathrm{Fil}^n\mathbb{B}_{\mathrm{dR}}}\right)_{s=k-1} & \overset{\alpha_*^{(1)} \bmod \mathrm{Fil}^n}{\longrightarrow} & H^1\left(L, \mathbb{W}^*(m) \otimes \dfrac{\mathbb{B}_{\mathrm{dR}}}{\mathrm{Fil}^n\mathbb{B}_{\mathrm{dR}}}\right)_{s=k-1} \\ \downarrow & & \downarrow \\ H^1\left(L, \mathbb{W}_k^*(m) \otimes_{\mathbb{Q}_p} \dfrac{\mathrm{Fil}^1 B_{\mathrm{dR}}}{\mathrm{Fil}^n B_{\mathrm{dR}}}\right) & \overset{\alpha_{k,*}^{(1)} \bmod \mathrm{Fil}^n}{\longrightarrow} & H^1\left(L, \mathbb{W}_k^*(m) \otimes_{\mathbb{Q}_p} \dfrac{B_{\mathrm{dR}}}{\mathrm{Fil}^n B_{\mathrm{dR}}}\right) \end{array}$$

is a commutative diagram with injective column arrows, at every weight $k \in \mathcal{U}_n$.

Remarks: (a) By Hypothesis(dR) each $\mathbb{W}_k^*(m)$ is de Rham for all $k \in \mathfrak{X}_{\mathrm{dR}}^+$, hence

$$\alpha_{k,*}^{(1)} : H^1\left(L, \mathbb{W}_k^*(m) \otimes_{\mathbb{Q}_p} \mathrm{Fil}^1 B_{\mathrm{dR}}\right) \longrightarrow H^1\left(L, \mathbb{W}_k^*(m) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}\right)$$

is injective by [BK, Lemma 3.8.1], and so too the bottom map in our diagram.

(b) This means $\mathbf{x} \bmod \mathrm{Fil}^n\mathbb{B}_{\mathrm{dR}}$ is divisible by infinitely many $\mathcal{P}_k \in \mathrm{Spec}\big(\mathbb{Q}_p\langle\langle s\rangle\rangle\big)$ i.e. it must be $\mathbb{Q}_p\langle\langle s\rangle\rangle$-torsion; consequently $\mathbf{x}$ lies in the submodule generated by both $H^1(L, \mathbb{W}^*(m) \otimes \mathrm{Fil}^n\mathbb{B}_{\mathrm{dR}})$ and $\mathrm{Tors}_{\mathbb{Q}_p\langle\langle s\rangle\rangle}\left(H^1\big(L, \mathbb{W}^*(m) \otimes \mathrm{Fil}^1\mathbb{B}_{\mathrm{dR}}\big)\right)$, in fact

$$\mathbf{x} \ \in \ H^1\big(L, \mathbb{W}^*(m) \otimes \mathrm{Fil}^n\mathbb{B}_{\mathrm{dR}}\big) + \mathrm{Tors}_{\mathbb{Q}_p\langle\langle s\rangle\rangle}\big(\mathrm{Im}(\partial)\big)$$

because $\mathbf{x} \in \mathrm{Ker}\big(\alpha_*^{(1)}\big) = \mathrm{Im}(\partial)$.

Lemma A.2. *(ii) For all integers $m \geq 0$, there exists $h_m(s) \in s \cdot \mathbb{Q}_p[s]$ such that*

$$\left(\mathbb{W}^*(m) \otimes \frac{\mathbb{B}_{\mathrm{dR}}}{\mathrm{Fil}^1\mathbb{B}_{\mathrm{dR}}}\right)^{G_L}_{(h_m)} \ \cong \ L\langle\langle s\rangle\rangle\big[h_m^{-1}\big]^{\oplus e_{0,0}(\mathbb{W})}$$

as an isomorphism of finitely-generated $\mathbb{Q}_p\langle\langle s\rangle\rangle[h_m^{-1}]$-torsion free modules.

The proof is given in the Appendix, together with various other local calculations.

Applying the above lemma, one sees immediately the image of the boundary mapping ∂ after localising at $h_m(s)$ must also be finitely-generated; in particular, the submodule $\mathrm{Tors}_{\mathbb{Q}_p\langle\langle s\rangle\rangle[h_m^{-1}]}\Big(\mathrm{Im}(\partial) \otimes \mathbb{Q}_p\langle\langle s\rangle\rangle[h_m^{-1}]\Big)$ will be of finite type over the ambient algebra, so is killed off by a polynomial $\widetilde{g}_m(s) \in \mathbb{Q}_p[s]$.

As a corollary, $\mathrm{Tors}_{\mathbb{Q}_p\langle\langle s\rangle\rangle}\big(\mathrm{Im}(\partial)\big)$ is killed off after localisation at the polynomial $g_m(s) := h_m(s) \times \widetilde{g}_m(s)$, therefore the image of $\mathbf{x}$ in $H^1\left(L, \mathbb{W}^*(m) \otimes \mathrm{Fil}^1\mathbb{B}_{\mathrm{dR}}\right)_{(g_m)}$ belongs to $H^1\left(L, \mathbb{W}^*(m) \otimes \mathrm{Fil}^n\mathbb{B}_{\mathrm{dR}}\right) \otimes \mathbb{Q}_p\langle\langle s\rangle\rangle\big[g_m^{-1}\big]$. Crucially g_m is independent of the index $n \geq 1$; allowing $n \to +\infty$, one discovers

$$\text{the image of } \mathbf{x} \ \in \ \mathbb{Q}_p\langle\langle s\rangle\rangle\big[g_m^{-1}\big] \otimes \Big(\bigcap_{n\geq 1} H^1\big(L, \mathbb{W}^*(m) \otimes \mathrm{Fil}^n\mathbb{B}_{\mathrm{dR}}\big)\Big) \ = \ \{0\}$$

via convergence of continuous cohomology, which completes the argument.

□

§§2.2 – A Duality Theorem over de Rham Points in $\mathbb{U}_{k_0}$

The invariant map of local class field theory furnishes us with a pairing

$$\big[-,-\big]_{K,\mathbb{W}} : H^1\big(K,\mathbb{W}\big) \times H^1\big(K,\mathbb{W}^*(1)\big) \stackrel{\cup}{\longrightarrow} H^2\big(K,\mathbb{Q}_p\langle\langle s\rangle\rangle(1)\big) \stackrel{\mathrm{inv}_K}{\longrightarrow} \mathbb{Q}_p\langle\langle s\rangle\rangle.$$

As the image of the exponential map provides a canonical subgroup $H^1_{\mathcal{E}}\big(K,\mathbb{W}^*(1)\big)$, we'll require a precise description for its orthogonal complement.

Definition 2.2.1. *The geometric part* $H^1_{\mathcal{G},\dagger}\big(K,\mathbb{W}\big)$ *is the* $\mathbb{Q}_p\langle\langle s\rangle\rangle$*-saturation of*

$$H^1_{\mathcal{G},\dagger,0}\big(K,\mathbb{W}\big) \ := \ \varinjlim_{\mathcal{U}\subset\mathfrak{X}^+_{\mathrm{dR}}} \ \mathrm{Ker}\Big(H^1\big(K,\mathbb{W}\big) \longrightarrow \bigoplus_{k\in\mathcal{U}} H^1\big(K,\mathbb{W}_k\otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}\big)\Big).$$

The direct limit above is taken with respect to inclusion, over the open subsets $\mathcal{U}$ in the cofinite topology on $\mathfrak{X}^+_{\mathrm{dR}}$. In fact $H^1_{\mathcal{G},\dagger}$ coincides with the $\mathbb{Q}_p\langle\langle s\rangle\rangle$-saturation of the kernel of $H^1\big(K,\mathbb{W}\big)\longrightarrow\bigoplus_{k\in\mathfrak{X}^+_{\mathrm{dR}}} H^1\big(K,\mathbb{W}_k\otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}\big)$, but we won't bother to prove this assertion here.

Theorem 2.2.2. *Under* $\big[-,-\big]_{K,\mathbb{W}}$ *the subgroups* $H^1_{\mathcal{G},\dagger}\big(K,\mathbb{W}\big)$ *and* $H^1_{\mathcal{E}}\big(K,\mathbb{W}^*(1)\big)$ *are exact annihilators of each other.*

Proof: The argument is a refinement of [DS, Prop 1]; however the latter proof was incomplete as it only showed the vanishing of $\big[H^1_{\mathcal{G}},H^1_{\mathcal{E}}\big]_{K,\mathbb{W}}$, so we expand on this.

Recall $H^1\big(K,\mathbb{W}\big)$ and $H^1\big(K,\mathbb{W}^*(1)\big)$ were of finite-type over $\mathbb{Q}_p\langle\langle s\rangle\rangle$, hence their $\mathbb{Q}_p\langle\langle s\rangle\rangle$-torsion submodules can be annihilated by localising at a polynomial $t(s)$. Setting $q(s) := t(s)\times f(s)$ where $f(s)\in s\cdot\mathbb{Q}_p\langle\langle s\rangle\rangle$ was the element in Lemma 2.1.3 which made the map $\widetilde{\mathbf{EXP}}$ surjective, the localised version

$$H^1\big(K,\mathbb{W}\big)_{(q)} \ \times \ H^1\big(K,\mathbb{W}^*(1)\big)_{(q)} \stackrel{\cup}{\longrightarrow} H^2\big(K,\mathbb{Q}_p\langle\langle s\rangle\rangle(1)\big)_{(q)} \stackrel{\mathrm{inv}_K}{\longrightarrow} \mathbb{Q}_p\langle\langle s\rangle\rangle\big[q^{-1}\big]$$

becomes a non-degenerate pairing of torsion free $\mathbb{Q}_p\langle\langle s\rangle\rangle_{(q)}$-modules.

Remarks: (a) The demonstration now reduces to establishing that $H^1_{\mathcal{G},\dagger}\big(K,\mathbb{W}\big)_{(q)}$ and $H^1_{\mathcal{E}}\big(K,\mathbb{W}^*(1)\big)_{(q)}$ are exact orthogonal complements under $\mathrm{inv}_K\circ\cup$.

(b) It's enough to prove the equivalent statement over $L = K_{n(\mathbb{W})}$, then use the Galois equivariance of the pairing $\big[-,-\big]_{L,\mathbb{W}}$ to deduce it over the ground field K.

Tensoring the fundamental exact sequence

$$0 \longrightarrow \mathbb{Q}_p\langle\langle s\rangle\rangle(1) \longrightarrow \mathbb{B}^{\varphi=p}_{\mathrm{cris}}\oplus\mathrm{Fil}^1\mathbb{B}_{\mathrm{dR}} \longrightarrow \mathbb{B}_{\mathrm{dR}} \longrightarrow 0$$

by $\mathbb{W}\otimes_{\mathbb{Q}_p\langle\langle s\rangle\rangle}\mathbb{W}^*$ and then taking G_L-invariants, yields a boundary homomorphism $\epsilon : H^1\big(L,\mathbb{W}\otimes\mathbb{W}^*\otimes\mathbb{B}_{\mathrm{dR}}\big)\longrightarrow H^1\big(L,\mathbb{W}\otimes\mathbb{W}^*(1)\big)$. The commutative diagram

$$\begin{array}{ccc} H^1\big(L,\mathbb{W}\big)_{(q)} \ \times \ \Big(\mathbb{W}^*(1)\otimes\mathbb{B}_{\mathrm{dR}}\Big)^{G_L}_{(q)} & \stackrel{\big(\mathrm{id},\widetilde{\mathbf{EXP}}\big)}{\longrightarrow} & H^1\big(L,\mathbb{W}\big)_{(q)}\times H^1\big(L,\mathbb{W}^*(1)\big)_{(q)} \\ \big\downarrow{\scriptstyle (-\otimes 1,\ \times t)} & & \big\downarrow{\scriptstyle \cup} \\ H^1\big(L,\mathbb{W}\otimes\mathbb{B}_{\mathrm{dR}}\big)_{(q)}\times H^0\big(L,\mathbb{W}^*\otimes\mathbb{B}_{\mathrm{dR}}\big)_{(q)} & \stackrel{\det(\varepsilon)\circ\cup}{\longrightarrow} & H^2\big(L,\mathbb{Q}_p\langle\langle s\rangle\rangle\big[q^{-1}\big](1)\big). \end{array}$$

therefore gives us an alternative way of computing the local pairing $\big[-,-\big]_{L,\mathbb{W}}$.

In other words, for localised elements $\mathbf{x} \in H^1\big(L, \mathbb{W}\big)_{(q)}$ and $\mathbf{y} \in \big(\mathbb{W}^*(1) \otimes \mathbb{B}_{\mathrm{dR}}\big)^{G_L}_{(q)}$ one has the formula $\big[\mathbf{x}\ ,\ \widetilde{\mathbf{EXP}}(\mathbf{y})\big]_{L,\mathbb{W}} = \ \mathrm{inv}_L \circ \det\big(\epsilon(\mathbf{x} \otimes 1 \cup t \cdot \mathbf{y})\big)$.

Lemma A.3. *Under the local pairing*

$$H^1\big(L, \mathbb{W} \otimes \mathbb{B}_{\mathrm{dR}}[s^{-1}]\big) \times H^0\big(L, \mathbb{W}^* \otimes \mathbb{B}_{\mathrm{dR}}[s^{-1}]\big) \ \stackrel{\det(\varepsilon)\circ\cup}{\longrightarrow}\ H^2\big(L, \mathbb{Q}_p\langle\langle s\rangle\rangle\big[s^{-1}\big](1)\big)$$

an element $\mathbf{x} \in H^1\big(L, \mathbb{W}\otimes\mathbb{B}_{\mathrm{dR}}[s^{-1}]\big)$ *is orthogonal to all of* $H^0\big(L, \mathbb{W}^* \otimes \mathbb{B}_{\mathrm{dR}}[s^{-1}]\big)$, *if and only if*

$$\mathbf{x} \in \ker^+_{\mathrm{dR}} \ := \ \varinjlim_{\mathcal{U}\subset\mathfrak{X}^+_{\mathrm{dR}}} \ \mathrm{Ker}\Big(H^1\big(L, \mathbb{W} \otimes \mathbb{B}_{\mathrm{dR}}[s^{-1}]\big) \longrightarrow \bigoplus_{k\in\mathcal{U}} H^1\big(L, \mathbb{W}_k \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}\big)\Big).$$

The demonstration is fairly tedious, and is supplied at the end of the Appendix.

The remainder of the proof for 2.2.2 is a consequence of the computation below. For every $\mathbf{x} \in H^1\big(L, \mathbb{W}\big)_{(q)}$ one calculates that

$$\begin{aligned}\Big[\mathbf{x}\ ,\ H^1_{\mathcal{E}}\big(K, \mathbb{W}^*(1)\big)_{(q)}\Big]_{L,\mathbb{W}} &\stackrel{\text{by } 2.1.3}{=} \Big[\mathbf{x}\ ,\ \mathrm{Im}\big(\mathbf{EXP}_{\mathbb{W}^*(1)}\big)_{(q)}\Big]_{L,\mathbb{W}} \\ &= \ \Big[\mathbf{x}\ ,\ \widetilde{\mathbf{EXP}}\big(H^0(L, \mathbb{W}^*(1) \otimes \mathbb{B}_{\mathrm{dR}})\big)_{(q)}\Big]_{L,\mathbb{W}} \\ &= \ \mathrm{inv}_L \circ \det\Big(\epsilon\Big(\mathbf{x} \otimes 1 \cup t \cdot H^0(L, \mathbb{W}^*(1) \otimes B_{\mathrm{dR}})_{(q)}\Big)\Big)\end{aligned}$$

is zero, if and only if $\mathbf{x}\otimes 1$ lies in $\Big(\ker^+_{\mathrm{dR}}\Big) \otimes_{\mathbb{Q}_p\langle\langle s\rangle\rangle[s^{-1}]} \mathbb{Q}_p\langle\langle s\rangle\rangle\big[q^{-1}\big]$ via Lemma A.3; the latter condition is equivalent to $\mathbf{x} \in H^1_{\mathcal{G},\dagger}(L, \mathbb{W})_{(q)}$, which of course means that $H^1_{\mathcal{G},\dagger}(L, \mathbb{W}) = H^1_{\mathcal{E}}\big(L, \mathbb{W}^*(1)\big)^{\perp}$ as asserted.

Conversely, since $H^1_{\mathcal{G},\dagger}(L, \mathbb{W})$ and $H^1_{\mathcal{E}}\big(L, \mathbb{W}^*(1)\big)$ are both $\mathbb{Q}_p\langle\langle s\rangle\rangle$-saturated in their respective cohomologies, $H^1_{\mathcal{G},\dagger}(L, \mathbb{W})^{\perp} = \big(H^1_{\mathcal{E}}\big(L, \mathbb{W}^*(1)\big)^{\perp}\big)^{\perp} = H^1_{\mathcal{E}}\big(L, \mathbb{W}^*(1)\big)$ which completes the proof of the theorem.

□

As Joel Bellaiche pointed out to us, the original definition in [DS] of the geometric subgroup '$H^1_{\mathcal{G}}(K, \mathbb{W})$' was, in general, too strong to obtain a correct orthogonality. By using instead local conditions imposed at a dense set of de Rham weights ≥ 2, the argument is easily repaired and yields a nice duality result.

In particular, the following computes the rank of these modules via specialisation.

Corollary 2.2.3. *There exists an open set* $\mathcal{U} \subset \mathfrak{X}^+_{\mathrm{dR}}$ *such that*

(i) $\mathrm{rank}_{\mathbb{Q}_p\langle\langle s\rangle\rangle}\Big(H^1_{\mathcal{G},\dagger}\big(L, \mathbb{W}\big)\Big) = \ \dim_{\mathbb{Q}_p}\Big(H^1_g\big(K, \mathbb{W}_k\big)\Big)$ *at every* $k \in \mathcal{U}$;

(ii) $\mathrm{rank}_{\mathbb{Q}_p\langle\langle s\rangle\rangle}\Big(H^1_{\mathcal{E}}\big(L, \mathbb{W}^*(1)\big)\Big) = \ \dim_{\mathbb{Q}_p}\Big(H^1_e\big(K, \mathbb{W}^*_k(1)\big)\Big)$ *at every* $k \in \mathcal{U}$.

Proof: See for instance [De, p104].

□

§§**2.3 – Examples Arising from Modular Forms**

Let E be a finite extension of $\mathbb{Q}_p$. Henceforth we shall consider two-dimensional $\mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$-representations $\mathbb{V}$ defined over $E\langle\langle s\rangle\rangle$ (which, one recalls, denoted the affinoid E-algebra of the disk $\mathbb{U}_{k_0} \subset \mathbb{Z}_p$ centred on a fixed base weight $k_0 \geq 2$). Assume that $\bigwedge^2 \mathbb{V} \cong E\langle\langle s\rangle\rangle\big(\epsilon\chi_{\mathrm{cy}}^{k_0-1}\langle\chi_{\mathrm{cy}}\rangle^{s+1-k_0}\big)$ for some finite order character ϵ, and moreover

$$\mathbb{V}_k \otimes_E \mathbb{C}_p \;\cong\; \mathbb{C}_p \oplus \mathbb{C}_p(k-1)$$

which means $\mathbb{V}$ has Hodge-Tate weights lying on the lines $s = 0$ and $s = k-1$. Because we are mainly interested in studying those $\mathbb{V}$'s interpolating families of modular forms, one further insists at almost all $k \in \mathbb{U}_{k_0} \cap \mathbb{Z}_{\geq 2}$:

$$\mathbb{V}_k \cong \mathrm{Hom}\big(V(f_{k,\epsilon}), E\big) \quad \text{for some classical eigenform } f_{k,\epsilon} \text{ of type } \big(k, \epsilon\omega^{k_0-k}\big).$$

Let's first determine the structure of the deformed tangent space associated to $\mathbb{V}$.

Proposition 2.3.1. *(i) The representation* $\mathrm{Res}_{E/\mathbb{Q}_p}(\mathbb{V})$ *satisfies Hypothesis*(dR)*;*
(ii) For all $i \geq 1$, $H^0\Big(K, \mathbb{V}(i) \otimes_{E\langle\langle s\rangle\rangle} \mathbb{B}_{\mathrm{dR}}/\mathrm{Fil}^0\mathbb{B}_{\mathrm{dR}}\Big)$ *is locally free of rank* $[K:\mathbb{Q}_p]$.

Proof: These calculations make frequent use of Sen's local representation theory, which is extensively applied (over multi-variable Tate algebras) throughout [BC,Ki]. The crux of the argument hinges on establishing a

Key Fact: There exists $\mathcal{F}_{\mathbb{V}}(s) \in \mathbb{Q}_p[s]$ such that $\Big(\mathbb{V}_{(\mathcal{F}_{\mathbb{V}})} \otimes_{E\langle\langle s\rangle\rangle} \mathbb{C}_p\langle\langle s\rangle\rangle\Big)^{G_K}$ becomes a free module of rank $[K:\mathbb{Q}_p]$ over the localised algebra $\mathbb{Q}_p\langle\langle s\rangle\rangle\big[1/\mathcal{F}_{\mathbb{V}}\big]$.

Deferring its proof for the moment, we quickly explain how 2.3.1(ii) follows directly from this claim. Fixing an integer $i \geq 1$,

$$H^0\Big(K, \mathbb{V}_{(\mathcal{F}_{\mathbb{V}})}(i) \otimes_{E\langle\langle s\rangle\rangle} \mathrm{Fil}^{-i}\mathbb{B}_{\mathrm{dR}}/\mathrm{Fil}^{1-i}\mathbb{B}_{\mathrm{dR}}\Big) \;\cong\; \Big(\mathbb{V}_{(\mathcal{F}_{\mathbb{V}})} \otimes_{E\langle\langle s\rangle\rangle} \mathbb{C}_p\langle\langle s\rangle\rangle\Big)^{G_K}$$

which is a free $\mathbb{Q}_p\langle\langle s\rangle\rangle\big[1/\mathcal{F}_{\mathbb{V}}\big]$-module of rank $[K:\mathbb{Q}_p]$, courtesy of the 'Key Fact'. Assume by induction on $j \geq i$ that $H^0\big(K, \mathbb{V}_{(\mathcal{F}_{\mathbb{V}})}(i) \otimes_{E\langle\langle s\rangle\rangle} \mathrm{Fil}^{-j}\mathbb{B}_{\mathrm{dR}}/\mathrm{Fil}^{1-i}\mathbb{B}_{\mathrm{dR}}\big)$ is $\mathbb{Q}_p\langle\langle s\rangle\rangle\big[1/\mathcal{F}_{\mathbb{V}}\big]$-free of rank equal to $[K:\mathbb{Q}_p]$. Tensoring the exact sequence

$$0 \longrightarrow \mathrm{Fil}^{-j}\mathbb{B}_{\mathrm{dR}}/\mathrm{Fil}^{1-i}\mathbb{B}_{\mathrm{dR}} \longrightarrow \mathrm{Fil}^{-j-1}\mathbb{B}_{\mathrm{dR}}/\mathrm{Fil}^{1-i}\mathbb{B}_{\mathrm{dR}} \longrightarrow \mathbb{C}_p\langle\langle s\rangle\rangle\big(\chi_{\mathrm{cy}}^{-j-1}\big) \longrightarrow 0$$

over $E\langle\langle s\rangle\rangle$ by $\mathbb{V}_{(\mathcal{F}_{\mathbb{V}})}(i)$ and taking its $\mathrm{Gal}\big(\overline{K}/K\big)$-invariants, yields a corresponding long exact sequence in G_K-cohomology.

Remark: The term $H^0\big(K, \mathbb{V}_{(\mathcal{F}_{\mathbb{V}})}(i) \otimes_{E\langle\langle s\rangle\rangle} \mathbb{C}_p\langle\langle s\rangle\rangle(-j-1)\big)$ vanishes as $i-j-1 \neq 0$, hence one obtains an isomorphism of free $\mathbb{Q}_p\langle\langle s\rangle\rangle\big[1/\mathcal{F}_{\mathbb{V}}\big]$-modules

$$\begin{aligned} &H^0\big(K, \mathbb{V}_{(\mathcal{F}_{\mathbb{V}})}(i) \otimes_{E\langle\langle s\rangle\rangle} \mathrm{Fil}^{-j}\mathbb{B}_{\mathrm{dR}}/\mathrm{Fil}^{1-i}\mathbb{B}_{\mathrm{dR}}\big) \\ &\qquad \xrightarrow{\sim} H^0\big(K, \mathbb{V}_{(\mathcal{F}_{\mathbb{V}})}(i) \otimes_{E\langle\langle s\rangle\rangle} \mathrm{Fil}^{-j-1}\mathbb{B}_{\mathrm{dR}}/\mathrm{Fil}^{1-i}\mathbb{B}_{\mathrm{dR}}\big) \quad \text{for } j > i-1, \end{aligned}$$

with both sharing a common rank $[K:\mathbb{Q}_p]$. Passing to the direct limit as $j \to \infty$, clearly we can replace $\mathrm{Fil}^{-j}\mathbb{B}_{\mathrm{dR}}$ above with the full period ring $\mathbb{B}_{\mathrm{dR}}$.

(Lastly, an identical simple induction on the index $m = 1-i$ in $\mathbb{B}_{\mathrm{dR}}/\mathrm{Fil}^{1-i}\mathbb{B}_{\mathrm{dR}}$, this time exploiting the short exact sequence

$$0 \longrightarrow \mathrm{Fil}^m\mathbb{B}_{\mathrm{dR}}/\mathrm{Fil}^{m+1}\mathbb{B}_{\mathrm{dR}} \longrightarrow \mathbb{B}_{\mathrm{dR}}/\mathrm{Fil}^{m+1} \overset{\mathrm{mod\ Fil}^m}{\longrightarrow} \mathbb{B}_{\mathrm{dR}}/\mathrm{Fil}^m \longrightarrow 0$$

allows us to descend from $\cdots/\mathrm{Fil}^{1-i}\mathbb{B}_{\mathrm{dR}}$ back down to $\cdots/\mathrm{Fil}^0\mathbb{B}_{\mathrm{dR}}$ in $(i-1)$-steps.)

Establishing the 'Key Fact'.
This is just a special case of [Ki, Prop 2.4] so we shall briefly sketch the argument. For simplicity, assume the coefficients $E \subset K$ (if not, work with $\mathrm{Res}_{E/\mathbb{Q}_p}\mathbb{V}$ instead). Without loss of generality, we also suppose that $\mathbb{U}_{k_0} = \mathbb{Z}_p$ which can always be achieved by an affine translation, followed by a scaling transformation.

Set $M = \mathbb{V} \otimes_{E\langle\langle s\rangle\rangle} \mathbb{C}_p\langle\langle s\rangle\rangle$ and write K_∞ for the cyclotomic $\mathbb{Z}_p$-extension of K, with corresponding Galois groups $\Gamma_K = \mathrm{Gal}\big(K_\infty/K\big)$ and $H_K = \mathrm{Gal}\big(\overline{K}/K_\infty\big)$. From [Se, Prop 6] there is an extension K_n/K inside K_∞ of degree $[K_n : K] = p^n$, such that $W_\infty := M^{H_K}$ is free of $K_\infty\langle\langle s\rangle\rangle$-rank two, and

$$W_\infty \otimes_{K_\infty\langle\langle s\rangle\rangle} \mathbb{C}_p\langle\langle s\rangle\rangle \ \cong\ M \quad \text{as an isomorphism of } \mathrm{Gal}\big(\overline{K}/K_n\big)\text{-modules.}$$

Enlarging the extension K_n if necessary, one can then find a $K_\infty\langle\langle s\rangle\rangle$-basis for W_∞ such that the free $K_n\langle\langle s\rangle\rangle$-submodule '$W$' which it generates is Γ_{K_n}-stable.

Remarks: (a) The operator $\Phi := \frac{\log \gamma^{p^{n'}}}{\log \chi_{\mathrm{cy}}\left(\gamma^{p^{n'}}\right)} \in \mathrm{End}_{K_n\langle\langle s\rangle\rangle}\big(W\big)$ where $\overline{<\gamma>} = \Gamma_K$, is easily seen to be independent of $n' \geq n$. The characteristic polynomial of $\Phi\Big|_W$ is $\mathcal{P}_{\mathrm{Sen},W}(\Phi, X) = X(X-s)$ since each $\mathbb{V}_k$ has Hodge-Tate weights 0 and $k-1$.

(b) As a corollary, the localisation $W^\natural = W \otimes_{K_n\langle\langle s\rangle\rangle} K_n\langle\langle s\rangle\rangle\big[1/s\big]$ decomposes into two separate Φ-eigenspaces, i.e. one has a splitting

$$W^\natural \ = \ W^\natural\Big|_{\Phi=0} \oplus W^\natural\Big|_{\Phi=s} \quad \text{as a direct sum of free } K_n\langle\langle s\rangle\rangle\big[1/s\big]\text{-modules.}$$

The action of $\gamma^{p^n} \in \Gamma_{K_n}$ on W is via the formula $\gamma^{p^n}\Big|_W = \exp\Big(\Phi \log \chi_{\mathrm{cy}}\big(\gamma^{p^n}\big)\Big)$; therefore γ^{p^n} acts trivially on the first summand of $W^\natural$, and through $\Psi(\gamma)^{p^n}$ on the second summand of $W^\natural$.

(c) Extending scalars on $W^\natural$ all the way to $\mathbb{C}_p\langle\langle s\rangle\rangle\big[1/s\big]$, then as G_{K_n}-modules

$$\mathbb{V} \otimes_{E\langle\langle s\rangle\rangle} \mathbb{C}_p\langle\langle s\rangle\rangle\big[1/s\big] \ \cong\ \mathbb{C}_p\langle\langle s\rangle\rangle\big[1/s\big] \ \oplus\ \mathbb{C}_p\langle\langle s\rangle\rangle\big[1/s\big] \otimes \Psi$$

i.e. $\mathrm{Res}_{E/\mathbb{Q}_p}(\mathbb{V})$ satisfies Hypothesis(dR)(i). To check it also obeys the second requirement needed in Hypothesis(dR), we simply point out that $\mathbb{V}_k \cong V(f_{k,\epsilon})^*$ when $k \in \mathbb{U}_{k_0} \cap \mathbb{Z}_{\geq 2}$, and $V(f_{k,\epsilon})$ is well known to be de Rham.

Now taking Galois invariants and observing that $H^0\big(K_n, \mathbb{C}_p\langle\langle s\rangle\rangle(\Psi)\big)$ vanishes, we may then conclude $\big(\mathbb{V} \otimes_{E\langle\langle s\rangle\rangle} \mathbb{C}_p\langle\langle s\rangle\rangle\big[1/s\big]\big)^{G_{K_n}}$ is free of $K_n\langle\langle s\rangle\rangle\big[1/s\big]$-rank one. The semi-linear action of $\mathrm{Gal}\big(K_n/K\big)$ is given by some one-cocycle x_n, thus we can choose $\mathcal{G}(s) \in K_n\langle\langle s\rangle\rangle$ so that the image of $x_n \in H^1\big(\mathrm{Gal}(K_n/K), K_n\langle\langle s\rangle\rangle^\times\big)$ under the map $H^1\big(\mathrm{Gal}(K_n/K), K_n\langle\langle s\rangle\rangle^\times\big) \longrightarrow H^1\big(\mathrm{Gal}(K_n/K), K_n\langle\langle s\rangle\rangle[1/\mathcal{G}]^\times\big)$ is zero.

By the Weierstrass preparation theorem, one may assume $\mathcal{G}$ is a polynomial in s (of course its coefficients lie in K_n). Defining $\mathcal{F}_{\mathbb{V}}(s) := s \times \mathrm{Norm}_{K_n/\mathbb{Q}_p}\big(\mathcal{G}(s)\big)$ then $\mathcal{F}_{\mathbb{V}}$ automatically has $\mathbb{Q}_p$-coefficients, with the requisite property

$$H^0\left(\mathrm{Gal}\big(K_n/K\big), \left(\mathbb{V}\otimes_{E\langle\langle s\rangle\rangle}\mathbb{C}_p\langle\langle s\rangle\rangle\big[1/\big(s\cdot \mathrm{N}_{K_n/\mathbb{Q}_p}\mathcal{G}\big)\big]\right)^{G_{K_n}}\right) \;\cong\; K\langle\langle s\rangle\rangle\big[1/\mathcal{F}_{\mathbb{V}}\big].$$

The right-hand side above is certainly $\mathbb{Q}_p\langle\langle s\rangle\rangle\big[1/\mathcal{F}_{\mathbb{V}}\big]$-free of rank equal to $[K:\mathbb{Q}_p]$, from which our 'Key Fact' follows readily.

□

In the specific context of ordinary representations over (CNL) deformation rings, Greenberg [Gr] gives a simpler definition for a p-local condition in his Selmer group; more precisely, he takes the saturation of those one-cocycles which trivialise after passing to the maximal unramified quotient.

Given that these deformation rings behave (locally!) like affinoid algebras, both Definitions 2.1.1 and 2.2.1 will generalise in an obvious fashion to such a situation. It is worthwhile asking whether the condition $H^1_{\mathcal{G},\dagger}(-,-)$ at primes above p agrees with the local condition used by Greenberg? In the special case of two-dimensional representations we shall shortly discover the answer is 'Yes', although we conjecture that it holds in dimension greater than two as well.

The Universal p-Ordinary Galois Representation.
Assume now that our prime $p \geq 5$, and write $\Lambda^{\mathrm{wt}} = \mathbb{Z}_p\big[\big[\Gamma^{\mathrm{wt}}\big]\big]$ for the weight algebra where $\Gamma^{\mathrm{wt}} \cong 1 + p\mathbb{Z}_p$. Let $\mathbb{T}_\infty$ denote the universal ordinary Galois representation of tame level $N \geq 1$, so that $\mathbb{T}_\infty$ is cut out of $\varprojlim_{r\geq 1} H^1_{\text{ét}}\big(X_1(Np^r)\otimes_{\mathbb{Q}}\overline{\mathbb{Q}}\;,\;\mathbb{Z}_p(1)\big)$ using Hida's p-ordinary and N-primitive projectors $\mathbf{e}_{\mathrm{ord}}, \mathbf{e}_{\mathrm{prim}}$ (e.g. see [Hi, §3]).

The abstract Hecke algebra acts on $\mathbb{T}_\infty$ through a local factor $\mathcal{R}$, which is a finite and flat extension of Λ^{wt}. As a local $G_{\mathbb{Q}_p}$-representation, there is a filtration

$$0 \longrightarrow \mathrm{Fil}^+\mathbb{T}_\infty \longrightarrow \mathbb{T}_\infty \longrightarrow \mathbb{T}_\infty/\mathrm{Fil}^+\mathbb{T}_\infty \longrightarrow 0$$

of $\mathcal{R}$-modules; the right-hand quotient term is unramified, and isomorphic to the dualising module $\omega_{\mathcal{R}} = \mathrm{Hom}\big(\mathcal{R}, \Lambda^{\mathrm{wt}}\big)$ if one ignores the Galois action.

Theorem 2.3.2. *For $i \geq 1$, the module $H^1_{\mathcal{G},\dagger}\big(K, \mathbb{T}_\infty(1-i)\big)$ is the $\mathcal{R}$-saturation of*

$$\mathrm{Im}\Big(H^1\big(K, \mathrm{Fil}^+\mathbb{T}_\infty(1-i)\big) \longrightarrow H^1\big(K, \mathbb{T}_\infty(1-i)\big)\Big),$$

and is of rank $[K:\mathbb{Q}_p]$ over the deformation ring $\mathcal{R}$.

This result fails if the integer $i < 1$ since $H^1_{\mathcal{G},\dagger}\big(K, \mathbb{T}_\infty(1-i)\big)$ is an $\mathcal{R}$-torsion module, whilst the image of $H^1\big(K, \mathrm{Fil}^+\mathbb{T}_\infty(1-i)\big)$ always has rank $[K:\mathbb{Q}_p]$.

Under the restriction that $i = 1$, it was shown in [DS, Thm 3] that the module $H^1_{\mathcal{G},\dagger}(K, \mathbb{T}_\infty)$ can be further identified with the $\mathcal{R}$-saturation of

$$\mathbf{e}_{\mathrm{prim}}\cdot\left(\left(\mathbf{e}_{\mathrm{ord}}\cdot\varprojlim_{r,m}\partial_{r,m}\Big(J_\infty(K)\widehat{\otimes}\mathbb{Z}_p\Big)\right)\otimes_\Lambda \mathrm{Frac}(\Lambda)\right)\cap H^1\big(K, \mathbb{T}_\infty\big)$$

where $\partial_{r,m} : \mathrm{jac}\big(X_1(Np^r)\big)(K)/p^m \longrightarrow H^1(K, \mathrm{jac}\big(X_1(Np^r)\big)[p^m])$ denotes the Kummer map on the Jacobian at level Np^r, and $J_\infty := \varprojlim_{r\geq 1}\mathrm{jac}\big(X_1(Np^r)\big)_{/\mathbb{Q}}$.

Proof of Theorem 2.3.2:
Let $\Omega_{\mathbb{T}_\infty(1-i)}$ denote the image of $H^1\big(K, \mathrm{Fil}^+\mathbb{T}_\infty(1-i)\big)$ inside of $H^1\big(K, \mathbb{T}_\infty(1-i)\big)$. Then the result will follow, provided one can establish three assertions:

(A) $\Omega_{\mathbb{T}_\infty(1-i)} \ \subset \ H^1_{\mathcal{G},\dagger}\big(K, \mathbb{T}_\infty(1-i)\big)$;

(B) $\mathrm{rank}_{\mathcal{R}}\left(\Omega_{\mathbb{T}_\infty(1-i)}\right) \ = \ [K:\mathbb{Q}_p]$;

(C) $\mathrm{rank}_{\mathcal{R}}\left(H^1_{\mathcal{G},\dagger}\big(K, \mathbb{T}_\infty(1-i)\big)\right) \ \leq \ [K:\mathbb{Q}_p]$.

To prove (A) we first recall that for a fixed base weight $k_0 \in \mathbb{Z}_{\geq 2}$ and character ϵ, there exists a (unique) Mellin transform $\mu_{k_0,\epsilon} : \mathcal{R} \hookrightarrow \widetilde{\mathcal{R}}^{\mathrm{norm}} \longrightarrow \overline{\mathbb{Q}}_p\langle\langle s\rangle\rangle$ extending the mapping which sends $\tau \in \mathbb{Z}_{p,N}^\times$ to the Iwasawa function $k \mapsto \epsilon\omega^{k_0}(\tau) < \tau^k >$. The image $\mu_{k_0,\epsilon}(\mathcal{R})$ consists of rigid analytic functions convergent on some disk $\mathbb{U}_{k_0}$ (since $\mathcal{R}$ is of finite-type over Λ^{wt}), with coefficients in a field $E = E(k_0,\epsilon)$ say.

Suppose that the big Galois representation $\rho_\infty : G_{\mathbb{Q}} \longrightarrow \mathrm{Aut}_{\mathcal{R}}(\mathbb{T}_\infty)$ is residually absolutely irreducible, so that $\mathcal{R}$ is Gorenstein and $\mathbb{T}_\infty/\mathrm{Fil}^+\mathbb{T}_\infty$ is free of rank one. At primes $\widetilde{P_{k,\epsilon}} \in \mathrm{Spec}\big(\widetilde{\mathcal{R}}^{\mathrm{norm}}\big)$ with $k \in \mathbb{U}_{k_0} \cap \mathbb{Z}_{\geq 2}$, there are commutative diagrams

$$\begin{array}{ccccccc} 0 \longrightarrow H^1\big(K, \mathrm{Fil}^+\mathbb{T}_\infty(1-i)\big) & \longrightarrow & H^1\big(K, \mathbb{T}_\infty(1-i)\big) & \longrightarrow & H^1\big(K, \mathbb{T}_\infty/\mathrm{Fil}^+(1-i)\big) \\ \big\downarrow \otimes_{\mathcal{R}} \widetilde{\mathcal{R}}^{\mathrm{norm}}/\widetilde{P_{k,\epsilon}} & & \big\downarrow \otimes_{\mathcal{R}} \widetilde{\mathcal{R}}^{\mathrm{norm}}/\widetilde{P_{k,\epsilon}} & & \big\downarrow \otimes_{\mathcal{R}} \widetilde{\mathcal{R}}^{\mathrm{norm}}/\widetilde{P_{k,\epsilon}} \\ H^1\big(K, \mathrm{Fil}^+\mathbb{V}_{k,\epsilon}(1-i)\big) & \longrightarrow & H^1\big(K, \mathbb{V}_{k,\epsilon}(1-i)\big) & \longrightarrow & H^1\big(K, \mathbb{V}_{k,\epsilon}/\mathrm{Fil}^+(1-i)\big) \end{array}$$

where the $G_{\mathbb{Q}_p}$-representations $\mathbb{V}_{k,\epsilon} := \mu_{k_0,\epsilon}(\mathbb{T}_\infty)\Big|_{s=k-1}$ are each of E-dimension 2 (note that $H^0\big(K, \mathbb{T}_\infty/\mathrm{Fil}^+(1-i)\big)$ vanishes under our assumption).

It follows that the specialisation $\big(\mu_{k_0,\epsilon}\big)_*\big(\Omega_{\mathbb{T}_\infty(1-i)}\big) \otimes_{s\mapsto k-1} E$ has zero image inside $H^1\big(K, \mathbb{V}_{k,\epsilon}/\mathrm{Fil}^+(1-i)\big)$, whence

$$\big(\mu_{k_0,\epsilon}\big)_*\big(\Omega_{\mathbb{T}_\infty(1-i)}\big)\Big|_{s=k-1} \ \subset \ \mathrm{Im}\Big(H^1\big(K, \mathrm{Fil}^+\mathbb{V}_{k,\epsilon}(1-i)\big) \longrightarrow H^1\big(K, \mathbb{V}_{k,\epsilon}(1-i)\big)\Big)$$

by the exactness of the bottom row in our diagram. If the weight $k \in \mathbb{Z}_{\geq i+1} \cap \mathbb{U}_{k_0}$, one can further surmise

$$\mathrm{Im}\Big(H^1\big(K, \mathrm{Fil}^+\mathbb{V}_{k,\epsilon}(1-i)\big) \to H^1\big(K, \mathbb{V}_{k,\epsilon}(1-i)\big)\Big) \ = \ H^1_g\big(K, \mathbb{V}_{k,\epsilon}(1-i)\big)$$

as $\mathbb{V}_{k,\epsilon}(1-i) \cong V\big(f^*_{k,\epsilon}\big)(k-i)$ is semistable, and satisfies a 'Panchiskin condition'. As a direct consequence, $\big(\mu_{k_0,\epsilon}\big)_*\big(\Omega_{\mathbb{T}_\infty(1-i)}\big) \otimes 1$ has trivial image in

$$\bigoplus_{k\in\mathcal{U}} H^1\Big(K, \ \mu_{k_0,\epsilon}(\mathbb{T}_\infty)(1-i)\Big|_{s=k-1} \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}\Big) \quad \text{once the weights } \mathcal{U} \subset \mathbb{Z}_{\geq i+1}.$$

Since $\mathbb{T}_\infty$ is completely covered by these affinoid representations $\mathbb{V} = \mu_{k_0,\epsilon}\big(\mathbb{T}_\infty\big)$, then taking the direct limit over subsets $\mathcal{U} \subset \mathfrak{X}^+_{\mathrm{dR}}$, assertion (A) is established.

Remark: If $\rho_\infty : G_{\mathbb{Q}} \to \mathrm{Aut}_{\mathcal{R}}(\mathbb{T}_\infty)$ is residually reducible, $H^0\big(K, \mathbb{T}_\infty/\mathrm{Fil}^+(1-i)\big)$ might not vanish at $i = 1$. However this H^0 will certainly be an $\mathcal{R}$-torsion module, therefore it plays no important rôle in the calculations.

In order to prove (B), we'll apply the local Euler-Poincaré characteristic formula. Indeed the $\mathcal{R}$-torsion of the H^0 implies that

$$\begin{aligned}\mathrm{rank}_{\mathcal{R}}\left(\Omega_{\mathbb{T}_\infty(1-i)}\right) &= \mathrm{rank}_{\mathcal{R}}\Big(H^1\big(K, \mathrm{Fil}^+\mathbb{T}_\infty(1-i)\big)\Big) \\ &= \mathrm{rank}_{\widetilde{\mathcal{R}}^{\mathrm{norm}}}\Big(H^1\Big(K, \big(\mathrm{Fil}^+\mathbb{T}_\infty\big)_{\widetilde{\mathcal{R}}^{\mathrm{norm}}}(1-i)\Big)\Big) \\ &= \mathrm{rank}_{\widetilde{\mathcal{R}}^{\mathrm{norm}}/\widetilde{P}}\Big(H^1\Big(K,\ \widetilde{P}\left(\mathrm{Fil}^+\mathbb{T}_\infty\right)(1-i)\Big)\Big)\end{aligned}$$

at all but finitely many bad primes $\widetilde{P} \in \mathrm{Spec}\big(\widetilde{\mathcal{R}}^{\mathrm{norm}}\big)$. Now using Euler-Poincaré the right-hand side is $[K:\mathbb{Q}_p] \times \dim_{E(\widetilde{P})}\big(\widetilde{P}\left(\mathrm{Fil}^+\mathbb{T}_\infty\right)\big)$, and claim (B) follows.

Finally to dispose of assertion (C), one again adopts a specialisation approach. Fixing a base weight $k_0 \in \mathbb{Z}_{\geq 2}$ and character ϵ, the representation $\mathbb{V} = \mu_{k_0,\epsilon}\big(\mathbb{T}_\infty\big)$ is defined over $E\langle\langle s\rangle\rangle$ say. Choosing an open set $\mathcal{U}' \subset \mathfrak{X}^+_{\mathrm{dR}}$ satisfying $\mathcal{U}' \subset \mathbb{Z}_{\geq i+1}$ and arguing as we did before, one deduces

$$\begin{aligned}\mathrm{rank}_{\widetilde{\mathcal{R}}^{\mathrm{norm}}}\Big(H^1_{\mathcal{G},\dagger}\big(K, \widetilde{\mathbb{T}_\infty}(1-i)\big)\Big) &\leq \dim_E\Big(H^1_g\big(K, \mathbb{V}_{k,\epsilon}(1-i)\big)\Big) \quad \text{for some } k \in \mathcal{U}' \\ &= 2 \times \big[K:\mathbb{Q}_p\big] - \dim_E\Big(\mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}}\big(\mathbb{V}_{k,\epsilon}(1-i)\big)\Big) \\ &= \big[K:\mathbb{Q}_p\big] \quad \text{since } \mathbb{V}_{k,\epsilon}(1-i) \cong V\big(f^*_{k,\epsilon}\big)(k-i).\end{aligned}$$

This completes the demonstration of part (C), and of the theorem too.

□

Remarks: (a) For every integer $i \geq 1$, one can similarly identify $H^1_{\mathcal{E}}\big(K, \mathbb{T}^*_\infty(i)\big)$ with the $\mathcal{R}$-saturation of the module $\mathrm{Ker}\Big(H^1\big(K, \mathbb{T}^*_\infty(i)\big) \to H^1\big(K, \big(\mathrm{Fil}^+\mathbb{T}_\infty\big)^*(i)\big)\Big)$, which is of rank $[K:\mathbb{Q}_p]$ by Euler-Poincaré.

(b) It's also easy to prove $\mathrm{Im}\big(H^1(K, \mathrm{Fil}^+\mathbb{T}_\infty) \to H^1(K, \mathbb{T}_\infty)\big)$ injects into the tower $\mathbf{e}_{\mathrm{ord}} \cdot \varprojlim_{r\geq 1} H^1_g\Big(K, \mathrm{Ta}_p\big(J_1(Np^r)\big)\Big)$; this can be checked just by specialising at primes $\widetilde{P_{2,\epsilon}} \in \mathrm{Spec}\big(\widetilde{\mathcal{R}}^{\mathrm{norm}}\big)$ of weight two, with ϵ ranging over $\mathrm{Hom}\big(\Gamma^{\mathrm{wt}}, \overline{\mathbb{Q}}_p^\times\big)[\mathrm{tors}]$.

(c) Lastly Theorem 2.3.2 confirms that for the universal $G_{\mathbb{Q}}$-representation $\mathbb{T}_\infty$, Greenberg's Selmer groups coincide with the Selmer groups studied in [De, §7-§10].

§3 – The Global Theory

Let F be a number field. We now assume $\mathbb{W}$ denotes a $\mathrm{Gal}(\overline{F}/F)$-representation free of finite rank over $\mathbb{Q}_p\langle\langle s\rangle\rangle$, and unramified outside a finite set Σ containing the primes above p and the archimedean places.

We impose the following pseudo-geometric conditions on $\mathbb{W}$ and Σ:

Hypothesis(PsGe). *(i) At all places $\nu \nmid p$, both $\mathbb{W}^{G_{F_\nu}}$ and $\big(\mathbb{W}^*(1)\big)^{G_{F_\nu}}$ are zero;*

(ii) At all places $\nu|p$, each restriction $\mathbb{W}\big|_{G_{F_\nu}}$ satisfies condition (dR) *with $K = F_\nu$.*

For example, a two-dimensional $G_{\mathbb{Q}}$-representation interpolating an eigenfamily $\mathcal{F}$ over the p-adic disk $\mathbb{U}_{k_0}$, exhibits the properties required in (PsGe) for any $F/\mathbb{Q}$. Moreover taking duals, direct sums and tensor products of representations satisfying (PsGe) yields further specimens (although (i) isn't closed under tensor products).

In general, given an order $\mathcal{O} = \mathbb{Z} + \pi^c \mathcal{O}_E$ in a local field E, one can always choose a G_F-stable lattice $\mathbb{L} \subset \mathbb{W} \otimes_{\mathbb{Q}_p} E$ defined over $\mathcal{O}\langle\langle s \rangle\rangle$, such that as $E\langle\langle s\rangle\rangle[G_F]$-modules

$$\mathbb{L} \otimes_{\mathcal{O}\langle\langle s\rangle\rangle} E\langle\langle s\rangle\rangle \;\cong\; \mathbb{W}[\pi].$$

Since these notes only deals with $\mathbb{Q}_p\langle\langle s\rangle\rangle$-coefficients, we take $\mathcal{O} = \mathbb{Z}_p$ throughout.

Notations: (a) Let F_Σ denote the maximal algebraic extension of F unramified outside Σ, and set $G_{F,\Sigma} := \mathrm{Gal}(F_\Sigma/F)$ so that $\mathbb{L} \subset \mathbb{W}$ are naturally $G_{F,\Sigma}$-modules.
(b) If $G = G_{F,\Sigma}$ or $G = G_{F_\nu}$ for some $\nu \in \Sigma$, then G satisfies the p-finiteness condition; in particular, the modules $H^j(G, \mathbb{L})$ over the noetherian base ring $\mathbb{Z}_p\langle\langle s\rangle\rangle$ have only finitely many associated primes – let's call this finite set '$\mathfrak{X}^j_{\mathrm{bad}}(G)$', say.
(c) At all $k \in \mathbb{U}_{k_0} - \mathfrak{X}^j_{\mathrm{bad}}(G)$, each term $\#\big(H^j(G,\mathbb{L})[\lambda_k]\big)$ will be finite and bounded.

Definition 3.1. *The subset $\mathfrak{X}_{\mathrm{bad}} \subset \mathbb{U}_{k_0}$ is given by the union of bad weights*

$$\mathfrak{X}_{\mathrm{bad}} \;\; := \;\; \bigcup_{j=1}^{3} \mathfrak{X}^j_{\mathrm{bad}}\big(G_{F,\Sigma}\big) \;\cup\; \bigcup_{j=1}^{3} \mathfrak{X}^j_{\mathrm{bad}}\big(G_{F_\nu}\big).$$

If we now examine the G-cohomology of $0 = \mathbb{L}[\lambda_k] \longrightarrow \mathbb{L} \overset{\times\lambda_k}{\longrightarrow} \mathbb{L} \longrightarrow \mathbb{L}/\lambda_k \longrightarrow 0$, one obtains a truncated exact sequence

$$0 \;\longrightarrow\; \frac{H^j(G,\mathbb{L})}{\lambda_k \cdot H^j(G,\mathbb{L})} \;\overset{\times\lambda_k}{\longrightarrow}\; H^j\big(G, \mathbb{L}/\lambda_k\big) \;\longrightarrow\; H^{j+1}(G,\mathbb{L})[\lambda_k] \;\longrightarrow\; 0.$$

As a consequence, for every $\lambda_k \in \mathbb{U}_{k_0} - \mathfrak{X}_{\mathrm{bad}}$ there are injective homomorphisms

$$\frac{H^j(G,\mathbb{L})}{\lambda_k \cdot H^j(G,\mathbb{L})} \;\hookrightarrow\; H^j\big(G, \mathbb{L}/\lambda_k\big) \quad \text{with } j = 0, 1, 2$$

having finite cokernel, bounded independently of the specialisation λ_k.

§§3.1 – Local Conditions

The main strategy is to prove analogous statements to the one above, but replacing the full cohomology groups instead with subgroups $H^1_\star(F_\nu, \mathbb{L})$ of local conditions.

Conditions at primes away from p.
Suppose that $\nu \nmid p$. If the $\mathrm{Gal}\big(\overline{F}_\nu/F_\nu\big)$-invariants $\mathbb{W}^{G_{F_\nu}} = 0$, the Frobenius element $\mathrm{Frob}_\nu \in \mathrm{Gal}(F_\nu^{\mathrm{nr}}/F_\nu)$ acting on $\mathbb{W}^{I_{F_\nu}}$ cannot have 1 occurring as an eigenvalue; it follows directly that the cohomology group $H^1\big(F_\nu^{\mathrm{nr}}/F_\nu, \mathbb{L}^{I_{F_v}}\big) \cong \frac{\mathbb{L}^{I_{F_v}}}{(\mathrm{Frob}_\nu - 1)\cdot \mathbb{L}^{I_{F_v}}}$ must be torsion over $\mathbb{Z}_p\langle\langle s\rangle\rangle$. Defining $H^1_{\mathrm{nr}}\big(F_\nu, \mathbb{L}\big)$ to be the $\mathbb{Z}_p\langle\langle s\rangle\rangle$-saturation of

$$\mathrm{Im}\Big(H^1\big(F_\nu^{\mathrm{nr}}/F_\nu, \mathbb{L}^{I_{F_v}}\big) \overset{\mathrm{inf}}{\longrightarrow} H^1\big(\overline{F}_\nu/F_\nu, \mathbb{L}\big)\Big)$$

clearly $H^1_{\mathrm{nr}}(F_\nu, \mathbb{L}) = \mathrm{Tors}_{\mathbb{Z}_p\langle\langle s\rangle\rangle}\big(H^1(F_\nu, \mathbb{L})\big)$. Moreover if $\big(\mathbb{W}^*(1)\big)^{G_{F_\nu}}$ is also zero, a simple Euler characteristic argument implies the $\mathrm{rank}_{\mathbb{Z}_p\langle\langle s\rangle\rangle}\big(H^1(F_\nu, \mathbb{L})\big) = 0$.
(*N.B.* We'll take H^1_{nr} to be our local condition at primes in Σ not lying above p.)

Conditions at the primes above p.
Suppose that $\nu\big|p$. Again let's abbreviate $\mathbb{L}/\lambda_k$ by $\mathbb{L}_k$, which is clearly a $\mathbb{Z}_p$-module. Writing $H^1_{\mathfrak{X}^+}(F_\nu,\mathbb{L})$ for the kernel of $H^1(F_\nu,\mathbb{L}) \longrightarrow \bigoplus_{k'\in\mathfrak{X}^+_{\mathrm{dR}}} H^1\big(F_\nu,\mathbb{L}_{k'}\otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}\big)$, the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1_{\mathfrak{X}^+}(F_\nu,\mathbb{L}) & \longrightarrow & H^1(F_\nu,\mathbb{L}) & \longrightarrow & \bigoplus_{k'\in\mathfrak{X}^+_{\mathrm{dR}}} H^1\big(F_\nu,\mathbb{L}_{k'}\otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}\big) \\ & & \big\downarrow \text{mod } \lambda_k & & \big\downarrow \text{mod } \lambda_k & & \big\downarrow \mathrm{proj}_k \\ 0 & \longrightarrow & H^1_g(F_\nu,\mathbb{L}_k) & \longrightarrow & H^1(F_\nu,\mathbb{L}_k) & \stackrel{-\otimes 1}{\longrightarrow} & H^1\big(F_\nu,\mathbb{L}_k\otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}\big) \end{array}$$

has exact rows (note that H^1_g is automatically p-saturated since B_{dR} is a field).

Remarks: (i) If $H^1_{\mathcal{G},\dagger}(F_\nu,\mathbb{L})$ is the pre-image of $H^1_{\mathcal{G},\dagger}(F_\nu,\mathbb{W})$ under $-\otimes_{\mathbb{Z}_p\langle\langle s\rangle\rangle}\mathbb{Q}_p\langle\langle s\rangle\rangle$ then using the previous diagram, one easily checks injectivity of the homomorphism

$$\frac{H^1_{\mathcal{G},\dagger}(F_\nu,\mathbb{L})}{\lambda_k\cdot H^1_{\mathcal{G},\dagger}(F_\nu,\mathbb{L})} \stackrel{\alpha_\nu^{(k)}}{\longrightarrow} H^1_g(F_\nu,\mathbb{L}_k) \quad \text{for all } k\in\mathfrak{X}^+_{\mathrm{dR}}-\mathfrak{X}^+_{\mathcal{G},\dagger}\big(G_{F_\nu}\big);$$

here '$\mathfrak{X}^+_{\mathcal{G},\dagger}$' denotes the weights appearing in the $\mathbb{Z}_p\langle\langle s\rangle\rangle$-torsion module $\frac{H^1_{\mathcal{G},\dagger}(F_\nu,\mathbb{L})}{H^1_{\mathfrak{X}^+}(F_\nu,\mathbb{L})}$.
(ii) As $H^1_{\mathcal{G},\dagger}(F_\nu,\mathbb{L})$ is $\mathbb{Z}_p\langle\langle s\rangle\rangle$-saturated the quotient $\frac{H^1(F_\nu,\mathbb{L})}{H^1_{\mathcal{G},\dagger}(F_\nu,\mathbb{L})}$ is $\mathbb{Z}_p\langle\langle s\rangle\rangle$-torsion free, so we obtain an exact sequence

$$0=\frac{H^1(F_\nu,\mathbb{L})}{H^1_{\mathcal{G},\dagger}(F_\nu,\mathbb{L})}[\lambda_k]\longrightarrow H^1_{\mathcal{G},\dagger}\big(F_\nu,\mathbb{L}\big)_k\longrightarrow H^1\big(F_\nu,\mathbb{L}\big)_k\longrightarrow\left(\frac{H^1(F_\nu,\mathbb{L})}{H^1_{\mathcal{G},\dagger}(F_\nu,\mathbb{L})}\right)_k\longrightarrow 0$$

at every p-adic weight $k\in\mathbb{U}_{k_0}$.

Hypothesis(Bnd). *For all $\nu\big|p$ and weights $k\in\mathbb{U}_{k_0}$, each module $Y_\nu=\frac{H^1(F_\nu,\mathbb{L})}{H^1_{\mathcal{G},\dagger}(F_\nu,\mathbb{L})}$ has the property that $\#\mathrm{Tors}_{\mathbb{Z}_p}\big(Y_\nu/\lambda_k\big)$ is finite and bounded independently of λ_k.*

We shall show that (Bnd) always holds for a two-dimensional p-ordinary family in the final section, but we conjecture it holds in greater generality at dimension two.

Proposition 3.1.1. *(a) For all weights $k\in\mathfrak{X}^+_{\mathrm{dR}}-\mathfrak{X}_{\mathrm{bad}}-\mathfrak{X}^+_{\mathcal{G},\dagger}\big(G_{F_\nu}\big)$, the mapping*

$$\alpha_\nu^{(k)}: H^1_{\mathcal{G},\dagger}(F_\nu,\mathbb{L})_k \longrightarrow H^1_g(F_\nu,\mathbb{L}_k)$$

is injective, with finite cokernel.

(b) For all weights $k\in\mathfrak{X}^+_{\mathrm{dR}}-\mathfrak{X}_{\mathrm{bad}}-\mathfrak{X}^+_{\mathcal{G},\dagger}\big(G_{F_\nu}\big)$, the induced quotient map

$$\overline{\beta}_\nu^{(k)}:\left(\frac{H^1(F_\nu,\mathbb{L})}{H^1_{\mathcal{G},\dagger}(F_\nu,\mathbb{L})}\right)_k \stackrel{\sim}{\longrightarrow} \frac{H^1(F_\nu,\mathbb{L})_k}{H^1_{\mathcal{G},\dagger}(F_\nu,\mathbb{L})_k} \stackrel{\text{mod } H^1_g}{\longrightarrow} \frac{H^1(F_\nu,\mathbb{L}_k)}{H^1_g(F_\nu,\mathbb{L}_k)}$$

has finite cokernel bounded independently of λ_k, and finite kernel.

(c) If one further assumes that $\mathbb{L}$ satisfies (Bnd)*, both the cokernel of $\alpha_\nu^{(k)}$ and the kernel of $\overline{\beta}_\nu^{(k)}$ are likewise bounded independently of λ_k.*

Proof: The following diagram (with exact rows) summarises the relevant maps:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1_{\mathcal{G},\dagger}(F_\nu,\mathbb{L})_k & \longrightarrow & H^1(F_\nu,\mathbb{L})_k & \longrightarrow & \left(\dfrac{H^1(F_\nu,\mathbb{L})}{H^1_{\mathcal{G},\dagger}(F_\nu,\mathbb{L})}\right)_k & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \alpha_\nu^{(k)}} & & \Big\downarrow{\scriptstyle \beta_\nu^{(k)}} & & \Big\downarrow{\scriptstyle \overline{\beta}_\nu^{(k)}} & & \\ 0 & \longrightarrow & H^1_g(F_\nu,\mathbb{L}_k) & \longrightarrow & H^1(F_\nu,\mathbb{L}_k) & \longrightarrow & \dfrac{H^1(F_\nu,\mathbb{L}_k)}{H^1_g(F_\nu,\mathbb{L}_k)} & \longrightarrow & 0 \end{array}$$

We have already seen that

(i) $\alpha_\nu^{(k)}$ is injective for every $k \in \mathfrak{X}^+_{\mathrm{dR}} - \mathfrak{X}^+_{\mathcal{G},\dagger}(G_{F_\nu})$;

(ii) $\beta_\nu^{(k)}$ is injective for every $k \in \mathbb{U}_{k_0} - \mathfrak{X}_{\mathrm{bad}}$;

(iii) $\mathrm{Coker}\big(\beta_\nu^{(k)}\big)$ is finite and bounded, for every $k \in \mathbb{U}_{k_0} - \mathfrak{X}_{\mathrm{bad}}$.

Applying the snake lemma to our diagram above, the cokernel of $\overline{\beta}_\nu^{(k)}$ will then automatically be finite and bounded at each weight $k \in \mathfrak{X}^+_{\mathrm{dR}} - \mathfrak{X}_{\mathrm{bad}} - \mathfrak{X}^+_{\mathcal{G},\dagger}(G_{F_\nu})$. **It therefore suffices to show that the kernel of $\overline{\beta}_\nu^{(k)}$ is finite, and bounded independently of λ_k whenever $\mathbb{L}$ satisfies (Bnd).**

Remark: Let Z_k denote the pre-image under $\beta_\nu^{(k)}$ of $H^1_g(F_\nu,\mathbb{L}_k) \cap \mathrm{Im}\big(H^1(F_\nu,\mathbb{L})_k\big)$, which coincides with the p-saturation of $H^1_{\mathcal{G},\dagger}(F_\nu,\mathbb{L})_k$ in $H^1(F_\nu,\mathbb{L})_k$. One finds

$$\mathrm{Ker}\big(\overline{\beta}_\nu^{(k)}\big) \;\cong\; \frac{\big\{\mathbf{y} \in H^1(F_\nu,\mathbb{L})_k \;\big|\; \beta_\nu^{(k)}(\mathbf{y}) \in H^1_g(F_\nu,\mathbb{L}_k)\big\}}{H^1_{\mathcal{G},\dagger}(F_\nu,\mathbb{L})_k} \;=\; \frac{Z_k}{H^1_{\mathcal{G},\dagger}(F_\nu,\mathbb{L})_k}$$

upon exploiting the earlier isomorphism $\left(\frac{H^1(F_\nu,\mathbb{L})}{H^1_{\mathcal{G},\dagger}(F_\nu,\mathbb{L})}\right)_k \cong \frac{H^1(F_\nu,\mathbb{L})_k}{H^1_{\mathcal{G},\dagger}(F_\nu,\mathbb{L})_k}$.

The problem reduces to showing that $H^1_{\mathcal{G},\dagger}(F_\nu,\mathbb{L})_k$ has finite index in its saturation. Recalling that $Y_\nu = \frac{H^1(F_\nu,\mathbb{L})}{H^1_{\mathcal{G},\dagger}(F_\nu,\mathbb{L})}$, then from the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1_{\mathcal{G},\dagger}(F_\nu,\mathbb{L})_k & \longrightarrow & H^1(F_\nu,\mathbb{L})_k & \longrightarrow & \big(Y_\nu\big)_k & \longrightarrow & 0 \\ & & \bigcap & & \bigcup & & & & \\ & & Z_k & \xrightarrow{=} & Z_k & & & & \end{array}$$

clearly the quotient $\frac{Z_k}{H^1_{\mathcal{G},\dagger}(F_\nu,\mathbb{L})_k}$ must be isomorphic to a submodule $\mathfrak{D}_k \subset \big(Y_\nu\big)_k$; in fact $\mathfrak{D}_k$ will be $\mathbb{Z}_p$-torsion, because Z_k equals the p-saturation of $H^1_{\mathcal{G},\dagger}(F_\nu,\mathbb{L})_k$. Furthermore $\big(Y_\nu\big)_k/\mathfrak{D}_k \cong H^1(F_\nu,\mathbb{L})_k/Z_k$ and the latter is certainly $\mathbb{Z}_p$-torsion free, hence so is $\big(Y_\nu\big)_k/\mathfrak{D}_k$. We conclude that $\mathfrak{D}_k = \mathrm{Tors}_{\mathbb{Z}_p}\big(Y_\nu\big)_k$, which is finite.

It follows $H^1_{\mathcal{G},\dagger}(F_\nu,\mathbb{L})_k$ indeed has finite index in its p-saturation, thus both the kernel of $\overline{\beta}_\nu^{(k)}$ and the cokernel of $\alpha_\nu^{(k)}$ are finite. Lastly Hypothesis(Bnd) guarantees that the size of $\mathrm{Tors}_{\mathbb{Z}_p}\big(Y_\nu\big)_k$ is bounded independently of λ_k, therefore the same will be true of $\mathrm{Ker}\big(\overline{\beta}_\nu^{(k)}\big)$ and $\mathrm{Coker}\big(\alpha_\nu^{(k)}\big)$. The proof is now complete.

□

§§3.2 – Dualities

Our next task is to pass to cohomologies taking coefficients in discrete modules. Let M denote any $\mathbb{Z}_p\langle\langle s\rangle\rangle$-module. One defines its $\chi_{\rm cy}$-twisted Pontryagin dual by

$$\mathcal{A}_M := \text{Hom}_{\rm cont}\big(M, \mu_{p^\infty}\big), \text{ and also } \mathcal{A}_{M/\lambda_k} := \text{Hom}_{\rm cont}\big(M/\lambda_k, \mu_{p^\infty}\big) \cong \mathcal{A}_M[\lambda_k]$$

provided the weight $k \in \mathbb{U}_{k_0}$.

(For example when $M = \mathbb{L}$, both $\mathcal{A}_{\mathbb{L}}$ and $\mathcal{A}_{\mathbb{L}/\lambda_k}$ will inherit the structure of $\mathbb{Z}_p\langle\langle s\rangle\rangle\big[G\big]$-modules, either for $G = G_{F,\Sigma}$, or with $G = G_{F_\nu}$ at some place $\nu \in \Sigma$.)

The arguments we now use are identical to those of Paul Smith in [De, Appx C] so instead of us reproducing the full proofs again, let's briefly sketch the details. As usual the method has both a local and global component.

The local situation $G = G_{F_\nu}$.

Suppose M and N are G-modules. Examining the first few terms in the spectral sequence $H^r\big(G, \text{Ext}^s(M, N)\big) \implies \text{Ext}^{r+s}_G(M, N)$, there is a long exact sequence

$$0 \longrightarrow H^1\big(G, \text{Hom}(M, N)\big) \longrightarrow \text{Ext}^1_G(M, N)$$
$$\longrightarrow H^0\big(G, \text{Ext}^1(M, N)\big) \longrightarrow H^2\big(G, \text{Hom}(M, N)\big) \longrightarrow \dots .$$

Firstly if $M = \mathbb{L}$ and $N = \mu_{p^\infty}$, one has an injection $H^1(F_\nu, \mathcal{A}_{\mathbb{L}}) \hookrightarrow \text{Ext}^1_G(\mathbb{L}, \mu_{p^\infty})$; analogously if $M = \mathbb{L}_k$ and $N = \mu_{p^\infty}$, we obtain $H^1(F_\nu, \mathcal{A}_{\mathbb{L}/\lambda_k}) \hookrightarrow \text{Ext}^1_G(\mathbb{L}_k, \mu_{p^\infty})$.

Remark: Applying the left exact functor $\text{Hom}_{\mathbb{Z}[G]}\big(-, \mu_{p^\infty}\big)$ to the multiplication by λ_k sequence on $\mathcal{A}_{\mathbb{L}}$, produces yet another long exact sequence

$$0 \to \mathcal{A}^G_{\mathbb{L}/\lambda_k} \to \mathcal{A}^G_{\mathbb{L}} \stackrel{\times\lambda_k}{\longrightarrow} \mathcal{A}^G_{\mathbb{L}} \to \text{Ext}^1_G\big(\mathbb{L}_k, \mu_{p^\infty}\big) \to \text{Ext}^1_G\big(\mathbb{L}, \mu_{p^\infty}\big) \stackrel{\times\lambda_k}{\longrightarrow} \text{Ext}^1_G\big(\mathbb{L}, \mu_{p^\infty}\big)$$

from which one obtains a map $\psi^{(k)}_\nu : \text{Ext}^1_G\big(\mathbb{L}_k, \mu_{p^\infty}\big) \longrightarrow \text{Ext}^1_G\big(\mathbb{L}, \mu_{p^\infty}\big)[\lambda_k]$.

Proposition 3.2.1. *For all weights $k \in \mathbb{U}_{k_0} - \mathfrak{X}_{\rm bad}$, this mapping*

$$\psi^{(k)}_\nu : \text{Ext}^1_G\big(\mathbb{L}_k, \mu_{p^\infty}\big) \longrightarrow \text{Ext}^1_G\big(\mathbb{L}, \mu_{p^\infty}\big)[\lambda_k]$$

is surjective, with finite kernel bounded independently of λ_k.

Deferring its proof momentarily, it is shown in [De, p271] there exists a diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & H^1(F_\nu, \mathcal{A}_{\mathbb{L}/\lambda_k}) & \longrightarrow & \text{Ext}^1_G\big(\mathbb{L}_k, \mu_{p^\infty}\big) & \longrightarrow & \text{Ext}^1\big(\mathbb{L}_k, \mu_{p^\infty}\big)^G \\ & \downarrow & & \downarrow \psi^{(k)}_\nu & & \| \\ 0 \longrightarrow & H^1(F_\nu, \mathcal{A}_{\mathbb{L}})[\lambda_k] & \longrightarrow & \text{Ext}^1_G\big(\mathbb{L}, \mu_{p^\infty}\big)[\lambda_k] & \longrightarrow & \text{Ext}^1\big(\mathbb{L}, \mu_{p^\infty}\big)^G[\lambda_k] \end{array}$$

with exact rows.

Writing $\gamma^{(k)}_\nu$ for the left-most vertical arrow, then the snake lemma applied to the diagram above yields the following

Corollary 3.2.2. *For all weights $k \in \mathbb{U}_{k_0} - \mathfrak{X}_{\rm bad}$, the homomorphism*

$$\gamma^{(k)}_\nu : H^1(F_\nu, \mathcal{A}_{\mathbb{L}/\lambda_k}) \longrightarrow H^1(F_\nu, \mathcal{A}_{\mathbb{L}})[\lambda_k]$$

is surjective, with finite kernel bounded independently of λ_k.

Proof of Proposition 3.2.1:
The surjectivity of $\psi_\nu^{(k)}$ is clear at all p-adic weights k. Before examining its kernel, one should point out the stated result holds more generally, for modules M that are of finite rank but not necessarily $\mathbb{Z}_p\langle\langle s\rangle\rangle$-torsion free. However we shall only need the result for $M = \mathbb{L}$ in this paper.

The Yoneda (cup-product) pairing on Ext-groups yields a perfect duality

$$\mathrm{Ext}^1_{G_{F_\nu}}\big(\mathbb{L}, \mu_{p^\infty}\big) \times H^1\big(F_\nu, \mathbb{L}\big) \overset{\cup}{\longrightarrow} H^2\big(F_\nu, \mu_{p^\infty}\big) \overset{\mathrm{inv}_{F_\nu}}{\longrightarrow} \mathbb{Q}_p/\mathbb{Z}_p$$

and on a p-adic level, $\mathrm{Ext}^1_{G_{F_\nu}}\big(\mathbb{L}_k, \mu_{p^\infty}\big) \times H^1\big(F_\nu, \mathbb{L}_k\big) \overset{\mathrm{inv}_{F_\nu}\circ\cup}{\longrightarrow} \mathbb{Q}_p/\mathbb{Z}_p$ for $k \in \mathbb{U}_{k_0}$. It is shown in [De, p270] that one has the diagram

$$\begin{array}{ccccc}
\mathcal{A}_{\mathbb{L}}^{G_{F_\nu}} & & 0 & & \\
\big\downarrow{\times\lambda_k} & & \big\uparrow & & \\
\mathcal{A}_{\mathbb{L}}^{G_{F_\nu}} & & H^2\big(F_\nu, \mathbb{L}\big)[\lambda_k] & & \\
\big\downarrow{\partial} & & \big\uparrow & & \\
\mathrm{Ext}^1_{G_{F_\nu}}\big(\mathbb{L}_k, \mu_{p^\infty}\big) & \times & H^1\big(F_\nu, \mathbb{L}_k\big) & \longrightarrow & \mathbb{Q}_p/\mathbb{Z}_p \\
\big\downarrow{\psi_\nu^{(k)}} & & \big\uparrow{`\eta'} & & \\
\mathrm{Ext}^1_{G_{F_\nu}}\big(\mathbb{L}, \mu_{p^\infty}\big) & \times & H^1\big(F_\nu, \mathbb{L}\big) & \longrightarrow & \mathbb{Q}_p/\mathbb{Z}_p
\end{array}$$

whose columns are exact.

Let $\mathfrak{C} \subset H^1\big(F_\nu, \mathbb{L}_k\big)$ denote the precise orthogonal complement of $\mathrm{Ker}\big(\psi_\nu^{(k)}\big)$; because both pairings are perfect, it follows that $\#\mathrm{Ker}\big(\psi_\nu^{(k)}\big) = \#\Big(H^1(F_\nu, \mathbb{L}_k)/\mathfrak{C}\Big)$. A diagram chase reveals that $\mathfrak{C} = \mathrm{Im}(\eta)$, whence $H^1(F_\nu, \mathbb{L}_k)/\mathfrak{C} \cong H^2(F_\nu, \mathbb{L})[\lambda_k]$ as the right-hand column is exact. Finally the latter group is finite and bounded at all p-adic weights outside of $\mathfrak{X}^2_{\mathrm{bad}}\big(G_{F_\nu}\big)$, therefore the same is true of $\mathrm{Ker}\big(\psi_\nu^{(k)}\big)$. □

The global situation $G = G_{F,\Sigma}$.
Recall that $G_{F_\nu} \subset G_{F,\Sigma}$ at each place $\nu \in \Sigma$. Given any $\mathbb{Z}_p\langle\langle s\rangle\rangle\big[G_{F,\Sigma}\big]$-module M, one defines subgroups

$$\text{Ш}^i\big(G_{F,\Sigma}, M\big) := \mathrm{Ker}\left(H^i\big(G_{F,\Sigma}, M\big) \overset{\oplus\mathrm{res}_\nu}{\longrightarrow} \bigoplus_{\nu\in\Sigma} H^i\big(F_\nu, M\big)\right) \quad \text{with } i = 0, 1, 2.$$

For instance if $M = \mathbb{L}$ and $i = 2$, there is an exact sequence

$$0 \to \mathcal{Z}(k) \to \text{Ш}^2\big(G_{F,\Sigma}, \mathbb{L}\big)_k \to H^2\big(G_{F,\Sigma}, \mathbb{L}\big)_k \to \bigoplus_{\nu\in\Sigma} \mathrm{res}_\nu\Big(H^2\big(G_{F,\Sigma}, \mathbb{L}\big)\Big)_k \to 0$$

where '$\mathcal{Z}(k)$' is a quotient of $\bigoplus_{\nu\in\Sigma} \mathrm{res}_\nu\Big(H^2\big(F_\nu, \mathbb{L}\big)\Big)[\lambda_k] \subset \bigoplus_{\nu\in\Sigma} H^2\big(F_\nu, \mathbb{L}\big)[\lambda_k]$. From this particular description, we deduce that $\#\mathcal{Z}(k)$ must certainly be finite and bounded independently of $k \in \mathbb{U}_{k_0} - \bigcup_{\nu\in\Sigma} \mathfrak{X}^2_{\mathrm{bad}}\big(G_{F_\nu}\big)$.

Remarks: (a) For every $\nu \in \Sigma$, there are isomorphisms $H^2(F_\nu, \mathbb{L})_k \cong H^2(F_\nu, \mathbb{L}_k)$ at all weights $k \in \mathbb{U}_{k_0}$, since the group G_{F_ν} has cohomological dimension two.
(b) The natural injections $H^2(G_{F,\Sigma}, \mathbb{L})_k \hookrightarrow H^2(G_{F,\Sigma}, \mathbb{L}_k)$ have finite and bounded cokernel for all $k \in \mathbb{U}_{k_0} - \mathfrak{X}^3_{\text{bad}}(G_{F,\Sigma})$, because $H^3(G_{F,\Sigma}, \mathbb{L})[\lambda_k]$ has this property.

For simplicity, assume now that our p-adic weight k lies outside the finite set $\mathfrak{X}_{\text{bad}}$. Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 \to \mathcal{Z}(k) \to & \mathbf{Ш}^2\big(G_{F,\Sigma}, \mathbb{L}\big)_k & \to & H^2\big(G_{F,\Sigma}, \mathbb{L}\big)_k & \to & \bigoplus_{\nu \in \Sigma} \mathrm{res}_\nu\Big(H^2\big(G_{F,\Sigma}, \mathbb{L}\big)\Big)_k & \to 0 \\
& \big\downarrow \theta_\Sigma^{(k)} & & \big\downarrow & & \big\downarrow & \\
0 \to & \mathbf{Ш}^2\big(G_{F,\Sigma}, \mathbb{L}_k\big) & \to & H^2\big(G_{F,\Sigma}, \mathbb{L}_k\big) & \to & \bigoplus_{\nu \in \Sigma} \mathrm{res}_\nu\Big(H^2\big(G_{F,\Sigma}, \mathbb{L}\big)_k\Big) & \to 0
\end{array}$$

with exact rows. Combining together (a) and (b) with the boundedness of $\#\mathcal{Z}(k)$, the snake lemma implies the map $\theta_\Sigma^{(k)} : \mathbf{Ш}^2\big(G_{F,\Sigma}, \mathbb{L}\big)_k \longrightarrow \mathbf{Ш}^2\big(G_{F,\Sigma}, \mathbb{L}_k\big)$ is finite and bounded independently of $k \in \mathbb{U}_{k_0} - \mathfrak{X}_{\text{bad}}$.

To pass to the discrete version, we exploit the fact that for a $G_{F,\Sigma}$-module M there is a perfect pairing $\langle -, - \rangle_M : \mathbf{Ш}^2\big(G_{F,\Sigma}, M\big) \times \mathbf{Ш}^1\big(G_{F,\Sigma}, \mathcal{A}_M\big) \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p$. In particular, both $M = \mathbb{L}$ and $M = \mathbb{L}_k$ are connected (see [De, p273]) via:

$$\begin{array}{ccccc}
\mathbf{Ш}^2\big(G_{F,\Sigma}, \mathbb{L}\big)_k & \times & \mathbf{Ш}^1\big(G_{F,\Sigma}, \mathcal{A}_{\mathbb{L}}\big)[\lambda_k] & \overset{\langle -,- \rangle_{\mathbb{L}}}{\longrightarrow} & \mathbb{Q}_p/\mathbb{Z}_p \\
\big\downarrow \theta_\Sigma^{(k)} & & \big\uparrow \widehat{\theta_\Sigma^{(k)}} & & \big\| \\
\mathbf{Ш}^2\big(G_{F,\Sigma}, \mathbb{L}_k\big) & \times & \mathbf{Ш}^1\big(G_{F,\Sigma}, \mathcal{A}_{\mathbb{L}/\lambda_k}\big) & \overset{\langle -,- \rangle_{\mathbb{L}_k}}{\longrightarrow} & \mathbb{Q}_p/\mathbb{Z}_p.
\end{array}$$

We conclude that the dual map $\widehat{\theta_\Sigma^{(k)}} : \mathbf{Ш}^1\big(G_{F,\Sigma}, \mathcal{A}_{\mathbb{L}/\lambda_k}\big) \longrightarrow \mathbf{Ш}^1\big(G_{F,\Sigma}, \mathcal{A}_{\mathbb{L}}\big)[\lambda_k]$ must also have finite and bounded kernel/cokernel outside of $\mathfrak{X}_{\text{bad}}$.

Proposition 3.2.3. *For all weights $k \in \mathbb{U}_{k_0} - \mathfrak{X}_{\text{bad}}$, the natural maps*

$$\gamma_\Sigma^{(k)} : H^1\big(G_{F,\Sigma}, \mathcal{A}_{\mathbb{L}/\lambda_k}\big) \longrightarrow H^1\big(G_{F,\Sigma}, \mathcal{A}_{\mathbb{L}}\big)[\lambda_k]$$

are surjective, with finite kernels bounded independently of λ_k.

Proof: This argument is straightforward. One simply applies the snake lemma to

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathbf{Ш}^1\big(G_{F,\Sigma}, \mathcal{A}_{\mathbb{L}/\lambda_k}\big) & \longrightarrow & H^1\big(G_{F,\Sigma}, \mathcal{A}_{\mathbb{L}/\lambda_k}\big) & \longrightarrow & \bigoplus_{\nu \in \Sigma} H^1\big(F_\nu, \mathcal{A}_{\mathbb{L}/\lambda_k}\big) \\
& \big\downarrow \widehat{\theta_\Sigma^{(k)}} & & \big\downarrow \gamma_\Sigma^{(k)} & & \big\downarrow \oplus\gamma_\nu^{(k)} \\
0 \longrightarrow & \mathbf{Ш}^1\big(G_{F,\Sigma}, \mathcal{A}_{\mathbb{L}}\big)[\lambda_k] & \longrightarrow & H^1\big(G_{F,\Sigma}, \mathcal{A}_{\mathbb{L}}\big)[\lambda_k] & \longrightarrow & \bigoplus_{\nu \in \Sigma} H^1\big(F_\nu, \mathcal{A}_{\mathbb{L}}\big)[\lambda_k]
\end{array}$$

then uses the fact the left-hand vertical arrow is bounded by the previous discussion, while the right-hand vertical arrow is bounded by Corollary 3.2.2.

□

§§3.3 – Controlling the Discrete Selmer Group

In order to define our Selmer group, we shall dualise the local conditions in §§3.1 under the perfect pairing

$$H^1\big(F_\nu,\mathbb{L}\big)\times H^1\big(F_\nu,\mathcal{A}_{\mathbb{L}}\big)\longrightarrow \mathbb{Q}_p/\mathbb{Z}_p.$$

If ν doesn't lie over p, we take $H^1_{\rm nr}\big(F_\nu,\mathcal{A}_{\mathbb{L}}\big)$ to be the orthogonal complement of the unramified cocycles in $H^1\big(F_\nu,\mathbb{L}\big)$; if $\nu\big|p$, we set $H^1_{\mathcal{E},\dagger}\big(F_\nu,\mathcal{A}_{\mathbb{L}}\big):=H^1_{\mathcal{G},\dagger}\big(F_\nu,\mathbb{L}\big)^{\perp}$.

Definition 3.3.1. *The discrete Selmer group associated to $\mathcal{A}_{\mathbb{L}}$ over F, is given by*

$$\mathrm{Sel}_{F,\Sigma}\big(\mathcal{A}_{\mathbb{L}}\big):=\mathrm{Ker}\left(H^1\big(G_{F,\Sigma},\mathcal{A}_{\mathbb{L}}\big)\overset{\oplus\mathrm{res}_\nu}{\longrightarrow}\bigoplus_{\nu\nmid p}\frac{H^1\big(F_\nu,\mathcal{A}_{\mathbb{L}}\big)}{H^1_{\rm nr}\big(F_\nu,\mathcal{A}_{\mathbb{L}}\big)}\oplus\bigoplus_{\nu|p}\frac{H^1\big(F_\nu,\mathcal{A}_{\mathbb{L}}\big)}{H^1_{\mathcal{E},\dagger}\big(F_\nu,\mathcal{A}_{\mathbb{L}}\big)}\right).$$

Recall for a $\mathbb{Z}_p$-adic representation $\mathbf{T}$, the Bloch-Kato Selmer group $H^1_{e,\Sigma}\big(F,\mathcal{A}_{\mathbf{T}}\big)$ consists of 1-cocycles unramified away from p, and orthogonal to $H^1_g(F_\nu,\mathbf{T})$ at the primes ν dividing p (i.e. those coycles lying in the exponential part $H^1_e\big(F_\nu,\mathcal{A}_{\mathbf{T}}\big)$).

The main achievement of this paper is the following result.

Theorem 3.3.2. *(i) For all weights $k\in\mathfrak{X}^+_{\rm dR}-\mathfrak{X}_{\rm bad}-\bigcup_{\nu\in\Sigma}\mathfrak{X}^+_{\mathcal{G},\dagger}\big(G_{F_\nu}\big)$, the maps*

$$H^1_{e,\Sigma}\big(F,\mathcal{A}_{\mathbb{L}/\lambda_k}\big)\ \longrightarrow\ \mathrm{Sel}_{F,\Sigma}\big(\mathcal{A}_{\mathbb{L}}\big)[\lambda_k]$$

have finite kernel bounded independently of λ_k, and finite cokernel.

(ii) If one further assumes that $\mathbb{L}$ satisfies (Bnd)*, then the cokernel of these maps is also bounded independently of λ_k.*

Proof: Using Proposition 3.1.1(a), if $\nu\big|p$ then the dual homomorphism

$$\widehat{\alpha^{(k)}_\nu}:\frac{H^1\big(F_\nu,\ \mathcal{A}_{\mathbb{L}/\lambda_k}\big)}{H^1_e\big(F_\nu,\ \mathcal{A}_{\mathbb{L}/\lambda_k}\big)}\ \longrightarrow\ \frac{H^1\big(F_\nu,\ \mathcal{A}_{\mathbb{L}}\big)}{H^1_{\mathcal{E},\dagger}\big(F_\nu,\ \mathcal{A}_{\mathbb{L}}\big)}[\lambda_k]$$

has finite cokernel bounded independently of the specialisation λ_k, and finite kernel. If $\nu\nmid p$ then $H^1_{\rm nr}(F_\nu,\mathbb{L})$ is the whole H^1. Therefore in the commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^1_{e,\Sigma}\big(F,\mathcal{A}_{\mathbb{L}/\lambda_k}\big) & \longrightarrow & H^1\big(G_{F,\Sigma},\mathcal{A}_{\mathbb{L}/\lambda_k}\big) & \longrightarrow & \bigoplus_{\nu\in\Sigma}\frac{H^1\big(F_\nu,\mathcal{A}_{\mathbb{L}/\lambda_k}\big)}{H^1_*\big(F_\nu,\mathcal{A}_{\mathbb{L}/\lambda_k}\big)} \\
 & & \downarrow & & \downarrow{\scriptstyle\gamma^{(k)}_\Sigma} & & \downarrow{\scriptstyle\oplus\widehat{\alpha^{(k)}_\nu}} \\
0 & \longrightarrow & \mathrm{Sel}_{F,\Sigma}\big(\mathcal{A}_{\mathbb{L}}\big)[\lambda_k] & \longrightarrow & H^1\big(G_{F,\Sigma},\mathcal{A}_{\mathbb{L}}\big)[\lambda_k] & \longrightarrow & \bigoplus_{\nu\in\Sigma}\frac{H^1\big(F_\nu,\ \mathcal{A}_{\mathbb{L}}\big)}{H^1_\star\big(F_\nu,\ \mathcal{A}_{\mathbb{L}}\big)}[\lambda_k]
\end{array}$$

the right-hand vertical map has finite kernel, with finite and bounded cokernel. Moreover by Corollary 3.2.2, the middle arrow is surjective with bounded kernel. Lastly, combining Hypothesis(Bnd) with 3.1.1(c) implies that the kernel of $\oplus\widehat{\alpha^{(k)}_\nu}$ is bounded independently of λ_k. The result follows on applying the snake lemma.

□

With appropriate modifications Theorem 3.3.2 works at non-de Rham weights too. For each $k \in \mathbb{U}_{k_0}$, one can define $\mathrm{Sel}_{F,\Sigma}\big(\mathcal{A}_{\mathbb{L}/\lambda_k}\big)$ to equal

$$\mathrm{Ker}\left(H^1\big(G_{F,\Sigma}, \mathcal{A}_{\mathbb{L}/\lambda_k}\big) \overset{\oplus \mathrm{res}_\nu}{\longrightarrow} \bigoplus_{\nu \nmid p} \frac{H^1\big(F_\nu, \mathcal{A}_{\mathbb{L}/\lambda_k}\big)}{H^1_{\mathrm{nr}}\big(F_\nu, \mathcal{A}_{\mathbb{L}/\lambda_k}\big)} \oplus \bigoplus_{\nu | p} \frac{H^1\big(F_\nu, \mathcal{A}_{\mathbb{L}/\lambda_k}\big)}{H^1_{e,\dagger}\big(F_\nu, \mathcal{A}_{\mathbb{L}/\lambda_k}\big)} \right)$$

where $H^1_{e,\dagger}\big(F_\nu, \mathcal{A}_{\mathbb{L}/\lambda_k}\big)$ denotes the orthogonal complement of $\big(H^1_{\mathcal{G},\dagger}(F_\nu, \mathbb{L})/\lambda_k\big)_{p\text{-sat}}$ under the p-adic pairing $H^1\big(F_\nu, \mathcal{A}_{\mathbb{L}/\lambda_k}\big) \times H^1\big(F_\nu, \mathbb{L}_k\big) \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p$. Then exactly the same argument (as in the proof of 3.3.2) establishes boundedness of the maps

$$\mathrm{Sel}_{F,\Sigma}\big(\mathcal{A}_{\mathbb{L}/\lambda_k}\big) \longrightarrow \mathrm{Sel}_{F,\Sigma}\big(\mathcal{A}_{\mathbb{L}}\big)[\lambda_k] \quad \text{for almost all } k \in \mathbb{U}_{k_0}.$$

Of course at almost all $k \in \mathfrak{X}^+_{\mathrm{dR}}$, both $\mathrm{Sel}_{F,\Sigma}\big(\mathcal{A}_{\mathbb{L}/\lambda_k}\big)$ and $H^1_{e,\Sigma}\big(F, \mathcal{A}_{\mathbb{L}/\lambda_k}\big)$ coincide!

Two-dimensional Galois representations revisited.
As in the Introduction, let $\mathcal{F}(s) = \sum_{n=1}^\infty A_n(s) q^n \in E\langle\langle s\rangle\rangle[[q]]$ denote the p-adic eigenfamily deforming the classical eigenform $f \in \mathcal{S}_{k_0}\big(\Gamma_0(Np), \epsilon\big)$ over $\mathbb{U}_{k_0} \subset \mathbb{Z}_p$. We'll assume Σ contains the primes above p and those dividing the tame level N.

Conjecture 3.3.3. *If the slope of $\mathcal{F}$ at p is $< k_0 - 1$, then $\mathbb{L}_{\mathcal{F}}$ satisfies* (Bnd).

Our principal evidence for this conjecture is that it is true for p-ordinary families. The demonstration utilises the fact that the big Galois representation corresponding to $\mathcal{F}$ can be cut out of the finite-rank Λ-module $\mathbb{T}_\infty$, which means the behaviour of each quotient $Y_\nu\big(\mathbb{L}_{\mathcal{F}}\big)$ can be determined via structure theory of these modules.

Lemma 3.3.4. *If $\mathcal{F}(s)$ has slope zero, then $\mathbb{L}_{\mathcal{F}}$ satisfies the Hypothesis*(Bnd).

Proof: By Theorem 2.3.2, the $\mathrm{rank}_{\mathcal{R}}\big(H^1_{\mathcal{G},\dagger}(F_\nu, \mathbb{T}_\infty)\big) = [F_\nu : \mathbb{Q}_p]$ at each place $\nu | p$, which is precisely half the rank of $H^1\big(F_\nu, \mathbb{T}_\infty\big)$. Since $H^1_{\mathcal{G},\dagger}\big(F_\nu, \mathbb{T}_\infty\big)$ is $\mathcal{R}$-saturated inside the full H^1, the quotient module $Y_{\mathbb{T},\nu} = \frac{H^1(F_\nu, \mathbb{T}_\infty)}{H^1_{\mathcal{G},\dagger}(F_\nu, \mathbb{T}_\infty)}$ will be $\mathcal{R}$-torsion free. From the structure theory, it sits inside an exact sequence

$$0 \longrightarrow Y_{\mathbb{T},\nu} \longrightarrow \mathcal{R}^{\oplus[F_\nu:\mathbb{Q}_p]} \longrightarrow \mathcal{B}_\nu \longrightarrow 0$$

where $\mathcal{B}_\nu$ is pseudo-null as an $\mathcal{R}$-module (and therefore finite).

Examining the effect of multiplication by $\lambda_k = s - k + 1$ on the various terms, we obtain the long exact sequence

$$\mu_{k_0,\epsilon}\big(\mathcal{R}^{\oplus[F_\nu:\mathbb{Q}_p]}\big)[\lambda_k] \longrightarrow \mu_{k_0,\epsilon}\big(\mathcal{B}_\nu\big)[\lambda_k] \longrightarrow \mu_{k_0,\epsilon}\big(Y_{\mathbb{T},\nu}\big)/\lambda_k \longrightarrow \mu_{k_0,\epsilon}\big(\mathcal{R}^{\oplus[F_\nu:\mathbb{Q}_p]}\big)/\lambda_k$$

whose left-most term is trivial since $\mu_{k_0,\epsilon}\big(\mathcal{R}\big)$ is $\mathbb{Z}_p\langle\langle s\rangle\rangle$-torsion free, and whose right-most term has trivial p-torsion as $\mu_{k_0,\epsilon}\big(\mathcal{R}\big)/\lambda_k$ is a discrete valuation ring. One key consequence is that $\#\mathrm{Tors}_{\mathbb{Z}_p}\Big(\mu_{k_0,\epsilon}\big(Y_{\mathbb{T},\nu}\big)/\lambda_k\Big)$ is bounded above by $\#\mathcal{B}_\nu$, which is certainly independent of λ_k.

Finally, at slope zero the lattice $\mathbb{L}_{\mathcal{F}}$ is cut out of $\mathbb{T}_\infty$ using idempotents in the Hecke algebra, which implies that $Y_\nu = \frac{H^1(F_\nu, \mathbb{L}_{\mathcal{F}})}{H^1_{\mathcal{G},\dagger}(F_\nu, \mathbb{L}_{\mathcal{F}})}$ is a summand of $\widetilde{\mathcal{R}}^{\mathrm{norm}} \otimes_{\mathcal{R}} Y_{\mathbb{T},\nu}$. In particular, one concludes $\#\mathrm{Tors}_{\mathbb{Z}_p}\big(Y_\nu/\lambda_k\big)$ is bounded above by $\big(\#\mathcal{B}_\nu\big)^{[\widetilde{\mathcal{R}}^{\mathrm{norm}}:\mathcal{R}]}$ hence $\mathbb{L}_{\mathcal{F}}$ satisfies (Bnd) as required.

□

§Appendix – Technicalities on the Local H^0's and H^1's

We can now supply the missing proofs to several lemmas quoted in the main text. As many of the arguments are well known to the experts, we shall aim to be brief.

Lemma A.1. *If K' is any finite extension of $\mathbb{Q}_p$, then*

$$H^0\Big(K', \mathbb{C}_p\langle\langle s\rangle\rangle \otimes \big(\Psi^i\chi_{\mathrm{cy}}^j\big)\Big) \;=\; \begin{cases} 0 & \text{if } (i,j)\neq(0,0) \\ K'\langle\langle s\rangle\rangle & \text{if } (i,j)=(0,0). \end{cases}$$

Proof: Clearly $\mathbb{C}_p^{G_{K'}} = \widehat{K'} = K'$ because the local field K' is p-adically complete, whence $H^0\big(K', \mathbb{C}_p\langle\langle s\rangle\rangle\big) = K'\langle\langle s\rangle\rangle$ as the Galois action is solely on the coefficients. It thus remains to show vanishing for $H^0\big(K', \mathbb{C}_p\langle\langle s\rangle\rangle \otimes \Psi^i\chi_{\mathrm{cy}}^j\big)$ when $(i,j)\neq(0,0)$.

Case(I): The pair $(i,j)\in\mathbb{N}\times\{0\}$.
Suppose there exists a non-zero $x = \sum_{n=0}^{\infty} c_n s^n \in H^0\big(K', \mathbb{C}_p\langle\langle s\rangle\rangle(\Psi^i)\big)$ with $i>0$. For any $\sigma\in\mathrm{Gal}\big(\overline{\mathbb{Q}}_p/K'\big)$, one has the expansion

$$\Psi^i(\sigma) \;=\; \left(\sum_{k=0}^{\infty} \frac{\log^k\big(\chi_{\mathrm{cy}}(\sigma)\big)}{k!} s^k\right)^i \;=\; \sum_{k=0}^{\infty} \left(\sum_{j_1+\dots+j_i=k} \frac{1}{j_1!\dots j_i!}\right) \log^k\big(\chi_{\mathrm{cy}}(\sigma)\big) s^k$$

in which case

$$\sigma(x) \;=\; \sum_{n=0}^{\infty} \left(\sum_{j+k=n} \sigma(c_j)\left(\sum_{j_1+\dots+j_i=k} \frac{1}{j_1!\dots j_i!}\right)\log^k\big(\chi_{\mathrm{cy}}(\sigma)\big)\right) \times s^n.$$

Let $n_0\geq 0$ denote the smallest index such that $c_{n_0}\neq 0$; since x is $G_{K'}$-invariant,

$$c_{n_0} \;=\; \sum_{j+k=n_0} \sigma(c_j)\left(\sum_{j_1+\dots+j_i=k} \frac{1}{j_1!\dots j_i!}\right)\log^k\big(\chi_{\mathrm{cy}}(\sigma)\big) \;=\; \sigma(c_{n_0})$$

(here the right-hand equality follows because the coefficients $\sigma(c_j)=0$ for $j<n_0$). In particular $c_{n_0}\in K'\backslash\{0\}$, and furthermore

$$\begin{aligned} c_{n_0+1} \;&=\; \sum_{j+k=n_0+1} \sigma(c_j)\left(\sum_{j_1+\dots+j_i=k} \frac{1}{j_1!\dots j_i!}\right)\log^k\big(\chi_{\mathrm{cy}}(\sigma)\big) \\ &=\; \sigma(c_{n_0+1}) \;+\; i\times\sigma(c_{n_0})\times\log\big(\chi_{\mathrm{cy}}(\sigma)\big). \end{aligned}$$

We conclude that for all $\sigma\in G_{K'}$,

$$\frac{c_{n_0+1}}{c_{n_0}} = \sigma\left(\frac{c_{n_0+1}}{c_{n_0}}\right) + i\times\log\big(\chi_{\mathrm{cy}}(\sigma)\big), \quad \text{i.e.} \quad (\sigma-1)\left(\frac{c_{n_0+1}}{c_{n_0}}\right) = -i\times\log\big(\chi_{\mathrm{cy}}(\sigma)\big).$$

However the one-cocycle $\xi:\sigma\mapsto -i\times\log\big(\chi_{\mathrm{cy}}(\sigma)\big)$ generates $H^1\big(K',\mathbb{C}_p\big)$ by [Ta], and thus cannot be a coboundary. Consequently such an $x\neq 0$ could not have existed in the first place!

Case(II): The pair $(i,j)\in -\mathbb{N}\times\{0\}$.
Let's now assume $i<0$. If $\Psi^i(\sigma)=\sum_{n=0}^{\infty}a_n s^n$ say, then $\Psi^i(\sigma)\cdot\Psi^{-i}(\sigma)=1$ implies

$$\Big(a_0+a_1s+a_2s^2+\dots\Big)\times\Big(1-i\log\big(\chi_{\rm cy}(\sigma)\big)+\dots\Big) \;=\; 1,$$

hence $a_0=1$ and $a_1=i\log\big(\chi_{\rm cy}(\sigma)\big)$. An entirely identical argument to Case(I) then allows us to deduce the vanishing of $H^0\big(K',\mathbb{C}_p\langle\langle s\rangle\rangle(\Psi^i)\big)$ for negative i.

Case(III): The pair $(i,j)\in\mathbb{Z}\times\mathbb{Z}\backslash\{0\}$.
Here we treat the situation where the power of the p-cyclotomic character $j\neq 0$. One may write $\Psi^i(\sigma)=\sum_{n=0}^{\infty}b_n(\sigma)s^n$ with $b_n(\sigma)\in\mathbb{C}_p$ for each n, e.g. $b_0(\sigma)=1$. As before, assume there exists a non-zero $x=\sum_{n=0}^{\infty}c_ns^n\in H^0\big(K',\mathbb{C}_p\langle\langle s\rangle\rangle(\Psi^i\chi_{\rm cy}^j)\big)$; then for all $\sigma\in G_{K'}$,

$$\begin{aligned}\sigma(x) \;&=\; \left(\sum_{n=0}^{\infty}\sigma(c_n)s^n\right)\times\left(\sum_{n=0}^{\infty}b_n(\sigma)s^n\right)\times\chi_{\rm cy}^j(\sigma)\\ &=\; \chi_{\rm cy}^j(\sigma)\times\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sigma(c_{n-k})b_k(\sigma)\right)\times s^n.\end{aligned}$$

Let $n_1\geq 0$ be the smallest index for which $c_{n_1}\neq 0$. Because x is $G_{K'}$-invariant,

$$c_{n_1} \;=\; \chi_{\rm cy}^j(\sigma)\times\sum_{k=0}^{n_1}\sigma(c_{n_1-k})b_k(\sigma) \;=\; \chi_{\rm cy}^j(\sigma)\sigma\big(c_{n_1}\big)b_0(\sigma) \;=\; \chi_{\rm cy}^j(\sigma)\sigma\big(c_{n_1}\big)$$

therefore $c_{n_1}\in H^0\big(K',\mathbb{C}_p\otimes\chi_{\rm cy}^j\big)$. However the latter group is trivial using [Ta] again which means x cannot exist, i.e. $H^0\big(K',\mathbb{C}_p\langle\langle s\rangle\rangle(\Psi^i\chi_{\rm cy}^j)\big)$ must be zero.

□

Recall $\mathbb{W}$ is a G_K-representation satisfying Hypothesis(dR), and that $L=K_{n(\mathbb{W})}$.

Lemma A.2. *(i) Provided $m\ll 0$, $H^0\big(L,\mathbb{W}^*\otimes\mathrm{Fil}^m\mathbb{B}_{\rm dR}\big)$ is independent of m;*
(ii) For all $m\geq 0$, there exists a polynomial $h_m(s)\in s\cdot\mathbb{Q}_p[s]$ such that

$$H^0\left(L,\mathbb{W}^*(m)\otimes\frac{\mathbb{B}_{\rm dR}}{\mathrm{Fil}^1\mathbb{B}_{\rm dR}}\right)_{(h_m)} \;\cong\; L\langle\langle s\rangle\rangle\big[h_m^{-1}\big]^{\oplus e_{0,0}(\mathbb{W})}$$

as an isomorphism of finitely-generated $\mathbb{Q}_p\langle\langle s\rangle\rangle\big[h_m^{-1}\big]$-torsion free modules.

Proof: Write $\mathcal{C}$ for the affinoid algebra $\mathbb{C}_p\langle\langle s\rangle\rangle[s^{-1}]$; by Hypothesis(dR) one knows

$$\mathbb{W}^*\otimes\mathcal{C} \;\cong\; \bigoplus_{j\in\mathbb{Z}}\mathcal{C}\big(\chi_{\rm cy}^j\big)^{\oplus e_{0,-j}(\mathbb{W})} \;\oplus\; \bigoplus_{i,j\in\mathbb{Z},\;i\neq 0}\mathcal{C}\big(\Psi^i\chi_{\rm cy}^j\big)^{\oplus e_{-i,-j}(\mathbb{W})}$$

viewed as a splitting of $\mathcal{C}[G_L]$-modules. In particular, exploiting Lemma A.1:

$$H^0\big(L,\mathbb{W}^*\otimes\mathcal{C}(m)\big) \;=\; \bigoplus_{j\in\mathbb{Z}}H^0\big(L,\mathcal{C}(m+j)\big)^{\oplus e_{0,-j}(\mathbb{W})} \;=\; L\langle\langle s\rangle\rangle[s^{-1}]^{\oplus e_{0,m}(\mathbb{W})}$$

which vanishes if either $m\gg 0$ or $m\ll 0$.

Examining closely the exact sequence

$$0 \longrightarrow \left(\mathbb{W}^* \otimes \mathrm{Fil}^{m+1}\mathbb{B}_{\mathrm{dR}}[s^{-1}]\right)^{G_L} \longrightarrow \left(\mathbb{W}^* \otimes \mathrm{Fil}^{m}\mathbb{B}_{\mathrm{dR}}[s^{-1}]\right)^{G_L} \longrightarrow \left(\mathbb{W}^* \otimes \mathcal{C}(m)\right)^{G_L}$$

we deduce that $H^0\big(L, \mathbb{W}^* \otimes \mathrm{Fil}^{m+1}\mathbb{B}_{\mathrm{dR}}\big) \cong H^0\big(L, \mathbb{W}^* \otimes \mathrm{Fil}^{m}\mathbb{B}_{\mathrm{dR}}\big)$ whenever $m \ll 0$, and assertion (i) must hold true.

Remark: To prove part (ii) we note by [Ki, Prop 2.3], for $r = 0$ and $r = 1$ the cohomologies $H^r\big(L, \mathbb{W}^* \otimes \mathbb{C}_p\langle\langle s\rangle\rangle(\chi_{\mathrm{cy}}^n)\big)$ are killed by the polynomial $\det\big(\Phi + n\big)$ (recall from the proof of 2.3.1 that $\Phi = \frac{\log \gamma^{p^{n'}}}{\log \chi_{\mathrm{cy}}(\gamma^{p^{n'}})}$ was an endomorphism of 'W', the free $L\langle\langle s\rangle\rangle$-submodule inside $W_\infty = \big(\mathbb{W} \otimes \mathbb{C}_p\langle\langle s\rangle\rangle\big)^{H_K}$ which is Γ_L-stable).

N.B. Let $J = J_{\mathbb{W}}$ denote the largest integer for which $e_{0,-J}(\mathbb{W}) \neq 0$. By choosing $\mathcal{P}_m(s) := \prod_{w=m}^{J+m} \det\big(\Phi + m - w\big)$ and $\mathcal{Q}_m(s) := \prod_{w=0}^{m-1} \det\big(\Phi + m - w\big)$, we claim:

- $\left(\mathbb{W}^*(m) \otimes \dfrac{\mathbb{B}_{\mathrm{dR}}}{\mathrm{Fil}^{1-m}\mathbb{B}_{\mathrm{dR}}}\right)^{G_L} \big[(s \times \mathcal{P}_m)^{-1}\big] \cong L\langle\langle s\rangle\rangle\big[(s \times \mathcal{P}_m)^{-1}\big]^{\oplus e_{0,0}(\mathbb{W})}$
- $\left(\mathbb{W}^*(m) \otimes \dfrac{\mathbb{B}_{\mathrm{dR}}}{\mathrm{Fil}^{1}\mathbb{B}_{\mathrm{dR}}}\right)^{G_L} \big[(s \times \mathcal{Q}_m)^{-1}\big] \cong \left(\mathbb{W}^*(m) \otimes \dfrac{\mathbb{B}_{\mathrm{dR}}}{\mathrm{Fil}^{1-m}}\right)^{G_L} \big[(s \times \mathcal{Q}_m)^{-1}\big].$

Assuming both claims are true, then Lemma A.2(ii) follows immediately by setting the polynomial $h_m(s)$ equal to $s \times \prod_{w=0}^{J+m} \mathrm{N}_{L/\mathbb{Q}_p}\Big(\det\big(\Phi + m - w\big)\Big)$.

To prove the first claim, we once more apply our decomposition over $L = K_{n(\mathbb{W})}$. One discovers (using Lemma A.1) there exist isomorphisms

$$\left(\mathbb{W}^*(m) \otimes \mathcal{C}(-n)\right)^{G_L} \cong \left(\bigoplus_{j \in \mathbb{Z}} \mathcal{C}(m + j - n)^{\oplus e_{0,-j}}\right)^{G_L} = \bigoplus_{j \in \mathbb{Z}} L\langle\langle s\rangle\rangle\big[s^{-1}\big]^{\oplus e_{0,m-n}(\mathbb{W})}.$$

If $n > J + m$ then $e_{0,m-n}(\mathbb{W}) = 0$, which implies that $\big(\mathbb{W}^*(m) \otimes \mathcal{C}(-n)\big)^{G_L} = 0$. Considering the exact sequence

$$0 \longrightarrow \left(\mathbb{W}^*(m) \otimes \frac{\mathrm{Fil}^{1-n}\mathbb{B}_{\mathrm{dR}}}{\mathrm{Fil}^{1-m}\mathbb{B}_{\mathrm{dR}}}\big[s^{-1}\big]\right)^{G_L}$$
$$\longrightarrow \left(\mathbb{W}^*(m) \otimes \frac{\mathrm{Fil}^{-n}\mathbb{B}_{\mathrm{dR}}}{\mathrm{Fil}^{1-m}\mathbb{B}_{\mathrm{dR}}}\big[s^{-1}\big]\right)^{G_L} \longrightarrow \left(\mathbb{W}^*(m) \otimes \mathcal{C}(-n)\right)^{G_L}$$

the right-most term vanishes if $n > J + m$, and is killed by $\det\big(\Phi + m - n\big)$ if the integer n satisfies $J + m \geq n \geq m$.

Therefore one concludes the localisation $\left(\mathbb{W}^*(m) \otimes \frac{\mathrm{Fil}^{-n}\mathbb{B}_{\mathrm{dR}}}{\mathrm{Fil}^{1-m}\mathbb{B}_{\mathrm{dR}}}\right)^{G_L}\big[(s \times \mathcal{P}_m)^{-1}\big]$ is independent of $n \geq m$; moreover

$$\left(\mathbb{W}^*(m) \otimes \frac{\mathrm{Fil}^{-n}\mathbb{B}_{\mathrm{dR}}}{\mathrm{Fil}^{1-m}\mathbb{B}_{\mathrm{dR}}}\right)^{G_L}\big[(s \times \mathcal{P}_m)^{-1}\big] \cong \left(\mathbb{W}^*(m) \otimes \frac{\mathrm{Fil}^{-m}\mathbb{B}_{\mathrm{dR}}}{\mathrm{Fil}^{1-m}\mathbb{B}_{\mathrm{dR}}}\right)^{G_L}\big[(s \times \mathcal{P}_m)^{-1}\big]$$

which coincides with $\big(\mathbb{W}^*(m) \otimes \mathcal{C}(-m)\big)^{G_L}\big[\mathcal{P}_m^{-1}\big] = L\langle\langle s\rangle\rangle\big[(s \times \mathcal{P}_m)^{-1}\big]^{\oplus e_{0,0}(\mathbb{W})}$. Allowing $n \to \infty$ and observing that $\mathbb{B}_{\mathrm{dR}} = \varinjlim_n \mathrm{Fil}^{-n}\mathbb{B}_{\mathrm{dR}}$, the first claim follows.

Remark: To prove the second claim, we instead use the long exact sequence

$$0 \longrightarrow \Big(\mathbb{W}^*(m)\otimes\mathcal{C}(-n)\Big)^{G_L} \longrightarrow \Big(\mathbb{W}^*(m)\otimes\frac{\mathbb{B}_{\rm dR}}{{\rm Fil}^{1-n}\mathbb{B}_{\rm dR}}\big[s^{-1}\big]\Big)^{G_L}$$

$$\longrightarrow \Big(\mathbb{W}^*(m)\otimes\frac{\mathbb{B}_{\rm dR}}{{\rm Fil}^{-n}\mathbb{B}_{\rm dR}}\big[s^{-1}\big]\Big)^{G_L} \longrightarrow H^1\Big(L,\mathbb{W}^*(m)\otimes\ \mathcal{C}(-n)\Big).$$

The left-most and right-most terms are both killed by $\det\big(\Phi+m-n\big)$, so inductively

$$\Big(\mathbb{W}^*(m)\otimes\frac{\mathbb{B}_{\rm dR}}{{\rm Fil}^1}\Big)^{G_L}\Big[\big(s\times\mathcal{Q}_m\big)^{-1}\Big] \ \cong\ \ldots\ \cong\ \Big(\mathbb{W}^*(m)\otimes\frac{\mathbb{B}_{\rm dR}}{{\rm Fil}^{1-m}}\Big)^{G_L}\Big[\big(s\times\mathcal{Q}_m\big)^{-1}\Big]$$

which is the statement we were required to show. The argument is now complete. □

It remains to establish the orthogonality result used in the proof of Theorem 2.2.2.

Lemma A.3. *If $L=K_{n(\mathbb{W})}$ as before, then under the local pairing*

$$H^1\big(L,\mathbb{W}\otimes\mathbb{B}_{\rm dR}[s^{-1}]\big)\times H^0\big(L,\mathbb{W}^*\otimes\mathbb{B}_{\rm dR}[s^{-1}]\big)\ \stackrel{\det(\varepsilon)\circ\cup}{\longrightarrow}\ H^2\big(L,\mathbb{Q}_p\langle\!\langle s\rangle\!\rangle[s^{-1}](1)\big)$$

an element $\mathbf{x}\in H^1\big(L,\mathbb{W}\otimes\mathbb{B}_{\rm dR}[s^{-1}]\big)$ is orthogonal to all of $H^0\big(L,\mathbb{W}^*\otimes\mathbb{B}_{\rm dR}[s^{-1}]\big)$, if and only if

$$\mathbf{x}\in\ker^+_{\rm dR}\ :=\ \varinjlim_{\mathcal{U}\subset\mathfrak{X}^+_{\rm dR}}\ {\rm Ker}\Big(H^1\big(L,\mathbb{W}\otimes\mathbb{B}_{\rm dR}[s^{-1}]\big)\longrightarrow\bigoplus_{k\in\mathcal{U}}H^1\big(L,\mathbb{W}_k\otimes_{\mathbb{Q}_p}B_{\rm dR}\big)\Big).$$

Proof: Since $\mathbb{B}_{\rm dR}=\varinjlim {\rm Fil}^{-m}\mathbb{B}_{\rm dR}$, it suffices to show the lemma with $\mathbb{B}_{\rm dR}$ replaced by ${\rm Fil}^{-m}\mathbb{B}_{\rm dR}$ for m sufficiently large (in fact, considering instead the Tate twist $\mathbb{W}(m)$ in place of $\mathbb{W}$, without loss of generality one may even assume that $m=0$). We must therefore verify the quotient pairing

$$\frac{H^1\big(L,\mathbb{W}\otimes\mathbb{B}^+_{\rm dR}[s^{-1}]\big)}{\Delta^{+,0}_{\rm dR}(\mathbb{W})}\times H^0\big(L,\mathbb{W}^*\otimes\mathbb{B}^+_{\rm dR}[s^{-1}]\big)\ \stackrel{\det(\varepsilon)\circ\cup}{\longrightarrow}\ H^2\big(L,\mathbb{Q}_p\langle\!\langle s\rangle\!\rangle[s^{-1}](1)\big)$$

is non-degenerate on the left, where the subgroup

$$\Delta^{+,0}_{\rm dR}(\mathbb{W})\ :=\ \varinjlim_{\mathcal{U}\subset\mathfrak{X}^+_{\rm dR}}\ {\rm Ker}\Big(H^1\big(L,\mathbb{W}\otimes\mathbb{B}^+_{\rm dR}[s^{-1}]\big)\longrightarrow\bigoplus_{k\in\mathcal{U}}H^1\big(L,\mathbb{W}_k\otimes_{\mathbb{Q}_p}B^+_{\rm dR}\big)\Big).$$

There are two steps: (1) assume the result holds modulo ${\rm Fil}^1$ then deduce it in full, and (2) establish the result indeed holds modulo ${\rm Fil}^1$.

Step 1: Let's make the initial assumption

Hypothesis(Fil^1). *Consider a fixed element $\mathbf{x}\in H^1\big(L,\mathbb{W}\otimes\mathbb{B}^+_{\rm dR}[s^{-1}]\big)-\Delta^{+,0}_{\rm dR}(\mathbb{W})$; if $\mathbf{x}$ is orthogonal to $H^0\big(L,\mathbb{W}^*\otimes\mathbb{B}^+_{\rm dR}[s^{-1}]\big)$, then $\mathbf{x}$ lies in $H^1\big(L,\mathbb{W}\otimes{\rm Fil}^1\mathbb{B}_{\rm dR}[s^{-1}]\big)$.*

The goal is to deduce the full result from this. Write $\mathbf{tw}_1$ to denote the isomorphism

$$H^1\big(L,\mathbb{W}\otimes{\rm Fil}^1\mathbb{B}_{\rm dR}[s^{-1}]\big)\ \stackrel{\sim}{\longrightarrow}\ H^1\big(L,\mathbb{W}(1)\otimes\mathbb{B}_{\rm dR}[s^{-1}]\big)$$

sending the one cocycle $\sigma\mapsto w_\sigma\otimes t\cdot b$ to the one cocycle $\sigma\mapsto w_\sigma\cdot(\zeta_{p^n})_n\otimes b$, where as usual the uniformiser $t=\log\big[(\zeta_{p^n})_n\big]_R$.

Suppose $\mathbf{x} \notin \Delta^{+,0}_{\mathrm{dR}}(\mathbb{W})$ and that $\mathbf{x}$ is orthogonal to all of $H^0\big(L, \mathbb{W}^* \otimes \mathbb{B}^+_{\mathrm{dR}}[s^{-1}]\big)$. Certainly Hypothesis($\mathrm{Fil}^1$) implies $\mathbf{x} \in H^1\big(L, \mathbb{W} \otimes \mathrm{Fil}^1\mathbb{B}_{\mathrm{dR}}[s^{-1}]\big)$, while we also know that $\mathbf{x}\big|_{s=k-1} \neq 0$ for infinitely many $k \in \mathfrak{X}^+_{\mathrm{dR}}$ by the definition of $\Delta^{+,0}_{\mathrm{dR}}(\mathbb{W})$. Clearly $\mathbf{tw}_1(\mathbf{x})\big|_{s=k-1} \neq 0$ at these very same weights k, thus $\mathbf{tw}_1(\mathbf{x}) \notin \Delta^{+,0}_{\mathrm{dR}}\big(\mathbb{W}(1)\big)$. Now consider the analogous pairing with $\mathbb{W}(1)$ in place of $\mathbb{W}$:

$$H^1\big(L, \mathbb{W}(1) \otimes \mathbb{B}^+_{\mathrm{dR}}[s^{-1}]\big) \times H^0\big(L, \mathbb{W}^*(-1) \otimes \mathbb{B}^+_{\mathrm{dR}}[s^{-1}]\big) \overset{\det(\varepsilon)\circ\cup}{\longrightarrow} H^2\big(L, \mathbb{Q}_p\langle\langle s\rangle\rangle[s^{-1}](1)\big).$$

Applying Hypothesis(Fil^1) instead to the twisted representation $\mathbb{W}(1)$ implies that $\mathbf{tw}_1(\mathbf{x}) \in H^1\big(L, \mathbb{W}(1) \otimes \mathrm{Fil}^1\mathbb{B}_{\mathrm{dR}}[s^{-1}]\big)$, in other words $\mathbf{x} \in H^1\big(L, \mathbb{W} \otimes \mathrm{Fil}^2\mathbb{B}_{\mathrm{dR}}[s^{-1}]\big)$. Repeating the argument indefinitely, one conclude

$$\mathbf{x} \in \bigcap_{n \geq 0} H^1\big(L, \mathbb{W} \otimes \mathrm{Fil}^n\mathbb{B}_{\mathrm{dR}}[s^{-1}]\big) = \{0\}$$

which contradicts the fact $\mathbf{x} \notin \Delta^{+,0}_{\mathrm{dR}}(\mathbb{W})$.

Conversely suppose $\mathbf{x} \in \Delta^{+,0}_{\mathrm{dR}}(\mathbb{W})$; then $\mathbf{x}\big|_{s=k-1} = 0$ for infinitely many $k \in \mathfrak{X}^+_{\mathrm{dR}}$. It follows that $\mathbf{x}$ must be $\mathbb{Q}_p\langle\langle s\rangle\rangle[s^{-1}]$-torsion, and so too must $\det(\varepsilon)\big(\mathbf{x} \cup \mathbf{y}\big)$ for all $\mathbf{y} \in H^0\big(L, \mathbb{W}^* \otimes \mathbb{B}^+_{\mathrm{dR}}[s^{-1}]\big)$. However $H^2\big(L, \mathbb{Q}_p\langle\langle s\rangle\rangle[s^{-1}](1)\big) \cong \mathbb{Q}_p\langle\langle s\rangle\rangle[s^{-1}]$ is free thus $\det(\varepsilon)\big(\mathbf{x} \cup \mathbf{y}\big)$ must be zero, and one obtains $\mathbf{x} \in H^0\big(L, \mathbb{W}^* \otimes \mathbb{B}^+_{\mathrm{dR}}[s^{-1}]\big)^{\perp}$.

Step 2: To prove the result modulo Fil^1, we employ the commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathbb{Q}_p\langle\langle s\rangle\rangle(1) & \longrightarrow & \big(\mathbb{B}^{\varphi=p}_{\mathrm{cris}} \cap \mathbb{B}^+_{\mathrm{dR}}\big) \oplus \mathrm{Fil}^1\mathbb{B}_{\mathrm{dR}} & \longrightarrow & \mathbb{B}^+_{\mathrm{dR}} & \to 0 \\
& \big\| & & \big\downarrow \text{mod } \mathrm{Fil}^1 & & \big\downarrow \text{mod } \mathrm{Fil}^1 & \\
0 \longrightarrow & \mathbb{Q}_p\langle\langle s\rangle\rangle(1) & \longrightarrow & \mathbb{B}^{\varphi=p}_{\mathrm{cris}} \cap \mathbb{B}^+_{\mathrm{dR}} & \longrightarrow & \mathbb{C}_p\langle\langle s\rangle\rangle & \to 0
\end{array}$$

with exact rows. If $\mathcal{C} = \mathbb{C}_p\langle\langle s\rangle\rangle[s^{-1}]$, let $\partial : H^1\big(L, \mathcal{C}\big) \longrightarrow H^2\big(L, \mathbb{Q}_p\langle\langle s\rangle\rangle[s^{-1}](1)\big)$ denote the boundary map in the cohomology of

$$0 \longrightarrow \mathbb{Q}_p\langle\langle s\rangle\rangle[s^{-1}](1) \longrightarrow \big(\mathbb{B}^{\varphi=p}_{\mathrm{cris}} \cap \mathbb{B}^+_{\mathrm{dR}}\big)[s^{-1}] \longrightarrow \mathcal{C} \to 0.$$

Remark: It's sufficient to show that under the pairing

$$\{\!\{-,-\}\!\}_{L,\mathbb{W}} : H^1\big(L, \mathbb{W} \otimes \mathcal{C}\big) \times H^0\big(L, \mathbb{W}^* \otimes \mathcal{C}\big) \overset{\partial\circ\cup}{\longrightarrow} H^2\big(L, \mathbb{Q}_p\langle\langle s\rangle\rangle[s^{-1}](1)\big)$$

an element $\mathbf{x} \in H^1\big(L, \mathbb{W} \otimes \mathcal{C}\big)$ is orthogonal to all of $H^0\big(L, \mathbb{W}^* \otimes \mathcal{C}\big)$, if and only if $\mathbf{x}$ belongs to $\mathfrak{K}_{\mathbb{W},\mathbb{C}_p} := \varinjlim_{\mathcal{U} \subset \mathfrak{X}^+_{\mathrm{dR}}} \mathrm{Ker}\Big(H^1\big(L, \mathbb{W} \otimes \mathcal{C}\big) \longrightarrow \bigoplus_{k \in \mathcal{U}} H^1\big(L, \mathbb{W}_k \otimes_{\mathbb{Q}_p} \mathbb{C}_p\big)\Big)$.

First assume $\mathbf{x}$ is orthogonal to $H^0\big(L, \mathbb{W}^* \otimes \mathcal{C}\big)$. For some open subset $\mathcal{U}'' \subset \mathfrak{X}^+_{\mathrm{dR}}$ we have isomorphisms $H^r\big(L, \mathbb{M} \otimes \mathcal{C}\big)_{s=k-1} \cong H^r\big(L, \mathbb{M}_k \otimes_{\mathbb{Q}_p} \mathbb{C}_p\big)$ at every $k \in \mathcal{U}''$, with $\mathbb{M} = \mathbb{W}, \mathbb{W}^*$ and $r = 0, 1$. Furthermore, each p-adic pairing

$$H^1\big(L, \mathbb{W}_k \otimes_{\mathbb{Q}_p} \mathbb{C}_p\big) \times H^0\big(L, \mathbb{W}^*_k \otimes_{\mathbb{Q}_p} \mathbb{C}_p\big) \overset{\partial_k\circ\cup}{\longrightarrow} H^2\big(L, \mathbb{Q}_p(1)\big)$$

is perfect by [BK, 3.8.8], as the underlying p-adic representations $\mathbb{W}_k$ are de Rham.

Therefore at all such $k \in \mathcal{U}''$, each specialisation $\mathbf{x}\big|_{s=k-1}$ will be orthogonal to $H^0\big(L, \mathbb{W}^* \otimes \mathcal{C}\big)_{s=k-1} = H^0\big(L, \mathbb{W}_k^* \otimes_{\mathbb{Q}_p} \mathbb{C}_p\big)$ under $\partial_k \circ \cup$; as a corollary $\mathbf{x}\big|_{s=k-1} = 0$, i.e. $\mathbf{x} \in \mathrm{Ker}\Big(H^1\big(L, \mathbb{W} \otimes \mathcal{C}\big) \longrightarrow \bigoplus_{k \in \mathcal{U}''} H^1\big(L, \mathbb{W}_k \otimes_{\mathbb{Q}_p} \mathbb{C}_p\big)\Big) \subset \mathfrak{K}_{\mathbb{W},\mathbb{C}_p}$.

Alternatively, if one fixes $\mathbf{x} \in \mathfrak{K}_{\mathbb{W},\mathbb{C}_p}$ then $\mathbf{x}\big|_{s=k-1} = 0$ at infinitely many $k \in \mathfrak{X}_{\mathrm{dR}}^+$, in which case $\mathbf{x} \in \mathrm{Tors}_{\mathbb{Q}_p\langle\langle s\rangle\rangle[s^{-1}]}\Big(H^1\big(L, \mathbb{W} \otimes \mathcal{C}\big)\Big)$. In particular, it is killed by some element $z_{\mathbf{x}}(s) \in \mathbb{Q}_p\langle\langle s\rangle\rangle[s^{-1}]$; one then calculates for each $\mathbf{y} \in H^0\big(L, \mathbb{W}^* \otimes \mathcal{C}\big)$,

$$z_{\mathbf{x}}(s) \cdot \big\{\!\!\big\{\mathbf{x}, \mathbf{y}\big\}\!\!\big\}_{L,\mathbb{W}} \;=\; \big\{\!\!\big\{z_{\mathbf{x}}(s) \cdot \mathbf{x}, \mathbf{y}\big\}\!\!\big\}_{L,\mathbb{W}} \;=\; \big\{\!\!\big\{0, \mathbf{y}\big\}\!\!\big\}_{L,\mathbb{W}} \;=\; 0$$

i.e. $\big\{\!\!\big\{\mathbf{x}, \mathbf{y}\big\}\!\!\big\}_{L,\mathbb{W}}$ is torsion inside $H^2\big(L, \mathbb{Q}_p\langle\langle s\rangle\rangle[s^{-1}](1)\big)$. But the latter is a free $\mathbb{Q}_p\langle\langle s\rangle\rangle[s^{-1}]$-module, whence $\big\{\!\!\big\{\mathbf{x}, \mathbf{y}\big\}\!\!\big\}_{L,\mathbb{W}} = 0$ at every element $\mathbf{y} \in H^0\big(L, \mathbb{W}^* \otimes \mathcal{C}\big)$.

As both directions are now established, one concludes Hypothesis(Fil^1) holds. □

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