$(\mathcal{F}, \mathcal{G})$-abundant semigroups

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Abstract

On a semigroup S, define the equivalence relation

$$\mathcal{F} = \{(a, b) \in S \times S \mid \forall x \in S : xa = x \Leftrightarrow xb = x\},$$

and define $\mathcal{G}$ dually. We say S is *$\mathcal{F}$-abundant* if there is an idempotent in every $\mathcal{F}$-class, and similarly for $\mathcal{G}$-abundance, and we say S is $(\mathcal{F}, \mathcal{G})$-abundant if it is both $\mathcal{F}$-abundant and $\mathcal{G}$-abundant. These concepts are analogous to the notions of regularity and one- and two-sided abundance, defined in terms of Green's relations $\mathcal{L}$ and $\mathcal{R}$, and their generalisations $\mathcal{L}^*$ and $\mathcal{R}^*$. We relate this new form of abundance to the earlier ones, considering in particular the analogs of superabundance and amiability.

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1 Introduction

First, we recall some notation. For a non-empty set X:

- denote the full transformation semigroup on X by $\mathrm{T}(X)$;
- denote the semigroup of partial transformations on X by $\mathrm{PT}(X)$;
- denote the (inverse) semigroup of injective partial transformations on X by $\mathrm{I}(X)$.

Denote by 1_X the identity element of all of the above semigroups, and let $\emptyset$ denote the empty function (a member of all but the first of them). For a semigroup S, denote its set of idempotents by $E(S)$, denote by S^0 the result of adjoining a zero 0 to S, and denote by S^1 the result of adjoining an identity element 1 to S.

The notion of regularity is a fundamental one in semigroup theory, and has several characterisations, the most important of which is in terms of Green's relations. Thus, a semigroup S is regular if and only if for all $a \in S$, there is $e \in E(S)$ such that $(a, e) \in \mathcal{L}$ (or equivalently, if for all $a \in S$, there is $e \in E(S)$ such that $(a, e) \in \mathcal{R}$). If one requires the idempotent in each $\mathcal{L}$-class and $\mathcal{R}$-class of a regular semigroup to be unique, one recovers the class of inverse semigroups. If each $\mathcal{H}$-class (where $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$) contains an idempotent, then it is unique, and we recover the class of completely regular semigroups. These two special classes of regular semigroups may be viewed as unary semigroups, where the unary operation is a kind of generalised inversion, and there are familiar equational axiomatisations. Regularity is a relatively common property for semigroups (for example, full transformation semigroups are regular), and much structure theory has been developed specifically for them.

Generalisations of regularity may be defined in terms of so-called *generalised Green's relations*. Thus for any semigroup S, we define the relation $\mathcal{R}^*$ via $(a,b) \in \mathcal{R}^*$ if and only if $(a,b) \in \mathcal{R}$ in some oversemigroup of S, or equivalently (as is well-known), for all $x, y \in S^1$, $xa = ya$ if and only if $xb = yb$. (We define $\mathcal{L}^*$ in the obvious dual way). Then $\mathcal{R} \subseteq \mathcal{R}^*$. The semigroup S is *abundant* if every $\mathcal{R}^*$-class of S and every $\mathcal{L}^*$-class of S contains an idempotent. This time, we must specify both the left and right-sided conditions: one does not imply the other; see [5]. Regularity implies abundance but not conversely. If the idempotents in each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class of an abundant semigroup are unique, we obtain *amiable* semigroups as in [1]. If each $\mathcal{H}^*$-class (where $\mathcal{H}^* = \mathcal{L}^* \cap \mathcal{R}^*$) contains an idempotent, then it is unique, and we get *superabundant* semigroups as in [5]. Again, in each of these cases, one can define unary operations by picking out the idempotent in the relevant class of each element to define unary or perhaps biunary semigroup structures which again have axiomatisations appearing in the literature (see Section 5.6 of [3] for both). A motivation for the definition of abundance is the structure theory developed for regular semigroups, and considerable prograss can be made with abundant semigroups; see for example [4], [5] and much work since.

An equivalent definition of regularity is as follows: the semigroup S is regular if for all $a \in S$, the left ideal S^1a is generated by an idempotent (equivalently if for all $a \in S$, the right ideal aS^1 is generated by an idempotent). In a similar way, one says that a semigroup with zero S is a *left Baer semigroup* if for all $a \in S$, the left annihilator $(0 : a) = \{s \in S \mid sa = 0\}$, which is a left ideal, is generated by an idempotent (and there are obvious right- and two-sided versions, called right Baer and Baer semigroups respectively).

All of the above notions also have some motivation from the theory of semigroups of partial functions with additional operations. Regular semigroups are generalisations of inverse semigroups, which model semigroups of one-to-one partial functions under composition and inversion (indeed there is a well-known "Cayley theorem" making explicit this connection: the Vagner-Preston theorem shows that every inverse semigroup emnbeds in $\mathrm{I}(X)$ for some X, where the embedding respects the inverse operation). Abundant semigroups and especially adequate semigroups (in which the idempotents commute) can be viewed as generalising semigroups of one-to-one partial functions in which inversion is not modelled but one does have notions of domain and range of a partial function: for $f \in \mathrm{I}(X)$, $D(f) = \{(x,x) \mid x \in \mathrm{dom}(f)\}$ is such that $(f, D(f)) \in \mathcal{L}^*$; similarly for the dually defined $R(f)$. Both D and R may also be defined on $\mathrm{PT}(X)$, although this semigroup is not abundant.

Examples of left Baer semigroups arise from semigroups of partial functions in which one has a notion of "antodomain", modelling the complement of the domain of a partial function. For $f \in \mathrm{PT}(X)$, define $A(f) = \{(x,x) \mid x \notin \mathrm{dom}(f)\}$; note that $A^2 = D$. Also observe that $\mathrm{PT}(X)$ has zero the empty partial function, and then $(0 : f)$ is generated as a left ideal by $A(f)$, so $\mathrm{PT}(X)$ is left Baer. An axiomatisation of semigroups of partial functions under composition and A appears in [8].

Another natural operation on partial functions is "fix", in which one defines $F(f) = \{(x,x) \mid xf = x\}$ for any $f \in \mathrm{PT}(X)$. This operation is considered in [7] and [9], and we return to it in more detail later. For now, we note that for all $g \in \mathrm{PT}(X)$, $gf = g$ if and only if f fixes $\mathrm{ran}(g)$, if and only if $gF(f) = g$.

Definition 1.1 *Let S be a semigroup. For $a \in S$, define* $\mathrm{fix}(a) = \{x \in S \mid xa = x\}$*, and define* $\mathrm{cofix}(a)$ *dually.*

Evidently, in any semigroup S, $\mathrm{fix}(a)$ is a left ideal of S if non-empty, and $\mathrm{fix}(a) \subseteq S^1 a$ with equality if and only if a is idempotent. The following definitions are obviously equivalent to the ones given in the abstract.

Definition 1.2 *Define the equivalence relation $\mathcal{F}$ on the semigroup S by setting $(a, b) \in \mathcal{F}$ whenever* $\mathrm{fix}(a) = \mathrm{fix}(b)$*. Define $\mathcal{G}$ dually using* cofix*. Define $\mathcal{K} = \mathcal{F} \cap \mathcal{G}$.*

The following terminologies are motivated by their use for the generalised Green's relations $\mathcal{L}^*$ and $\mathcal{R}^*$.

Definition 1.3 *Let $\mathcal{A}, \mathcal{B}$ be equivalence relations defined on all semigroups. The semigroup S is* $\mathcal{A}$-abundant *(*$\mathcal{A}$-amiable*) if every $\mathcal{A}$-class of S contains an idempotent (a unique idempotent), and is* $(\mathcal{A}, \mathcal{B})$*-abundant (*$(\mathcal{A}, \mathcal{B})$-amiable*) if it is both $\mathcal{A}$-abundant and $\mathcal{B}$-abundant ($\mathcal{A}$-amiable and $\mathcal{B}$-amiable).*

Using this notation, the following are clear for the semigroup S:

- S is regular if and only if it is $\mathcal{L}$-abundant (equivalently, $\mathcal{R}$-abundant);
- S is completely regular if and only if it is $\mathcal{H}$-abundant;
- S is inverse if and only if it is $(\mathcal{L}, \mathcal{R})$-amiable;
- S is right abundant if and only if it is $\mathcal{L}^*$-abundant, and abundant if and only if it is $(\mathcal{L}^*, \mathcal{R}^*)$-abundant;
- S is superabundant if and only if it is $\mathcal{H}^*$-abundant;
- S is amiable if and only if it is $(\mathcal{L}^*, \mathcal{R}^*)$-amiable.

Our goal in what follows is to study the versions of abundance and amiability defined in terms of $\mathcal{F}$ and $\mathcal{G}$, and relate them to the more familiar versions just listed.

The next obsevation is immediate from the fact that for an idempotent e in a semigroup E, $\mathrm{fix}(e) = Se$, and makes clear the parallels with not just regular semigroups but also one and two-sided Baer semigroups.

Proposition 1.4 *The semigroup S is $\mathcal{F}$-abundant if and only if, for all $a \in S$,* $\mathrm{fix}(a)$ *is non-empty and generated as a left ideal by an idempotent. Similarly for $\mathcal{G}$-abundant and $(\mathcal{F}, \mathcal{G})$-abundant.*

From the above observation for $\mathrm{PT}(X)$ that $gf = g$ if and only if $gF(f) = g$, it follows that $\mathrm{PT}(X)$ is $\mathcal{F}$-abundant. (See also Proposition 3.5 below.)

The plan of this paper is as follows. After some necessary preliminaries on idempotent elements of a semigroup in Section 2, we show in Section 3 that in the multiplicative semigroups of rings with identity, $\mathcal{R}^*$-abundance, $\mathcal{F}$-abundance and the left Baer property all agree. The canonical example of a regular semigroup is $\mathrm{T}(X)$, and we shall show that with adjoined zero, it is $(\mathcal{F},\mathcal{G})$-abundant. Moreover $\mathrm{PT}(X)$ and $\mathrm{I}(X)$ are also shown to be $(\mathcal{F},\mathcal{G})$-abundant. We show that $(\mathcal{F},\mathcal{G})$-abundant semigroups may be neither $\mathcal{L}^*$- nor $\mathcal{R}^*$-abundant.

In Section 4, we consider $\mathcal{K}$-abundant semigroups, which are the analogs of completely regular and superabundant semigroups but defined using $\mathcal{F}$ and $\mathcal{G}$. We give examples, show that the idempotent in each $\mathcal{K}$-class is unique, and note that they form a proper quaivariety when viewed as unary semigroups.

In Section 5, we compare $(\mathcal{F},\mathcal{G})$-amiable semigroups to inverse and $(\mathcal{L}^*,\mathcal{R}^*)$-amiable semigroups. We show that $(\mathcal{F},\mathcal{G})$-amiability forces $\mathcal{F}$ and $\mathcal{G}$ to be equal, and so they are also $\mathcal{K}$-abundant (although the converse fails), and we characterise them as unary semigroups. We show that, unlike the amiable case considered in [2], when S is $(\mathcal{F},\mathcal{G})$-amiable there is no small number of excluded cases that can be used to characterise when $E(S)$ commutes. Despite this, the set of idempotents of any $(\mathcal{F},\mathcal{G})$-amiable semigroup are shown to form a semilattice under the natural order (something not true for amiable semigroups), and we describe the meet operation. We show that a regular semigroup is $(\mathcal{F},\mathcal{G})$-amiable if and only if it is isomorphic to an inverse semigroup S of one-to-one partial functions in which the identity map on the fix-set of every element of S is itself in S.

2 Some preliminaries on idempotent elements

Let S be a semigroup. The *natural right quasiorder* on $E(S)$ is given by $e \leq_r f$ if and only if $e = ef$, and the *natural left quasiorder* on $E(S)$ is given by $e \leq_l f$ if and only if $e = fe$. Denote by $\sim_r$ and $\sim_l$ the respective induced equivalence relations. It is easily seen that $(e,f) \in \mathcal{L}$ if and only if $(e,f) \in \mathcal{L}^*$, if and only if $e \sim_r f$. The *natural order* $\leq$ on $E(S)$ is the intersection of $\leq_l$ and $\leq_r$, so that $e \leq f$ if and only if $e = ef = fe$, and is a partial order on $E(S)$.

Let E be a nonempty subset of $E(S)$. All of the following concepts are defined in [10], although many of them were first defined earlier than that. We say E is *right pre-reduced* if $\leq_r$ is a partial order (that is, $e = ef$ and $f = fe$ imply $e = f$), and *left pre-reduced* if $\leq_l$ is a partial order. Clearly, $E \subseteq E(S)$ is right pre-reduced if and only if no two elements of E are related by $\sim_r$. We say E is *pre-reduced* if it is both left and right pre-reduced.

We say E is *right reduced* if $\leq_r \subseteq \leq_l$ when restricted to E, in which case E is right pre-reduced and $\leq_r$ is the natural order on E. We say E is *left reduced* if $\leq_l \subseteq \leq_r$ on E, and E is *reduced* if it is both left and right reduced. We say E *commutes* if $ef = fe$ for all $e, f \in E$; such E is clearly reduced. If $E(S)$ commutes then it is a semilattice.

From Proposition 2.11 and Corollary 2.12 in [10] together with a familiar fact about regular semigroups, we obtain the folllowing.

Result 2.1 *Suppose S is a semigroup. If $E(S)$ is (right) pre-reduced then $E(S)$ is (right) reduced. If S is regular and $E(S)$ is reduced then $E(S)$ commutes and S is an inverse semigroup.*

The following is also clear.

Proposition 2.2 *Suppose a left ideal I of the semigroup S is generated by the idempotent e. Then the idempotent f also generates I if and only if $e \sim_r f$.*

3 $\mathcal{F}$-abundance and $(\mathcal{F}, \mathcal{G})$-abundance

The following is easily seen.

Proposition 3.1 *Let S be a semigroup, with $e, f \subseteq E(S)$. Then $(e, f) \in \mathcal{F}$ if and only if $e \sim_r f$. Hence $\mathcal{L}, \mathcal{L}^*$ and $\mathcal{F}$ are all equal when restricted to $E(S)$, and similarly for $\mathcal{R}, \mathcal{R}^*$ and $\mathcal{G}$, as well as $\mathcal{H}, \mathcal{H}^*$ and $\mathcal{K}$.*

Evidently, if a semigroup is commutative, the relations $\mathcal{F}$ and $\mathcal{G}$ agree, and this applies to $\mathcal{L}$ and $\mathcal{R}$ as well as $\mathcal{L}^*$ and $\mathcal{R}^*$. Although $\mathcal{L}$ and $\mathcal{R}$ are typically different in an inverse semigroup, we have the following.

Lemma 3.2 *For an inverse semigroup S and any $a \in S$, $\mathrm{fix}(a) = \mathrm{fix}(a')$, so $\mathcal{F} = \mathcal{G}$ on S.*

Proof. Now for $s \in S$, if $sa = s$ then because S is an inverse semigroup,

$$sa' = saa' = ss'saa' = saa's's = sa(sa)'s = ss's = s,$$

so $\mathrm{fix}(a) \subseteq \mathrm{fix}(a')$; by symmetry, they are equal, and dualising gives $\mathrm{cofix}(a) = \mathrm{cofix}(a')$. So if $(a, b) \in \mathcal{F}$, then $(a', b') \in \mathcal{G}$ (since $xa = x$ if and only if $a'x' = x'$ for all $x, a \in S$), so $\mathrm{cofix}(a) = \mathrm{cofix}(a') = \mathrm{cofix}(b') = \mathrm{cofix}(b)$, and so $(a, b) \in \mathcal{G}$. Hence $\mathcal{F} \subseteq \mathcal{G}$. By symmetry, we have equality. □

One of the motivations for considering the $\mathcal{F}$-abundant and $(\mathcal{F}, \mathcal{G})$-abundant properties is that they coincide with the one- and two-sided abundant properties (and indeed the left Baer and Baer properties) on the multiplicative semigroup of a ring with identity. First note that in such a ring R, $e \in E(R)$ if and only if $1 - e \in E(R)$.

Theorem 3.3 *Suppose R is a ring with identity. Considering only the multiplicative semigroup of R, the following are equivalent: R is $\mathcal{R}^*$-abundant ($(\mathcal{L}^*, \mathcal{R}^*)$-abundant); R is $\mathcal{F}$-abundant ($(\mathcal{F}, \mathcal{G})$-abundant); R is left Baer (Baer).*

Proof. Pick $a, b \in R$. The following are equivalent:

- for all $x \in R$, $xa = 0$ if and only if $xb = 0$;
- for all $x \in R$, $x(1 - a) = x$ if and only if $x(1 - b) = x$;
- for all $x, y \in R$, $(x - y)(1 - a) = x - y$ if and only if $(x - y)(1 - b) = x - y$;
- for all $x, y \in R$, $xa = ya$ if and only if $xb = yb$.

Hence $(a, b) \in \mathcal{R}^*$ if and only if $(1 - a, 1 - b) \in \mathcal{F}$, and so for $a \in S$ and $e \in E(R)$, $(a, e) \in \mathcal{R}^*$ if and only if $(1 - a, 1 - e) \in \mathcal{F}$. It follows that R is $\mathcal{R}^*$-abundant if and only if it is $\mathcal{F}$-abundant. Moreover $(a, e) \in \mathcal{F}$ if and only if $(0 : a) = S(1 - e)$, so S is $\mathcal{F}$-abundant if and only if it is left Baer. The two-sided versions follow by dualising. □

A ring with identity whose multiplicative semigroup is (left) Baer is called a *(left) Rickart ring* in the literature (the term "Baer ring" referring to something more specialised).

Amongst the most familiar regular semigroups are $\mathrm{T}(X), \mathrm{PT}(X)$, and $\mathrm{I}(X)$ (the third of which is inverse). But $\mathrm{T}(X)$ is not generally $\mathcal{F}$-abundant, since there can be an element a of $\mathrm{T}(X)$ for which $\mathrm{fix}(a) = \emptyset$; for example, if $|X| > 1$ and a is any permutation of X that fixes no elements of X, then $\mathrm{fix}(a) = \emptyset$. However, this is easily remedied by adjoining a zero element.

Proposition 3.4 $\mathrm{T}(X)^0$ *is $\mathcal{F}$-abundant. Moreover* $\mathrm{T}(X)$ *is $\mathcal{G}$-abundant, whence so is* $\mathrm{T}(X)^0$*, and so* $\mathrm{T}(X)^0$ *is $(\mathcal{F}, \mathcal{G})$-abundant.*

Proof. Let $S = \mathrm{T}(X)^0$. Now $e \in E(S)$ if and only if $e = 0$ or e is a projection onto a subset of X. For $s \in S$, if s fixes elements of X, let e be a projection onto the elements of X fixed by s; otherwise let $e = 0$. For $t \in S$, the following are equivalent: $ts = t$; $t = 0$ or else s fixes the range of t; $t = 0$ or the range of t is contained in the range of e; $te = t$. Thus $(s, e) \in \mathcal{F}$, and so S is $\mathcal{F}$-abundant.

For $s \in \mathrm{T}(X)$, let f be any projection whose kernel equals the equivalence relation generated by the pairs $\{(xs, x) \mid x \in X\}$; then for any $x \in X$, $xsf = xf$, and so $sf = f$ by definition. Hence if $ft = t$ then $st = t$. Moreover if $st = t$ for some $t \in \mathrm{T}(X)$, then for all $x \in X$, $(xs)t = xt$, so $\ker(t)$ contains $\{(xs, x) \mid x \in X\}$, and so it also contains $\ker(f) = \{(x, y) \mid xf = yf\}$, which is the equivalence relation generated by $\{(xf, x) \mid x \in X\}$ because f is a projection, and so $xft = xt$ for all $x \in X$, so $ft = t$ by definition. So $(s, f) \in \mathcal{G}$. This shows $\mathrm{T}(X)$ is $\mathcal{G}$-abundant. It follows easily that $\mathrm{T}(X)^0$ is as well. □

Proposition 3.5 $\mathrm{PT}(X)$ *is $(\mathcal{F}, \mathcal{G})$-abundant.*

Proof. For $s \in \mathrm{PT}(X)$, let e_s denote the identity map on the set of elements fixed by s, an idempotent element (possibly equal to zero). Then for all $t \in \mathrm{PT}(X)$, $ts = t$ if and only if s fixes all elements in the range of t, that is, $\mathrm{ran}(t) \subseteq \mathrm{dom}(e_s)$, or equivalently, $te_s = t$. So $(s, e_s) \in \mathcal{F}$. Hence $\mathrm{PT}(X)$ is $\mathcal{F}$-abundant.

For $s \in \mathrm{PT}(X)$, consider the subset

$$X_s = \{x \in X \mid x \in \mathrm{dom}(s^n) \text{ for all } n \geq 0\}.$$

Then if $x \in X_s$, so is xs^n for all $n \geq 0$. Now let f_s be any projection mapping into its own domain whose kernel equals the equivalence relation ρ_s on X_s generated by the pairs $\{(xs, x) \mid x \in X_s\} \subseteq X_s \times X_s$; note that $X_{f_s} = X_s$ since f_s is idempotent. Also, for any $x \in X_s$, $xsf_s = xf_s$, and so $sf_s = f_s$ by definition. Hence if $f_st = t$ then $st = t$.

Conversely, suppose $st = t$ for some $t \in \mathrm{PT}(X)$. Then $\mathrm{dom}(t) \subseteq \mathrm{dom}(s)$, so $\mathrm{dom}(t) \subseteq X_s$. Hence for $x \in \mathrm{dom}(t)$, since $x(st) = xt$, we have that $xs \in \mathrm{dom}(t)$. As $(xs)t = xt$,

$\ker(t)$, viewed as an equivalence relation on $\mathrm{dom}(t)$, contains

$$\begin{aligned}\{(xs,x) \mid x \in \mathrm{dom}(t)\} &= \{(xs,x) \mid x \in X_s\} \cap (\mathrm{dom}(t) \times \mathrm{dom}(t)) \\ &= \{(xs,x) \mid x \in X_s\}|_{\mathrm{dom}(t)}.\end{aligned}$$

Note also that $x \in \mathrm{dom}(t)$ if and only if $xs \in \mathrm{dom}(t)$, since $st = t$, so $\mathrm{dom}(t) \subseteq X_s$ is a union of ρ_s-classes (that is, $\ker(f_s)$-classes) of X_s.

Now for $x \in \mathrm{dom}(t)$, $x \in \mathrm{dom}(f_s) = X_s$, and $(xf_s)f_s = xf_s$, so $(xf_s, x) \in \ker(f_s) = \rho_s$, so $xf_s \in \mathrm{dom}(t)$. And if $x \in \mathrm{dom}(f_s t)$ then $x \in \mathrm{dom}(f_s)$ and $xf_s \in \mathrm{dom}(t) \subseteq X_s$, so as $(x, xf_s) \in \rho_s$, we have $x \in \mathrm{dom}(t)$. So $\mathrm{dom}(t) = \mathrm{dom}(f_s t)$.

But for $x \in \mathrm{dom}(t)$, xf_s is in the ρ_s-class of x restricted to $\mathrm{dom}(t)$ and is therefore in $\mathrm{dom}(t)$. Hence $(xf_s, x) \in \ker(f_s)|_{\mathrm{dom}(t)} \subseteq \ker(t)$ from above. Hence $xf_s t = xt$ for all $x \in \mathrm{dom}(t)$. So $f_s t = t$ by definition.

Hence $st = t$ if and only if $f_s t = t$ for all $t \in \mathrm{PT}(X)$, and so $(s, f_s) \in \mathcal{G}$. This shows $\mathrm{PT}(X)$ is $\mathcal{G}$-abundant, hence $(\mathcal{F}, \mathcal{G})$-abundant. □

Proposition 3.6 $\mathrm{I}(X)$ *is* $(\mathcal{F}, \mathcal{G})$*-abundant.*

Proof. For $s \in \mathrm{I}(X)$, define e_s as in the proof of Proposition 3.5. Then arguing as before, $(s, e_s) \in \mathcal{F}$, which equals $\mathcal{G}$ by Lemma 3.2, so $\mathrm{I}(X)$ is $(\mathcal{F}, \mathcal{G})$-abundant. □

So every inverse semigroup embeds in a $(\mathcal{F}, \mathcal{G})$-abundant inverse semigroup, and is $(\mathcal{F}, \mathcal{G})$-abundant if and only if it is $\mathcal{F}$-abundant by Lemma 3.2.

Example 3.7 *An inverse semigroup which is not* $\mathcal{F}$*-abundant even after adjoining a zero.*
Let $X = \{x, y, z, w\}$ and let $S = \{a, g, h, 1_X, \emptyset\} \subseteq \mathrm{I}(X)$, where

$$a = \{(x,y), (y,x), (z,z), (w,w)\}, g = \{(z,z)\}, h = \{(w,w)\}, 0 = \emptyset, 1 = 1_X.$$

Then S is a semigroup under composition, with multiplication table as follows.

$\cdot$	a	g	h	1	0
a	1	g	h	a	0
g	g	g	0	g	0
h	h	0	h	h	0
1	a	g	h	1	0
0	0	0	0	0	0

S is an inverse subsemigroup of $\mathrm{I}(X)$ since it is closed under taking inverses: all elements are self-inverse. It is easy to check that $\mathrm{fix}(a)$ is different for each a, so $\mathcal{F}$ separates the elements of S, which is therefore not $\mathcal{F}$-abundant since $\{a\}$ contains no idempotent. Being commutative (or indeed by Proposition 3.2), it is not $\mathcal{G}$-abundant either. Adjoining a zero element does not affect this since all the subsets $\mathrm{fix}(a)$ remain distinct.

In the other direction, we have examples like the following.

Example 3.8 *A commutative $\mathcal{F}$-abundant semigroup which is neither $\mathcal{L}^*$- nor $\mathcal{R}^*$-abundant.*

Let $X = \{w, x, y, z\}$ and let $S = \{a, b, c, 0, 1\} \subseteq \mathrm{PT}(X)$, where

$$a = \{(w, y), (x, z)\}, b = \{(w, z), (x, y)\}, c = \{(w, x), (x, w), (y, z), (z, y)\}, 0 = \emptyset, 1 = 1_X.$$

Then S is a commutative submonoid of $\mathrm{PT}(X)$, having the following multiplication table.

$\cdot$	a	b	c	0	1
a	0	0	b	0	a
b	0	0	a	0	b
c	b	a	1	0	c
0	0	0	0	0	0
1	a	b	c	0	1

Now $E(S) = \{0, 1\}$, and because for all $s, t \in S$, $st = s$ if and only if $s = 0$ or $t = 1$, it is not hard to see that the $\mathcal{F}$-classes of S are $\{0, a, b, c\}$ and $\{1\}$, and hence that S is $\mathcal{F}$-abundant (hence $(\mathcal{F}, \mathcal{G})$-abundant). S is neither $\mathcal{L}^*$- nor $\mathcal{R}^*$-abundant, since the $\mathcal{L}^*$-classes (and hence the $\mathcal{R}^*$-classes) are as follows: $\{a, c\}, \{b, 1\}$ and $\{0\}$, the first of which contains no idempotent.

$(\mathcal{F}, \mathcal{G})$-abundant semigroups are not closed under homomorphic images.

Example 3.9 *A commutative $(\mathcal{F}, \mathcal{G})$-abundant semigroup with a quotient which is not even $\mathcal{F}$-abundant.*

For S as in Example 3.8, define the equivalence relation θ on S by setting $(a, b) \in \theta$, with all other pairs of distinct elements unrelated. This is easily seen to be a congruence, and $S' = S/\theta$ has multiplication table as follows (writing the congruence class $\{a, b\}$ as a, and so on).

$\cdot$	a	c	0	1
a	0	a	0	a
c	a	1	0	b
0	0	0	0	0
1	a	b	0	1

Now the $\mathcal{F}$-classes of S' as follows: $\{0, a\}, \{c\}, \{1\}$. Note that $\{c\}$ has no idempotent, so the result cannot be $\mathcal{F}$-abundant.

4 $\mathcal{K}$-abundant semigroups

Recall that for a semigroup S, we defined $\mathcal{K} = \mathcal{F} \cap \mathcal{G}$. From Proposition 3.1 and its dual, and the fact that for idempotents e, f, if $e \sim_r f$ and $e \sim_l f$, then $e = f$, we have the following.

Proposition 4.1 *If S is a semigroup, then each $\mathcal{K}$-class contains at most one idempotent of S.*

Of course, if S is $\mathcal{K}$-abundant, then it is $\mathcal{F}$-abundant and $\mathcal{G}$-abundant. Trivially, every band S is $\mathcal{K}$-abundant, and by Proposition 4.1, it follows that $\mathcal{K}$ separates the elements of S.

For completely regular semigroups, each $\mathcal{H}$-class is a group, whilst in $\mathcal{H}^*$-abundant (superabundant) semigroups, each $\mathcal{H}^*$-class is a cancellative semigroup of a special kind. This leads to decomposition theorems for these classes of semigroups. However, the $\mathcal{K}$-classes of a $\mathcal{K}$-abundant semigroup are not even subsemigroups in general.

Example 4.2 *A small $\mathcal{K}$-abundant semigroup.*

Let $S = \{0, 1, a\}$ be the commutative semigroup in which 0 is a zero, 1 is an identity, and $a^2 = 1$. Then $\text{fix}(0) = \{0\} = \text{fix}(a)$ and $\text{fix}(1) = S$, so $\mathcal{F} = \mathcal{G} = \mathcal{K}$ partitions S into $\{0, a\}, \{1\}$, and so S is $\mathcal{K}$-abundant, but $\mathcal{K}$ is not a congruence on S, and $\{0, a\}$ is not a subsemigroup of S.

By Proposition 4.1, there is a natural way to view $\mathcal{K}$-abundant semigroups as unary semigroups: for $a \in S$, define $K(a)$ to be the unique $e \in E(S)$ for which $(s, e) \in \mathcal{K}$.

Proposition 4.3 *The class of unary semigroups arising from $\mathcal{K}$-abundant semigroups in the way just described is a proper quasivariety given by the following laws (in addition to associativity):*

1. $K(s)s = s$ *and* $sK(s) = s$*;*
2. $st = s$ *implies* $sK(t) = s$*;*
3. $ts = s$ *implies* $K(t)s = s$*.*

Proof. It is clear that these laws all hold in a $\mathcal{K}$-abundant semigroup with K defined as above.

Suppose S satisfies the above laws. Together these imply that $(s, F(s)) \in \mathcal{K}$ for all s. By the first law, $F(s)s = s$, so by the third, $F(s)F(s) = F(s)$, so $F(s) \in E(S)$. Hence S is $\mathcal{K}$-abundant with $K(s)$ the unique $e \in E(S)$ for which $(s, e) \in \mathcal{K}$.

Example 3.9 establishes that the class is a proper quasivariety: since S there is commutative and $(\mathcal{F}, \mathcal{G})$-abundant, $\mathcal{F} = \mathcal{G}$ and so it is $\mathcal{K}$-abundant, with $K(1) = 1$ with $K(s) = 0$ for $s \neq 1$. The same congruence as before is therefore still a congruence (the algebra has not altered), clearly respecting K, and the resulting quotient semigroup is not even $\mathcal{F}$-abundant. □

5 $\mathcal{F}$–amiable and $(\mathcal{F}, \mathcal{G})$-amiable semigroups

Not surprisingly, $\mathcal{F}$-amiability and $\mathcal{G}$-amiability are independent properties.

Example 5.1 *An $\mathcal{F}$-amiable semigroup that is not $\mathcal{G}$-amiable.*

Consider the right zero semigroup S with elements e, f. It is easily seen that $\mathcal{F}$ is the diagonal relation on S, so S is $\mathcal{F}$-amiable, whilst $\mathcal{G}$ is the full relation on S, so it is not $\mathcal{G}$-amiable.

In an $(\mathcal{L},\mathcal{R})$-amiable semigroup (that is, an inverse semigroup), $\mathcal{L}$ and $\mathcal{R}$ are in general different (and are equal if and only if the semigroup is Clifford); this therefore applies to $(\mathcal{L}^*,\mathcal{R}^*)$-amiable semigroups as well. But for $\mathcal{F}$ and $\mathcal{G}$, the behaviour is different. First, a useful result.

Lemma 5.2 *Suppose S is an $\mathcal{F}$-amiable semigroup, with $s \in S$ and $e \in E(S)$ such that $(s,e) \in \mathcal{F}$. Then $se = e$.*

Proof. Now $es = e$ (since $ee = e$). and it is clear that $se \in E(S)$ with $(se)e = se$ and $e(se) = e$ so $se \sim_r e$ and so $(se,e) \in \mathcal{F}$ by Proposition 3.1, and so $se = e$ because S is $\mathcal{F}$-amiable. □

Proposition 5.3 *Suppose S is an $\mathcal{F}$-abundant semigroup. Then S is $(\mathcal{F},\mathcal{G})$-amiable if and only if $\mathcal{F} = \mathcal{G}$.*

Proof. If $\mathcal{F} = \mathcal{G}$ in S, then trivially S is $(\mathcal{F},\mathcal{G})$-abundant. But if $e, f \in E(S)$ and $(e,f) \in \mathcal{F} = \mathcal{G}$ then $e \sim_r f$ and $e \sim_l f$ by Proposition 3.1 and its dual, so $e = f$. So S is $(\mathcal{F},\mathcal{G})$-amiable.

Conversely, suppose S is $(\mathcal{F},\mathcal{G})$-amiable. Pick e in $E(S)$ and $s \in S$ such that $(s,e) \in \mathcal{F}$, and let $f \in E(S)$ be such that $(s,f) \in \mathcal{G}$. Then by Lemma 5.2 and its dual, $se = e$ and $fs = f$. But $(s,f) \in \mathcal{G}$, so $se = e$ implies that $fe = e$, but also $(s,e) \in \mathcal{F}$ so $fs = f$ implies that $fe = f$. Hence $e = fe = f$. It follows that $(s,e) \in \mathcal{F}$ implies $(s,e) \in \mathcal{G}$. Since this is true for all idempotents e, $\mathcal{F} \subseteq \mathcal{G}$. By symmetry, they are equal. □

The classes of completely regular and inverse semigroups are independent: neither is contained in the other. (Their intersection is the class of Clifford semigroups.) This also shows that the classes of $\mathcal{H}^*$-abundant and $(\mathcal{L}^*,\mathcal{R}^*)$-amiable semigroups are independent. By contrast, we have the following.

Corollary 5.4 *The class of $(\mathcal{F},\mathcal{G})$-amiable semigroups is a proper subclass of the class of $\mathcal{K}$-abundant semigroups.*

Proof. If S is $(\mathcal{F},\mathcal{G})$-amiable then by Proposition 5.3, $\mathcal{F} = \mathcal{G}$, which therefore also equals $\mathcal{K}$, and so S is $\mathcal{K}$-abundant. For failure of the converse, let $X = \{x,y\}$, and consider the rectangular band S with underlying set $X \times X$, equipped with the product given by $(a,b)(c,d) = (a,d)$ for all $a,b,c,d \in X$. Letting $a = (x,x), b = (x,y), c = (y,x), d = (y,y)$, it has the following multiplication table.

$\cdot$	a	b	c	d
a	a	b	a	b
b	a	b	a	b
c	c	d	c	d
d	c	d	c	d

Then $\mathrm{fix}(a) = \mathrm{fix}(c) = \{a,c\}$, $\mathrm{fix}(b) = \mathrm{fix}(d) = \{b,d\}$, so the $\mathcal{F}$-classes of S are $\{a,c\},\{b,d\}$. In the same way we see that the $\mathcal{G}$-classes of S are $\{a,b\},\{c,d\}$. So $\mathcal{F} \cap \mathcal{G}$ is the diagonal relation, and so S is $\mathcal{K}$-abundant. However, it is evidently neither $\mathcal{F}$-amiable nor $\mathcal{G}$-amiable. $\square$

An example of a semigroup that is $(\mathcal{F},\mathcal{G})$-amiable but neither $\mathcal{L}^*$- nor $\mathcal{R}^*$-abundant is $\{0,a\}$ with all products equal to 0: then $\mathcal{F}=\mathcal{G}$ is the full relation on S, whilst $\mathcal{L}^*=\mathcal{R}^*$ is the diagonal relation (so $\{a\}$ is in a class by itself). An example of a regular semigroup that is neither $\mathcal{F}$- nor $\mathcal{G}$-abundant is $\{1,a\}$ with 1 an identity element and $a^2=1$: then $\mathcal{L}=\mathcal{R}$ is the full relation on S, whilst $\mathcal{F}=\mathcal{G}$ is the diagonal relation (so $\{a\}$ is in a class by itself).

It is well-known that a regular semigroup S is inverse if and only if $E(S)$ is a semilattice (in which case it is reduced). In other cases we have the following easy consequence of Result 2.1 together with Proposition 3.1 and its dual.

Proposition 5.5 *If S is an $\mathcal{F}$-abundant (resp. $\mathcal{L}^*$-abundant) semigroup, then it is $\mathcal{F}$-amiable (resp. $\mathcal{L}^*$-amiable) if and only if $E(S)$ is right reduced, and if S is an $(\mathcal{F},\mathcal{G})$-abundant (resp. $(\mathcal{L}^*,\mathcal{R}^*)$-abundant) semigroup, then it is $(\mathcal{F},\mathcal{G})$-amiable (resp. $(\mathcal{L}^*,\mathcal{R}^*)$-abundant) if and only if $E(S)$ is reduced.*

Hence if S is $(\mathcal{L}^*,\mathcal{R}^*)$-abundant, then $E(S)$ being a semilattice is sufficient to ensure $(\mathcal{L}^*,\mathcal{R}^*)$-amiability. But it is not necessary: it was shown in [4] that there is an infinite $(\mathcal{L}^*,\mathcal{R}^*)$-amiable semigroup that is not adequate (that is, its idempotents do not commute). It was later shown in [1] that there are finite amiable semigroups that are not adequate.

Similarly, if S is $(\mathcal{F},\mathcal{G})$-abundant, then $E(S)$ being a semilattice is sufficient to ensure $(\mathcal{F},\mathcal{G})$-amiability. But again, it is not necessary: the example given in [1] is also $(\mathcal{F},\mathcal{G})$-amiable, as we now show.

Example 5.6 *An $(\mathcal{F},\mathcal{G})$-amiable semigroup M in which $E(M)$ does not commute.*

As in [1], but using a slightly different notation, let $M=\{e,f,0,a\}$ be the semigroup with multiplication as follows.

$\cdot$	e	f	0	a
e	e	0	0	0
f	a	f	0	a
0	0	0	0	0
a	a	0	0	0

Now $E(M)=\{e,f,0\}$. Note that $\mathrm{fix}(e)=\{e,a,0\}$, $\mathrm{fix}(f)=\{f,0\}$, and $\mathrm{fix}(0)=\mathrm{fix}(a)=\{0\}$, so the $\mathcal{F}$-classes are as follows: $\{0,a\}$, $\{e\}$ and $\{f\}$, and so M is $\mathcal{F}$-abundant. Similarly, $\mathrm{cofix}(e)=\{e,0\}$, $\mathrm{cofix}(f)=\{f,0,a\}$, $\mathrm{cofix}(0)=\mathrm{cofix}(a)=\{0\}$, so $\mathcal{G}=\mathcal{F}$, and so M is $(\mathcal{F},\mathcal{G})$-amiable by Proposition 5.3. But as noted in [1], its idempotents do not commute: $ef=0$, while $fe=a\notin E(S)$.

In [2], the authors show that an $(\mathcal{L}^*,\mathcal{R}^*)$-amiable semigroup is adequate (that is, $E(S)$ is a semilattice) if and only if it contains no copy of either M as in Example 5.6 or the infinite example S as in Example 1.4 of [4], which as noted in [2] is nothing but a copy of the semigroup S defined by the presentation $S=\langle e,f \mid e^2=e, f^2=f\rangle$.

Regarding $(\mathcal{F},\mathcal{G})$-amiable semigroups in which $E(S)$ does not commute, the situation is more complicated than for non-adequate $(\mathcal{L}^*,\mathcal{R}^*)$-amiable semigroups.

Proposition 5.7 *There is an infinite family of finite $(\mathcal{F},\mathcal{G})$-amiable semigroups, none of which contains a copy of any of the others and none of which has commuting idempotents.*

Proof. Let S denote the semigroup defined by the presentation

$$S = \langle e, f \mid e^2 = e, f^2 = f\rangle.$$

Factor out the ideal consisting of all strings of length $2n$ or more ($n > 2$) to give S_n (the evenness of length here is not esential but is notationally convenient). Then $S_n = \{e, f, ef, fe, efe, fef, \ldots, e(fe)^{n-1}, f(ef)^{n-1}, 0\}$, where 0 represents all strings of length $2n$ or more. It is not hard to see that no copy of S_n is in S_m if $n < m$, $E(S_n) = \{e, f, 0\}$, and clearly $ef \neq fe$. To complete the proof, we show each S_n is $(\mathcal{F},\mathcal{G})$-amiable.

Pick $s \in S_n\backslash E(S_n)$. Then $\text{fix}(0) = S$, $\text{fix}(e) = \{e(fe)^{k-1} \mid 1 \leq k \leq n\}$, $\text{fix}(f) = \{f(ef)^{k-1} \mid 1 \leq k \leq n\}$, and if s is not idempotent, $\text{fix}(s) = \{0\}$; hence the $\mathcal{F}$-classes are $\{e\}, \{f\}$ and $S\backslash\{e, f\}$ (with 0 in the third of these); the $\mathcal{G}$-classes are the same as is easily seen. □

It would not be difficult to describe all of the finite $(\mathcal{F},\mathcal{G})$-amiable semigroups generated by two non-commuting idempotents; it would include all of the above as well as M in Example 5.6, and many more besides. Note that S as in the proof of Proposition 5.7 is not $(\mathcal{F},\mathcal{G})$-amiable (since $\text{fix}(s)$ is empty if $s \neq e, f$), but appending a zero to S is easily seen to yield an infinite $(\mathcal{F},\mathcal{G})$-amiable example whose idempotents ($\{e, f, 0\}$) do not commute: the argument is very similar to the above proof that S_n is always $(\mathcal{F},\mathcal{G})$-amiable.

Analogous to the axiomatisation of $\mathcal{L}^*$-amiable and $(\mathcal{L}^*,\mathcal{R}^*)$-amiable semigroups as quasivarieties of unary/biunary semigroups as in [3], observe that every $\mathcal{F}$-amiable semigroup S can be turned into a unary semigroup in which every element $s \in S$ has associated to it the unique idempotent in its $\mathcal{F}$-class, denoted by $F(s)$, and similarly for $\mathcal{G}$-amiable semigroups in terms of $\mathcal{G}$, with the unique idempotent in the $\mathcal{G}$-class of S denoted by $G(s)$.

Proposition 5.8 *If S is $\mathcal{F}$-amiable, with F as just defined, the following laws are satisfied for all $s, t \in S$:*

1. $F(s)s = s$
2. $st = s$ *implies* $sF(t) = s$
3. $s^2 = s$ *implies* $F(s) = s$
4. $F(s)F(t) = F(s)$ *and* $F(t)F(s) = F(t)$ *imply* $F(s) = F(t)$.

Indeed the above three laws axiomatise unary semigroups that arise from $\mathcal{F}$-amiable semigroups in this way.

Proof. If S is $\mathcal{F}$-amiable, the first two laws follow because $(s, F(s)) \in \mathcal{F}$ for all $s \in S$, whilst the third follows because each idempotent is in an $\mathcal{F}$-class of its own, and the final law follows from the fact that no two distinct idempotents are related under $\mathcal{F}$, so $F(S) = E(S)$ is right reduced by Proposition 3.1 and Result 2.1.

Conversely, assume S is a unary semigroup satisfying the above laws. For $s \in S$, by the first law $F(s)s = s$, so by the second, $F(s)F(s) = F(s)$, so $F(s) \in E(S)$. The first two laws give that $(s, F(s)) \in \mathcal{F}$, so S is $\mathcal{F}$-abundant. If also $(s, e) \in \mathcal{F}$ for some $e \in E(S)$, then $(e, F(s)) \in \mathcal{F}$, and $e = F(e)$ by the third law, so $F(e) \sim_r F(s)$ and so by the fourth law, $e = F(e) = F(s)$, so S is $\mathcal{F}$-amiable. $\square$

There is an obvious variant of the above for the $\mathcal{G}$-amiable property in terms of the unary operation G. But for an $(\mathcal{F}, \mathcal{G})$-amiable semigroup, $F = G$ by Proposition 5.3, so we only need the one unary operation to axiomatise them. Axioms are those as in Proposition 5.8 together with their duals. These classes are clearly quasivarieties, with the $(\mathcal{F}, \mathcal{G})$-amiable semigroups forming a proper subclass of the quasivariety of $\mathcal{K}$-abundant semigroups viewed as unary semigroups. But none are varieties, as Example 3.9 shows: there, $\mathcal{F} = \mathcal{G}$, but the quotient by θ is not even $\mathcal{F}$-abundant.

It follows from a basic fact about inverse semigroups and Proposition 5.5 that if the semigroup relations $\mathcal{A}$ and $\mathcal{B}$ are any of (i) $\mathcal{L}$ and $\mathcal{R}$, (ii) $\mathcal{L}^*$ and $\mathcal{R}^*$, or (iii) $\mathcal{F}$ and $\mathcal{G}$, $E(S)$ in an $(\mathcal{A}, \mathcal{B})$-amiable semigroup is reduced, and the partial orders $\leq_l, \leq_r$ are both equal to the natural order $\leq$. In Case (i) of inverse semigroups, $E(S)$ is a semilattice and therefore meets exist with respect to the natural order. In Case (ii), namely what are normally called amiable semigroups, note that for S as in the proof of Proposition 5.7 above, S has only two idempotents e and f, and neither $e \leq f$ nor $f \leq e$ holds, so $E(S)$ does not have meets. So in an amiable semigroup S, $E(S)$ need not have meets. The situation for Case (iii) is intermediate between the first two cases. To see this, we first give a one-sided result.

Proposition 5.9 *Suppose S is an $\mathcal{F}$-amiable semigroup in which $E(S)$ is reduced, with associated unary operation F. Then $E(S)$ is a meet-semilattice with respect to the natural order, in which $e \wedge f = F(ef)$ for all $e, f \in E(S)$.*

Proof. For $e, f \in E(S)$, clearly $F(ef) \in E(S)$. Now $F(ef)ef = F(ef)$, so it follows immediately that $F(ef)f = F(ef)$, so $F(ef) \leq f$. But $(ef, F(ef)) \in \mathcal{F}$, so by Lemma 5.2, $efF(ef) = F(ef)$, and so $eF(ef) = F(ef)$, so $F(ef) \leq e$. If $g \in E(S)$ is such that $g \leq e$ and $g \leq f$, then $ge = gf = g$, so $gef = g$ and so $gF(ef) = g$, so $g \leq F(ef)$. $\square$

The assumption that $E(S)$ is reduced cannot be dropped here. Letting S be the right zero semigroup on $\{e, f\}$ as in Example 5.1, $E(S) = S$ is right reduced but not reduced, the partial order $\leq_r$ on S is the natural order, and neither $e \leq f$ nor $f \leq e$ holds, so e, f have no meet.

Corollary 5.10 *Suppose S is a $(\mathcal{F}, \mathcal{G})$-amiable semigroup, with associated unary operation F. Then $E(S)$ is a meet-semilattice with respect to the natural order, in which $e \wedge f = F(ef)$ for all $e, f \in E(S)$.*

In Example 5.6, note that $F(ef) = F(0) = 0 = F(a) = F(fe)$, confirming the claim of Corollary 5.10. Similarly, in $S = \langle e, f \mid e^2 = e, f^2 = f \rangle$, $E(S) = \{e, f\}$, and e and f have no meet in $E(S) = \{e, f\}$, giving an easy demonstration that S is not $(\mathcal{F}, \mathcal{G})$-amiable.

A non-trivial left zero semigroup is $(\mathcal{F}, \mathcal{G})$-abundant (being a band) but has no zero. Any semilattice without zero is a (necessarily infinite) $(\mathcal{F}, \mathcal{G})$-amiable semigroup; however, any finite $(\mathcal{F}, \mathcal{G})$-amiable semigroup S has a zero because $F(S)$ is a meet-semilattice under $\leq$ and so has a smallest element e, which is easily seen to be a zero of the semigroup S.

Proposition 3.6 asserted that the inverse semigroup $\mathrm{I}(X)$ is an $(\mathcal{F},\mathcal{G})$-abundant semigroup, hence $(\mathcal{F},\mathcal{G})$-amiable. By the proof of Theorem 3.6, $F(a)=e_a$ for all $a\in \mathrm{I}(X)$, where e_a is the identity map on the set of elements fixed by a. Semigroups of injective partial functions equipped with F inherited from $\mathrm{I}(X)$ also inherit the laws for $(\mathcal{F},\mathcal{G})$-amiable semigroups. However, as for the partial function case, no finite set of laws is enough to axiomatise algebras of injective partial functions under F, as follows from Theorem 8.2 in [9]. Adding inversion to the signature changes this: it turns out we obtain nothing but inverse semigroups that are $\mathcal{F}$-abundant; by Proposition 5.5, these may be equivalently described as regular $(\mathcal{F},\mathcal{G})$-amiable semigroups, or indeed as regular $(\mathcal{F},\mathcal{G})$-abundant semigroups with reduced set of idempotents.

Proposition 5.11 *A regular semigroup S is $(\mathcal{F},\mathcal{G})$-amiable if and only if it is isomorphic to an inverse semigroup of injective partial functions equipped with unary operation F for which $F(a)$ is the identity map on the set of elements fixed by a.*

Proof. We have already argued that an inverse semigroup of injective partial functions with F as described is $(\mathcal{F},\mathcal{G})$-amiable. Conversely, suppose S is a regular semigroup equipped with F satisfying the laws for $(\mathcal{F},\mathcal{G})$-amiable semigroups (those in Proposition 5.8 and their duals); it is inverse by Proposition 5.5. Represent it as in the Vagner-Preston theorem, with $a\in S$ represented as $\psi_a : Saa'\mapsto Sa'a$, via $s\psi_a = sa$, and note that $\psi_{F(a)}$ is the identity map on $SF(a)$ since $F(a)\in E(S)$, and has domain $SF(a)$, which is all $x\in S$ for which $xF(a)=x$, that is, $xa=x$, or equivalently, $xa=x$ and $xaa'=x$ (where a' is the inverse of a), which is to say $x\in dom(\psi_a)$ and $x\psi_a=x$. This shows that $\psi_{F(a)}=F(\psi(a))$. $\Box$

The isomorphism class of inverse semigroups of injective partial functions equipped with the operation F such that $F(a)$ is as in the statement of Proposition 5.11 is given a finite equational axiomatisation in [7] (see Proposition 7.2 and Theorem 7.3), as the class of so-called *DII-semigroups*. Hence when viewed as a class of biunary semigroups equipped with inversion and F, the class of inverse $(\mathcal{F},\mathcal{G})$-amiable semigroups is a finitely based variety.

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References

[1] Araújo, J, Kinyon, M.: On a Problem of M. Kambites Regarding Abundant Semigroups. Comm. Algebr **40**, 4439–4447 (2012)

[2] Araújo, J, Kinyon, M., Malheiro, A.: A characterisation of adequate semigroups by forbidden subsemigroups. Proc. R. Soc. A **143**, 1115–1122 (2013)

[3] Araújo, J, Kinyon, M., Robert, Y.: Varieties of regular semigroups with uniquely defined inversion. Portugaliae Mathematica **76**, 205–228 (2020)

[4] Fountain, J.: Adequate semigroups. Proc. Edinburgh Math. Soc. **22**, 113–125 (1979)

[5] Fountain, J.: Abundant semigroups. Proc. London Math. Soc. **44**, 103–129 (1982)

[6] McCune, W.: Prover9 and Mace4, version LADR-Dec-2007, (http://www.cs.unm.edu/ mccune/prover9/).

[7] Jackson, M., Stokes, T.: Agreeable semigroups. J. Algebra **266**, 393–417 (2003)

[8] Jackson, M., Stokes, T.: Modal restriction semigroups: towards an algebra of functions', Int. J. Algebra Comput. **21**, 1053–1095

[9] Jackson, M., Stokes, T.: On representing semigroups with subsemilattices. J. Algebra **376**, 228–260 (2013)

[10] Stokes, T.: Generalised domain and E-inverse semigroups. Semigroup Forum **97**, 32–52 (2018)