APPROXIMATION OF INVARIANT MEASURES FOR A CLASS OF MAPS WITH INDIFFERENT FIXED POINTS

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Abstract. Certain dynamical systems on the interval with neutrally stable repelling points admit invariant probability measures which are absolutely continuous with respect to Lebesgue measure. These maps are often used as a model of intermittent dynamics, since they exhibit polynomial rather than exponential decay of correlations (due to the absence of a spectral gap in the underlying transfer operator). This paper presents a class of these maps which are expanding (with convex branches) for which the invariant probability measures can be rigorously approximated by Ulam's method (a sequence of finite rank approximations to the transfer operator). L^1–convergence of the scheme is proved, and some numerical experiments are reported.

1. Introduction

It is well known that expanding maps with indifferent fixed points (or periodic orbits) with local tangencies of $O(x^{1+\alpha})$ $(0 < \alpha < 1)$ can admit absolutely continuous invariant probability measures (ACIPMs); see [3, 11, 10, 13] and the references contained therein. These maps were originally considered in the study of intermittency in turbulent flows [14], and are interesting because they exhibit polynomial, rather than exponential, decay of correlations [3, 11, 13]. This is intimately connected with the absence of a spectral gap in the corresponding transfer operators (Frobenius–Perron (FP) operators), and is in sharp contrast to the situation for uniformly expanding maps [6, 1]. The densities of the ACIPMs of uniformly expanding maps are highly amenable to numerical approximation by projection methods, since the spectral gap in the FP operator (and eigenvector at 1) persists under the small perturbations induced by suitable approximation schemes [5, 4, 12]. Maps with indifferent fixed points do not have a spectral gap, and this makes the convergence of invariant measure approximations a delicate business [10]. In this paper, we prove convergence of Ulam's method [15] for approximating the densities of ACIPMs of a class of maps with an indifferent fixed point. The method is reminiscent of Li's original proof [9] of convergence of Ulam's method for uniformly expanding transformations of Lasota–Yorke type [7]. In that setting, the FP operator preserves a cone of non-negative BV functions in L^1, and Li's convergence result followed from the observation that the Ulam–type projections of the FP operator also preserved that cone. The method of the current paper also relies on the invariance of certain relatively compact (cone-like)

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subsets of L^1 under the action of both the FP operator, and its Ulam approximations. Unfortunately, the L^1–functions we consider need to have power law singularities near the indifferent fixed point, and discretization effects appear to preclude uniform estimates on the Ulam approximations. Consequently, some of the calculations for the convergence result are rather detailed, and the class of maps to which the results apply is somewhat restricted. Nonetheless, it is interesting to obtain a convergence result for Ulam's method in the absence of a strong regularity condition (such as BV [9, 12]) and its stability is not assured by the spectral picture for the FP operator [5].

Some numerical computations are presented in the final section of the paper. The convergence to the invariant density appears to be of power law type, with the exponent depending on the value of α (although a convincing explanation for this rate of convergence is lacking).

Class of maps. We consider expanding maps of the unit interval which have two onto branches and an indifferent fixed point at 0. The maps must be C^1, and have convex branches (in lieu of higher-order regularity). More precisely, let $0 < \alpha < 1$, and let $\mathcal{T}_\alpha$ be the class of maps T satisfying the following conditions:

(1) $T(0) = 0$ and there is a $\gamma \in (0,1)$ such that $T : [0,\gamma) \xrightarrow{\text{onto}} [0,1)$, $T : [\gamma,1] \xrightarrow{\text{onto}} [0,1]$.
(2) Each branch of T is increasing, convex, and is C^1 (or, in the case of the first branch, can be extended to a C^1 function on $[0,\gamma]$); $T'(x) \geq 1$ for all $x \in (0,\gamma) \cup (\gamma,1)$. The intervals $[a_N, b_N]$ on which the branches of T^N can be extended to be a C^1 diffeomorphism onto $[0,1]$ will be called *monotonicity intervals* of T^N.
(3) There is a constant $C \in (0,\infty)$ such that
$$T'(x) = 1 + C\,x^\alpha + o(x^\alpha)$$
in the neighbourhood of zero ($g(x) = o(h(x))$ means that $\lim_{x\to 0} \frac{g(x)}{h(x)} = 0$).

Examples.

(1) The Pommeau–Manneville map discussed in [13, 3]
$$T(x) = x\,(1 + x^\alpha) \qquad (\text{mod } 1).$$
(2) A variant of the Pommeau–Manneville map discussed by [11, 10]
$$T(x) = \begin{cases} x\,(1 + (2\,x)^\alpha) & \text{if } x \in [0, 1/2), \\ 2\,x - 1 & \text{if } x \in [1/2, 1]. \end{cases}$$
(3) Let $\varphi_t(x_0)$ by the solution of the differential equation $x' = x^{1+\alpha}$, $x(0) = x_0$. Pick τ such that $\varphi_\tau(1) = 2$ and put
$$T(x) = \varphi_\tau(x) \qquad (\text{mod } 1).$$
In this case, one can readily compute $\tau = \frac{1-2^{-\alpha}}{\alpha}$ and $\gamma = (2 - 2^{-\alpha})^{-1/\alpha}$. Since $\varphi_t(x) = x\,(1 - \alpha\,x^\alpha\,t)^{-1/\alpha}$, it is easy to obtain precise estimates on the rate of approach of pre-images of x to the indifferent repeller at 0. Moreover, since exact formulas are available for the inverse branches, this map is a good choice for numerical computations, and these are reported in Section 4.

Invariant densities. The main result of this paper is that for $0 < \alpha < 1$, maps in the class $\mathcal{T}_\alpha$ admit ACIPMs whose densities can be approximated by Ulam's method with L^1–convergence. The existence of the ACIPMs can be deduced from the results of many authors (including [3, 11, 13]), but establishing existence in our setting gives a useful preparation for the analysis of Ulam's method.

Let $[0,1]$ be equipped with the Borel σ–algebra, and denote Lebesgue measure by λ. A Borel measure μ on $[0,1]$ is absolutely continuous (AC) with respect to λ if $\mu(A) > 0 \Rightarrow \lambda(A) > 0$. A measure μ is an *invariant measure* if $\mu = \mu \circ T^{-1}$. We are particularly interested in finite AC invariant measures; these can be normalized to obtain probability measures (ACIPMs). By the Radon–Nikodym theorem, an ACIPM has an L^1 *density function* $f = \frac{d\mu}{d\lambda}$, so that $\mu(A) = \int_A f\, d\lambda$. ACIPMs can be characterized as fixed points of a transfer operator on L^1: Since maps $T \in \mathcal{T}_\alpha$ are expanding, $\mu \circ T^{-1}$ is AC whenever μ is AC, so the invariance condition can be written as

$$\int_A f\, d\lambda = \mu(A) = \mu\left(T^{-1}(A)\right) = \int_{T^{-1}(A)} d\mu = \int_A d\mu \circ T^{-1} = \int_A \frac{d\mu \circ T^{-1}}{d\lambda}\, d\lambda.$$

The *Frobenius–Perron* operator [6, 1] $\mathcal{P} : L^1[0,1] \to L^1[0,1]$ is defined by $\mathcal{P}\left(\frac{d\mu}{d\lambda}\right) = \frac{d\mu \circ T^{-1}}{d\lambda}$; one seeks non-negative fixed points of $\mathcal{P}$. When normalized, such a function is the density of an ACIPM for T. $\mathcal{P}$ is a *Markov* operator (*ie.* is linear, monotone and preserves integrals). Moreover,

$$\mathcal{P}f(y) = \sum_{\{y_i | T(y_i) = y\}} \frac{f(y_i)}{|T'(y_i)|}.$$

Maps in $\mathcal{T}_\alpha$ give exactly two pre-images to each $y \in (0,1)$; we will adopt the convention that these are $y_1 \in [0, \gamma)$ and $y_2 \in [\gamma, 1)$.

For each $A > 0$ define

$$\mathcal{C}_A = \{f \in L^1 | f \geq 0, f \text{ decreasing}, f(x) \leq (\textstyle\int_0^1 f\, d\lambda)\, A\, x^{-\alpha}\}.$$

The key step in proving the existence of an ACIPM with a density in $\mathcal{C}_A$ is to establish that for large enough A, $\mathcal{C}_A$ is invariant by the FP operator (Proposition 1.1). The calculations which establish this fact are deferred to the next section. Similar (but more careful) calculations will then be done in Section 3 to show that the finite rank Ulam approximations to $\mathcal{P}$ also leave certain of these cones invariant (Proposition 1.3). The convergence of Ulam's method then follows by a more or less standard argument (Theorem 2).

Proposition 1.1. *For large enough* A, $\mathcal{P} : \mathcal{C}_A \to \mathcal{C}_A$.

The proof of the proposition is given in Section 2.

Lemma 1.2. *For each* $A > 0$, *let* $\mathcal{C}_A^{\text{dens}} = \mathcal{C}_A \cap \{\|f\|_{L^1} = 1\}$. *Each* $\mathcal{C}_A^{\text{dens}}$ *is norm–compact as a subset of* $L^1[0,1]$.

Proof. Embed $\mathcal{C}_A^{\text{dens}}$ in $L^1(\mathbb{R})$ by defining $\tilde{f}|_{[0,1]} = f$ and $\tilde{f}|_{\mathbb{R}\setminus[0,1]} = 0$ for each $f \in \mathcal{C}_A^{\text{dens}}$ and put $\tilde{\mathcal{C}} = \{\tilde{f} | f \in \mathcal{C}_A^{\text{dens}}\}$. Then, since each $f \in \mathcal{C}_A^{\text{dens}}$ is decreasing,

$$\int_{\mathbb{R}} |\tilde{f}(x+h) - \tilde{f}(x)|\, d\lambda(x) = \int_0^h f(x)\, d\lambda(x) + \int_{1-h}^1 f(x)\, d\lambda(x) \le 2\, \frac{A}{1-\alpha}\, h^{1-\alpha} \overset{\text{uniformly}}{\to} 0$$

as $h \to 0$. Then $\tilde{\mathcal{C}} \subset L^1(\mathbb{R})$ is relatively compact by Theorem IV.8.20 of [2]. Relative compactness of $\mathcal{C}_A^{\text{dens}}$ in $L^1[0,1]$ follows immediately, and the lemma follows since each $\mathcal{C}_A^{\text{dens}}$ is closed. □

Theorem 1. *Let $T \in \mathcal{T}_\alpha$. Then T has a unique ACIPM whose density is a decreasing function satisfying $f(x) \le A\, x^{-\alpha}$ for large enough A.*

Proof. Let A^* be large enough that Proposition 1.1 holds and put $\mathcal{C}_* = \mathcal{C}_{A^*}^{\text{dens}}$. Since $\mathcal{C}_*$ is convex, it contains a fixed point of $\mathcal{P}$ by the Markov–Kakutani fixed point theorem [2, Theorem V.10.6]. The proof of uniqueness of the ACIPM is deferred until Section 2. □

Ulam approximations. Ulam's method [15] consists in replacing the FP operator by a sequence of finite rank discretizations whose fixed points are relatively easy to compute. For each $n > 0$, let $\xi_n = \left\{ \left[\frac{i}{n}, \frac{i+1}{n}\right) \right\}_{i=0}^{n-1}$ be the partition of $[0,1)$ into uniform subintervals and $\mathcal{E}_n$ be the projection operator on L^1 which acts by taking expectations:

$$\mathcal{E}_n f = \sum_{i=0}^{n-1} \frac{\int f\, \mathbf{1}_{[\frac{i}{n}, \frac{i+1}{n})}\, d\lambda}{\lambda[\frac{i}{n}, \frac{i+1}{n})}\, \mathbf{1}_{[\frac{i}{n}, \frac{i+1}{n})}.$$

The Ulam approximations to $\mathcal{P}$ are $\mathcal{P}_n = \mathcal{E}_n \circ \mathcal{P}$, and the Ulam approximations to densities of ACIPMs satisfy $f_n = \mathcal{P}_n f_n$.

Proposition 1.3. *Let $\mathcal{C}_{A,n} = \mathcal{C}_A^{\text{dens}} \cap \text{Range}(\mathcal{E}_n)$. For large enough A and n, $\mathcal{P}_n : \mathcal{C}_{A,n} \to \mathcal{C}_{A,n}$.*

Lemma 1.4. *Let $f \in \mathcal{C}_A$. Then $\|\mathcal{E}_n f - f\|_{L^1} \le \frac{3-\alpha}{1-\alpha}\, A\, \|f\|_{L^1}\, n^{\alpha-1}$.*

Proof. Since $0 \le f(x) \le A\, \|f\|\, x^{-\alpha}$ we have

$$\int_0^{1/n} |\mathcal{E}_n(f) - f|\, d\lambda \le \int_0^{1/n} \mathcal{E}_n(f)\, d\lambda + \int_0^{1/n} f\, d\lambda = 2 \int_0^{1/n} f\, d\lambda \le 2\, \frac{A}{1-\alpha} \|f\| n^{\alpha-1}.$$

Since f is decreasing, $\text{var}_{[1/n,1)} f = f(\frac{1}{n}) - f(1) \le A\, \|f\|_{L^1} (\frac{1}{n})^{-\alpha}$. Thus,

$$\int_{1/n}^1 |\mathcal{E}_n(f) - f|\, d\lambda \le \frac{1}{n} \text{var}_{[1/n,1)} f \le A\, \|f\|_{L^1}\, n^{\alpha-1}$$

(the first inequality is a standard property of $\mathcal{E}_n$). □

Theorem 2 (Convergence of Ulam's method)**.** *Let $T \in \mathcal{T}_\alpha$. Let f be the density of the unique ACIPM. Then, for large enough n, the finite rank operator $\mathcal{E}_n \circ \mathcal{P}$ has a unique non-negative, normalized fixed point f_n. Moreover, $\|f_n - f\|_{L^1} \to 0$ as $n \to \infty$.*

Proof. Let A be large enough that Propositions 1.1 and 1.3 hold. As in the proof of Theorem 1, for large enough n, $\mathcal{E}_n \circ \mathcal{P}$ has a fixed point $f_n \in \mathcal{C}_{A,n} \subset \mathcal{C}_A^{\text{dens}}$. By Lemma 1.2, every subsequence of $\{f_n\}$ contains an L^1 convergent subsequence. Let $f_{n_i} \xrightarrow{L^1} f \in \mathcal{C}_A^{\text{dens}}$. Then (since $f_{n_i} = \mathcal{E}_n \mathcal{P} f_{n_i}$),

$$\|f - \mathcal{P}f\|_{L^1} \le \|f - f_{n_i}\|_{L^1} + \|\mathcal{E}_{n_i}\mathcal{P}f_{n_i} - \mathcal{P}f_{n_i}\|_{L^1} + \|\mathcal{P}(f_{n_i} - f)\|_{L^1}.$$

Since all three terms on the right converge to 0 as $i \to \infty$, $f = \mathcal{P}f$. Since T admits a unique ACIPM, f is its density, and all subsequences of $\{f_n\}$ have f as their common limit point. Thus $f_n \xrightarrow{L^1} f$. □

2. Invariance of $\mathcal{C}_A$ and uniqueness of the ACIPM

Expansivity estimates. The maps $T \in \mathcal{T}_\alpha$ are not uniformly expanding on $[0,1]$, but do have this property on $[\delta, \gamma]$ for any $\delta > 0$. Given $\epsilon > 0$, let $\delta > 0$ be such that

$$|T'(x) - (1 + C\,x^\alpha)| \le \epsilon\, x^\alpha \tag{1}$$

if $x \in (0, \delta)$. Equation (1) will be used for estimates near 0.

Lemma 2.1. *Let $T \in \mathcal{T}_\alpha$, $\epsilon_0 > 0$ be given, and δ_0 be such that (1) holds. For every $\delta_1 \le \delta_0$ there is a constant $\kappa_1 > 1$ such that $\kappa_1 < \frac{T(x)}{x} \le T'(x)$ for $x \in [\delta_1, \gamma]$ and*

$$1 + \frac{C - \epsilon_0}{1+\alpha}\, x^\alpha \le \frac{T(x)}{x} \le \min\{T'(x), \kappa_1\}$$

if $x \in [0, \delta_1)$.

Proof. First of all, if $y \le \delta_1 \le \delta_0$ then equation (1) ensures that $T'(y) \ge 1 + (C - \epsilon_0)\, y^\alpha$ so that

$$T(x) = T(0) + \int_0^x T'(y)\, dy \ge 0 + x + \frac{C - \epsilon_0}{1+\alpha}\, x^{1+\alpha},$$

and the lower estimate on $\frac{T(x)}{x}$ follows. Next, since $T|_{[0,\gamma]}$ is a convex function, T' is increasing so $T'(y) \le \frac{T(x)-T(y)}{x-y} \le T'(x)$ if $y < x$. Using $y = 0$ gives $\frac{T(x)}{x} \le T'(x)$ on $[0, \gamma]$. We can also write:

$$T(x) = T(y) + \tfrac{T(x)-T(y)}{x-y}(x-y) \ge T(y) + T'(y)\,(x-y) \ge T(y) + \tfrac{T(y)}{y}(x-y) = \tfrac{T(y)}{y}x,$$

so that $\frac{T(x)}{x}$ is an increasing function of x. Put $\kappa_1 = \frac{T(\delta_1)}{\delta_1}$. □

Corollary 2.2. *If $x \le \delta_1$ and $\eta \in [0,1]$ then*

$$\left(\frac{x}{T(x)}\right)^\eta \le 1 - \eta\, \frac{C - \epsilon_0}{1+\alpha}\, \frac{[T(x)]^\alpha}{{\kappa_1}^{1+\alpha}}.$$

Proof. By Lemma 2.1,

$$\frac{T(x) - x}{T(x)} \ge \frac{\frac{C-\epsilon_0}{1+\alpha}\, x^{1+\alpha}}{T(x)} \ge \frac{\frac{C-\epsilon_0}{1+\alpha}\left[\frac{T(x)}{\kappa_1}\right]^{1+\alpha}}{T(x)} = \frac{C - \epsilon_0}{1+\alpha}\, \frac{[T(x)]^\alpha}{{\kappa_1}^{1+\alpha}}.$$

The corollary follows since $(\frac{x}{T(x)})^\eta = (1 - \frac{T(x)-x}{T(x)})^\eta \le 1 - \eta\, \frac{T(x)-x}{T(x)}$. □

Lemma 2.3. *Choose* $0 < \rho' \le \rho < 1-\alpha$ *and* $\epsilon_0, \epsilon_1 > 0$ *such that*

$$(C+\epsilon_0)(1+\epsilon_1)(1+\alpha-\rho') = (C-\epsilon_0)(1+\alpha).$$

Let δ_0 *satisfy (1) for* ϵ_0 *and if* $\alpha > \rho$ *choose* $0 < \delta_1 \le \delta_0$ *such that* $\frac{T(\delta_1)}{\delta_1} \le 1 + \frac{2\,\epsilon_1}{\alpha-\rho}$ *(otherwise* $\delta_0 = \delta_1$*). Let* $\epsilon_2 = \frac{\rho-\rho'}{1+\alpha-\rho'}$*. Then if* $\xi, y \le \delta_1$ *and* $(1-\epsilon_2)\, y^\alpha \le \xi^\alpha$ *we have*

$$\frac{1}{T'(\xi)} \le \left(\frac{y}{T(y)}\right)^{1+\alpha-\rho}.$$

Proof. Note first that when $x > 0$,

$$(1+x)^{1+\alpha-\rho} = 1 + (1+\alpha-\rho)\, x + (1+\alpha-\rho)\frac{\alpha-\rho}{2}(1+z)^{\alpha-\rho-1}\, x^2 \tag{2}$$

for some $z \in [0, x]$. We put $x = \frac{T(y)-y}{y}$. If $\alpha > \rho$ then $\frac{\alpha-\rho}{2} x \le \epsilon_1$ (recall that $y \le \delta_1$ and $\frac{T(y)}{y}$ is an increasing function of y). Since $\alpha < 1$, $(1+z)^{\alpha-\rho-1} \le 1$ and the last term on the RHS of (2) is bounded by $(1+\alpha-\rho)\,\epsilon_1\, x$. [If $\alpha \le \rho$ the same bound holds.] Putting this bound and the expression for x in (2) gives:

$$\left(\frac{T(y)}{y}\right)^{1+\alpha-\rho} \le 1 + (1+\alpha-\rho)\,(1+\epsilon_1)\,\frac{T(y)-y}{y}.$$

However, a calculation similar to Lemma 2.1 establishes that $\frac{T(y)-y}{y} \le \frac{C+\epsilon_0}{1+\alpha}\, y^\alpha$, so

$$\begin{aligned} \left(\frac{T(y)}{y}\right)^{1+\alpha-\rho} &\le 1 + (1+\alpha-\rho)\,(1+\epsilon_1)\,\frac{C+\epsilon_0}{1+\alpha}\, y^\alpha \\ &= 1 + (1+\alpha-\rho')\,(1-\epsilon_2)\,(1+\epsilon_1)\,\frac{C+\epsilon_0}{1+\alpha}\, y^\alpha \\ &= 1 + (C-\epsilon_0)\,(1-\epsilon_2)\, y^\alpha \\ &\le 1 + (C-\epsilon_0)\,\xi^\alpha \le T'(\xi). \end{aligned} \tag{3}$$

The lemma follows. □

Note that the estimates in Lemma 2.3 are finer than are needed for the proof of Proposition 1.1 where we take $\rho = \rho'$ (the gap between ρ and ρ' is used to account for the discretization effects of Ulam's method in the proof of Proposition 1.3).

Lemma 2.4. *Let* $f \in \mathcal{C}_A$ *and let* $y_2 \in (\gamma, 1]$*. Then* $\frac{f(y_2)}{T'(y_2)} \le \frac{\int f\, d\lambda}{\gamma}$.

Proof. Since f is decreasing and non–negative, $x\, f(x) \le \int f\, d\lambda$ so $f(y_2) \le f(\gamma) \le \frac{\int_0^1 f\, d\lambda}{\gamma}$. The lemma follows since $T' \ge 1$. □

Proof of Proposition 1.1. Choose $0 < \rho' = \rho < 1-\alpha$ and $0 < \epsilon_0, \epsilon_1, \delta_1$ as in Lemma 2.3. Let κ_1 be as in Lemma 2.1. Let

$$\sigma = \min\left\{ \frac{C-\epsilon_0}{1+\alpha}\,\frac{1-\rho}{{\kappa_1}^{1+\alpha}}, 1-{\kappa_1}^{\alpha-1} \right\}$$

and let $A_{\min} = A_{\min}(\rho)$ be such that

$$A_{\min}\,\sigma = \frac{1}{\gamma}.$$

Choose $A \geq A_{\min}$. Let $f \in \mathcal{C}_A$. Since $\mathcal{P}$ is a Markov operator, $\mathcal{P}f \geq 0$ and $\int \mathcal{P}f\,d\lambda = \int f\,d\lambda$. There is no loss of generality in assuming that $\int f\,d\lambda = 1$, since then we need only prove that (i) $\mathcal{P}f$ is decreasing; and (ii) $\mathcal{P}f(y) \leq Ay^{-\alpha}$. In the notation established above,

$$\mathcal{P}f(y) = \frac{f(y_1)}{T'(y_1)} + \frac{f(y_2)}{T'(y_2)}. \tag{4}$$

To see that (i) is true, note that both branches of T are increasing, and by convexity, $1/T'$ is decreasing. Thus, since f is decreasing, so too is $\mathcal{P}f$. To prove (ii) we apply the lemmas.

(*Contribution of* y_1) Since $f \in \mathcal{C}_A^{\text{dens}}$, we have

$$\frac{f(y_1)}{T'(y_1)} \leq \frac{A\,y_1^{-\alpha}}{T'(y_1)} = A\,y^{-\alpha}\left(\frac{y_1}{T(y_1)}\right)^{-\alpha}\frac{1}{T'(y_1)} \tag{5}$$

(since $T(y_1) = y$). Now, if $y_1 \in [\delta_1, \gamma)$ we use Lemma 2.1 to estimate $\frac{1}{T'(y_1)} \leq \frac{y_1}{T(y_1)} \leq \frac{1}{\kappa_1}$ so (5) becomes

(6)

$$\frac{f(y_1)}{T'(y_1)} \leq A\,y^{-\alpha}\left(\frac{y_1}{T(y_1)}\right)^{1-\alpha} \leq A\,y^{-\alpha}\left(\frac{1}{\kappa_1}\right)^{1-\alpha} \leq A\,y^{-\alpha} - A\,(1-{\kappa_1}^{\alpha-1}) \leq A\,y^{-\alpha} - A\,\sigma.$$

(the second to last inequality is because $y^{-\alpha} \geq 1$). On the other hand, if $y_1 \in [0, \delta_1]$ we can apply Lemma 2.3 (with $\rho = \rho'$ so $y_1 = \xi$) to estimate $\frac{1}{T'(y_1)} \leq \left(\frac{y_1}{T(y_1)}\right)^{1+\alpha-\rho}$. Then (5) becomes

$$\frac{f(y_1)}{T'(y_1)} \leq A\,y^{-\alpha}\left(\frac{y_1}{T(y_1)}\right)^{1-\rho} \leq A\,y^{-\alpha}\left(1 - \frac{C-\epsilon_0}{1+\alpha}\,\frac{1-\rho}{{\kappa_1}^{1+\alpha}}\,y^{\alpha}\right)$$

(the last inequality is by Corollary 2.2 since $T(y_1) = y$). Thus:

$$\frac{f(y_1)}{T'(y_1)} \leq A\,y^{-\alpha} - A\,\sigma. \tag{7}$$

Comparing equations (6) and (7) with the choice of A:

$$\frac{f(y_1)}{T'(y_1)} \leq A\,y^{-\alpha} - \frac{1}{\gamma} - (A - A_{\min})\,\sigma. \tag{8}$$

(*Contribution of* y_2) By Lemma 2.4, $\frac{f(y_2)}{T'(y_2)} \leq \frac{1}{\gamma}$. By (4) and (8) we have

$$\mathcal{P}f(y) \leq A\,y^{-\alpha} - (A - A_{\min})\,\sigma, \tag{9}$$

from which Proposition 1.1 follows.

Uniqueness of the density. Let $f_* \in \mathcal{C}_{A^*}$ be the invariant density from the proof of Theorem 1. We will prove that f_* is the unique invariant density by showing that the corresponding ACIPM is equivalent to Lebesgue measure, and ergodic.

Lemma 2.5. *Suppose that for some $\tilde{A}$, $f \in \mathcal{C}_{\tilde{A}}$, $f = \mathcal{P}f$ and $0 < \int f\,d\lambda$. Then $d\mu = f\,d\lambda$ is equivalent to λ.*

Proof. Clearly, μ is AC with respect to λ. To show that λ is AC with respect to μ we will show that $f > 0$ λ–a.e. To this end, let δ be such that $\int_0^\delta \tilde{A}\,x^{-\alpha}dx = 1$. Then, since f is decreasing and bounded by $\tilde{A}\int f\,d\lambda\,x^{-\alpha}$, $f\mathbf{1}_{(0,\delta)} > 0$. By Lemma 2.1, the sequence $y_0 = \gamma$, $y_n = T^{-1}(y_{n-1}) \cap [0,\gamma)$ is decreasing, and indeed converges to 0. Thus, there is an N such that $y_N < \delta$, so $f|_{[0,y_N]} \geq f(y_N) > 0$. But

$$f(x) = \mathcal{P}^N f(x) = \sum_{T^N(x_i)=x} \frac{f(x_i)}{{T^N}'(x_i)} \geq \frac{f(y_N)}{{T^N}'(y_N)} + \sum_{\substack{T^N(x_i)=x\\ x_i>y_N}} \frac{f(x_i)}{{T^N}'(x_i)} \geq \frac{f(y_N)}{{T^N}'(y_N)}$$

(we have used the fact that $[0,y_N)$ is the first monotonicity interval of T^N, and $1/{T^N}'$ has a decreasing continuous extension to $[0,y_N]$). Thus, f is bounded away from 0. □

Lemma 2.6. *If $0 \leq f = \mathcal{P}f$ and $E = T^{-1}E$ λ–a.e. then $f\mathbf{1}_E$ is decreasing, and a fixed point of $\mathcal{P}$.*

Proof. Write $f_E = f\,\mathbf{1}_E$. Consider the simple functions

$$\Delta_N = \{f = \sum a_{B_N}\,\mathbf{1}_{B_N} | B_N \text{ is a monotonicity interval of } T^N\}.$$

By an argument similar to [8], $\bigcup_{N=1}^\infty \Delta_N$ is dense in L^1. Let $f_D \in \Delta_N$. Then,

$$\mathcal{P}^N f_D = \sum_{B_N} a_{B_N} \frac{\mathbf{1}_{B_N} \circ T^{-N}}{{T^N}' \circ T^{-N}} = \sum_{B_N} \frac{a_{B_N}}{{T^N}' \circ T^{-N}}$$

since each $\mathbf{1}_{B_N} \circ T^{-N} = \mathbf{1}$ (all branches of T^N are onto). Since the branches of T^N are convex, $\mathcal{P}^N f_D$ is a decreasing function. Next, observe that $\mathbf{1}_E = \mathbf{1}_{T^{-1}E} = \mathbf{1}_E \circ T$ (λ–a.e.) so

$$\mathcal{P}f_E(x) = \sum_{\{T(x_i)=x\}} \frac{f(x_i)\mathbf{1}_E(x_i)}{T'(x_i)} = \sum_{\{T(x_i)=x\}} \frac{f(x_i)\mathbf{1}_E(T(x_i))}{T'(x_i)} = [\mathcal{P}f(x)]\,\mathbf{1}_E(x) = f_E(x).$$

We also have $\mathcal{P}^N f_E = f_E$ and hence

$$\|f_E - \mathcal{P}^N f_D\|_{L^1} = \|\mathcal{P}^N f_E - \mathcal{P}^N f_D\|_{L^1} = \|\mathcal{P}^N(f_E - f_D)\|_{L^1} \leq \|(f_E - f_D)\|_{L^1}.$$

Thus, f_E is an L^1–density point of a sequence of decreasing functions, so is decreasing. □

Let $d\mu = f_*\,d\lambda$. By Lemma 2.5 (with $\tilde{A} = A^*$), μ is equivalent to λ. Now suppose that E is a measurable set such that $E = T^{-1}(E)$ μ–a.e. and $\mu(E) > 0$. Then, since μ is equivalent to λ, $E = T^{-1}(E)$ λ–a.e. and $\lambda(E) > 0$. Now put $f_E = f_*\mathbf{1}_E$. Then $\|f_E\|_{L^1} = \mu(E) > 0$. By Lemma 2.6, $f_E = \mathcal{P}f_E$ and f_E is decreasing. We therefore know that $f_E \in \mathcal{C}_{\tilde{A}}$ where $\tilde{A} = \frac{A^*}{\mu(E)}$. By Lemma 2.5, $f_E > 0$ λ–a.e. This shows that $E = [0,1]$ (except possibly for a set of measure zero), so μ is ergodic. Any other ergodic invariant measure is singular with respect to μ (and hence λ), so μ is unique amongst the ACIMs.

3. Convergence of Ulam's method

Let $\mathcal{P}_n = \mathcal{E}_n \mathcal{P}$ be the nth Ulam approximation to $\mathcal{P}$. It is easy to check that $\mathcal{E}_n$ is monotone, preserves integrals, and the decreasing property of functions. Since $\mathcal{P}$ also preserves these properties, $\mathcal{P}_n$ does too. Therefore, all of the work in proving Proposition 1.3 reduces to checking that if A and n are large enough then

$$f = \sum_{i=0}^{n-1} \pi_i \, \mathbf{1}_{[\frac{i}{n}, \frac{i+1}{n})} \in \mathcal{C}_A \Rightarrow \mathcal{P}_n f \in \mathcal{C}_A \tag{10}$$

(note that $\mathcal{P}_n f = \mathcal{E}_n \mathcal{P} f = {\mathcal{E}_n}^2 \mathcal{P} f = \mathcal{E}_n \mathcal{P}_n f \in \text{Range}(\mathcal{E}_n)$).

Remark: A "uniform" application of Proposition 1.1 is not sufficient to prove Proposition 1.3, since $\mathcal{E}_n$ does not preserve $\mathcal{C}_A$: if $g \in \mathcal{C}_A$ and $g|_{[0,1/n)} = A\,x^{-\alpha}$ then $\mathcal{E}_n(g)|_{[0,1/n)} = \frac{A}{1-\alpha}\left(\frac{1}{n}\right)^{-\alpha}$. However, away from 0, and for $A > A_{\min}$, equation (9) suggests that there is some slack in which $\mathcal{E}_n$ can act; this is used in Lemma 3.5 to prove bounds on $\mathcal{P}_n f(y)$ of the type required by (10) for values of y away from 0. Near to 0, more explicit calculations are needed. □

Action of $\mathcal{P}_n$ on $\mathcal{C}_{A,n}$. The action of $\mathcal{P}_n$ on Range($\mathcal{E}_n$) is:

$$\mathcal{P}_n \left(\sum_{i=0}^{n-1} \pi_i \, \mathbf{1}_{\left[\frac{i}{n}, \frac{i+1}{n}\right)} \right) = \sum_{j=0}^{n-1} \left(\sum_{i=0}^{n-1} \pi_i \, P_{ij} \right) \mathbf{1}_{\left[\frac{j}{n}, \frac{j+1}{n}\right)}$$

where P is the matrix with entries

$$P_{ij} = \frac{\lambda\left(\left[\frac{i}{n}, \frac{i+1}{n}\right) \cap T^{-1}\left[\frac{j}{n}, \frac{j+1}{n}\right)\right)}{\lambda\left(\left[\frac{i}{n}, \frac{i+1}{n}\right)\right)}.$$

To prove (10) we need to show that if A and n are large enough then

$$\pi_i \le A\left(\frac{i+1}{n}\right)^{-\alpha} \; \forall i \qquad \Rightarrow \qquad \sum_{i=0}^{n-1} \pi_i \, P_{ij} \le A \left(\frac{j+1}{n}\right)^{-\alpha} \; \forall j.$$

As in the proof of Proposition 1.1, most of the effort relates to controlling the contribution of the left branch of T. The argument below treats the cases where $P_{ii} > 0$ and $P_{ii} = 0$ separately, since slightly different estimates seem to be required. Note that the contribution of the left branch to P_{ii} will be zero if $T^{-1}\left[\frac{i}{n}, \frac{i+1}{n}\right)$ lies wholly to the left of $\left[\frac{i}{n}, \frac{i+1}{n}\right)$.

Lemma 3.1. *Let $f = \sum_{i=0}^{n-1} \pi_i \mathbf{1}_{[\frac{i}{n}, \frac{i+1}{n})}$ be a decreasing density. For every $j < n$ there exists an $l \le j$ such that*

$$\sum_{i=0}^{n-1} \pi_i \, P_{ij} \le \pi_{l-1} P_{(l-1)\,j} + \pi_l \, P_{l\,j} + \frac{1}{\gamma}$$

(if $j = 0$ adopt the convention that $P_{(-1)\,0} = 0$).

Proof. Since T is expanding, each connected component of $T^{-1}\left[\frac{j}{n}, \frac{j+1}{n}\right)$ has length less than or equal to $1/n$. Thus, the left component intersects at most two consecutive intervals $\left[\frac{l-1}{n}, \frac{l}{n}\right)$ and $\left[\frac{l}{n}, \frac{l+1}{n}\right)$. The contribution of the right branch is got by averaging $\frac{f(y_2)}{T'(y_2)}$ over $y \in \left[\frac{j}{n}, \frac{j+1}{n}\right)$. A similar argument to Lemma 2.4 bounds this contribution by $1/\gamma$. □

Notation. Let $0 < \rho' < \rho < 1 - \alpha$ and let $\epsilon_0, \epsilon_1, \epsilon_2 > 0$ be as in Lemma 2.3. Let $\delta_1 > 0$ and κ_1 be as in Lemmas 2.3 and 2.1 (respectively). Let T^{-1} denote the left branch of the inverse map of T, and for fixed n denote

$$y_j = T^{-1}(\tfrac{j}{n}).$$

We will also adopt the notation of Lemma 3.1, in respect of the relation between l and j.

Lemma 3.2 (Cell containing 0)**.** *Fix $A, n > 0$ and let $(\pi_i)_{i=0}^{n-1}$ be as in the left hand side of (10). Let P be the matrix representation of $\mathcal{P}_n$. If $\frac{1}{n} < \delta_1$ then*

$$\pi_0\, P_{00} \le A\left(\frac{1}{n}\right)^{-\alpha} - A\,\frac{C - \epsilon_0}{(1+\alpha)\,{\kappa_1}^{1+\alpha}}.$$

Proof. We have $\pi_0 \le A\,\left(\frac{1}{n}\right)^{-\alpha}$, and

$$P_{00} = \frac{\lambda(T^{-1}[0, 1/n))}{1/n} = \frac{\lambda[0, y_1)}{T(y_1)} = \frac{y_1}{T(y_1)} \le 1 - \frac{C - \epsilon_0}{1+\alpha}\,\frac{1}{{\kappa_1}^{1+\alpha}}\,(T(y_1))^{\alpha},$$

by Corollary 2.2. The lemma follows immediately. □

Lemma 3.3 (Cells with self–intersections)**.** *Fix $A, n > 0$ and let $(\pi_i)_{i=0}^{n-1}$ be as in the left hand side of (10). Let P be the matrix representation of $\mathcal{P}_n$. If $\frac{j-1}{n} \le y_j \le \frac{j}{n} \le y_{j+1} \le \delta_1$ then*

$$\pi_{j-1}\, P_{(j-1)j} + \pi_j\, P_{jj} \le A\left(\frac{j+1}{n}\right)^{-\alpha} - A\,\frac{1-\alpha}{2^{\alpha}\,(1+\alpha)}\,\frac{C-\epsilon_0}{{\kappa_1}^{1+\alpha}}.$$

Proof. We have

$$\begin{aligned}
\pi_{j-1}P_{(j-1)\,j} + \pi_j\, P_{jj} &\le A\left(\frac{j}{n}\right)^{-\alpha} P_{(j-1)\,j} + A\left(\frac{j+1}{n}\right)^{-\alpha} P_{jj} \\
&= A\left(\frac{j+1}{n}\right)^{-\alpha}\left(\left(\frac{j+1}{j}\right)^{\alpha} - 1\right) P_{(j-1)\,j} \\
&\qquad + A\left(\frac{j+1}{n}\right)^{-\alpha}(P_{(j-1)\,j} + P_{jj}). \qquad (11)
\end{aligned}$$

First of all, since $T^{-1}[\frac{j}{n}, \frac{j+1}{n}) = [y_j, y_{j+1})$ we have

$$P_{(j-1)j} = \frac{\lambda\left([y_j, y_{j+1}) \cap [\frac{j-1}{n}, \frac{j}{n})\right)}{1/n} = \frac{\lambda[y_j, \frac{j}{n})}{1/n} = \frac{T(y_j) - y_j}{\frac{1}{j}\,T(y_j)} \qquad (12)$$

and

$$
\begin{aligned}
P_{(j-1)j} + P_{jj} &= \frac{\lambda\left([y_j, y_{j+1}) \cap [\frac{j-1}{n}, \frac{j}{n})\right)}{1/n} + \frac{\lambda\left([y_j, y_{j+1}) \cap [\frac{j}{n}, \frac{j+1}{n})\right)}{1/n} \\
&= \frac{\lambda\left([y_j, y_{j+1}) \cap (\frac{j-1}{n}, \frac{j+1}{n})\right)}{1/n} = \frac{\lambda[y_j, y_{j+1})}{1/n} = \frac{y_{j+1} - y_j}{T(y_{j+1}) - T(y_j)}. \qquad (13)
\end{aligned}
$$

Next, we bound $(1 + \frac{1}{j})^\alpha \leq 1 + \frac{\alpha}{j}$ and use (12) to bound the first term on the RHS of (11) by

$$
A \left(\frac{j+1}{n}\right)^{-\alpha} \alpha \frac{T(y_j) - y_j}{T(y_j)}.
$$

As in the proof of Lemma 2.1 we have

$$
\frac{y_{j+1} - y_j}{T(y_{j+1}) - T(y_j)} \leq \frac{1}{T'(y_j)} \leq \frac{y_j}{T(y_j)} = 1 - \frac{T(y_j) - y_j}{T(y_j)}.
$$

Together with (13), this bounds the second term on the RHS of (11), so that

$$
\pi_{j-1} P_{(j-1)\,j} + \pi_j \, P_{jj} \leq A \left(\frac{j+1}{n}\right)^{-\alpha} \left(1 - (1-\alpha)\frac{T(y_j) - y_j}{T(y_j)}\right).
$$

As in the proof of Corollary 2.2,

$$
\frac{T(y_j) - y_j}{T(y_j)} \geq \frac{C - \epsilon_0}{(1+\alpha)\,{\kappa_1}^{1+\alpha}} \left(\frac{j}{n}\right)^{\alpha}.
$$

The lemma follows since $\left(\frac{j}{j+1}\right)^{\alpha} \geq \frac{1}{2^\alpha}$. □

Lemma 3.4 (Cells near 0 without self intersections)**.** *Fix $A, n > 0$ and let $(\pi_i)_{i=0}^{n-1}$ be as in the left hand side of (10). Let P be the matrix representation of $\mathcal{P}_n$. If $\frac{l-1}{n} \leq y_j \leq \frac{l}{n} \leq y_{j+1} \leq \delta_1$ and $l < j$ then*

$$
\pi_{l-1} \, P_{(l-1)j} + \pi_l \, P_{lj} \leq A \left(\frac{j+1}{n}\right)^{-\alpha} - A \, \frac{1 - \alpha - \rho}{(1+\alpha)} \, \frac{C - \epsilon_0}{{\kappa_1}^{1+\alpha}}
$$

provided that $\epsilon_n \leq \epsilon_2$ where $\epsilon_n = \alpha \left(\frac{C+\epsilon_0}{1+\alpha}\right)^{1/(1+\alpha)} n^{-\alpha/(1+\alpha)}$.

Proof. By the conditions on π, (13) and the mean value theorem,

$$
\begin{aligned}
\pi_{l-1} \, P_{l-1\,j} + \pi_l \, P_{l\,j} &\leq A \left(\frac{l}{n}\right)^{-\alpha} P_{(l-1)\,j} + \left(\frac{l+1}{n}\right)^{-\alpha} P_{l\,j} \\
&\leq A \left(\frac{l}{n}\right)^{-\alpha} (P_{(l-1)\,j} + P_{l\,j}) \\
&= A \left(\frac{l}{n}\right)^{-\alpha} \frac{y_{j+1} - y_j}{T(y_{j+1}) - T(y_j)} = A \left(\frac{l}{n}\right)^{-\alpha} \frac{1}{T'(\xi_j)} \qquad (14)
\end{aligned}
$$

for some $\xi_j \in T^{-1}\left[\frac{j}{n}, \frac{j+1}{n}\right)$. To assemble a bound on the RHS of (14), note that

$$\begin{aligned}\left(\frac{l}{n}\right)^{-\alpha} &= \left(\frac{j+1}{n}\right)^{-\alpha} \left(\frac{y_{j+1}}{l/n}\right)^{\alpha} \left(\frac{(j+1)/n}{y_{j+1}}\right)^{\alpha} \\ &\leq \left(\frac{j+1}{n}\right)^{-\alpha} \left(\frac{T(l/n)}{l/n}\right)^{\alpha} \left(\frac{(j+1)/n}{y_{j+1}}\right)^{\alpha} \\ &\leq \left(\frac{j+1}{n}\right)^{-\alpha} \left(\frac{T(y_{j+1})}{y_{j+1}}\right)^{2\alpha} \qquad (15)\end{aligned}$$

since $T(\frac{l}{n}) \geq T(y_j) = \frac{j}{n} \geq y_{j+1}$ and $\frac{T(y)}{y}$ is increasing (proof of Lemma 2.1).

Claim: $\frac{1}{T'(\xi_j)} \leq \left(\frac{y_{j+1}}{T(y_{j+1})}\right)^{1+\alpha-\rho}$.

Proof of claim: If ${y_{j+1}}^{\alpha} - {\xi_j}^{\alpha} \leq \epsilon_2\, {y_{j+1}}^{\alpha}$, then the claim follows from Lemma 2.3. When $y \geq \xi$, the mean value theorem gives

$$y^{\alpha} - \xi^{\alpha} \leq \alpha\, \xi^{\alpha-1}\,(y-\xi).$$

Since T is expanding, if $y, \xi \in [y_j, y_{j+1}]$ then $y - \xi \leq y_{j+1} - y_j \leq T(y_{j+1}) - T(y_j) = \frac{1}{n}$. Next, by the conditions on l, j and Lemma 2.1,

$$\frac{1}{n} \leq \frac{j}{n} - \frac{l}{n} \leq T(y_j) - y_j \leq \frac{C+\epsilon_0}{1+\alpha} {y_j}^{1+\alpha}.$$

Thus, ${y_j}^{-1} \leq \left(\frac{C+\epsilon_0}{1+\alpha}\right)^{1/(1+\alpha)} n^{1/(1+\alpha)}$ so that

$${y_{j+1}}^{\alpha} - {\xi_j}^{\alpha} \leq \alpha\, {y_{j+1}}^{\alpha}\, {y_j}^{-1}\, \tfrac{1}{n} \leq {y_{j+1}}^{\alpha}\, \epsilon_n.$$

The claim follows from Lemma 2.3 since $\epsilon_n \leq \epsilon_2$. ☐

By (14), (15) and the claim,

$$\pi_{l-1}\, P_{l-1\,j} + \pi_l\, P_{l\,j} \leq \left(\frac{j+1}{n}\right)^{-\alpha} \left(\frac{y_{j+1}}{T(y_{j+1})}\right)^{1-\alpha-\rho}.$$

The lemma follows from Corollary 2.2. ☐

Lemma 3.5. *Fix $A, n > 0$ and let $(\pi_i)_{i=0}^{n-1}$ be as in the left hand side of (10). Let P be the matrix representation of $\mathcal{P}_n$. If $y_{j+1} > \delta_1$ then*

$$(\pi\, P)_j \leq A\left(\frac{j+1}{n}\right)^{-\alpha}$$

provided that $A > A_{\min}$ and $\frac{1}{n} \leq \min\left\{(\kappa_1 - 1)\,\delta_1, \frac{{\delta_1}^{1+\alpha}}{\alpha}\left(1 - \frac{A_{\min}}{A}\right)\sigma\right\}$.

Proof. Let $f = \sum_{i=0}^{n-1} \pi_i\, \mathbf{1}_{[\frac{i}{n}, \frac{i+1}{n})}$ so that $f \in \mathcal{C}_A$. By (9),

$$\mathcal{P}f(y) \leq A\, y^{-\alpha} - (A - A_{\min})\,\sigma.$$

Next, observe that $(\pi\, P)_j$ is the value assumed by $\mathcal{P}_n f$ on $[\frac{j}{n}, \frac{j+1}{n})$. But $\mathcal{P}_n f = \mathcal{E}_n(\mathcal{P}f)$ so

$$(\pi\, P)_j = \frac{\int_{j/n}^{(j+1)/n} \mathcal{P}f\, d\lambda}{1/n} \leq n \int_{j/n}^{(j+1)/n} A\, y^{-\alpha}\, dy - (A - A_{\min})\,\sigma. \qquad (16)$$

Now, the first term on the RHS of (16) is

$$\frac{A\,n}{1-\alpha}\left(\left(\tfrac{j+1}{n}\right)^{1-\alpha}-\left(\tfrac{j}{n}\right)^{1-\alpha}\right)\le\frac{A\,n}{1-\alpha}\left(\frac{j}{n}\right)^{1-\alpha}\frac{1-\alpha}{j}=A\left(\frac{j+1}{n}\right)^{-\alpha}\left(1+\frac{1}{j}\right)^{\alpha}.$$

Now, $\left(1+\frac{1}{j}\right)^{\alpha}\le 1+\frac{\alpha}{j}$, and

$$\frac{j}{n}=\frac{j+1}{n}-\frac{1}{n}=T(y_{j+1})-\frac{1}{n}\ge\kappa_1\,y_{j+1}-\frac{1}{n}>\kappa_1\,\delta_1-\frac{1}{n}\ge\delta_1,$$

by the first condition on n. Consequently,

$$\frac{\alpha}{j}\le\frac{\alpha}{n\,\delta_1}\le\delta_1^{\alpha}\left(1-\frac{A_{\min}}{A}\right)\sigma,$$

by the second condition on n. Since we also have $\frac{j+1}{n}>\delta_1$ we can combine the above estimates to get

$$n\,A\int_{j/n}^{(j+1)/n}y^{-\alpha}\,dy\le A\left(\frac{j+1}{n}\right)^{-\alpha}+A\,{\delta_1}^{-\alpha}\frac{\alpha}{j}\le A\left(\frac{j+1}{n}\right)^{-\alpha}+(A-A_{\min})\,\sigma.$$

The lemma now follows from (16). □

Proof of Proposition 1.3. Let $\rho=(1-2^{-\alpha})\,(1-\alpha)$, $B=\frac{1-\rho}{1-\rho-\alpha}$ and choose $A\ge B\,A_{\min}$ and $0<\rho'<\rho$. Let $\epsilon_0,\epsilon_1,\epsilon_2,\delta_1,\kappa_1$ be chosen as in Lemmas 2.3 and 2.1. Let n_0 be large enough that the hypotheses on n in Lemmas 3.2, 3.4 and 3.5 hold for $n\ge n_0$. We need to establish (10). Let π be as in the LHS of (10) let $0\le j<n$. By Lemma 3.5, (10) holds for those j with $y_{j+1}\ge\delta_1$. Therefore, we suppose that

$$\frac{l-1}{n}\le y_j\le y_{j+1}\le\delta_1.$$

By Lemma 3.1 we need to show that

$$\pi_{l-1}\,P_{(l-1),j}+\pi_l\,P_{lj}\le A\left(\frac{j+1}{n}\right)^{-\alpha}-\frac{1}{\gamma}=A\left(\frac{j+1}{n}\right)^{-\alpha}-A_{\min}\,\sigma.$$

Several possibilities occur as to the relation of l and j.

($l=j=0$) By Lemma 3.2,

$$\pi_0\,P_{00}\le A\left(\frac{1}{n}\right)^{-\alpha}-A\,\frac{C-\epsilon_0}{1+\alpha}\frac{1}{{\kappa_1}^{1+\alpha}}\le A\left(\frac{1}{n}\right)^{-\alpha}-A\,\sigma\le A\left(\frac{1}{n}\right)^{-\alpha}-A_{\min}\,\sigma.$$

($l=j>0$, $P_{jj}>0$) In this case, T is insufficiently expanding for $T^{-1}[\frac{j}{n},\frac{j+1}{n})$ to lie wholly to the left of $[\frac{j}{n},\frac{j+1}{n})$. We are thus in the situation described by Lemma 3.3. Note that $\frac{1-\alpha}{2^{\alpha}}=1-\alpha-\rho=\frac{1-\rho}{B}$. Then, by Lemma 3.3 (using $l=j$),

$$\pi_{l-1}\,P_{(l-1)j}+\pi_l\,P_{lj}\le A\left(\frac{j+1}{n}\right)^{-\alpha}-A\,\frac{1-\alpha}{2^{\alpha}\,(1+\alpha)}\,\frac{C-\epsilon_0}{{\kappa_1}^{1+\alpha}}\le A\left(\frac{j+1}{n}\right)^{-\alpha}-A\,\frac{\sigma}{B}.$$

($l=j>0$, $P_{jj}=0$) In this case, $T^{-1}[\frac{j}{n},\frac{j+1}{n})\subset[\frac{j-1}{n},\frac{j}{n})$, and the same estimates as in Lemma 3.3 hold.

$(l < j)$ In this case, $T^{-1}(\frac{j+1}{n}) = y_{j+1} < \frac{l+1}{n} \leq \frac{j}{n}$ so $T^{-1}[\frac{j}{n}, \frac{j+1}{n})$ lies to the left of $\frac{j}{n}$. In case both $P_{(l-1)j}$ and P_{lj} are non-zero, we can apply Lemma 3.4 to get

$$\pi_{l-1}\, P_{(l-1)j} + \pi_l\, P_{lj} \leq A \left(\frac{j+1}{n}\right)^{-\alpha} - A\,\frac{1-\alpha-\rho}{(1+\alpha)}\,\frac{C-\epsilon_0}{{\kappa_1}^{1+\alpha}} \leq A \left(\frac{j+1}{n}\right)^{-\alpha} - A\,\frac{\sigma}{B}.$$

The other possibility is that $[y_j, y_{j+1}) = T^{-1}[\frac{j}{n}, \frac{j+1}{n}) \subset [\frac{l-1}{n}, \frac{l}{n})$ (so that $P_{lj} = 0$). In this case, the factor $(\frac{y_{j+1}}{l/n})^{\alpha}$ in the derivation of (15) can be bounded by 1, and the conclusion of Lemma 3.5 can be amended to replace the term $(1-\alpha-\rho)$ with $(1-\rho)$. Thus, the estimate of the previous case is slightly improved, but in particular is still valid.

Since $-A\,\frac{\sigma}{B} \leq -A_{\min}\,\sigma$, this completes the proof.

4. Numerical experiments and convergence rates

As noted in the introduction, Ulam's method is relatively easy to implement on a computer: for a given n, one must calculate the sets $T^{-1}[\frac{j}{n}, \frac{j+1}{n})$, and compute their intersections with intervals $[\frac{i}{n}, \frac{i+1}{n})$. The measure of these intersections gives the entries of the (extremely sparse) matrix P, and a left-eigenvector can be computed efficiently and accurately using sparse matrix methods (such as `eigs` in Matlab).

Example 3. Using the maps from Example 3 in the introduction, Ulam's method has been applied for a sequence of values of n and α. Explicitly, let $f_{n,\alpha}$ denote the normalized fixed point of $\mathcal{P}_n$; for each of $\alpha = 0.20, 0.35, 0.50, 0.65, 0.80$ and $n = 25\times 2^i$ $(i = 0, \cdots, 10)$. By Theorem 2, one expects each $f_{n,\alpha}$ to be decreasing, supported on $[0,1]$ and dominated by a function of the form $A\,x^{-\alpha}$. Indeed, one expects a power law singularity in the of order α near zero (see [3, 11, 13]). This is easily revealed by log–log plots of the densities $f_{n,\alpha}$. See Figure 1.

Rate of convergence. It is natural to ask about the accuracy of Ulam's method for a given n, as well as how quickly the "approximation errors" converge to 0 as n increases. Since explicit formulae for the invariant densities $f_{*,\alpha}$ are unknown, a direct computation of L^1 discrepancies $\epsilon_{n,\alpha} = \|f_{*,\alpha} - f_{n,\alpha}\|_{L^1}$ is not possible. However, the speed of convergence should be indicated by the L^1 difference between successive Ulam approximations for an increasing sequence of values of n. In view of Lemma 1.4, one expects $\epsilon_{n,\alpha} \geq O(n^{\alpha-1})$. We will assume that $\epsilon_{n,\alpha} \approx O(n^{\rho_\alpha})$ for some $\rho_\alpha \geq \alpha - 1$ and examine the data

$$\tilde{\epsilon}_n = \|f_{n,\alpha} - f_{n/2,\alpha}\|_{L^1},$$

with the hope of detecting the exponent ρ_α. A selection of data for five values of α and $n = 25\times 2^i$ $(i = 1, \ldots, 10)$ is presented in Table 1. For each value of α, a least squares linear fit was done to the log–log data, resulting in an estimated power law $\tilde{\epsilon}_n \approx c\,n^{\rho_\alpha}$. The calculated exponents are displayed in the second column of Table 1, and the data are depicted in Figure 2.

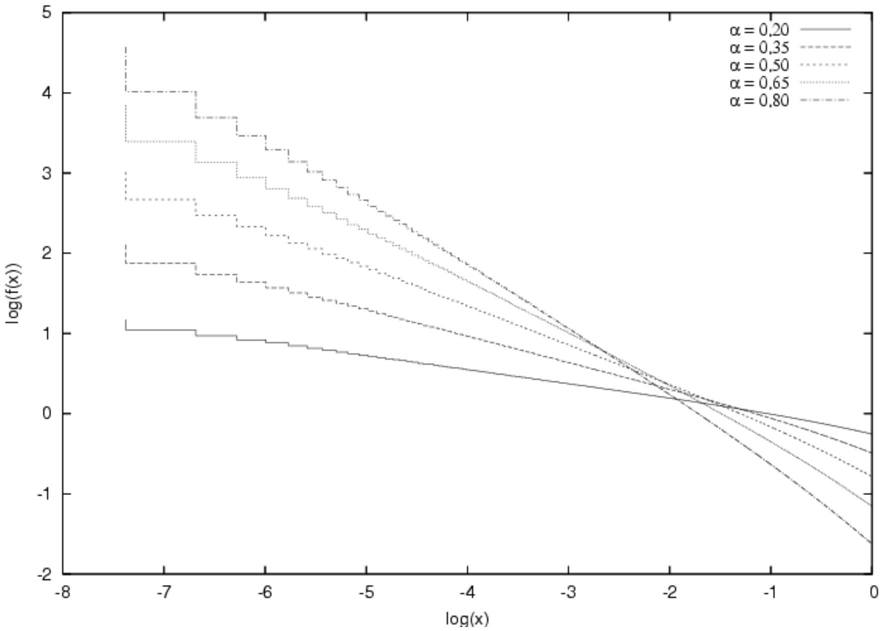

FIGURE 1. Plots of the $n = 1600$ Ulam approximations to the invariant densities from Example 3 of the introduction with $\alpha = 0.2, 0.35, 0.5, 0.65, 0.8$. The approximate densities are denoted $f(x)$ and are presented on a log–log scale.

α	ρ_α	σ_α	$\tilde{\epsilon}_{100}$	$\tilde{\epsilon}_{400}$	$\tilde{\epsilon}_{1600}$	$\tilde{\epsilon}_{6400}$	$\tilde{\epsilon}_{25600}$
0.20	-0.7285	-0.2172	0.007899	0.002931	0.001060	0.000378	0.000134
0.35	-0.5907	-0.3869	0.019507	0.009167	0.004019	0.001743	0.000750
0.50	-0.4544	-0.5501	0.041188	0.022236	0.012038	0.006342	0.003327
0.65	-0.3221	-0.7098	0.072785	0.047605	0.030299	0.019221	0.012233
0.80	-0.2167	-0.8640	0.120132	0.086702	0.064404	0.048262	0.035106

TABLE 1. The convergence rate of Ulam's method is indicated by $\tilde{\epsilon}_n = \|f_{n,\alpha} - f_{n/2,\alpha}\|_{L^1}$. A selection of these data are in columns 4 to 8 of the table. The exponent ρ_α is the numerically determined convergence rate $\tilde{\epsilon}_n = O(n^{\rho_\alpha})$ (see also Figure 2); the exponent σ_α is the numerically determined rate of decay of the spectral gap $(1 - \lambda_{n,\alpha}) = O(n^{\sigma_\alpha})$ (cf. Figure 3).

Convergence analysis. Conventional methods for analyzing the rate of convergence rely on a spectral gap in the FP operator [5, 12]. Since there is no such gap for maps in the class $\mathcal{T}_\alpha$, an alternative is to use spectral information from the $\mathcal{E}_n \circ \mathcal{P}$. These finite rank Ulam approximations are represented by ergodic, stochastic matrices, so have a unique eigenvalue with modulus 1 (whose eigenvector corresponds to the Ulam approximate density f_n), with the rest of the spectrum strictly contained in the unit circle in the complex plane. If $f_n = \mathcal{P}_n f_n$ and $f = \mathcal{P} f$ are the Ulam approximation and invariant densities (respectively) then

$$\|f - f_n\|_{L^1} = \|(Id - \mathcal{P}_n)^{-1}(f - \mathcal{E}_n f)\|_{L^1},$$

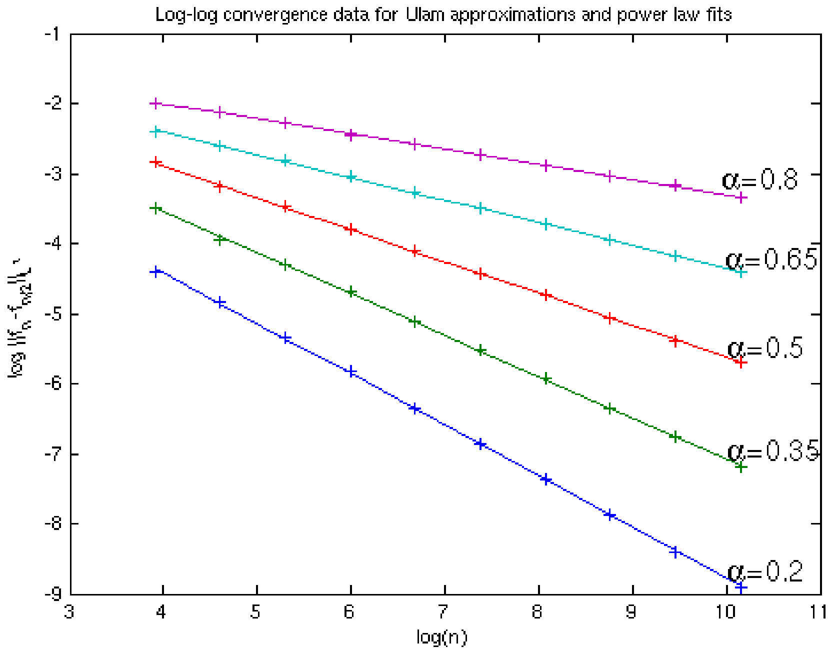

FIGURE 2. Plots of the convergence data Ulam approximations to the invariant densities from Example 3 of the introduction with $\alpha = 0.2, 0.35, 0.5, 0.65, 0.8$. For each α, f_n denotes the $n = 25\times 2^i$ $(i = 1, \cdots, 10)$ Ulam approximate density; the successive descrepancies $\|f_n - f_{n/2}\|_{L^1}$ provide an indication of the approximation error. These data are plotted on a log–log scale (with +), and the least squares linear fits (with solid lines) indicate power-law type convergence.

where the action of $(Id - \mathcal{P}_n)$ is restricted to those $g \in L^1$ for which $\int g\, d\lambda = 0$. By Lemma 1.4, the *projection component*, $\|f - \mathcal{E}_n f\|_{L^1} = O(n^{\alpha-1})$. The component of the error due to $(Id - \mathcal{P}_n)^{-1}$ is dominated by the spectral gap in the matrix representation of $\mathcal{P}_n$, and should be expected to be of $O((1 - \lambda_{n,\alpha})^{-1})$ (where $\lambda_{n,\alpha}$ denotes the second largest (by modulus) eigenvalue of $\mathcal{P}_n$). Good analytic estimates on the dependence of λ_n on n are difficult to obtain; by contrast, accurate and fast numerical computations can be done in Matlab. Heuristically, as n increases, the spectral gap $(1 - \lambda_n)$ should disappear, as the $\mathcal{P}_n$ become better and better approximations to the FP operator. Computations for a selection of values of α and n are displayed in Figure 3. The data have been used to used to estimate a decay rate on the spectral gap:

$$(1 - \lambda_{n,\alpha}) = O(n^{\sigma_\alpha});$$

these estimates are presented in the third column of Table 1. Interestingly, the computed exponents are $\sigma_\alpha \approx -\alpha$, for which a variety of heuristic explanations are possible (since $\mathcal{E}_n$ averages the dynamics over $[0, 1/n)$, the polynomial escape rate from the neighbourhood of 0 is replaced by a very weak exponential escape rate of $O(n^{-\alpha})$). Combining the spectral and projection components of the errors produces an error bound of $O(n^{\alpha-1-\sigma_\alpha})$. However, the least squares fit exponents in Table 1 suggest that this overestimates the

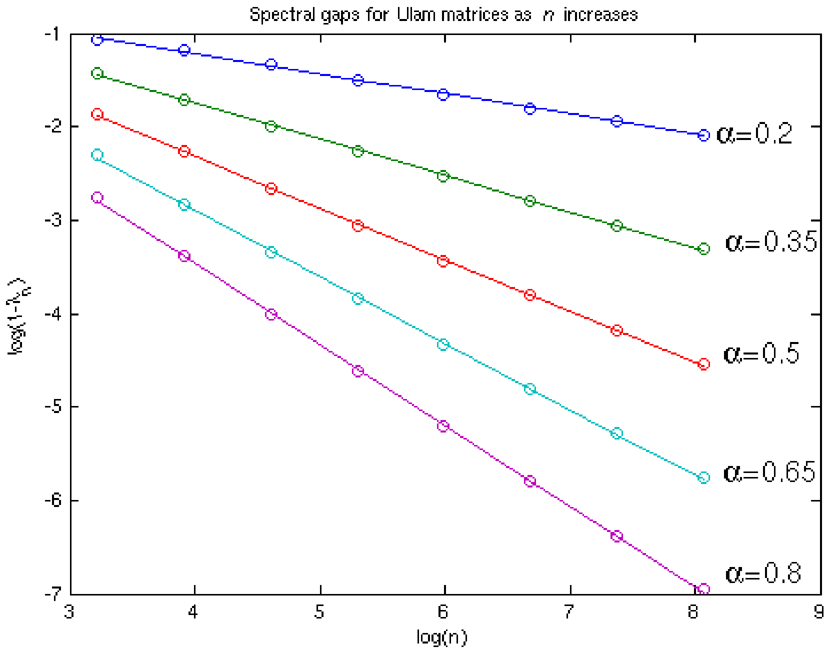

FIGURE 3. For each $\alpha = 0.2, 0.35, 0.5, 0.65, 0.8$, λ_n denotes the second largest eigenvalue of $\mathcal{P}_n$ ($n = 25 \times 2^i$, $i = 0, \ldots, 7$). The decaying spectral gap $(1 - \lambda_n)$ is depicted on a log–log scale (with $\circ$), and the least squares linear fits (with solid lines) indicate power-law type decay.

approximation error (especially for $\alpha > \frac{1}{2}$ where the error estimate would not imply convergence). A rigorous understanding of why the spectral gap decays as $n^{-\alpha}$, and why the spectrum based computation over-estimates the observed approximation error would be interesting.

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