

Good intermediate-rank lattice rules based on the weighted star discrepancy

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We study the problem of constructing good intermediate-rank lattice rules in the sense of having a low weighted star discrepancy. The intermediate-rank rules considered here are obtained by “copying” rank-1 lattice rules. We show that such rules can be constructed using a component-by-component technique and prove that the bound for the weighted star discrepancy achieves the optimal convergence rate.

Keywords: Intermediate-rank lattice rules, weighted star discrepancy, component-by-component construction.

1. Introduction

Integrals over the d -dimensional unit cube given by

$$I_d(f) = \int_{[0,1]^d} f(\mathbf{x}) \, d\mathbf{x}$$

may be approximated by rank-1 lattice rules. These are quadrature rules defined by

$$\frac{1}{n} \sum_{k=0}^{n-1} f\left(\left\{\frac{k\mathbf{z}}{n}\right\}\right). \quad (1)$$

Here, $\mathbf{z} \in \mathbb{Z}^d$ is the generating vector having all the components conveniently assumed to be relatively prime with n , while the braces around the vector indicate that we take the fractional part of each component of the vector.

In general terms, the “rank” of a lattice rule represents the minimum number of generating vectors required to produce the quadrature points. For d -dimensional integrals, lattice rules may have rank up to d . Further

details on the definition and the representation of lattice rules can be found in [13] and [14].

In some practical applications, the first variables are the most important. Hence, it seems natural to consider lattice rules obtained by “copying” rank-1 lattice rules. If $\ell \geq 1$ is an integer satisfying $\gcd(\ell, n) = 1$ and r is a fixed integer taken from the set $\{0, 1, \dots, d\}$, then we can define the following lattice rule:

$$Q_{N,d}(f) = \frac{1}{\ell^r n} \sum_{m_r=0}^{\ell-1} \cdots \sum_{m_1=0}^{\ell-1} \sum_{k=0}^{n-1} f \left(\left\{ \frac{kz}{n} + \frac{(m_1, \dots, m_r, 0, \dots, 0)}{\ell} \right\} \right). \quad (2)$$

For $r \geq 1$, this lattice rule is a rank- r lattice rule or “intermediate-rank lattice rule”. Let’s remark that the lattice rule (2) has $N = \ell^r n$ distinct points and is obtained by copying the rank-1 lattice rule (1) ℓ times in each of the first r dimensions. It is easy to observe that when $r = 0$ or $\ell = 1$, the lattice rule (2) is reduced to the rank-1 lattice rule (1).

Such intermediate-rank lattice rules have been previously studied in [5], [7], and [13]. Here, in order to construct these intermediate-rank lattice rules, we employ the “weighted star discrepancy” as a measure of “goodness”. An unweighted star discrepancy (corresponding to an L_∞ maximum error) has been previously used in [3] and in more general works such as [10] or [13], while the weighted star discrepancy has been used in [1], [4], and [12].

2. Bounds for the weighted star discrepancy

Let’s observe first that the quadrature points of the lattice rule (2) can be rewritten as:

$$\left\{ \frac{kz}{n} + \frac{(m_1, \dots, m_r, 0, \dots, 0)}{\ell} \right\} = \frac{\mathbf{y}_t}{N},$$

where \mathbf{y}_t/N , $0 \leq t \leq N - 1$, are in $[0, 1)^d$. Of course, these points are a reordering of the N -points of the rank- r lattice rule defined by (2). Hence the lattice rule (2) may be rewritten as

$$Q_{N,d}(f) = \frac{1}{N} \sum_{t=0}^{N-1} f \left(\frac{\mathbf{y}_t}{N} \right).$$

In order to introduce the weighted star discrepancy, let the set of quadrature points $\{\mathbf{y}_t/N, 0 \leq t \leq N - 1\}$ be denoted by P_N . Then the star discrepancy

of P_N is defined by

$$D_N^*(P_N) := \sup_{\mathbf{x} \in [0,1]^d} |\text{discr}(\mathbf{x}, P_N)|,$$

where $\text{discr}(\mathbf{x}, P_N)$ is the local discrepancy given by

$$\text{discr}(\mathbf{x}, P_N) := \frac{A([\mathbf{0}, \mathbf{x}], P_N)}{N} - \prod_{j=1}^d x_j.$$

Here $A([\mathbf{0}, \mathbf{x}], P_N)$ represents the counting function, namely the number of points in P_N which lie in $[\mathbf{0}, \mathbf{x}]$ with $\mathbf{x} = (x_1, x_2, \dots, x_d)$. The star discrepancy gives a measure of the uniformity of the distribution of the quadrature points.

Let now \mathbf{u} be an arbitrary subset of $\mathcal{D} := \{1, 2, \dots, d-1, d\}$ and denote its cardinality by $|\mathbf{u}|$. For the vector $\mathbf{x} \in [0, 1]^d$, let $\mathbf{x}_{\mathbf{u}}$ denote the vector from $[0, 1]^{|\mathbf{u}|}$ containing the components of \mathbf{x} whose indices belong to \mathbf{u} . By $(\mathbf{x}_{\mathbf{u}}, \mathbf{1})$ we mean the vector from $[0, 1]^d$ whose j -th component is x_j if $j \in \mathbf{u}$ and 1 if $j \notin \mathbf{u}$. Now let us introduce a set of non-increasing positive weights $\{\gamma_j\}_{j=1}^{\infty}$ which describes the decreasing importance of the successive coordinates x_j and set

$$\gamma_{\mathbf{u}} = \prod_{j \in \mathbf{u}} \gamma_j.$$

From Zaremba's identity (see for instance [15] or [16]) and by applying Hölder's inequality for integrals and sums, we obtain

$$\begin{aligned} |Q_{N,d}(f) - I_d(f)| &\leq \left(\sum_{\mathbf{u} \subseteq \mathcal{D}} \sup_{\mathbf{x}_{\mathbf{u}} \in [0,1]^{|\mathbf{u}|}} \gamma_{\mathbf{u}} |\text{discr}((\mathbf{x}_{\mathbf{u}}, \mathbf{1}), P_N)| \right) \\ &\quad \times \sup_{\mathbf{u} \subseteq \mathcal{D}} \gamma_{\mathbf{u}}^{-1} \int_{[0,1]^{|\mathbf{u}|}} \left| \frac{\partial^{|\mathbf{u}|}}{\partial \mathbf{x}_{\mathbf{u}}} f((\mathbf{x}_{\mathbf{u}}, \mathbf{1})) \right| d\mathbf{x}_{\mathbf{u}}. \end{aligned}$$

Thus we can define a weighted star discrepancy $D_{N,\gamma}^*(P_N)$ by

$$D_{N,\gamma}^*(P_N) := \sum_{\mathbf{u} \subseteq \mathcal{D}} \gamma_{\mathbf{u}} \sup_{\mathbf{x}_{\mathbf{u}} \in [0,1]^{|\mathbf{u}|}} |\text{discr}((\mathbf{x}_{\mathbf{u}}, \mathbf{1}), P_N)|. \quad (3)$$

From [10], we make use of Theorem 3.10 and Lemma 5.21, together with the arguments leading to Theorem 5.6, to obtain the following inequality:

$$\sup_{\mathbf{x}_{\mathbf{u}} \in [0,1]^{|\mathbf{u}|}} |\text{discr}((\mathbf{x}_{\mathbf{u}}, \mathbf{1}), P_N)| \leq 1 - (1 - 1/N)^{|\mathbf{u}|} + \frac{R_N(P_N, \mathbf{u})}{2}, \quad (4)$$

where

$$R_N(P_N, \mathbf{u}) = \frac{1}{N} \sum_{t=0}^{N-1} \prod_{j \in \mathbf{u}} \left(1 + \sum'_{-\frac{N}{2} < h \leq \frac{N}{2}} \frac{e^{2\pi i h y_{t,j}/N}}{|h|} \right) - 1.$$

In the above $y_{t,j}$ is the j -th coordinate of \mathbf{y}_t , while the $'$ in the sum indicates we omit the $h = 0$ term.

Let us mention here that from the general theory on lattice rules (for example, see [10] or [13]), it will follow that $R_N(P_N, \mathbf{u}) \geq 0$ for any $\mathbf{u} \subseteq \mathcal{D}$. From (3) and (4), we see that the general weighted star discrepancy satisfies the inequality

$$D_{N,\gamma}^*(P_N) \leq \sum_{\mathbf{u} \subseteq \mathcal{D}} \gamma_{\mathbf{u}} \left(1 - (1 - 1/N)^{|\mathbf{u}|} + \frac{R_N(P_N, \mathbf{u})}{2} \right). \quad (5)$$

Further bounds on the weighted star discrepancy may be obtained by making use of (5). If the weights γ_j are summable, that is,

$$\sum_{j=1}^{\infty} \gamma_j < \infty,$$

then from [4, Lemma 1], we obtain:

$$\sum_{\mathbf{u} \subseteq \mathcal{D}} \gamma_{\mathbf{u}} \left(1 - (1 - 1/N)^{|\mathbf{u}|} \right) \leq \frac{\max(1, \Gamma)}{N} \prod_{j=1}^{\infty} (1 + \gamma_j) \leq \frac{\max(1, \Gamma)}{\ell^r n} e^{\sum_{j=1}^{\infty} \gamma_j},$$

where

$$\Gamma := \sum_{j=1}^{\infty} \frac{\gamma_j}{1 + \gamma_j} < \infty.$$

The complete proof of this result may be found in [4]. Thus we obtain

$$\sum_{\mathbf{u} \subseteq \mathcal{D}} \gamma_{\mathbf{u}} \left(1 - (1 - 1/N)^{|\mathbf{u}|} \right) = O(n^{-1}), \quad (6)$$

where the implied constant depends on ℓ , r and the weights.

We have from [4] that

$$\sum_{\mathbf{u} \subseteq \mathcal{D}} \gamma_{\mathbf{u}} R_N(P_N, \mathbf{u}) = \frac{1}{N} \sum_{t=0}^{N-1} \prod_{j=1}^d \left(\beta_j + \gamma_j \sum'_{-\frac{N}{2} < h \leq \frac{N}{2}} \frac{e^{2\pi i h y_{t,j}/N}}{|h|} \right) - \prod_{j=1}^d \beta_j,$$

where $\beta_j = 1 + \gamma_j$. If we set

$$e_{N,d}^2(\mathbf{z}) = \sum_{\mathbf{u} \subseteq \mathcal{D}} \gamma_{\mathbf{u}} R_N(P_N, \mathbf{u}),$$

then we see that we have

$$e_{N,d}^2(\mathbf{z}) = \frac{1}{N} \sum_{t=0}^{N-1} \prod_{j=1}^d \left(\beta_j + \gamma_j \sum'_{-\frac{N}{2} < h \leq \frac{N}{2}} \frac{e^{2\pi i h y_{t,j}/N}}{|h|} \right) - \prod_{j=1}^d \beta_j. \quad (7)$$

Let's remark that the dependency on \mathbf{z} in $e_{N,d}^2(\mathbf{z})$ makes sense as the vectors \mathbf{y}_t actually depend on \mathbf{z} .

In research papers such as [2] or [5], it was proved that when n is prime, the quantity (7) is identical to a quadrature error obtained from applying a rank-1 lattice rule to a certain integrand. Working with such a quadrature error simplifies in general the analysis of the problem and also has some computational advantages. Using the techniques from the mentioned papers, it is relatively easy to prove that

$$e_{N,d}^2(\mathbf{z}) = \frac{1}{n} \sum_{k=0}^{n-1} \prod_{j=1}^d \left(\beta_j + \tilde{\gamma}_j \sum'_{-\frac{\tilde{N}_j}{2} < h \leq \frac{\tilde{N}_j}{2}} \frac{e^{2\pi i h k \hat{z}_j/n}}{|h|} \right) - \prod_{j=1}^d \beta_j. \quad (8)$$

In the above, the following notations have been introduced:

$$\tilde{\gamma}_j = \begin{cases} \gamma_j/\ell, & 1 \leq j \leq r, \\ \gamma_j, & r+1 \leq j \leq d. \end{cases}$$

Next,

$$\tilde{N}_j = \begin{cases} N/\ell = \ell^{r-1}n, & 1 \leq j \leq r, \\ N, & r+1 \leq j \leq d. \end{cases}$$

Finally, $\hat{\mathbf{z}} = (\hat{z}_1, \hat{z}_2, \dots, \hat{z}_d)$, with

$$\hat{z}_j = \begin{cases} \ell z_j, & 1 \leq j \leq r, \\ z_j, & r+1 \leq j \leq d. \end{cases}$$

Then by denoting

$$f_N(\mathbf{x}) = \prod_{j=1}^d \left(\beta_j + \tilde{\gamma}_j \sum'_{-\frac{\tilde{N}_j}{2} < h \leq \frac{\tilde{N}_j}{2}} \frac{e^{2\pi i h x_j}}{|h|} \right),$$

it is easy to observe that

$$e_{N,d}^2(\mathbf{z}) = \frac{1}{n} \sum_{k=0}^{n-1} f_N\left(\frac{k}{n}\hat{\mathbf{z}}\right) - \prod_{j=1}^d \beta_j.$$

Now it is clear that $e_{N,d}^2(\mathbf{z})$ (which is based on a rank- r lattice rule with $N = \ell^r n$ points) can be obtained from applying a modified n -point rank-1 lattice rule to f_N .

Next, we are looking to obtain a result for the mean of the quantities $e_{N,d}^2$. Such a result, together with (5) and (6), will allow us to deduce a certain bound for the weighted star discrepancy. This mean will be taken over all possible values of $\hat{\mathbf{z}}$. Because $\hat{\mathbf{z}}$ is known when \mathbf{z} is known, the mean will be actually considered for all possible values for \mathbf{z} . Each component z_j , $1 \leq j \leq d$, of the vector \mathbf{z} can be taken from the set $\mathcal{Z}_n := \{1, 2, \dots, n-1\}$ because we only take the fractional part of each component of the vector. Thus, for prime n , the mean $M_{N,d,\gamma}$ is defined by

$$M_{N,d,\gamma} := \frac{1}{(n-1)^d} \sum_{\mathbf{z} \in \mathcal{Z}_n^d} e_{N,d}^2(\mathbf{z}).$$

An expression for $M_{N,d,\gamma}$ is given in the next theorem.

Theorem 2.1. *If n is prime, ℓ is a positive integer such that $\gcd(\ell, n) = 1$ and r is an integer chosen such that $1 \leq r \leq d$, then*

$$\begin{aligned} M_{N,d,\gamma} &= \frac{1}{n} \prod_{j=1}^d \left(\beta_j + \tilde{\gamma}_j S_{\tilde{N}_j} \right) \\ &\quad + \frac{n-1}{n} \prod_{j=1}^d \left(\beta_j - \frac{\tilde{\gamma}_j}{n-1} \left(S_{\tilde{N}_j} - S_{\tilde{N}_j/n} \right) \right) - \prod_{j=1}^d \beta_j, \end{aligned} \quad (9)$$

where

$$S_n = \sum'_{-\frac{n}{2} \leq h < \frac{n}{2}} \frac{1}{|h|}.$$

Proof. Using the definition of the mean and separating out the $k = 0$ term in (8), we obtain:

$$M_{N,d,\gamma} = \frac{1}{n} \prod_{j=1}^d \left(\beta_j + \tilde{\gamma}_j S_{\tilde{N}_j} \right) + \Theta_{N,\gamma} - \prod_{j=1}^d \beta_j, \quad (10)$$

where

$$\begin{aligned} \Theta_{N,\gamma} &= \frac{1}{n(n-1)^d} \sum_{\mathbf{z} \in \mathcal{Z}_n^d} \sum_{k=1}^{n-1} f_N \left(\frac{k}{n} \hat{\mathbf{z}} \right) \\ &= \frac{1}{n} \sum_{k=1}^{n-1} \prod_{j=1}^d \left(\frac{1}{n-1} \sum_{z_j=1}^{n-1} \left(\beta_j + \tilde{\gamma}_j \sum'_{-\frac{\tilde{N}_j}{2} < h \leq \frac{\tilde{N}_j}{2}} \frac{e^{2\pi i h k z_j / n}}{|h|} \right) \right) \end{aligned}$$

$$= \frac{1}{n} \sum_{k=1}^{n-1} \prod_{j=1}^d \left(\beta_j + \frac{\tilde{\gamma}_j}{n-1} \sum_{z_j=1}^{n-1} \sum'_{-\frac{\tilde{N}_j}{2} < h \leq \frac{\tilde{N}_j}{2}} \frac{e^{2\pi i h k \hat{z}_j / n}}{|h|} \right).$$

For $1 \leq k \leq n-1$ and for any $j \geq 1$, consider now

$$T_n(k, j) = \sum_{z_j=1}^{n-1} \sum'_{-\frac{\tilde{N}_j}{2} < h \leq \frac{\tilde{N}_j}{2}} \frac{e^{2\pi i h k \hat{z}_j / n}}{|h|}. \quad (11)$$

By separating out the terms for which $h \equiv 0 \pmod{n}$ and replacing h by nq , we obtain

$$\begin{aligned} T_n(k, j) &= \sum_{z_j=1}^{n-1} \sum'_{\substack{-\frac{\tilde{N}_j}{2} < h \leq \frac{\tilde{N}_j}{2} \\ h \equiv 0 \pmod{n}}} \frac{1}{|h|} + \sum_{z_j=1}^{n-1} \sum'_{\substack{-\frac{\tilde{N}_j}{2} < h \leq \frac{\tilde{N}_j}{2} \\ h \not\equiv 0 \pmod{n}}} \frac{e^{2\pi i h k \hat{z}_j / n}}{|h|} \\ &= \sum_{z_j=1}^{n-1} \sum'_{-\frac{\tilde{N}_j}{2} < nq \leq \frac{\tilde{N}_j}{2}} \frac{1}{n|q|} + \sum'_{\substack{-\frac{\tilde{N}_j}{2} < h \leq \frac{\tilde{N}_j}{2} \\ h \not\equiv 0 \pmod{n}}} \frac{1}{|h|} \sum_{z_j=1}^{n-1} \left(e^{2\pi i h k / n} \right)^{\hat{z}_j}. \end{aligned}$$

If $\hat{z}_j = \ell z_j$, then

$$\sum_{z_j=1}^{n-1} \left(e^{2\pi i h k / n} \right)^{\hat{z}_j} = \sum_{z_j=1}^{n-1} \left(e^{2\pi i h k \ell / n} \right)^{z_j}.$$

Since n is prime and $\gcd(\ell, n) = 1$, then when $h \not\equiv 0 \pmod{n}$, it follows that $h k \ell \not\equiv 0 \pmod{n}$. It is then easy to check that

$$\sum_{z_j=1}^{n-1} \left(e^{2\pi i h k \ell / n} \right)^{z_j} = -1.$$

When $\hat{z}_j = z_j$, the sum is the above with $\ell = 1$ and has the same value of -1 . Replacing in the expression of $T_n(k, j)$ we obtain:

$$T_n(k, j) = \frac{n-1}{n} S_{\tilde{N}_j/n} - \sum'_{\substack{-\frac{\tilde{N}_j}{2} < h \leq \frac{\tilde{N}_j}{2} \\ h \not\equiv 0 \pmod{n}}} \frac{1}{|h|}.$$

The last term of the sum may be written as:

$$\sum'_{\substack{-\frac{\tilde{N}_j}{2} < h \leq \frac{\tilde{N}_j}{2} \\ h \not\equiv 0 \pmod{n}}} \frac{1}{|h|} = \sum'_{-\frac{\tilde{N}_j}{2} < h \leq \frac{\tilde{N}_j}{2}} \frac{1}{|h|} - \sum'_{-\frac{\tilde{N}_j}{2} < nq \leq \frac{\tilde{N}_j}{2}} \frac{1}{n|q|}$$

$$= S_{\tilde{N}_j} - \frac{1}{n} \sum'_{-\frac{\tilde{N}_j}{2n} < q \leq \frac{\tilde{N}_j}{2n}} \frac{1}{|q|} = S_{\tilde{N}_j} - \frac{1}{n} S_{\tilde{N}_j/n}.$$

Thus we obtain:

$$T_n(k, j) = \frac{n-1}{n} S_{\tilde{N}_j/n} - S_{\tilde{N}_j} + \frac{1}{n} S_{\tilde{N}_j/n} = S_{\tilde{N}_j/n} - S_{\tilde{N}_j}. \quad (12)$$

Using now (12), we see that

$$\Theta_{N, \gamma} = \frac{1}{n} \sum_{k=1}^{n-1} \prod_{j=1}^d \left(\beta_j + \frac{\tilde{\gamma}_j}{n-1} (S_{\tilde{N}_j/n} - S_{\tilde{N}_j}) \right),$$

and by replacing in (10), we obtain the desired result. \square

From this theorem, we can deduce the following:

Corollary 2.1. *If n is a prime number, ℓ is a positive integer such that $\gcd(\ell, n) = 1$ and r satisfies $1 \leq r \leq d$, then there exists a $\mathbf{z} \in \mathcal{Z}_n^d$ such that*

$$e_{N, d}^2(\mathbf{z}) \leq \frac{1}{n} \prod_{j=1}^d \left(\beta_j + \tilde{\gamma}_j S_{\tilde{N}_j} \right) \leq \frac{1}{n} \prod_{j=1}^d \left(\beta_j + 2\tilde{\gamma}_j \ln \tilde{N}_j \right).$$

Proof. Since $\beta_j = 1 + \gamma_j$ for any $1 \leq j \leq d$, it will follow from [9, Lemmas 1 and 2] and the arguments used in [4] that

$$\frac{n-1}{n} \prod_{j=1}^d \left(\beta_j - \frac{\tilde{\gamma}_j}{n-1} (S_{\tilde{N}_j} - S_{\tilde{N}_j/n}) \right) - \prod_{j=1}^d \beta_j \leq 0.$$

Using this in (9) together with the fact that $S_{\tilde{N}_j} \leq 2 \ln \tilde{N}_j$ for any $\tilde{N}_j \geq 2$ (see also [4] and [9]), we obtain

$$M_{N, d, \gamma} \leq \frac{1}{n} \prod_{j=1}^d \left(\beta_j + \tilde{\gamma}_j S_{\tilde{N}_j} \right) \leq \frac{1}{n} \prod_{j=1}^d \left(\beta_j + 2\tilde{\gamma}_j \ln \tilde{N}_j \right).$$

Clearly there must be a vector $\mathbf{z} \in \mathcal{Z}_n^d$ such that

$$e_{N, d}^2(\mathbf{z}) \leq M_{N, d, \gamma}.$$

This, together with the previous inequalities completes the proof. \square

From (5), (6) and Corollary 2.1, it follows that there exists a generating vector \mathbf{z} such that

$$D_{N, \gamma}^*(\mathbf{z}) \leq O(n^{-1}) + \frac{1}{2n} \prod_{j=1}^d \left(\beta_j + 2\tilde{\gamma}_j \ln \tilde{N}_j \right),$$

with the implied constant depending on ℓ , r and the weights, but independent of the dimension. As the above bound has a $\ln n$ dependency, it would appear that the weighted star discrepancy has the order of magnitude of $O(n^{-1}(\ln n)^d)$, a result which is widely believed to be the best possible in an unweighted setting (see [8] or [10] for details). However, in our case, under the assumption that the weights are summable, it follows from [1, Lemma 3] or [4, Lemma 2] that there exists a generating vector \mathbf{z} such that the weighted star discrepancy achieves the strong tractability error bound

$$D_{N,\gamma}^*(\mathbf{z}) = O(n^{-1+\delta}),$$

for any $\delta > 0$, where the implied constant depends on δ , ℓ , r and the weights but is independent of n and d .

3. Component-By-Component Construction Of The Generating Vector

In this section we show that intermediate-rank lattice rules of the form (2) that have good bounds for the weighted star discrepancy, can be obtained by making use of the so-named ‘‘component-by-component’’ (CBC) construction of the vector \mathbf{z} . This idea has been successfully used in several research papers such as [3], [4], [7], and [12] and is based on finding each component one at a time. The result is based on the following:

Theorem 3.1. *Consider n a prime number, ℓ a positive integer such that $\gcd(\ell, n) = 1$ and r chosen such that $1 \leq r \leq d$. Assume there exists a vector \mathbf{z} in \mathcal{Z}_n^d such that*

$$e_{N,d}^2(\mathbf{z}) \leq \frac{1}{n-1} \prod_{j=1}^d (\beta_j + \tilde{\gamma}_j S_{\tilde{N}_j}).$$

Then there exists a $z_{d+1} \in Z_n$ such that:

$$e_{N,d+1}^2(\mathbf{z}, z_{d+1}) \leq \frac{1}{n-1} \prod_{j=1}^{d+1} (\beta_j + \tilde{\gamma}_j S_{\tilde{N}_j}).$$

Such a z_{d+1} can be found by minimizing $e_{N,d+1}^2(\mathbf{z}, z_{d+1})$ over \mathcal{Z}_n .

Proof. When we add a new component, we obtain from (8) that

$$e_{N,d+1}^2(\mathbf{z}, z_{d+1}) = \frac{1}{n} \sum_{k=0}^{n-1} \prod_{j=1}^{d+1} \left(\beta_j + \tilde{\gamma}_j \sum'_{-\frac{\tilde{N}_j}{2} < h \leq \frac{\tilde{N}_j}{2}} \frac{e^{2\pi i h k \hat{z}_j / n}}{|h|} \right) - \prod_{j=1}^{d+1} \beta_j$$

$$\begin{aligned}
&= \frac{1}{n} \sum_{k=0}^{n-1} \prod_{j=1}^d \left(\beta_j + \tilde{\gamma}_j \sum'_{-\frac{\tilde{N}_j}{2} < h \leq \frac{\tilde{N}_j}{2}} \frac{e^{2\pi i h k \hat{z}_j / n}}{|h|} \right) \\
&\quad \times \left(\beta_{d+1} + \tilde{\gamma}_{d+1} \sum'_{-\frac{\tilde{N}_{d+1}}{2} < h \leq \frac{\tilde{N}_{d+1}}{2}} \frac{e^{2\pi i h k \hat{z}_{d+1} / n}}{|h|} \right) - \prod_{j=1}^{d+1} \beta_j.
\end{aligned}$$

From (8) and by separating out the $k = 0$ term in the above, we see that we can write

$$\begin{aligned}
e_{N,d+1}^2(\mathbf{z}, z_{d+1}) &= \beta_{d+1} e_{N,d}^2(\mathbf{z}) + \frac{\tilde{\gamma}_{d+1} S_{\tilde{N}_{d+1}}}{n} \prod_{j=1}^d (\beta_j + \tilde{\gamma}_j S_{\tilde{N}_j}) \\
&\quad + \frac{\tilde{\gamma}_{d+1}}{n} \sum_{k=1}^{n-1} \prod_{j=1}^d \left(\beta_j + \tilde{\gamma}_j \sum'_{-\frac{\tilde{N}_j}{2} < h \leq \frac{\tilde{N}_j}{2}} \frac{e^{2\pi i h k \hat{z}_j / n}}{|h|} \right) \\
&\quad \times \left(\sum'_{-\frac{\tilde{N}_{d+1}}{2} < h \leq \frac{\tilde{N}_{d+1}}{2}} \frac{e^{2\pi i h k \hat{z}_{d+1} / n}}{|h|} \right).
\end{aligned}$$

We next average $e_{N,d+1}^2(\mathbf{z}, z_{d+1})$ over all possible values of $z_{d+1} \in \mathcal{Z}_n$ and consider:

$$\text{Avg}(e_{N,d+1}^2(\mathbf{z}, z_{d+1})) = \frac{1}{n-1} \sum_{z_{d+1}=1}^{n-1} e_{N,d+1}^2(\mathbf{z}, z_{d+1}).$$

As the other terms that occur in the expression of the average are independent of z_{d+1} , we next focus on the quantity

$$\frac{1}{n-1} \sum_{z_{d+1}=1}^{n-1} \sum'_{-\frac{\tilde{N}_{d+1}}{2} < h \leq \frac{\tilde{N}_{d+1}}{2}} \frac{e^{2\pi i h k \hat{z}_{d+1} / n}}{|h|} = \frac{1}{n-1} (S_{\tilde{N}_{d+1}/n} - S_{\tilde{N}_{d+1}}),$$

where we made use of (11) and (12). By replacing this equality in the expression of the average, we see that $\text{Avg}(e_{N,d+1}^2(\mathbf{z}, z_{d+1}))$ is given by:

$$\begin{aligned}
&\beta_{d+1} e_{N,d}^2(\mathbf{z}) + \frac{\tilde{\gamma}_{d+1} S_{\tilde{N}_{d+1}}}{n} \prod_{j=1}^d (\beta_j + \tilde{\gamma}_j S_{\tilde{N}_j}) \\
&+ \frac{\tilde{\gamma}_{d+1} (S_{\tilde{N}_{d+1}} - S_{\tilde{N}_{d+1}/n})}{n(n-1)} \times \left[- \sum_{k=1}^{n-1} \prod_{j=1}^d \left(\beta_j + \tilde{\gamma}_j \sum'_{-\frac{\tilde{N}_j}{2} < h \leq \frac{\tilde{N}_j}{2}} \frac{e^{2\pi i h k \hat{z}_j / n}}{|h|} \right) \right].
\end{aligned}$$

Next,

$$\begin{aligned}
& -\frac{1}{n} \sum_{k=1}^{n-1} \prod_{j=1}^d \left(\beta_j + \tilde{\gamma}_j \sum_{-\frac{\tilde{N}_j}{2} < h \leq \frac{\tilde{N}_j}{2}} \frac{e^{2\pi i h k \hat{z}_j / n}}{|h|} \right) \\
&= -\frac{1}{n} \sum_{k=0}^{n-1} \prod_{j=1}^d \left(\beta_j + \tilde{\gamma}_j \sum_{-\frac{\tilde{N}_j}{2} < h \leq \frac{\tilde{N}_j}{2}} \frac{e^{2\pi i h k \hat{z}_j / n}}{|h|} \right) + \frac{1}{n} \prod_{j=1}^d (\beta_j + \tilde{\gamma}_j S_{\tilde{N}_j}) \\
&= -e_{N,d}^2(\mathbf{z}) - \prod_{j=1}^d \beta_j + \frac{1}{n} \prod_{j=1}^d (\beta_j + \tilde{\gamma}_j S_{\tilde{N}_j}) \leq \frac{1}{n} \prod_{j=1}^d (\beta_j + \tilde{\gamma}_j S_{\tilde{N}_j}).
\end{aligned}$$

In the last step we used $e_{N,d}^2(\mathbf{z}) \geq 0$, as $R_N(P_N, \mathbf{u}) \geq 0$ for any $\mathbf{u} \subseteq \mathcal{D}$ (see the previous section). Using also that $S_{\tilde{N}_{d+1}} - S_{\tilde{N}_{d+1}/n} \leq S_{\tilde{N}_{d+1}}$ and the hypothesis, we now obtain:

$$\begin{aligned}
& \text{Avg}(e_{N,d+1}^2(\mathbf{z}, z_{d+1})) \\
&\leq \beta_{d+1} e_{N,d}^2(\mathbf{z}) + \frac{\tilde{\gamma}_{d+1} S_{\tilde{N}_{d+1}}}{n} \prod_{j=1}^d (\beta_j + \tilde{\gamma}_j S_{\tilde{N}_j}) \\
&\quad + \frac{\tilde{\gamma}_{d+1} S_{\tilde{N}_{d+1}}}{n(n-1)} \prod_{j=1}^d (\beta_j + \tilde{\gamma}_j S_{\tilde{N}_j}) \\
&= \beta_{d+1} e_{N,d}^2(\mathbf{z}) + \frac{\tilde{\gamma}_{d+1} S_{\tilde{N}_{d+1}}}{n} \prod_{j=1}^d (\beta_j + \tilde{\gamma}_j S_{\tilde{N}_j}) \left(1 + \frac{1}{n-1} \right) \\
&\leq \frac{\beta_{d+1}}{n-1} \prod_{j=1}^d (\beta_j + \tilde{\gamma}_j S_{\tilde{N}_j}) + \frac{\tilde{\gamma}_{d+1} S_{\tilde{N}_{d+1}}}{n-1} \prod_{j=1}^d (\beta_j + \tilde{\gamma}_j S_{\tilde{N}_j}) \\
&= \frac{1}{n-1} \prod_{j=1}^d (\beta_j + \tilde{\gamma}_j S_{\tilde{N}_j}) (\beta_{d+1} + \tilde{\gamma}_{d+1} S_{\tilde{N}_{d+1}}).
\end{aligned}$$

Clearly, the $z_{d+1} \in \mathcal{Z}_n$ chosen to minimize $e_{N,d+1}^2(\mathbf{z}, z_{d+1})$ will satisfy

$$e_{N,d+1}^2(\mathbf{z}, z_{d+1}) \leq \text{Avg}(e_{N,d+1}^2(\mathbf{z}, z_{d+1})).$$

This, together with the previous inequality completes the proof. \square

From this theorem we can deduce the following:

Corollary 3.1. *Consider n a prime number, ℓ a positive integer such that $\gcd(\ell, n) = 1$ and r chosen such that $1 \leq r \leq d$. Then for any $m =$*

$1, 2, \dots, d$, there exists a $\mathbf{z} \in \mathcal{Z}_n^m$ such that

$$e_{N,m}^2(z_1, z_2, \dots, z_m) \leq \frac{1}{n-1} \prod_{j=1}^m \left(\beta_j + \tilde{\gamma}_j S_{\tilde{N}_j} \right),$$

where

$$e_{N,m}^2(z_1, z_2, \dots, z_m) = \frac{1}{n} \sum_{k=0}^{n-1} \prod_{j=1}^m \left(\beta_j + \tilde{\gamma}_j \sum_{-\frac{\tilde{N}_j}{2} < h \leq \frac{\tilde{N}_j}{2}} \frac{e^{2\pi i h k z_j / n}}{|h|} \right) - \prod_{j=1}^m \beta_j.$$

We can set $z_1 = 1$ and for every $2 \leq m \leq d$, z_m can be chosen by minimizing $e_{N,m}^2(z_1, z_2, \dots, z_m)$ over the set \mathcal{Z}_n .

Proof. If $m = 1$, then by expanding the expression of $e_{N,1}^2(z_1)$ and using well-known results for geometrical series, we obtain that $e_{N,1}^2(z_1) = 0$ for any $z_1 \in \mathcal{Z}_n$. The result then follows straight from Theorem 3.1. \square

Component-by-component (CBC) algorithm

The generating vector $\mathbf{z} = (z_1, z_2, \dots, z_d)$ of a lattice rule (2) that satisfies the bound from Corollary 3.1 can be constructed as follows:

1. Set the value for the first component of the vector, say $z_1 := 1$.
2. For $m = 2, 3, \dots, d$, find $z_m \in \mathcal{Z}_n$ such that $e_{N,m}^2(z_1, z_2, \dots, z_m)$ is minimized.

Clearly each $e_{N,m}^2(z_1, z_2, \dots, z_m)$ can be evaluated in $O(n^2 m)$ operations with a constant depending also on ℓ and r . This cost can be reduced to $O(nm)$ by using asymptotic techniques as presented in [6] (see also [4, Appendix A]). Thus the total complexity of the algorithm will be $O(n^2 d^2)$. This can be reduced to $O(n^2 d)$ if we store the products during the construction at an extra expense of $O(n)$ storage. In fact, this order of complexity can be further reduced to $O(nd \log n)$ by making use of the fast CBC algorithm proposed by Nuyens and Cools in [11]. Their approach was based on minimizing a function of the form

$$\frac{1}{n} \sum_{k=0}^{n-1} \prod_{j=1}^d \left(1 + \gamma_j \omega \left(\left\{ \frac{k z_j}{n} \right\} \right) \right) - 1.$$

From (8), we know that $e_{N,d}^2(\mathbf{z})$ is obtained by applying a rank-1 lattice rule to a modified function, so the techniques used in [11] will also work here with some modifications.

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