On EQ-monoids

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Abstract

An EQ-monoid A is a monoid with distinguished subsemilattice L with $1 \in L$ and such that any $a, b \in A$ have a largest right equalizer in L. The class of all such monoids equipped with a binary operation that identifies this largest right equalizer is a variety. Examples include Heyting algebras, Cartesian products of monoids with zero, as well as monoids of relations and partial maps on sets. The variety is 0-regular (though not ideal determined and hence congruences do not permute), and we describe the normal subobjects in terms of a global semilattice structure. We give representation theorems for several natural subvarieties in terms of Boolean algebras, Cartesian products and partial maps. The case in which the EQ-monoid is assumed to be an inverse semigroup with zero is given particular attention. Finally, we define the derived category associated with a monoid having a distinguished subsemilattice containing the identity (a construction generalising the idea of a monoid category), and show that those monoids for which this derived category has equalizers in the semilattice constitute a variety of EQ-monoids.

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1 The definition, examples and basic properties.

Throughout, all semilattices will be viewed as meet-semilattices, equipped with the usual partial order given by $e \leq f$ if and only if $e = ef$.

Definition 1.1 *An* EQ-monoid A *is a monoid with additional binary operation* $\bowtie$ *and a distinguished semilattice* L_A *for which, for all* a, b, $(a \bowtie b) = max\{\alpha \in L_A \mid a\alpha = b\alpha\}$.

As a general convention to be used throughout, for a fixed EQ-monoid A, we shall use lower case Greek letters to denote elements of L_A. Thus if one of $\alpha, \beta, \ldots$ appears in an expression without qualification, it should be assumed to be an element of L_A rather than an arbitrary element of A; letters in conventional font are assumed to be such general elements.

Example 1.2 *Boolean algebras.*

If $(B, \vee, \wedge, 0, 1, ')$ is a Boolean algebra, then $(B, \wedge, 1)$ is a monoid, and is an EQ-monoid if $L_B = B$. In this case $(a \bowtie b)$ is $a \leftrightarrow b = (a \wedge b) \vee (a' \wedge b')$, the "if-and-only-if" connective, as is easily checked.

Example 1.3 *Topological spaces.*

If X is a topological space with $\mathcal{O}$ its collection of open sets, then for $S, T \subseteq X$, $S \leftrightarrow T = max\{U \in \mathcal{O} \mid S \cap U = T \cap U\}$ exists and is the interior of $(S \cup T') \cap (S' \cup T)$ (where S' is the complement of S in X). Now $\mathcal{O}$ is a subsemilattice of the semilattice with top element $(2^X, \cap, X)$, which is therefore an EQ-monoid in which $L_{2^X} = \mathcal{O}$.

Example 1.4 *Relations.*

Let X be a topological space with $A = \mathcal{R}(X)$ the monoid of all relations on X under composition $\circ$, with identity 1 the identity function on X. Letting L consist of restrictions of the identity function to open sets, clearly L is a submonoid which is a semilattice, and $(f \bowtie g) = max\{\alpha \in L \mid f \circ \alpha = g \circ \alpha\}$ exists for all $f, g \in A$ and is the restriction of the identity to the interior of the subset of X on which f, g do not disagree. Thus $(\mathcal{R}(X), \circ)$ is an EQ-monoid in which $L_{\mathcal{R}(X)} = L$.

Important sub-EQ-monoids are $\mathcal{P}(X)$ and $\mathcal{I}(X)$, the monoids of partial and one-to-one partial maps on X; each contains L and so each is closed under $\bowtie$. In particular, it is worth noting that every previously studied operation on $\mathcal{P}(X)$ is derivable from (that is, is a term function in) $\bowtie$ (or possibly its range-defined dual), at least if a nullary operation representing the empty map is added to the signature (see [12] for a number of examples).

Another example we refer to later is $\mathcal{C}(X)$, the monoid of continuous partial maps $X \to X$.

Example 1.5 *Direct products of monoids with right zeros.*

Let M be a monoid with right zero element 0 (satisfying $a0 = 0$ for all $a \in M$). Then M is an EQ-monoid if one lets $L_M = \{0, 1\}$: then $(a \bowtie a) = 1$ for all $a \in M$, and $(a \bowtie b) = 0$ for all unequal $a, b \in M$. Let $\{M_x | x \in X\}$ be a family of such monoids, X a topological space. Let $M = \Pi_{x \in X} M_x$ be the direct product of the M_x. Then M is an EQ-monoid if L_M consists of the elements of M whose entries are 1 on an open subset of X and 0 elsewhere (a submonoid since open sets are closed under intersection); in this case, $(a \bowtie b)$ is the "characteristic function" of the interior of the subset of X on which $a, b \in M$ agree.

Example 1.6 *Closure rings.*

If R is a ring, the *adjoint operation* $\circ$ is defined by $a \circ b = a + b - ab$ for all $a, b \in R$, is associative, has 0 as an identity, and is commutative if and only if the ring multiplication in R is. (In a Boolean ring, this is exactly the join operation in the corresponding Boolean algebra.) If R is a ring in which $(L_R, \circ)$ is a semilattice which is a submonoid of $(R, \circ)$ with $\circ$ viewed as meet in L_R, and if $C(a) = max\{\alpha \in L_R \mid a \circ \alpha = \alpha\}$ exists for all $a \in R$, then R together with C is a *closure ring* (see [4]). Then a straightforward calculation shows that $C(a - b) = max\{\alpha \in L_A \mid a \circ \alpha = b \circ \alpha\}$, and so defining $(a \bowtie b) = C(a - b)$ for all $a, b \in R$ makes $(R, \circ)$ an EQ-monoid. Indeed, for any ring R, $(R, \circ)$ is an EQ-monoid if and only if it is a closure ring with the same L_R, and with $C(a) = (a \bowtie 0)$. These are the same as the E-rings considered in [3] in the ring with identity case.

In [4], the class of closure rings is characterised equationally. Likewise it is possible to characterise general EQ-monoids equationally: hence they form a variety.

Lemma 1.7 *In the EQ-monoid A, $(a \bowtie b)\alpha = (a\alpha \bowtie b)\alpha = (a\alpha \bowtie b\alpha)\alpha$.*

Proof. Let $a, b \in A$ and $\alpha \in L_A$. Then

$$a\alpha(a \bowtie b)\alpha \;=\; a(a \bowtie b)\alpha \;=\; b(a \bowtie b)\alpha \;=\; b\alpha(a \bowtie b)\alpha,$$

so $(a \bowtie b)\alpha \leq (a\alpha \bowtie b\alpha)$, and so $(a \bowtie b)\alpha \leq (a\alpha \bowtie b\alpha)\alpha$. Conversely,

$$a(a\alpha \bowtie b\alpha)\alpha \;=\; a\alpha(a\alpha \bowtie b\alpha) \;=\; b\alpha(a\alpha \bowtie b\alpha) \;=\; b(a\alpha \bowtie b\alpha)\alpha,$$

so $(a\alpha \bowtie b\alpha)\alpha \leq (a \bowtie b)$, and so $(a\alpha \bowtie b\alpha)\alpha \leq (a \bowtie b)\alpha$, proving equality. □

It now follows from Theorem 1 in [3] that EQ-monoids can be characterised as monoids with additional binary $\bowtie$ satisfying the following axioms: for all $a, b, c, d \in A$ and all $\alpha, \beta \in L_A = \{(a \bowtie b) \mid a, b \in A\}$,

1. $(a \bowtie a) = 1$;
2. $\alpha\beta = \beta\alpha$;
3. $(a\alpha \bowtie a)\alpha = \alpha$; and
4. $f(a)(a \bowtie b) = f(b)(a \bowtie b)$ for all derived unary term functions f on A not involving the monoid product.

The final law, called the *replacement rule*, is strictly an axiom scheme rather than a single axiom, but it turns out that finitely many laws can be used to define EQ-monoids (see [3]). One way to do it (though there are more elegant ways) is to list laws that force L_A to be a semilattice such that $(a \bowtie b)$ has the desired maximum property. The following finite set of laws does this. Note that we are adopting the convention that Greek letters stand for elements of the form $(a \bowtie b)$.

- $\alpha^2 = \alpha$
- $\alpha\beta = \beta\alpha$
- $(\alpha\beta \bowtie 1) = \alpha\beta$
- $(a \bowtie a) = 1$
- $a(a \bowtie b) = b(a \bowtie b)$
- $(a \bowtie b)\alpha = (a\alpha \bowtie b\alpha)\alpha$

These together with the usual monoid laws are necessary and sufficient to specify the class of EQ-monoids, which is therefore a finitely based variety of algebras. The necessity of each of these laws follows from previously stated results. For sufficiency, clearly the first three laws imply that $L_A = \{(a \bowtie b) \mid a, b \in A\}$ is a semilattice submonoid of A, while if $a\alpha = b\alpha$ for some $\alpha \in L_A$, then the remaining laws imply that

$$(a \bowtie b)\alpha = (a\alpha \bowtie b\alpha)\alpha = 1\alpha = \alpha,$$

so $(a \bowtie b)$ is the largest $\alpha \in L_A$ for which $a\alpha = b\alpha$.

We shall use these laws freely in equational deductions, along with

- $(a \bowtie b) = (b \bowtie a)$
- $(a \bowtie b)(b \bowtie c) = (a \bowtie c)(b \bowtie c)$

The first of these is obvious, while the second is an easy application of the replacement rule, where $f(x) = (a \bowtie x)$.

The variety of closure rings as in Example 1.6 and the variety of rings with binary $\bowtie$ for which $(R, \circ, \bowtie)$ is an EQ-monoid are term equivalent under the correspondence $(a \bowtie b) \Leftrightarrow C(a - b)$, $C(a) \Leftrightarrow (a \bowtie 0)$, as is shown in [3].

Definition 1.8 *The EQ-monoid A is* strong *if the equation $\alpha a \alpha = a\alpha$ holds in A.*

Examples 1.2, 1.3 and 1.5 are all strong. From Proposition 3 and Corollary 4 of [3], it follows easily that strong EQ-monoids can be characterised amongst monoids with additional binary $\bowtie$ by the rather simpler identities

1. $(a \bowtie a) = 1$;

2. $\alpha\beta = \beta\alpha$; and

3. $f(a)(a \bowtie b) = f(b)(a \bowtie b)$ for f any derived unary term function (in the full language of EQ-monoids) on A.

Let A be an EQ-monoid. For all $a, b \in A$, define $a \wedge b := a(a \bowtie b) = b(a \bowtie b)$.

Proposition 1.9 *$(A, \wedge)$ is a semilattice, extending the semilattice structure of L_A. Let $\leq$ be the usual induced partial order on A, defined by setting $a \leq b$ if $a = a \wedge b$. Then $L_A = \{a \in A \mid a \leq 1\}$. Moreover, $a \leq b$ if and only if $a = b\alpha$ for some $\alpha \in L_A$.*

Proof. For $a, b, c \in A$,

$$a \wedge a = a(a \bowtie a) = a1 = a;$$

$$a \wedge b = a(a \bowtie b) = b(a \bowtie b) = b(b \bowtie a) = b \wedge a;$$

and finally associativity:

$$\begin{aligned}
(a \wedge b) \wedge c &= a(a \bowtie b)(a(a \bowtie b) \bowtie c) \\
&= a(a(a \bowtie b) \bowtie c)(a \bowtie b) \\
&= a(a(a \bowtie b)^2 \bowtie c(a \bowtie b))(a \bowtie b) \\
&= a(a(a \bowtie b) \bowtie c(a \bowtie b))(a \bowtie b) \\
&= a(a \bowtie c)(a \bowtie b),
\end{aligned}$$

so

$$\begin{aligned}
a \wedge (b \wedge c) &= (b \wedge c) \wedge a \\
&= b(b \bowtie a)(b \bowtie c) \\
&= a(b \bowtie a)(b \bowtie c) \\
&= a(b \bowtie c)(b \bowtie a) \\
&= a(a \bowtie c)(b \bowtie a) \\
&= (a \wedge b) \wedge c.
\end{aligned}$$

Hence $(A, \wedge)$ is a semilattice.

Now $\alpha \in L_A$ if and only if $\alpha = 1(\alpha \bowtie 1) = 1 \wedge \alpha$, so $L_A = \{a \in A \mid a \leq 1\}$. Moreover, for all $\alpha, \beta \in L_A$, $\alpha \wedge \beta = \alpha(\alpha \bowtie \beta) = \alpha(\alpha \bowtie \beta\alpha) = \alpha(1 \bowtie \beta) = \alpha\beta$, showing that L_A is a subsemilattice of $(A, \wedge)$, in which $\alpha \wedge \beta = \alpha\beta$ for all $\alpha, \beta \in L_A$.

If $a \leq b$ then $a = a \wedge b = b(a \bowtie b)$; conversely if $a = b\alpha$ for some $\alpha \in L$ then $a \wedge b = (b\alpha) \wedge b = b\alpha(b\alpha \bowtie b) = b\alpha(b\alpha \bowtie b\alpha) = b\alpha = a$ so $a \leq b$. □

In fact the only things in A comparable with 1 under $\leq$ are elements of L_A since 1 is easily seen to be maximal. The order $\leq$ is stable under left multiplication but not necessarily right.

The EQ-monoid idea is also explored in [3], albeit in a more general setting, where the focus was on varieties of EQ-monoids with additional operations satisfying a regularity condition. The results presented here are generally independent of those appearing in [3].

2 Congruences on EQ-monoids.

Any variety $\mathcal{V}$ of EQ-monoids is *0-regular*, in the sense that any congruence on any algebra in $\mathcal{V}$ is determined by the congruence class containing the distinguished element 1. (The term $d(x, y) = (x \bowtie y)$ satisfies the conditions of Corollary 1.7 in Gumm and Ursini [5].) Thus EQ-monoids share some of the properties of groups (and more generally loops), rings, modules and Heyting algebras. In particular, as follows from a result of Hagemann in [6], any variety of EQ-monoids is modular.

Generally, in a 0-regular variety, we call a congruence class containing 0 a *normal*; let ρ_N be the congruence associated with the normal N. In the 0-regular variety of groups, normals can be characterised as normal subgroups, and in rings they are ideals. In this section we give an analogous characterisation of congruence classes containing 1 for EQ-monoids.

2.1 Congruences and normal filters.

For EQ-monoids, because $d(x, y) = (x \bowtie y)$ satisfies $d(x, y) = 1$ if and only if $x = y$, it is easy to see that for a normal N of A, $a\, \rho_N\, b$ if and only if $(a \bowtie b) \in N$.

Definition 2.1 *A* normal filter *N of the EQ-monoid A is a filter of $(A, \wedge)$ which contains the identity and in which $(a \bowtie b) \in N$ implies $(ac \bowtie bc) \in N$ for all $a, b, c \in A$.*

The second condition in this definition is superfluous if A is strong: $(a \bowtie b) \leq (ac \bowtie bc)$ for all $a, b, c \in A$, as is easily seen. Because $\{1\}$ is a sub-EQ-monoid of A, it follows easily that every normal filter of A is a sub-EQ-monoid.

Theorem 2.2 *The normals of the EQ-monoid A are exactly its normal filters.*

Proof. Certainly any normal N must be a filter containing 1 since $1 \wedge 1 = 1$ and $a \geq 1$ implies $1 = 1(a \bowtie 1)$, so $a = 1$. Conversely, given N is such a filter, we show that ρ_N given by $a\, \rho_N\, b$ if $(a \bowtie b) \in N$ is a congruence with associated normal equal to N. It is an equivalence relation because, repeatedly using the reflexive and replacement rules, we see that $(a \bowtie a) = 1$, $(a \bowtie b) = (b \bowtie a)$ and $(a \bowtie b)(b \bowtie c) \geq (a \bowtie c)$. Moreover ρ_N respects $\bowtie$ because $(a \bowtie b)(c \bowtie d) \leq ((a \bowtie c) \bowtie (b \bowtie d))$, and it respects the monoid product because $(bc \bowtie bd) \geq (c \bowtie d)$, and if $(a \bowtie b), (c \bowtie d) \in N$ then $(ac \bowtie bc) \in N$, so

$$(ac \bowtie bd) \geq (ac \bowtie bc)(bc \bowtie bd) \geq (ac \bowtie bc)(c \bowtie d) \in N,$$

so $(ac \bowtie bd) \in N$. Now $a\, \rho_N\, 1$ if and only if $a \wedge 1 = 1(a \bowtie 1) \in N$ if and only if $a \in N$, since $a \geq a \wedge 1$. Hence N is the normal corresponding to ρ_N. □

Denote by A/N the factor EQ-monoid associated with the normal filter N, and let $M \vee N$ be the join of the normal filters M and N in the lattice of normal filters of A. It is clear that if N is a normal filter of A and B is a sub-EQ-monoid of A for which $N \subseteq B \subseteq A$, then N is a normal filter of B also.

As a special case of a more general notion, we say two congruences ρ, θ on A *permute at unity* if $(a, 1) \in \rho \circ \theta$ implies $(a, 1) \in \theta \circ \rho$ and vice versa; equivalently, $(a, 1) \in \rho \vee \theta$ implies the existence of $b \in A$ for which $(b, 1) \in \rho$ and $(a, b) \in \theta$, and similarly with ρ, θ swapped. (The idea is more usually expressed in terms of a zero element for general algebras.)

Proposition 2.3 *Suppose M and N are normal filters of the EQ-monoid A for which ρ_M and ρ_N permute at unity. Then $(M \vee N)/M \cong M/(M \cap N)$.*

Proof. First note that both sides of the isomorphism are well-defined. Define $f : M \to (M \vee N)/N$ by setting $f(a) = a/N$, where a/N is defined to be the congruence class containing a in $(M \vee N)/N$. This is easily seen to be a homomorphism with kernel $M \cap N$. It remains to prove surjectivity. We must show that for any $b \in M \vee N$, there is $a \in M$ such that $(a, b) \in \rho_N$; that is, for any $(b, 1) \in \rho_{M \vee N} = \rho_M \vee \rho_N$, there is $(a, 1) \in \rho_M$ such that $(a, b) \in \rho_N$. This is immediate from the fact that ρ_M, ρ_N permute at unity. □

It is indirectly shown in [5] that all algebras in a variety with distinguished nullary 0 have congruences permuting at zero if and only if there is a binary term $s(x, y)$ in the variety for which $s(x, x) = 0$ and $s(x, 0) = x$. Such varieties are called *subtractive* in [14]. In the 0-regular case, they are exactly ideal determined varieties in the sense of [5]. Next we show that the isomorphism theorem $(M \vee N)/M \cong M/(M \cap N)$ fails to hold globally for EQ-monoids, or indeed even for commutative (and hence strong) EQ-monoids. Thus the variety of EQ-monoids does not have congruences permuting at unity (and therefore certainly is not Malcev).

Let $A = Fr[x, y]$ be the free commutative EQ-monoid on two generators x, y. Let $I = \langle (x \bowtie 1) \rangle$ and $J = \langle (x \bowtie y) \rangle$, the principal (necessarily normal) filters generated by $(x \bowtie 1)$ and $(x \bowtie y)$. Then $\alpha = (x \bowtie 1)(x \bowtie y) = (x \bowtie 1)(y \bowtie 1) \in I \vee J$. It follows from the replacement rule and the fact that all operations preserve 1 that $f\alpha = \alpha$ for all $f \in A$. Hence $\langle \alpha \rangle = A = I \vee J$. Hence $(I \vee J)/I$ is A with x made equal to 1; and hence is isomorphic to the free EQ-monoid on one generator, say y. Now a typical element of A has the form $t\gamma$, where $\gamma \in L_A$ and $t = 1$ or is a product of one or more of x, y. If $t\gamma \in J$, then $t\gamma \geq (x \bowtie y)$, and so $(x \bowtie y) = (x \bowtie y) \wedge (t\gamma) = (x \bowtie y)((x \bowtie y) \bowtie t\gamma) = (x \bowtie y)(t\gamma \bowtie 1)$. Thus $(x \bowtie y) = (x \bowtie y)(t\gamma \bowtie 1)$ will be an identity for EQ-monoids and holds if $x = y$, giving $t'\gamma' = 1$ (where t', γ' are the result of replacing y by x in t, γ respectively). In turn, $t'\gamma' = 1$ must be an identity for EQ-monoids in the variable x. This says that $1 \leq t'$ in all EQ-monoids, so that

$$1 = 1 \wedge t' = 1(1 \bowtie t') = (1 \bowtie t'),$$

and so $t' = 1$. Hence also $t = 1$. This shows that J contains only elements of L_A, and so it satisfies the identity $a = (a \bowtie 1)$; hence so will $J/(I \cap J)$. However $(I \vee J)/I$ does not since it is free and so $y \neq (y \bowtie 1)$, with $L_J = \{1, (y \bowtie 1)\}$, showing that $(I \vee J)/I \ncong J/(I \cap J)$.

2.2 Congruences and A-normal filters of L_A.

In an EQ-monoid A, congruences on A can also be described in terms of filters of L_A.

Definition 2.4 *A filter F of L_A is A-*normal *if $(a \bowtie b) \in F$ implies $(ac \bowtie bc) \in F$, for all $a, b, c \in A$.*

This is in general stronger than saying that F is a normal filter in the EQ-monoid L_A. If A is strong then every filter of L_A is A-normal.

For the A-normal filter F, let $\rho_F = \{(a, b) \mid (a \bowtie b) \in F\}$, and for the congruence ρ on A, let $F_\rho = \{\alpha \in L_A \mid \alpha\, \rho\, 1\}$.

Theorem 2.5 *Let A be an EQ-monoid. The A-normal filters of L_A are in one-to-one correspondence with the congruences on A, via the correspondences $F \leftrightarrow \rho_F$, $\rho \leftrightarrow F_\rho$, where F is any A-normal filter of L_A and ρ is any congruence on A.*

Proof. For F an A-normal filter of L_A, the fact that ρ_F is a congruence on A has a proof very similar to the proof that ρ_N is a congruence for N a normal filter, as above. Conversely, F_ρ is easily seen to be an A-normal filter of L_A, if ρ is a congruence on A. If $\rho_F = \rho_G$ for A-normal filters F, G of L_A, and if $\alpha \in F$, then $(\alpha \bowtie 1) \in F$, so $\alpha\, \rho_F\, 1$ and so $\alpha\, \rho_G\, 1$, and so $\alpha = (\alpha \bowtie 1) \in G$, so $F = G$. Likewise, if $F_\rho = F_\theta$ for congruences ρ, θ on A, and if $a\, \rho\, b$ then $(a \bowtie b)\, \rho\, (a \bowtie a) = 1$, so $(a \bowtie b)\, \theta\, 1$, so $a\, \theta\, b$, and so $\rho = \theta$, completing the proof. □

It follows easily that the A-normal filters in L_A are exactly the intersections of the normal filters of A with L_A, and that the normal filter associated with ρ_F (F an A-normal filter) is exactly $\{a \in A \mid (a \bowtie 1) \in F\}$.

We might as well allow the notation A/F where F is an A-normal filter of A, rather than A/ρ_F.

2.3 Extending to E-structures.

If there are finitely many additional operations $p_1, p_2, \ldots$ on an EQ-monoid, these become part of an *E-structure* on the EQ-monoid A (in the sense of [3]) if each such p_j of arity n is *regular* over $\bowtie$, that is, if

$$(a_1 \bowtie b_1) \cdots (a_n \bowtie b_n) \leq (p_j(a_1, \ldots, a_n) \bowtie p_j(b_1, \ldots, b_n))$$

for all $a_i, b_i \in A$; this is simply saying that $(A, p_1, p_2, \ldots)$ is an E-algebra under $\bowtie$ taking values in L_A; see [2].

A simpler equivalent form for the regularity condition, given in [3] and convenient for verification in examples, is

$$p(a_1, \ldots, a_n)\alpha = p(a_1\alpha, \ldots, a_n\alpha)\alpha$$

for all $a_i \in A$, $\alpha \in L_A$. The replacement rule works for all f defined in terms of any regular operations on an EQ-monoid; indeed this provides another means to define regularity.

We mention that most of the earlier examples have natural additional operations which are regular. Viewing the Boolean algebra $(B, \vee, \wedge, 0, 1, ')$ as in Example 1.2 as an EQ-monoid, distributivity guarantees that $\vee$ is regular. Similarly, in Example 1.3, distributivity ensures that $\cup$ is regular in 2^X. Both relational union and intersection are regular in the EQ-monoid $(\mathcal{R}(X), \circ, \bowtie)$ of Example 1.4; moreover if there is algebraic structure on X, the natural pointwise operations on $\mathcal{R}(X)$ are also regular. Again, if each monoid with right zero M_x as in Example 1.5 has an additional operation p, then in the Cartesian product $M = \Pi\{M_x \mid x \in X\}$, p is regular. (See [3] for all the details.) If R is a closure ring, then for $a, b \in R$ and $\alpha \in L_R$, it is easily checked that $(a + b) \circ \alpha = (a \circ \alpha + b \circ \alpha) \circ \alpha$, so $+$ is regular. Note that in any EQ-monoid A, $\bowtie$ itself satisfies the regularity condition, as does the monoid product if and only if A is strong.

E-structures were the main subject matter of [3]. Our interest in E-structures here is that the results in this section immediately generalise to them.

Proposition 2.6 *Any congruence on the EQ-monoid A extends to any other regular operation on A.*

Proof. Let N be a normal filter of A. Now $a \, \rho_N \, b$ if and only if $(a \bowtie b) \in N$. Hence if p is a regular operation of arity n on A, with $a_j, b_j \in A$ and $(a_j \bowtie b_j) \in N$ for all $j = 1, 2, \ldots, n$, then

$$(p(a_1, \ldots, a_n) \bowtie p(b_1, \ldots, b_n)) \ \geq \ (a_1 \bowtie b_1) \cdots (a_n \bowtie b_n) \in N.$$

Thus ρ_N respects p also. □

Most of the results of this section now immediately generalise to E-structures, including all those concerning normal filters. But also, if an E-structure A is *1-idempotent*, in the sense that for each regular operation p on A, $p(1, 1, \ldots, 1) = 1$ (that is, $\{1\}$ is a sub-E-structure), then normal filters will be sub-E-structures. (The same condition is required in the definition of a multi-operator group to guarantee that ideals are subalgebras: see [10].) Most of our earlier examples are 1-idempotent. Proposition 2.3 and the remarks around it now carry over easily to 1-idempotent E-structures.

A simple corollary to Proposition 2.6 is that for closure rings, any congruence respecting $(R, \circ, C)$ where R is a closure ring is automatically a ring congruence since it will respect the regular operation of addition and hence also multiplication (since $ab = a + b - a \circ b$).

3 Some varieties of EQ-monoids.

In any EQ-monoid A, the sub-EQ-monoid L_A is a *Brouwerian semilattice*, in the sense that for all $\alpha, \beta \in L_A$, there exists a largest $\gamma \in L_A$, namely $\alpha \to \beta = (\alpha\beta \bowtie \alpha)$, for which $\alpha\gamma \leq \beta$; note also that $(\alpha \bowtie \beta) = (\alpha \to \beta)(\beta \to \alpha)$. The converse is true also: if A is a Brouwerian semilattice in which $a \to b$ is the relative pseudocomplement of b in a, then defining $(a \bowtie b) = (a \to b)(b \to a)$ makes A an EQ-monoid in which $L_A = A$ since $(a \bowtie 1) = a$ for all $a \in A$, and moreover $(a \to b) = (ab \bowtie a)$. Hence EQ-monoids satisfying the identity $a = (a \bowtie 1)$ are nothing but

Brouwerian semilattices: the varieties are term equivalent via $(x \bowtie y) \Leftrightarrow (x \to y)(y \to x)$ and $x \to y \Leftrightarrow (x \bowtie xy)$. This extends easily to Heyting algebras, and a familiar correspondence between filters and congruences for Heyting algebras follows as a special case of Theorem 2.2; see [9, Lemma VI.2.10] for example.

For EQ-monoids which are semilattices under their multiplication, it turns out that the example of the open subsets of a topological space is generic: every EQ-semilattice is embeddable in the EQ-semilattice of subspaces of some topological space X, a result shown in [3].

In this section we describe all EQ-monoids in three fairly natural varieties of (possibly enriched) EQ-monoids: those in which the equality operation is associative, those for which L_A is a Boolean algebra, and those for which the monoid is an inverse semigroup. We then consider some important identities satisfied by the EQ-monoids $\mathcal{R}(X)$ and $\mathcal{P}(X)$ (X a set equipped with the discrete topology) in preparation for the final section.

3.1 Associative EQ-monoids.

Definition 3.1 *An EQ-monoid is* associative *if* $\bowtie$ *is an associative operation.*

If A is a Boolean algebra, defining $(a \bowtie b) = (a \leftarrow b)(a \to b)$ for all $a, b \in A$ makes A a strong EQ-monoid, as in Example 1.2; moreover, it is associative as is easily checked. Now every Boolean algebra $(R, 0, 1, \vee, \wedge, ')$ may be viewed as a Boolean ring with identity by defining $a + b = a \leftrightarrow b$ and $ab = a \vee b$. Then the additive identity of R is $0' = 1$ and the multiplicative identity is $1' = 0$. (This is the dual of the more common way of doing things.) Then we can view $\wedge$ as a derived operation given by $a \wedge b = a + b + ab$. In this way, $(R, \wedge, +)$ is an associative EQ-monoid.

We can generalise this. Given any Boolean ring $(R, +, \times)$, with or without identity, defining $a \wedge b = a + b + a \times b$ for all $a, b \in R$, $(R, \wedge, \times)$ is a Boolean algebra, and it is easy to see that $(R, \wedge, +)$ is an associative EQ-monoid in which $L_R = R$: for all $a, b \in R$, $a \wedge (a + b) = b \wedge (a + b)$, and if $a \wedge c = b \wedge c$ then $(a + b) \wedge c = c$, as is easily checked.

Theorem 3.2 *The variety of associative EQ-monoids is term equivalent to the variety of Boolean rings, under the correspondence* $(a \bowtie b) \Leftrightarrow a + b$, $ab \Leftrightarrow a + b + a \times b$.

Proof. Suppose $(A, \cdot, \bowtie)$ is an EQ-monoid in which $\bowtie$ is associative. Then for all $a \in A$,

$$(a \bowtie (a \bowtie 1)) = ((a \bowtie a) \bowtie 1) = (1 \bowtie 1) = 1,$$

so $a = (a \bowtie 1)$ and so $A = L_A$. Letting $a + b = (a \bowtie b)$, it is now clear that $(A, +)$ is an abelian group (of characteristic 2) with identity 1. Moreover for all $a, b, c \in A$, $a^2 = a, ab = ba$ and

$a(b \bowtie c) = a(ab \bowtie ac)$, so

$$\begin{aligned}
a(b+c) &= a(b \bowtie c) \\
&= a(b \bowtie c \bowtie 1) \\
&= a(ab \bowtie ac \bowtie a) \\
&= (ab \bowtie ac)(ab \bowtie ac \bowtie a) \\
&= ((ab \bowtie ac \bowtie a)a(ab) \bowtie (ab \bowtie ac \bowtie a)a(ac))(ab \bowtie ac \bowtie a) \\
&= ((ab \bowtie ac \bowtie a)(ab \bowtie ac)ab \bowtie (ab \bowtie ac \bowtie a)(ab \bowtie ac)ac)(ab \bowtie ac \bowtie a) \\
&= ((ab \bowtie ac)ab \bowtie (ab \bowtie ac)ac)(ab \bowtie ac \bowtie a) \\
&= ((ab \bowtie ac)ac \bowtie (ab \bowtie ac)ac)(ab \bowtie ac \bowtie a) \\
&= (ab \bowtie ac \bowtie a) \\
&= ab + ac + a.
\end{aligned}$$

Letting $a \times b = a + b + ab$, it is now a simple mechanical exercise to verify that $a \times a = a$, $a \times (b \times c) = (a \times b) \times c$ and $a \times (b + c) = a \times b + a \times c$. Hence $(A, +, \times)$ is a Boolean ring. We have already dealt with the converse. □

Corollary 3.3 *Let $(H, \vee, \wedge, \to)$ be a Heyting algebra. Then H is a Boolean algebra if and only if $\leftrightarrow$, given by $a \leftrightarrow b = (a \to b) \wedge (b \to a)$, is associative.*

3.2 Classical EQ-monoids.

The main examples of strong EQ-monoids are of the direct product kind as featured in Example 1.5, in which a direct product of EQ-monoids with right zeros is given a natural EQ-monoid structure. If the index set of the product is endowed with the discrete topology, then it is possible to characterize subalgebras of such examples, providing 0 is admitted as a nullary operation (representing the element of the direct product consisting entirely of zeros).

Definition 3.4 *An EQ-monoid A is* classical *if it is strong and L_A is a Boolean algebra.*

Thus the EQ-monoids described in Example 1.5 are classical, as are any subalgebras containing 0. This is because the condition that L_A be a Boolean algebra can be captured equationally by introducing only one new nullary 0 which is the bottom of L_A, and then the complement of $\alpha \in L_A$ is $(\alpha \bowtie 0)$; that L_A is a Boolean algebra amounts to the identity $((\alpha \bowtie 0) \bowtie 0) = \alpha$ for all $\alpha \in L_A$.

First we characterise simple strong EQ-monoids. In accord with standard algebraic practice, we shall say that the *EQ*-monoid A is *simple* if it has no non-trivial congruences, and that a congruence ρ on A is *maximal* if it is a proper subset of $A \times A$ contained in no other proper congruences on A. Of course a trivial EQ-monoid is simple. The usual connection between simplicity and maximality applies: factoring out a congruence from a non-trivial EQ-monoid gives a simple EQ-monoid if and only if the congruence is maximal. (See [1, Theorem 8.9] for example.)

The description of simple rings is much easier in the commutative case than it is in general: they are fields. Similarly, there is an easy description of simple EQ-monoids in the strong case. The following generalises a result appearing in [4] applying to the closure ring case, which in turn generalised a result appearing in [11] applying to Boolean algebras (equivalently, Boolean rings with identity) equipped with a topological closure operation.

Proposition 3.5 *Let A be a non-trivial strong EQ-monoid. Then A is simple if and only if $L_A = \{0, 1\}$ for some $0 \in A$. Hence non-trivial simple strong EQ-monoids are classical.*

Proof. By Theorem 2.5, A is simple if and only if the only filters of L_A are $\{0\}$ for some smallest element $0 \in L_A$, and L_A, that is, if and only if $L_A = \{0, 1\}$. □

Again generalising a result in [4], we have

Theorem 3.6 *Every classical EQ-monoid is a subdirect product of simple EQ-monoids.*

Proof. Let A be a classical EQ-monoid. If it is trivial, the result follows immediately. So assume it is not.

Every filter of L_A is A-normal since A is strong, so congruences on A correspond to filters of L_A by Theorem 2.5. Hence if F is an ultrafilter (that is, a maximal filter) of L_A, then A/F is simple.

Let $\delta = \bigcap\{\rho_F \mid F \text{ an ultrafilter of } L_A\}$. Thus $(a, b) \in \delta$ if and only if $(a \bowtie b) \in F$ for all ultrafilters F of L_A. Suppose $(a, b) \in \delta$, with $a \neq b$. If $(a \bowtie b) = 0$ then 0 is in every ultrafilter of L_A so $L_A = \{0\}$, whence $0 = 1$ and so $A = \{1\}$, a contradiction, so $(a \bowtie b) \neq 0$. Then, because $(a \bowtie b) \neq 1$ (since $a \neq b$), the complement α of $(a \bowtie b)$ in L_A is non-zero, and $\{\beta \in L_A \mid \beta \geq \alpha\}$ is a filter which extends to an ultrafilter G containing α and therefore not containing $(a \bowtie b)$, so $(a, b) \notin \rho_G$ and so $(a, b) \notin \delta$, a contradiction. Hence $(a, b) \in \delta$ implies $a = b$. Thus A is a subdirect product of the A/F, F an ultrafilter of A. □

A special case occurs when $A = L_A$ is a Boolean algebra: then A is a subdirect product of copies of the two-element Boolean algebra.

Note that the converse to Theorem 3.6 fails, essentially because the classical EQ-monoids do not form a variety; in particular, a subdirect product of simple EQ-monoids may not have L_A being a Boolean algebra (it may not have a smallest element) and hence may not be classical. To rectify this we need to equationally characterise classical EQ-monoids.

An *EQ-monoid with right zero* is an algebra which is an EQ-monoid having nullary operation 0 for which $a0 = 0$ for all $a \in A$ and $0\alpha = \alpha 0$ for all $\alpha \in L_A$.

Proposition 3.7 *Let A be an EQ-monoid with $0 \in A$. The following are equivalent.*

1. $0(a \bowtie b) = 0$ *for all* $a, b \in A$*;*

2. 0 *is the smallest element of* L_A*;*

3. A is an EQ-monoid with right zero.

Proof. Assuming 1 above, $(0 \bowtie 1) = 1(0 \bowtie 1) = 0(0 \bowtie 1) = 0$, so $0 \in L_A$ and clearly $0 \leq \alpha$ for all $\alpha \in L_A$, so 2 holds. Assuming 2, $0 \leq (a \bowtie 0)$ so $a0 = 0 \cdot 0 = 0$ for all $a \in A$, so 3 holds. Assuming 3, $0(a \bowtie b) = (a \bowtie b)0 = 0$ for all $a, b \in A$, so 1 holds. $\square$

In each of the examples given earlier, L_A has a smallest member, so all can be viewed as EQ-monoids with right zero.

The condition that L_A is a Boolean algebra is equivalent to requiring that $\bowtie$ be associative on L_A and that L_A have a smallest element $0 \neq 1$, as follows from Theorem 3.2 (since Boolean algebras correspond to Boolean rings with identity). Thus from Proposition 3.7, the class of classical EQ-monoids is just the obvious reduct of the variety of EQ-monoids with right zero in which $\bowtie$ is associative on L_A; we call this the variety of *classical EQ-monoids with right zero*, $\mathcal{V}$.

Clearly all simple strong EQ-monoids are in $\mathcal{V}$, and these are exactly the simple algebras in $\mathcal{V}$ by Proposition 3.5. Thus following from Theorem 3.6, we have

Corollary 3.8 *The algebras in $\mathcal{V}$ are exactly the subdirect products of simple algebras in $\mathcal{V}$.*

This says that the algebras in $\mathcal{V}$ are embeddable in examples as in Example 1.5, since the operation $\bowtie$ on the direct product of simple algebras as in the previous corollary coincides with the operation as defined in Example 1.5 on the direct product of monoids with right zeros.

3.3 Varieties associated with $\mathcal{R}(X)$ and $\mathcal{P}(X)$.

Note than in any EQ-monoid A, $(a \bowtie b)c = c$ implies $ac = bc$ for all $a, b, c \in A$. However, it is not hard to see that $\mathcal{R}(X)$ (with X endowed with the discrete topology) satisfies the reverse implication as well: $ac = bc$ implies $c = (a \bowtie b)c$.

Definition 3.9 *An EQ-monoid is* full *if it satisfies the implication*

$$ac = bc \Rightarrow c = (a \bowtie b)c.$$

Proposition 3.10 *If A is an EQ-monoid, then A is full if and only if for all $a, b, c \in A$, $(a \bowtie b)c(ac \bowtie bc) = c(ac \bowtie bc)$.*

Proof. If A is full, then because $ac(ac \bowtie bc) = bc(ac \bowtie bc)$, it follows that $(a \bowtie b)c(ac \bowtie bc) = c(ac \bowtie bc)$. Conversely, if the identity is satisfied and $ac = bc$, then $(ac \bowtie bc) = 1$ and so $(a \bowtie b)c = (a \bowtie b)c(ac \bowtie bc) = c(ac \bowtie bc) = c(ac \bowtie ac) = c$. $\square$

Because $a \leq b$ if and only if $a = a \wedge b$, the above proposition shows that the class of full EQ-monoids is a variety.

In an EQ-monoid A, for $\alpha \in L_A$, $a\alpha = b\alpha$ if and only if $\alpha \leq (a \bowtie b)$. For full EQ-monoids we can say more.

Theorem 3.11 *The following are equivalent for the monoid A with distinguished semilattice submonoid L.*

1. *For any $a, b \in A$, there exists $\alpha \in L$ for which $a\alpha = b\alpha$, and such that if $ac = bc$ for some c, then $\alpha c = c$.*
2. *A is a full EQ-monoid with $L_A = L$.*

Proof. Suppose the first condition holds. Then certainly A is an EQ-monoid with $L_A = L$ and $\alpha = (a \bowtie b)$ in the condition, as can be seen by setting $c = \beta \in L$. Moreover, if $ac = bc$ then $(a \bowtie b)c = c$, so A is full. The converse is immediate. □

One property possessed by $\mathcal{P}(X)$ with the discrete topology, but also by the continuous partial maps $X \to X$, $\mathcal{C}(X)$, on the topological space X, is that for all $a \in A$ and $\alpha \in L_A$, there exists $\beta \in L_A$ for which $\alpha a = a\beta$.

Definition 3.12 *For any EQ-monoid A, we call any $a \in A$ for which, for all $\alpha \in L_A$ there exists $\beta \in L_A$ for which $\alpha a = a\beta$* translucent*; thus a is translucent if and only if $\alpha \leq a$ for all $\alpha \in L_A$. If all elements of A are translucent, we say A is* translucent.

Let $\mathcal{T}(A)$ be the set of all translucent elements of A, a sub-EQ-monoid of A as is easily checked. So A is translucent if and only if $\mathcal{T}(A) = A$. Use of the term "translucent" here is consistent with its usage in a slightly different setting in [8].

Of course $c \in \mathcal{T}(A)$ if and only if $(a \bowtie b)c \leq c$ for all $a, b \in A$. It is possible to give an apparently stronger condition which is still equivalent to translucence.

Theorem 3.13 *For any EQ-monoid A, $c \in \mathcal{T}(A)$ if and only if $(a \bowtie b)c \leq c(ac \bowtie bc)$ for all $a, b \in A$.*

Proof. Suppose $c \in \mathcal{T}(A)$. Then for $a, b \in A$, there exists $\alpha \in L_A$ for which $(a \bowtie b)c = c\alpha$. Then $ac\alpha = a(a \bowtie b)c = b(a \bowtie b)c = bc\alpha$, so $\alpha \leq (ac \bowtie bc)$. Hence $(a \bowtie b)c = c\alpha \leq c(ac \bowtie bc)$.

For the converse, if $c \in A$ is such that $(a \bowtie b)c \leq c(ac \bowtie bc)$ for any $a, b \in A$, then for any $\beta \in L_A$, $\beta c = (\beta \bowtie 1)c \leq c(\beta c \bowtie c)$, so $\beta c \leq c$, and so $c \in \mathcal{T}(A)$. □

Again, the above implies that the class of translucent EQ-monoids is a variety, globally satisfying $(a \bowtie b)c \leq c(ac \bowtie bc)$. This is obviously the opposite inequality to that which defines the variety of full EQ-monoids, and so an EQ-monoid satisfying both satisfies the identity $(a \bowtie b)c = c(ac \bowtie bc)$.

Definition 3.14 *An EQ-monoid satisfying the identity $(a \bowtie b)c = c(ac \bowtie bc)$ is* E-deterministic.

We can obtain a representation theorem for E-deterministic EQ-monoids with zero in terms of the partial maps on certain subsets of the EQ-monoids, although it is not necessarily faithful in general. We restrict the action of elements in the representation to *sinks* – non-zero elements x

for which $\alpha x \in \{x, 0\}$ for all $\alpha \in L_A$. (In $\mathcal{P}(X)$ with X discrete, sinks are relations with a single range element.)

Let A be an EQ-monoid with zero 0. Let $S(A)$ be the set of all sinks in A. We say $S(A)$ is *dense in* A if, for all $\alpha \in L_A$, if $\alpha x = x$ for all $x \in S(A)$, then $\alpha = 1$. Clearly the set of sinks in $\mathcal{P}(X)$, an EQ-monoid with zero the empty set, is dense if X is discrete.

Proposition 3.15 *If A is E-deterministic with zero 0, then A is homomorphically representable in $\mathcal{P}(S(A))$ with $S(A)$ discrete, faithfully if $S(A)$ is dense.*

Proof. Let $a, b \in A$, $x \in S(A)$.

For all $a \in A$, define $\psi_a(x) = ax$ if $ax \neq 0$, undefined otherwise. Clearly $(\psi_a \circ \psi_b)(x) = \psi_{ab}(x)$ if both are defined. Yet $\psi_a \circ \psi_b$ is defined for all $x \in S(A)$ for which $bx \neq 0$ and $a(bx) \neq 0$, that is, $(ab)x \neq 0$, which is exactly when ψ_{ab} is defined, so the two partial maps are equal.

We must show that $\bowtie$ is respected. Now $\psi_{(a \bowtie b)}$ is defined at x if $(a \bowtie b)x \neq 0$, that is, $(a \bowtie b)x = x$, that is, $ax = bx$ by the definition of E-deterministicity, then $\psi_{(a \bowtie b)}(x) = (a \bowtie b)x = x$. On the other hand, $(\psi_a \bowtie \psi_b)$ is defined providing $ax = bx \neq 0$ or $ax = bx = 0$, that is, $ax = bx$, in which case it maps x to itself. Thus the two maps are equal. Hence the map $a \mapsto \psi_a$ is a homomorphism mapping A into $\mathcal{P}(S(A))$.

Now suppose $S(A)$ is dense. Suppose $\psi_a = \psi_b$ for some $a, b \in A$. Then $ax = bx$ for all $x \in S(A)$ for which neither ax nor bx is 0; but also for $x \in S(A)$, $ax = 0$ if and only if $bx = 0$. Hence $ax = bx$ for all $x \in S(A)$, so $(a \bowtie b)x = x$ for all $x \in S(A)$ by the E-deterministic property, and so $(a \bowtie b) = 1$. Hence $a = b$, and the mapping is an isomorphic embedding. □

4 Inverse EQ-monoids.

A semigroup A is *inverse* if for all $a \in A$ there exists $a' \in A$ for which $aa'a = a$ and $a'aa' = a'$. It can be shown that $(a')' = a$ for all a in the inverse semigroup A. Thus when viewed as semigroups with an additional unary operation $'$, the class of inverse semigroups is a variety containing the variety of groups. Semigroup congruences on an inverse semigroup automatically respect $'$, generalising a familiar fact about group congruences; see [7] for example.

The set $E(A)$ of idempotents in the inverse semigroup A is the set

$$\{a'a \mid a \in A\} = \{aa' \mid a \in A\},$$

and the elements of $E(A)$ commute with one-another; hence $E(A)$ is a semilattice under the semigroup multiplication.

There is a variant of the regular representation of a group for inverse semigroups, called the Vagner-Preston representation. Using this, A is faithfully represented within the inverse semigroup $\mathcal{I}_A$ of all one-to-one partial maps on A under composition, with f' the inverse partial mapping of $f \in \mathcal{I}_A$. The embedding $\psi : A \to \mathcal{I}_A$ is such that ψ_a (the image of $a \in A$ under ψ) has domain

$a'A = \{a'b \mid b \in A\}$ and $\psi_a(x) = ax$ for all $x \in a'A$; moreover idempotents in A map to restrictions of the identity in $\mathcal{I}_A$ and ψ_a has as its domain the set on which the restriction of the identity $\psi_{a'a}$ is defined.

Definition 4.1 *An* inverse EQ-monoid *is an EQ-monoid which is an inverse semigroup.*

The class of all inverse EQ-monoids is a variety if we view $'$ as a unary operation. Because all semigroup congruences on an inverse semigroup respect $'$, normal filters of inverse EQ-monoids are in one-to-one correspondence with congruences respecting all operations, including inversion. The canonical example of an inverse EQ-monoid is the sub-EQ-monoid $\mathcal{I}(X)$ of $\mathcal{R}(X)$, the EQ-monoid of all one-to-one partial maps on the topological space X, as in Example 1.4; for all $f, g \in \mathcal{I}(X)$, $(f \bowtie g)$ is the restriction of the identity map on X to the interior of $\{x \in X \mid f(x) = g(x)$ or neither is defined at $x\}$.

Theorem 4.2 *Any inverse EQ-monoid A is embeddable in one of the form $\mathcal{I}(X)$ (X a topological space) as above.*

Proof. Represent A as an inverse semigroup of partials on A via ψ as in the Vagner-Preston representation above. Then $1 \in A$ maps to the identity map on A. Moreover, because ψ is an isomorphism, $(a \bowtie b)$ maps to the largest restriction of the identity of the form ψ_α, $\alpha \in L_A$, on which ψ_a and ψ_b agree; calling this $(\psi_a \bowtie \psi_b)$ makes $(Im(\psi), \circ, \bowtie)$ into an EQ-monoid isomorphic to A. The partial maps ψ_α $(\alpha \in L_A)$ are closed under composition, which means their domains are closed under finite intersection and so form a base for the collection $\mathcal{O}$ of open subsets of a topology on A, and $\psi_\alpha \leq \psi_\beta$ means the domain of ψ_α is contained in the domain of ψ_β.

Let the domain of ψ_α be X_α. Thus $\psi_\alpha = \mathrm{id}_{X_\alpha}$, the restriction of the identity to X_α. Then

$$\begin{aligned}
& \mathrm{Int}(\{x \in A \mid \psi_a(x) = \psi_b(x) \text{ or neither is defined at } x\}) \\
= \; & \cup\{X_\alpha \mid \psi_a(x) = \psi_b(x) \text{ or neither is defined at } x, \text{ for all } x \in X_\alpha, \alpha \in L_A\} \\
= \; & \cup\{X_\alpha \mid \psi_a \circ \psi_\alpha = \psi_b \circ \psi_\alpha, \alpha \in L_A\} \\
= \; & \cup\{X_\alpha \mid \psi_{a\alpha} = \psi_{b\alpha}, \alpha \in L_A\} \\
= \; & \cup\{X_\alpha \mid a\alpha = b\alpha, \alpha \in L_A\} \\
= \; & \cup\{X_\alpha \mid \alpha \leq (a \bowtie b), \alpha \in L_A\} \\
= \; & X_{(a \bowtie b)},
\end{aligned}$$

so $\psi_{(a\bowtie b)} = \mathrm{id}_{X_{(a\bowtie b)}}$ has the desired form. □

It is possible to characterise those EQ-monoids which can be represented within $\mathcal{I}(X)$ where X is given the discrete topology (so that $(f \bowtie g)$ is the identity restricted to the entire region on which f, g agree or are both undefined), via a result due to Boris Schein given in [13]. In that paper, Schein considers so-called "subtraction semigroups". These are semigroups equipped with an additional binary difference operation $-$, satisfying the following laws:

1. $x(y-z) = xy - xz$

2. $(x-y)z = xz - yz$

3. $x - (y-x) = x$

4. $x - (x-y) = y - (y-x)$

5. $(x-y) - z = (x-z) - y$

6. $x - x = 0$.

Note that $\mathcal{I}(X)$ is a subtraction semigroup when endowed with the operations of composition and set difference, as is routine to verify. Schein shows in Theorem 2 (as a corollary of a more general result) that all subtraction semigroups are embeddable in examples of the form $\mathcal{I}(X)$.

Let us call a subtraction semigroup S which is an inverse monoid an *inverse subtraction monoid*. Any faithful representation of an inverse subtraction monoid S within $\mathcal{I}(X)$ for some X, via the injection ψ, will be such that $\psi(a')$ is the inverse of $\psi(a)$, since $\psi(a)\psi(a')\psi(a) = \psi(aa'a) = \psi(a)$, and similarly $\psi(a')\psi(a)\psi(a') = \psi(a'aa') = \psi(a')$; only the inverse of $\psi(a)$ satisfies these conditions. Likewise if 1 is the identity element in S, then $\psi(1)$ is automatically represented as the identity map on X, since $\psi(a)\psi(1) = \psi(a1) = \psi(a)$ for all $a \in S$, so because $\psi(a)$ is an injective partial map, $\psi(1)(x) = x$ for all x in the domain of $\psi(a)$, for all a; we can without loss of generality assume that every $x \in X$ is in the domain of some $\psi(a)$ (hence in the range of some $\psi(a)$ since we have closure under inverses). Hence inverse subtraction monoids can be represented faithfully in $\mathcal{I}(X)$ for some X. On the other hand, $\mathcal{I}(X)$ is of course an inverse subtraction monoid, hence so are all its subalgebras.

It is possible to translate this result into the language of inverse EQ-monoids with a zero element. Thus in $\mathcal{I}(X)$, where X is a set endowed with the discrete topology, $(f \bowtie g)$ is the restriction of the identity to the entire subset of X on which f, g either agree or are both undefined; let 0 be the empty partial map and 1 the identity map. Note that the set-theoretic difference of f and g as relations can be defined in terms of $\bowtie$ and 0:

$$f \backslash g = f((f \bowtie g) \bowtie 0).$$

So this formula can be used to convert any inverse EQ-monoid embeddable in $\mathcal{I}(X)$ (X given the discrete topology) into an inverse subtraction monoid.

Conversely, in $\mathcal{I}(X)$ it is possible to express $\bowtie$ in terms of the inversion operation and set difference:

$$(f \bowtie g) = 1\backslash[((f\backslash g)^{-1}(f\backslash g)) \cup ((g\backslash f)^{-1}(g\backslash f))],$$

where $\cup$ is defined for restrictions of the identity α, β by

$$\alpha \cup \beta = 1\backslash((1\backslash\alpha)(1\backslash\beta)).$$

So again, any inverse subtraction monoid (each of which is embeddable in $I(X)$ for some X) can be converted into an inverse EQ-monoid. With some additional argument we obtain the following result.

Theorem 4.3 *The variety of inverse subtraction monoids is term equivalent to the variety of inverse EQ-monoids with zero defined by the additional identities:*

1. $((\alpha \bowtie 0) \bowtie 0) = \alpha$ *(i.e. L_A is a Boolean algebra)*
2. $(x \bowtie y)z = z(xz \bowtie yz)$ *(the E-deterministic law)*
3. $xy(xy \bowtie xz) = xy(y \bowtie z)$
4. $x(\alpha \bowtie \beta) = x(x\alpha \bowtie x\beta)$
5. $(x'x \bowtie 1) = x'x$ *(i.e. $E(A) = L_A$).*

Hence this latter variety consists exactly of those EQ-monoids representable within $\mathcal{I}(X)$ for some X endowed with the discrete topology.

Proof. Starting with an inverse subtraction monoid A, define $e \vee f = 1 - ((1-e)(1-f))$ for all idempotents $e, f \in A$. Then defining

$$(a \bowtie b) = 1 - [((a-b)'(a-b)) \vee ((b-a)'(b-a))],$$

and letting ϕ be an embedding of A into $\mathcal{I}(X)$ with X given the discrete topology, $(a \bowtie b)$ will be represented as the restriction of the identity to where the images of a, b agree or are both undefined, so A is an EQ-monoid under $\bowtie$. But in the inverse EQ-monoid $\mathcal{I}(X)$, the identities in the statement of the theorem may routinely be checked to hold; hence they hold for the subalgebra $Im(\phi)$ and hence for A. Call the inverse EQ-monoid with zero obtained from A in this way $F(A)$.

Conversely, starting with an inverse EQ-monoid with zero A and setting $a - b = a((a \bowtie b) \bowtie 0)$, we verify the subtraction semigroup laws in turn. Thus let $x, y, z \in A$. For the first law for subtraction semigroups,

$$\begin{aligned} xy - xz &= xy((xy \bowtie xz) \bowtie 0) \\ &= xy(xy(xy \bowtie xz) \bowtie xy0) \text{ by (4) in the theorem statement} \\ &= xy(xy(y \bowtie z) \bowtie xy0) \text{ by (3)} \\ &= xy((y \bowtie z) \bowtie 0) \text{ by (4)} \\ &= x(y - z). \end{aligned}$$

For the second law,

$$\begin{aligned}
(x-y)z &= x((x \bowtie y) \bowtie 0)z \\
&= xz((x \bowtie y)z \bowtie 0z) \text{ by (2)} \\
&= xz(z(xz \bowtie yz) \bowtie z0) \text{ by (2)} \\
&= xz((xz \bowtie yz) \bowtie 0) \text{ by (4)} \\
&= xz - yz.
\end{aligned}$$

For the third,

$$\begin{aligned}
x-(y-x) &= x((x \bowtie y((y \bowtie x) \bowtie 0)) \bowtie 0) \\
&= x(x(x \bowtie y((y \bowtie x) \bowtie 0)) \bowtie x0), \text{ by (4)} \\
&= x(y((y \bowtie x) \bowtie 0)[x \bowtie y((y \bowtie x) \bowtie 0)] \bowtie x0), \text{ by the EQ-monoid law } a(a \bowtie b) = b(a \bowtie b) \\
&= x(y((y \bowtie x) \bowtie 0)[x \bowtie y] \bowtie 0), \text{ by Lemma 1.7} \\
&= x(y(1 \bowtie 0)(x \bowtie y) \bowtie 0), \text{ by Lemma 1.7} \\
&= x(y0 \bowtie 0) \\
&= x(0 \bowtie 0) \\
&= x.
\end{aligned}$$

The fourth:

$$\begin{aligned}
y-(y-x) &= y((y \bowtie y((y \bowtie x) \bowtie 0)) \bowtie 0) \\
&= y((1 \bowtie ((y \bowtie x) \bowtie 0)) \bowtie 0), \text{ by (4)} \\
&= y(((y \bowtie x) \bowtie 0) \bowtie 0) \\
&= y(y \bowtie x), \text{ by (1)} \\
&= x(x \bowtie y) \\
&= x-(x-y), \text{ by symmetry.}
\end{aligned}$$

The fifth:

$$\begin{aligned}
(x-y)-z &= x((x \bowtie y) \bowtie 0)((x(x \bowtie y) \bowtie 0) \bowtie z) \bowtie 0) \\
&= x((x \bowtie y) \bowtie 0)((x \bowtie z) \bowtie 0), \text{ by Lemma 1.7} \\
&= (x-z)-y, \text{ by symmetry in } y, z.
\end{aligned}$$

Finally, the sixth law: $x - x = x((x \bowtie x) \bowtie 0) = x(1 \bowtie 0) = x0 = 0$.

Call the inverse subtraction monoid obtained from A in this way $G(A)$.

Finally, we show that these constructions are mutually inverse: $F(G(A)) = A$ for all inverse EQ-monoids with zero A and $G(F(A)) = A$ for all inverse subtraction monoids A. But if A is an inverse subtraction monoid, then A can be embedded in $\mathcal{I}(X)$ say; the same function is an embedding of $F(A)$ into $\mathcal{I}(X)$ as an inverse EQ-monoid with zero. But then the subtraction operation defined on the inverse EQ-monoid $F(A)$ in terms of $\bowtie$ and 0 gives set-theoretic difference, so in particular the subtraction in $G(F(A))$ is represented as set-theoretic difference, which means that the subtraction agrees with the original one in A: $G(F(A)) = A$.

Conversely, if A is an inverse EQ-monoid with zero satisfying the laws in the statement of the theorem, then $L_A = E(A)$ (the set of idempotents of A) is a complemented semilattice having bottom 0, top 1, and with the complement of $\alpha \in L_A$ given by $(\alpha \bowtie 0)$: this follows because $(0 \bowtie 0) = 1$, $\alpha(\alpha \bowtie 0) = 0(\alpha \bowtie 0) = 0$, and $((\alpha \bowtie 0) \bowtie 0) = \alpha$. Hence L_A is a Boolean algebra with join $\vee$ given by $\alpha \vee \beta = ((\alpha \bowtie 0)(\beta \bowtie 0) \bowtie 0)$ (and meet given by multiplication). Moreover, for all $a \in A$,

$$\begin{aligned}
(a \bowtie 0) &= (a'0 \bowtie 0)(a \bowtie 0) \\
&= (a'a \bowtie 0)(a \bowtie 0) \\
&= (a'a \bowtie 0)(aa'a \bowtie 0) \\
&= (a'a \bowtie 0)(aa'a(a'a \bowtie 0) \bowtie 0) \\
&= (a'a \bowtie 0)(a0 \bowtie 0) \\
&= (a'a \bowtie 0).
\end{aligned}$$

Hence for all $x, y \in A$,

$$\begin{aligned}
(x-y)'(x-y) &= (x((x \bowtie y) \bowtie 0))'(x((x \bowtie y) \bowtie 0)) \\
&= ((x \bowtie y) \bowtie 0)'x'x((x \bowtie y) \bowtie 0) \\
&= x'x((x \bowtie y) \bowtie 0)^2, \text{ since } \alpha' = \alpha \text{ for all } \alpha \in L_A \text{ and } x'x \in L_A \\
&= x'x((x \bowtie y) \bowtie 0).
\end{aligned}$$

Hence $((x-y)'(x-y) \bowtie 0) = (x'x \bowtie 0) \vee (x \bowtie y)$ by de Morgan's laws in $L_A = E(A)$.

Now in $G(A)$, for $\alpha, \beta \in L_A = E(A)$, $1 - \alpha = 1((1 \bowtie \alpha) \bowtie 0) = \alpha \bowtie 0$, the complement of α in the Boolean algebra L_A, so $1 - (1-\alpha)(1-\beta)$ is just $\alpha \vee \beta$ in this Boolean algebra; also, $1 - (1 - \alpha) = \alpha$ for all $\alpha \in L_A$. Hence $1 - (\alpha \vee \beta) = (1-\alpha)(1-\beta)$.

Now letting $*$ be the EQ-monoid operation in $F(G(A))$, we must show $*$ and the original $\bowtie$ on A agree. But for all $x, y \in A$,

$$\begin{aligned}
x * y &= 1 - ((x-y)'(x-y) \vee (y-x)'(y-x)) \\
&= (1 - (x-y)'(x-y))(1 - (y-x)'(y-x))) \\
&= ((x-y)'(x-y) \bowtie 0)((y-x)'(y-x) \bowtie 0) \\
&= ((x'x \bowtie 0) \vee (x \bowtie y))((y'y \bowtie 0) \vee (y \bowtie x)) \\
&= ((x'x \bowtie 0)(y'y \bowtie 0)) \vee (x \bowtie y) \\
&= ((x \bowtie 0)(y \bowtie 0)) \vee (x \bowtie y) \\
&= (x \bowtie y)
\end{aligned}$$

since $(x \bowtie 0)(y \bowtie 0)(x \bowtie y) = (x \bowtie 0)(y \bowtie 0)(0 \bowtie 0) = (x \bowtie 0)(y \bowtie 0)$, so $(x \bowtie y) \geq (x \bowtie 0)(y \bowtie 0)$ in L_A. So the two varieties are term equivalent.

Now the fact that any EQ-monoid A satisfying the laws in the theorem statement is representable follows easily: $G(A)$ is representable in this way, hence so is $F(G(A))$ because of the way the EQ-monoid operations are defined in terms of those on $G(A)$; but $A = F(G(A))$. Conversely, any representable EQ-monoid satisfies the stated laws as mentioned earlier. □

5 A connection with categories.

In an attempt to model elementary reasoning with partial functions (with a view to implementations), the current author jointly developed with Desmond Fearnley-Sander the idea of an L-monoid. These are simply monoids having a distinguished subsemilattice L containing the identity. Generalising the idea of a monoid category, the derived category associated with an L-monoid was defined. The idea is to associate with each element of L an object and have the monoid elements act as arrows between these.

Of course, an EQ-monoid A is an L_A-monoid, and sometimes an L-monoid A will be an EQ-monoid with $L_A = L$, namely when $max\{\alpha \in L \mid a\alpha = b\alpha\}$ exists for all $a, b \in A$. When we assert that an L-monoid is an EQ-monoid, we mean with $L_A = L$, unless stated otherwise. By Theorem 3.11, we can speak of a *full L-monoid* as one satisfying the first condition in that theorem, and which will thus automatically be a full EQ-monoid.

The following is routine to verify.

Definition/Theorem 5.1 *Let A be an L-monoid. Then A defines a category, the* derived category of A, *with objects the elements of L, and with arrows all triples $\langle \alpha, f, \beta \rangle$ where $f \in A$ and $\alpha, \beta \in L$ satisfy $f = \beta f \alpha$, and with domain, codomain, identity and composition defined as follows:*

- *$dom[\langle \alpha, f, \beta \rangle] = \alpha$;*
- *$codom[\langle \alpha, f, \beta \rangle] = \beta$;*
- *$1_\alpha = \langle \alpha, \alpha, \alpha \rangle$;*
- *$\langle \beta, g, \gamma \rangle \circ \langle \alpha, f, \beta \rangle = \langle \alpha, gf, \gamma \rangle$.*

We note in passing that there is a natural partial order on all arrows in this derived category, inherited from the L-monoid natural partial order: $\langle \alpha, f, \beta \rangle \leq \langle \gamma, g, \delta \rangle$ if and only if $f \leq g$. Indeed if the L-monoid is translucent (defined as for EQ-monoids), this partial order makes the category into a poset category (one in which all hom-sets are partially ordered in a way compatible with the category composition).

An *equalizer* in a category for two arrows $f : A \to B$ and $g : A \to B$ is an arrow $h : C \to A$, together with its domain C, such that $fh = gh$, and whenever $fk = gk$ for any arrow $k : D \to A$, there is a unique arrow $l : D \to C$ for which $k = hl$. C is determined uniquely up to isomorphism and correspondingly for h, which must be monic.

The category SET with objects all sets and arrows all maps between sets has equalizers, as is well-known. Similarly, for some fixed set X, one can consider the category $\mathcal{C}_X$, whose objects are all subsets of X and whose arrows are the partial maps between subsets of X; then the equalizer of $f, g : A \to B$ is $h : C \to A$, where $C = \{x \in A \mid f(x) = g(x)$, or $f(x)$ and $g(x)$ are both undefined$\}$, and h is the inclusion map into A. The category $\mathcal{C}_X$ is isomorphic to the derived category of the L-monoid of all partial maps $X \to X$ with L all restrictions of the identity, the main example of a full L-monoid (and hence full EQ-monoid). Equalizers in this L-monoid category therefore always exist and the above shows that they are are all in L. In fact full L-monoids can be characterised by this property.

Theorem 5.2 *The L-monoid A is full if and only if the derived category of A has equalizers in L, in which case the equalizer of $f = \langle \alpha, a, \beta \rangle$ and $g = \langle \alpha, b, \beta \rangle$ is $h = \langle (a \bowtie b)\alpha, (a \bowtie b)\alpha, \alpha \rangle$, where $\bowtie$ is the associated EQ-monoid operation on A.*

Proof. Let A be a full L-monoid. Now if $f = \langle \alpha, a, \beta \rangle$ and $g = \langle \alpha, b, \beta \rangle$ are arrows $\alpha \to \beta$, consider $h = \langle (a \bowtie b)\alpha, (a \bowtie b)\alpha, \alpha \rangle$, an arrow $(a \bowtie b)\alpha \to \alpha$, with $f \circ h = g \circ h$. For any $k = \langle \gamma, d, \alpha \rangle$ for which $f \circ k = g \circ k$, we have that $ad = bd$, and so $(ad \bowtie bd) = 1$, so $(a \bowtie b)d = (a \bowtie b)d(ad \bowtie bd) = d(ad \bowtie bd) = d$, so $d = (a \bowtie b)\alpha d\gamma$.

Now the following are equivalent:

- there is an arrow q such that $k = h \circ q$
- $q = \langle \gamma, p, (a \bowtie b)\alpha \rangle$, with $p = (a \bowtie b)\alpha p\gamma$, and $d = (a \bowtie b)\alpha p$
- $q = \langle \gamma, d, (a \bowtie b)\alpha \rangle$.

The equivalence of the first two conditions is direct from the definition of composition of arrows. The third condition clearly implies the second: simply let $p = d$. Finally, assuming the truth of the second condition, we have that

$$p = (a \bowtie b)\alpha p\gamma = d\gamma = ((a \bowtie b)\alpha d\gamma)\gamma = (a \bowtie b)\alpha d\gamma = d,$$

and the third is immediate. This establishes that there exists a unique arrow q such that $h \circ q = k$, and hence that h is the equalizer of f, g, and is representable by an element of L.

Conversely, suppose the derived L-monoid category of A has equalizers in L. Then for all $a, b \in A$, in the derived category there exist arrows of the form $f = \langle 1, a, 1 \rangle$ and $g = \langle 1, b, 1 \rangle$, and so there exists $h = \langle \beta, \alpha, 1 \rangle$ (with $\beta \geq \alpha$) such that $f \circ h = g \circ h$, that is, $a\alpha = b\alpha$. Then for any $c \in A$ for which $ac = bc$, again there is an arrow $k = \langle 1, c, 1 \rangle$ for which $f \circ k = g \circ k$, so there exists a (unique) $q = \langle \gamma, d, 1 \rangle$ for which $k = h \circ q$, that is, $c = \alpha d$, so $c = \alpha c$. Hence A is a full L-monoid. $\square$

If A is translucent, then fullness implies E-determinism. Hence we have

Corollary 5.3 *The derived category of the translucent L-monoid A has equalizers in L if and only if A is an E-deterministic EQ-monoid with $L = \{(a \bowtie b) \mid a, b \in A\}$.*

Consider an L-monoid A in which $L = \{1\}$ only. Thus A is simply a monoid and the derived category is exactly a monoid category, and it is possible to identify arrows with elements of A. If $a, b \in A$ have an equalizer $c \in A$ then c must be monic, meaning that $cx = cy$ implies $x = y$. Hence there are as many elements of the form cx as there are elements of A, so if A is finite, there exists $y \in A$ for which $cy = 1$. Hence $ac = bc$ implies $a = acy = bcy = b$ and so $A = L = \{1\}$. Trivially then, for a finite monoid category, if they always exist, all equalizers are in $L = \{1\}$.

However, not all L-monoid categories with equalizers have them in L. Let R be the reals, and let M be the set of (total) maps $f : R \to R$ such that $f(x) = x$ for sufficiently large x; thus $f(x) = x$ for uncountably many x. (Note that by "uncountable" we mean the cardinality of the continuum.) Then M is indeed closed under composition and the identity map is in M, so M is a monoid.

For $f, g \in M$, let $(f \bowtie g) = \{x \in R \mid f(x) = g(x)\}$, an uncountable set always. Now for $f, g \in M$, if f agrees with the identity on at least $[A, \infty)$ and g on at least $[B, \infty)$, then $[A^2 + B^2 + 1, \infty)$ is a subset of $(f \bowtie g)$, and there are still uncountably many values below this where they agree, so let h map $(-\infty, A^2 + B^2 + 1)$ injectively onto these values (this is possible because of their equal cardinalities), and $[A^2 + B^2 + 1, \infty)$ onto itself via the identity.

There are many other possible ways to define h, but the point is that $h \in M$ and h is injective. Moreover $fh = gh$ since h maps into (indeed onto) the set of domain points where f, g agree, namely $(f \bowtie g)$. If also $fk = gk$, then k must also map into $(f \bowtie g)$. Now define q such that, if $k(x) = y$, then $q(x) = h^{-1}(y)$, which exists since y is in $(f \bowtie g)$. Then $hq(x) = h(h^{-1}(y)) = y = k(x)$ for all x, so $hq = k$, and q is obviously unique with this property (which follows anyhow since h is monic). Hence h is indeed an equalizer of f, g.

Thus we have an example of an L-monoid whose derived category has equalizers which generally are not in L, and so the assumption that the equalizer is in L in the previous theorem is necessary.

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