On the bipenalty method: why is it advantageous to add stiffness and mass

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In a recent paper, Askes et al [1] proposed the simultaneous use of stiffness and inertia of large magnitude to model constraints in time domain analysis. From a frequency domain perspective, as stiffness and inertia have opposite effects on the natural frequencies, this seems counter-intuitive. With increasing stiffness, the natural frequencies either increase or remain unchanged, whereas the opposite is true for inertia. However, it can be shown, through very simple illustrative examples, that the natural frequencies and modes of continuous systems can be found in this way, and that there are advantages in using both stiffness and mass simultaneously.

The "artificial stiffness" method of Courant [2] has gone through some changes recently, thanks to some debate generated at the first ISVCS [3]. At this symposium, Ilanko proposed the use of masses, instead of stiffness, as a way to model constraints, so that true upperbound solutions to frequencies could be obtained using the Rayleigh-Ritz Method. However, with large masses, introduction of very low frequencies with modes which violate the constraints and the difficulty in selling the idea of enforcing continuity conditions with large masses that vibrated at the differential velocity of the connecting points, shifted the focus on a different strategy using "negative stiffness" instead of mass [4,5]. It may be worth noting here, that the idea of using negative stiffness occurred to the Lead Author, as a result of what he learnt from a mistake in the sign of a mass term in his PhD research [6,7]. The switching of the sign did not affect the results for the limiting case of a very large mass used as a test in verifying the accuracy of the computer program. This pointed to the fact that if the mass is sufficiently large as to prevent the motion of a point, then the sign of the mass (whether it is right or wrong) will have no effect on the frequencies. Using positive and negative stiffness, it is possible to determine and control any error due to violation of the constraints, but it is necessary to ensure that the magnitude of the stiffness is greater than the highest magnitude of the critical penalty stiffness values associated with negative stiffness. Subsequently, the legitimacy of using positive and negative mass to modelling constraints was established for frequency analysis and it was also used successfully in time domain analysis [8-10]. However, it has been found that when using inertial type penalty parameters, while the higher modes converge well even with very small penalty masses, the magnitude of inertial penalty needed to enforce constraints at lower modes can cause numerical problems for some very high modes [11, 12]. The most recently introduced bipenalty method, which has been developed for time domain analysis, seems to offer a solution that addresses all the problems listed above. By using both stiffness and mass at two carefully tuned combinations, it is possible to obtain bounded results for the natural frequencies of constrained system.

Fig 1. Cantilever with spring-mass restraint

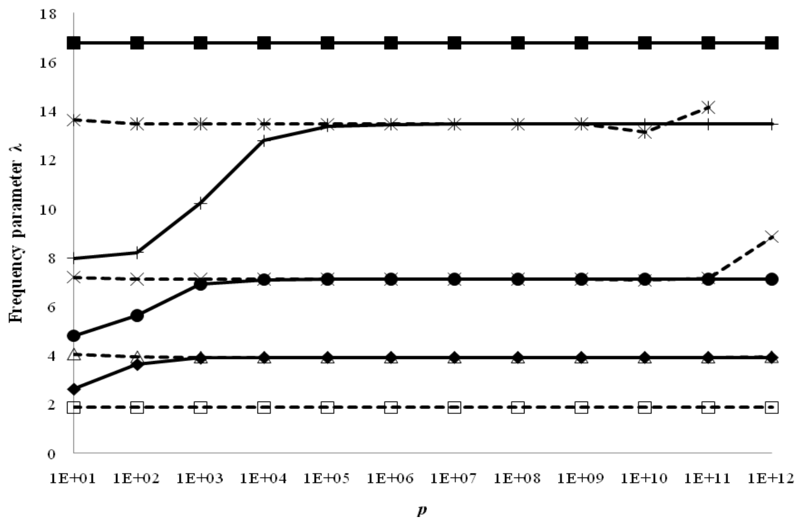

Figure 2. Frequency parameter λ vs penalty parameter $\log_{10} p$ for the propped

Fig. 1 shows an Euler-Bernoulli beam of length *L*, flexural rigidity *EI* and mass per unit length *m*, clamped at the left end and attached to a spring and mass at the right end. The stiffness of the spring is $p\alpha_s$ as where $\alpha_s = EI/L^3$ and *p* is the penalty parameter. The magnitude of mass is $p\alpha_m$ where $\alpha_m = \alpha_s/r$. The ratio *r* is the tuning ratio that changes the relative dominance of stiffness and mass. If *r* is very small then the system behaves as if it is inertially penalised and if *r* is very large it behaves like an elastically penalised system. With a four term assumed displacement of the form $f = \sum_{i=1}^{4} a_i x^{i+1}$ in a Rayleigh-Ritz method, results were generated for various values for the tuning ratio *r*. Figure 2 shows the variation of the non-dimensional frequency parameter of the beam $\lambda = L\left(m\omega^2/(EI)\right)^{1/4}$ with the penalty parameter *p* for two special cases. The solid line shows the results for a stiffness dominated penalised system with $r = \omega_4^2$, and the dotted lines shows the results for an inertia dominated case with $r = \omega_1^2$, where ω_i is the ith frequency of the cantilever beam (unconstrained at the right end). It may be seen that the three natural frequencies of the propped cantilever are approached from opposite directions by the natural frequencies of the penalised system with the two different tuning ratios. The highest frequency of the stiffness dominated system and the lowest frequency of the inertia dominated system remain unchanged; these are in fact equal to the highest and lowest frequencies of the constrained system, respectively. The way to tune the penalty parameter and the reasons for this behaviour will be presented at the symposium.

References

1. Askes, H., Caramés-Saddler, M. & Rodríguez-Ferran, A. 2010 Bipenalty method for time domain computational dynamics. *Proc. R. Soc. A* **466**, pp 1389-1408. (doi:10:1098/rspa.2009.0350).
2. Courant, R. 1943 Variational methods for the solution of problems of equilibrium and vibration. *Bulletin of the American Mathematical Society* **49**, 1-23.
3. Ilanko, S, 1997, On the application of Rayleigh's Principle and the Ritz method - Some Points to Ponder, Proceedings of the International Symposium on Vibrations of Continuous Systems, Colorado, pp 27-8.
4. Ilanko, S. & Dickinson, S.M. 1999 Asymptotic modelling of rigid boundaries and connections in the Rayleigh-Ritz Method. *Journal of Sound and Vibration* **219**, 370-378.
5. Ilanko, S. 2002a Existence of natural frequencies of systems with imaginary restraints and their convergence in asymptotic modeling. *Journal of Sound and Vibration* **255**, 883-898.
6. Ilanko, S and Dickinson, S.M., 1994, Corrigendum: The vibration and post-buckling of geometrically imperfect, simply supported, rectangular plates under uni-axial loading, part I: Theoretical approach; part II: Experimental investigation (S. Ilanko and S. M. Dickinson 1987 Journal of Sound and Vibration **118**, 313-351), *Journal of Sound and Vibration* **139,** 519-521
7. Ilanko, S, 1985, Vibration behaviour of in-plane loaded rectangular plates with initial geometrical imperfections, PhD Thesis, The University of Western Ontario, Canada.
8. Ilanko, S. 2005, Introducing the use of positive and negative inertial functions in asymptotic modeling. *Proc. R. Soc. A* **461**, 2545–2562.
9. Williams, F.W. & Ilanko, S. 2006 The use of the reciprocals of positive and negative inertial functions in asymptotic modelling. *Proc. R. Soc. A* **462**, 1909-1915.
10. Hetherington, J. & Askes, H. 2009 Penalty methods for time domain computational dynamics based on positive and negative inertia. *Computers and Structures* **87**, 1474-1482.
11. Monterrubio, L.E. 2009 The use of positive and negative penalty functions in solving constrained optimization problems and partial differential equations. PhD Thesis, University of Waikato.
12. Monterrubio, L.E. 2009 Free vibration of shallow shells using the Rayleigh–Ritz method and penalty parameters. *Proc. of the Inst. of Mech. Eng., Part C* 223, 2041-2983.