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THE DEVELOPMENT OF MULTIPLICATIVE  
THINKING AND PROPORTIONAL  
REASONING:  
MODELS OF CONCEPTUAL LEARNING  
AND TRANSFER

A thesis submitted in fulfilment of the requirements for the  
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## ABSTRACT

This thesis considers the development of multiplicative thinking and proportional reasoning from two perspectives. Firstly, it examines the research literature on progressions in conceptual understanding to create a Hypothetical Learning Trajectory (HLT). Secondly, it surveys modern views of how transfer by learners occurs in and between situations, contrasting object views of abstraction with knowledge in pieces views.

Case studies of six students aged 11-13 years illustrate conceptual changes that occur during the course of a school year. The students are involved in a design experiment in which I (the researcher) co-teach with the classroom teacher. The students represent a mix of gender, ethnicity and level of achievement. Comparison of the HLT with the actual learning trajectory for each student establishes its validity as a generic growth path.

Examination of the data suggests that two models of learning and by inference, transfer, describe the conceptual development of the students. There is consideration of students' use of anticipated actions on physical and imaged embodiments as objects of thought with a focus on the significance of object creation for conceptual growth, and the encapsulation, completeness and contextual detachment of objects.

There is broad consistency in students' progress through the phases of the HLT within each sub-construct though the developmental patterns of individual are variable and temporal alignment across the sub-constructs does not uniformly hold. Some consistency of order effect in concept development is noted. Discussion on the limitations of the HLT includes the difference between knowledge types from a pedagogical perspective, absence of significant model-representation-situation transfer, and order relations in conceptual development.

Considerable situational variation occurs as students solve problems that involve applications of the same concepts. Partial construction of concepts is common. This was true of all learners, irrespective of level of achievement. High-achieving students more readily anticipate actions and trust these anticipations as objects of thought than middle and low achievers. The data supports knowledge in pieces views of conceptual development. Complexity for learners in observing affordances in situations, and in co-ordinating the fine-grained knowledge required, explains the difficulty of transfer. While supporting the anticipation of action as significant from a learning perspective the research suggests that expertise in applying concepts involves a process of noticing similarity across contextually bound situations and cueing appropriate knowledge resources.



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# CHAPTER ONE: INTRODUCTION

## 1.1 Rationale

As a national co-ordinator from the outset of the New Zealand Numeracy Development Projects in 1999, I was interested in the learning of number. A small team of colleagues and I created the Number Framework (Ministry of Education, 2008) which has been used as a key resource in the professional development of teachers for the last eleven years. The framework maps a trajectory of progression in students' strategies for solving number problems and the aligned knowledge that is required to enact the strategies.

The Hypothetical Learning Trajectory (HLT) is a construct first proposed by Simon (1995). He saw the teacher as a field researcher who iteratively adapted learning goals, a predicted conceptual growth path, and activities on the basis of data about their students. Simon's localised interpretation of an HLT is far from proposition of a homogenous, generic growth path for all students as suggested by the Number Framework. Critics of HLTs suggest that variation between students and in the responses of the same student to different situations make the idea of a single, linear growth path both untenable and potentially restrictive (Watson & Mason, 2006; Lesh & Yoon, 2004).

Yet our initial work on the Number Framework assumed a high degree of consistency in the way students learned number concepts. Data supported the hypothesis of broad stages of progression (Young-Loveridge, 2006; Young-Loveridge & Wright, 2002). There was a possibility that the consistency of the data was due to strong alignment between teaching approaches and the measures of achievement. The consistency of students' progression through an HLT for multiplicative thinking and proportional reasoning is one of two critical issues under investigation in this research.

The second critical issue for this research is the process by which learners become more sophisticated in their use of number which is possibly of more significance educationally than the path of progression. The signal of increased sophistication is the ability to apply concepts with more consistency to a wider variety of situations. Transfer of learning refers to the ability of a learner to recognise similarity between a new situation and those they have previously encountered, in order to apply their existing knowledge productively (Royer, Mestre, & Dufresne, 2005). Transfer implies that students detect common structure in the face of contextual variation. Structure refers to a set of knowledge elements and relations that exist between those elements (di Sessa, 2008).

Two theories of conceptual development as means of transfer apply to this study, object theory, and co-ordination class theory. Object theory describes the process of creating abstractions that are seemingly, to an observer, applied consistently across variable situations (Sfard, 1991, 1998). Scholars use terms like encapsulation and

reification to name a stage at which a concept becomes an object of thought detached from base actions on embodiments and available for application to new contexts (Tall, Thomas, Davis, Grey & Simpson, 2000; Tall, 2008). For these theorists transfer indicates the existence of mathematical objects. In object theory abstraction of mathematical objects is the source of transfer. Upon abstraction, application to new contexts is a tidy and consistent process, though some theorists suggest a “folding back” process between levels of abstraction (Pirie & Kieren, 1994). Tidy generic stage models for the development of concepts are common in the literature. Do such tidy models reflect the reality of concept construction?

In contrast, situated learning theory suggests other possibilities for the process of construction. Situated theory shifts attention to learners and their interpretation of situations. Actor-orientated perspectives of situated theory advocate structure as perceived by learners not an inherent property of a situation. Rather a situation offers affordance for a learner to see that a concept is applicable. Knowledge that the learner already possesses affects what they see in the situation and provides the elements required to enact chosen courses of action. Co-ordination class theory belongs to the situated, actor-oriented genre. It offers stark contrast to object views in that it attributes transfer to recognition by the learner that a situation is similar to those already encountered. Successful experience cues knowledge resources to features of situations. Contextual information provides significant knowledge elements so concepts are connected, not detached, from the situations from which they arise. The creation of concepts that are co-ordination classes involves reading out concept-relevant features in situations and co-ordinating fine-grained knowledge elements.

Contrasting object theory and co-ordination class theory as models for transfer aligns easily with conceptual development of multiplicative thinking and proportional reasoning. These fields are complex and difficult for teachers to teach and students to learn. Understanding requires connection between models, representations and situations. These connections offer considerable opportunities for transfer. Considering a HLT and transfer simultaneously is worthwhile and seeks to inform an area of theory and practice lacking in the current body of research. The extent to which a growth path for a complex conceptual field is predictable in a generalised way for all learners contrasts with the natural variability in how individual learners transfer in and between different situations.

## **1.2 Research Questions**

The research considers this question:

**How do students develop their multiplicative thinking and proportional reasoning?**

In doing so it addresses two issues:

- a. **Does a Hypothetical Learning Trajectory (HLT) reflect the actual learning trajectory of students?**
- b. **What model of conceptual learning, object theory or co-ordination class theory, best represents the growth of multiplicative thinking and proportional reasoning, and the transfer of knowledge within and between situations?**

This question contrasts the HLT with the actual learning trajectories of the students in the study. At issue is whether the HLT holds as a framework for progression or whether differences in the conceptual growth of individual students suggest that a generic growth model is invalid. Other considerations are temporal patterns of conceptual growth, consistency of students' growth across the sub-constructs of the HLT and inclusion and exclusion of key ideas.

Good models are explanatory and provide frameworks for anticipation. The research considers the extent to which the principles of object theory and co-ordination class theory match the evidence of transfer. It also considers the utility of the two theories in terms of their explanatory power for the construction of concepts, not just the endpoint possession, and discusses the implications of the models for learning and teaching.

### **1.3 Research design**

I chose a design experiment as the methodology for this study. Design experiments are iterative research processes of testing a conjecture within a situated environment. The proposed HLT was that initial conjecture. Inquiry involved an on-going cycle of uncovering student thinking to inform instruction. In this situation, I was co-teacher and researcher. There were two reasons for the choice of design. I wanted the data to reflect learning that occurred within a classroom environment with a group of students from a range of backgrounds and achievement levels. I also recognised that the demands of teaching multiplicative thinking and proportional reasoning were high. I felt my involvement as a teacher, alongside another experienced colleague, would give the students their best possible opportunity for conceptual growth.

I identified nine students as subjects for case studies. The case study students provided a cross-section of gender, ethnicity and achievement. The selection of a limited sample from the whole class was pragmatic given the limitations of time as I acted as both teacher and researcher. Data came from work samples, interviews, written tests, anecdotal teaching notes, plans, observations by the class teacher, and group modelling books. For reasons of space, this thesis described the progress of six of the nine case study students.

To investigate the first research issue, the HLT, a summative map was created for each student at four times during the year; early Term One, end of Term One, late Term Three and late Term Four. In New Zealand terms are blocks of teaching weeks that make up the school year. I derived the maps from all of the data available on the students at each point in time. A graphic form of the HLT compared the growth of

individual students at different time points and the patterns of growth between students.

To investigate the second research issue, conceptual development, I examined the data for each student to look for evidence of transfer between situations, models and representations. These data were organised initially by sub-constructs in the HLT then synthesised to look for consistency or non-consistency with object theory and co-ordination class theory. In a further round of analysis, I compared the evidence from individual students to look for patterns in dispositions, preferences, common conceptual obstacles and patterns of achievement.

## **1.4 Thesis outline**

A comprehensive literature review forms the next three chapters of this thesis. One chapter each is devoted to research on learning and transfer, multiplicative thinking and proportional reasoning. Chapter 5: Methodology discusses the methodological issues around design experiments and describes the research methods in depth.

Presentation of the evidence in relation to the research question comprises Chapters Six – Ten through four in-depth case studies and two abbreviated individual student case studies. Each case describes:

- Background of the student, i.e. age, gender, ethnicity, interests, attitudes and social behaviour;
- Data on achievement available at the beginning of the year from interviews and tests;
- Descriptions of conceptual growth during the year in the sub-constructs of the HLT and other constructs of interest, i.e. multiplication and division of whole numbers, relationships in number pairs, part-whole and measures, quotients, operators, rates and ratios, probability, graphs, decimals and percentages;
- Summaries of the data in relation to object and co-ordination class theories;
- Growth in the HLT shown graphically using the Learning Trajectory Maps.

Chapter 11: Discussion synthesises the evidence across the six case studies in addressing each research question. It compares the HLT with the actual learning trajectories of the six students and discusses the strengths and limitations of the HLT as a generic growth path for multiplicative thinking and proportional reasoning.

Comparison of the principles of object theory and co-ordination class theory as models for transfer to the data on students also contributes to Chapter 11. There is discussion of the relative merits of the two theories and implications for research and pedagogy. Chapter 11 also presents possibilities for future research work.

## CHAPTER TWO: TRANSFER OF LEARNING AND THEORIES OF LEARNING

### 2.1 Why Take a Transfer Perspective?

The theoretical framework for this thesis derives from two main sources. Firstly theories about learning, in particular how it occurs, provide an epistemological perspective. There is an assumption of transfer of knowledge from one situation to another in most theories about learning that is not always explicitly stated. Learning and transfer are the subjects of this chapter. The second source for the theoretical framework is research in the mathematical domains of multiplicative thinking and proportional reasoning. These sources provide an ontological perspective about the structure of concepts applied by learners in transfer. Multiplicative thinking and proportional reasoning are the topics of Chapters Three and Four.

It seems an obvious assumption that when people learn their previous experiences inform new ones. This means that people identify similarities between new situations and those previously encountered in order to apply existing knowledge and develop new knowledge. Transfer is the application of knowledge learned in one situation to a new situation. Yet the weight of research suggests that transfer is difficult (Royer, Mestre, & Dufresne, 2005). Understanding how learning occurs and facilitates transfer constitutes a fundamental problem in education.

A focus on transfer of learning appeals for several reasons. Firstly, the topic of this thesis, the development of learners' multiplicative thinking and proportional reasoning is rich in connections. The connections between different contexts, between representational forms, particularly symbols, diagrams and materials, and between constructs pose challenges for learners in transfer (Kieren, 1980). Lamon (2007, p. 629) described the complexity of proportional reasoning:

Of all the topics in the school curriculum, fractions, ratios, and proportions arguably hold the distinction of being the most protracted in terms of development, the most difficult to teach, the most mathematically complex, the most cognitively challenging, the most essential to success in higher mathematics and science, and one of the most compelling research sites.

Secondly, transfer of learning is a significant topic for research and a source of some contestability. Situated learning theory casts considerable doubt over traditional views of transfer and contests the usefulness of broad theories of cognitive development (Lobato, 2006; Schliemann & Carraher, 2002). At the heart of the controversy is the extent to which people detach concepts from the situations in which they are developed. A situated perspective sees concepts and situations as irrevocably linked, while some learning theories view concepts as abstractions stripped away from specific situations. The body of work on transfer is in need of further examples that occur within the classroom environment and specifically detail the journey of students' learning of particular mathematical concepts.

Thirdly, a debate currently rages around the tidiness of learning, sometimes referred to as the coherency versus in pieces debate (diSessa, 2008). Coherent theories

describe learning as a process of progression through distinguishable stages or learning trajectories. In pieces theories describe the unpredictability and messiness of learning as people attempt to transfer old knowledge to new situations. Research into conceptual change demands greater attendance to the process of change as opposed to snapshots that contrast novice and expert performance. This also requires a fine-grained analysis of the knowledge resources and processes used by people in transfer. Attitudes and dispositions also have a contributory impact on transfer through providing resources (Perkins & Saloman, 1994) and enabling the learner to foresee their potential in meeting the demands of a situation (Mason & Spence, 1999). The significance of attitudes and dispositions as contributing to, and reflective of, success in mathematics is commonly accepted (Fitzpatrick, Swafford & Findell, 2001).

Finally, variation of learning between individuals and variation of individuals on tasks different in context but alike in structure warrants attention. Some literature on transfer presents an explanatory framework for this variation, allowing the possibility of finding consistency amongst the variation.

The next section surveys some of the literature on conceptual change. Theories about the development of concepts have shaped research into transfer, particularly studies in the traditional/classical genre. Transfer is implied but seldom explicitly stated in these theories, yet transfer seems essential in both the creation of new concepts and application of existing concepts.

## **2.2 Theories of Learning**

The value of a theory is that it informs the interpretation of situations. Good theories model key features in a way that allows prediction of effect. In this sense theories are judged by their utility and generalisability, not their correctness as an absolute. Pirie and Kieren (1994, p. 77) cite the following quote from Einstein:

A theory is a self-sharpening tool whose warrants and value in the end rest on this, that they permit the co-ordination of experience, 'with dividends'.

This section considers two theories of learning. The first theory is about abstraction through the creation of objects for thought. The second theory is about the process of learning as knowledge in pieces. Both theories provide insights about learning but provide different temporal and distance views of the process.

### **2.2.1 CREATION OF MATHEMATICAL OBJECTS**

Considerable commonality exists in the views of prominent learning theorists regarding the encapsulation of a process as an object for further thought. In essence, the common view is that for knowledge to progress the results of actions must be anticipated, then captured and encoded to become accessible for more complex tasks (Tall, Thomas, Davis, Gray, & Simpson, 2000). For example, a learner for whom combining eight sets of three is an action of repeated addition comes to see it contained in the symbols  $8 \times 3 = 24$  and the words "eight times three". Over time the

learner thinks with  $8 \times 3$  as an idea to derive new ideas like  $8 \times 3 = 4 \times 6$  and  $24 \div 3 = 8$  without need to carry out the original action or the actions of the new problems.

Piaget and Inhelder took an adaptive perspective of conceptual change in describing two processes; assimilation and accommodation (Piaget & Inhelder, 1969). Assimilation involves provocation for change to existing knowledge from a new situation. Accommodation occurs when the learner changes their mental structures to encompass the new knowledge. Following accommodation, the learner returns to a state of equilibrium, a common property of biological systems. Piaget's initial work was in biology. According to Piaget actions become objects of thought through a process of reflective abstraction. The result of a process is taken as anticipated and acted on as a given (Cobb, Boufi, McClain, & Whitenack, 1995). Piaget's adaptive view of conceptual change was partially shared with Vygotsky who believed that meta-cognition and language played pivotal roles in conceptual change (Vygotsky, 1962). Vygotsky saw conceptual change as mediated through cultural practices. Socially accepted elements, most noticeably oral language, symbols and artefacts act as tools that simultaneously enhance and constrain thought (Schliemann & Carraher, 2002). Social mores influence both the ideas that are considered and the processes by which they are considered.

Sfard (1991) accepted the mediation of culture but suggested a uniform development of ideas, irrespective of the cultures in which the ideas developed. She proposed three sequential stages of concept development, drawing on analysis of the history of mathematics and on learning research. *Interiorisation* is the first stage in which a learner becomes familiar with processes that will form the new concept. Sfard implied an ability to carry out the process mentally which could be considered to involve two or three stages through which the learner represents physical actions as figurative (imaged) material (Pirie & Kieren, 1994). The second stage is *condensation* in which the individual learns to deal with a process as an anticipated action, to regard it as an input-output relation without the need to enact it. Thirdly, *reification* is the stage in which the process is converted into an object that is compact and self-contained. Reification is required before the learner can use the object in a structural way, to create new concepts.

In support of her theory Sfard (1991) offered two natures of mathematical entities, operational and structural. Operational involves seeing the entity as a process and structural involves seeing it as an object with which to think. Sfard believes these dual natures to be essentially incompatible yet complementary. It is hard to conceive of a person easily thinking with an idea while still carrying it out as a process. In situations where multiple ideas need co-ordination this would create cognitive crowding and overload. Yet the person needs access to certain elements of the process at times to make sense of the problem at hand. For example, a person may view the fraction two-thirds as an object, a number between zero and one. In using two-thirds as an operator to find  $\frac{2}{3} \times 36$  the person needs to access aspects of two-thirds as a process, such as seeing it as two quantities of one-third or as two out of every three.

Many issues arise from Sfard's theory. She portrayed the stages as distinct and progressive. Is a person clearly at one stage at a given point in time or are they in a state of flux between stages? Other theorists depict abstraction as a dynamic process of folding back between stages (Pirie & Kieren, 1994) or as stages nested within one another (Hershkowitz, Schwarz, & Dreyfus, 2001). Is it possible to distinguish which stage a learner is at through observation? Do situations matter in terms of the person's ability to enact the concept as an object or are people consistent across situations? Does new learning in another related concept affect progress in learning a concept? Is the effect always progressive or can it be regressive?

Tall, Thomas, Davis, Gray, and Simpson (2000) summarised an extensive body of work relating to object theory. They hypothesised several different kinds of mathematical objects. One type of object is a procept, the combination of a process and an object of abstraction in its own right. This idea is similar to Sfard's duality of process and structure, though Tall *et al.* stressed the significance of symbols as unifying markers between the two. They point to the fact that a concept like 5 is a conceived idea that cannot be embodied in a physical representation in the same way a square can. The symbol 5 is a marker that represents the idea of five it so it can be talked about. If Tall *et al.* are correct, symbols potentially play a key role in transfer through providing markers for concepts and in providing common representations across perceptually different situations. The view is consistent with Vygotskian perspectives of words and symbols as mediating tools. Tall *et al.* also discussed percepts, which are empirical abstractions that are formed without definition from actions on real world objects. This idea is similar to phenomenological primitives or p-prims (diSessa, 2002) which will be discussed later in this chapter. Object theories have an attractive elegance and uniformity about them but are susceptible to evidence of variation in the way conceptual change occurs for individuals. Siegler (2000, 2007) highlighted the within child variability between assessments over short time periods. Learning in the long-term may exhibit a tidier trajectory but in the short-term there is considerable messiness.

Object theory clearly delineates stages but in practice the finer details of what occurs as an individual progresses through them are not described. Distinguishing a point at which a learner moves from one stage to the other seems difficult to establish. For instance, what signals that a concept is reified or encapsulated? Little is said in object theory about the stability of reified objects, only that they are available and used in other applications. There is a probability of enhancement of objects in response to new knowledge. For example, a reified idea of 5 that was developed from counting, and adequate for addition of five and five, must be embellished from learning about odd and prime numbers. A reified object must also be subject to change. An object being sufficient for purpose rather than complete seems more significant than the notion of encapsulation though these are not contradictory ideas. There is also an assumption that all learning of mathematical concepts is the same irrespective of the nature of the concepts.

The ambiguity of vocabulary used to describe object acquisition borders on tautology. Object refers to both physical objects and mental abstractions. di Sessa (2002) criticised the lack of clarity around definition of concept. It seems fundamental that

debate about how concepts develop begins from a common understanding of the entity under development. That is far from the case. For example, Tall, Thomas, Davis, Gray, and Simpson (2002) said that 5 is a concept, a conceived idea imposed on but not embodied by physical objects. Their point is informative as it is clear that people create concepts and think about them in a way that does not require actions on physical objects. For example, a person can consider the result of sharing a set of objects among five people without actually doing it. However, 5 may be considered a small concept while whole numbers may be considered a bigger concept that requires the use of five as an object of thought. It is not clear whether a learner must reify 5 in order to understand the concept of whole number. The object cannot be static and fixed since it must be amenable to new information. So when is a person's concept of 5 as a small concept sufficient for it to be used in creation of the bigger concept of whole numbers?

There is confusion around the distinction between scheme and concept in the literature. Vergnaud (1998, p. 168) helpfully defined a scheme as "the invariant organisation of behaviour for a certain class of situations." Schemes are generalised actions while concepts are conceived ideas used in these actions. However, in practice there is assumption of the existence of concepts from observed actions.

Object theory leaves much unsaid about the nature of development for specific concepts and assumes uniformity of process for all learners. This seems at odds with natural variability and the diverse nature of knowledge in different domains. There is an assumption of transfer between old and new knowledge in the construction of concepts and between the concept and other concepts upon acquisition of object status. Understanding the mechanisms by which transfer occurs is fundamental to understanding the construction of concepts as objects for further thought and the access and application of knowledge from established concepts to create concepts that are more advanced. The next section presents a theory of conceptual learning in pieces, not as tidy abstractions that are reliably applied. The picture is of considerable contextual variation as individual learners make strategic choices that are greatly influenced by their possession and co-ordination of fine-grained knowledge elements.

### **2.2.2 KNOWLEDGE IN PIECES**

In the same way that Piaget's theory of conceptual development, particularly the focus on adaptation and equilibrium, reflects his experience as a biologist, the work of Andrea diSessa and his colleagues resides in the tradition of physics education. A grand theory that connects small space and large space remains elusive in modern physics, data are often complex and unpredictable and results obtained in quantum mechanics are often counter-intuitive (Bryson, 2003). diSessa's co-ordination class theory (diSessa, 2002, 2008; diSessa & Wagner, 2005) seeks to model the relationship between transfer and the development of concepts and to explain why transfer can be elusive and unpredictable. His depiction of knowledge in pieces came from diSessa's early studies of students' views of forces (diSessa, 1993). diSessa sought to explain the erratic nature of learners' thinking. Knowledge used creditably

in one setting was either not used or used incorrectly in another. Learners held contradictory knowledge simultaneously without evidence of resolution or perturbation. The metaphor of in pieces was to portray a more chaotic view of concept development than that suggested by object theories.

Fundamental to the in pieces view of conceptual change is that transfer facilitates abstraction, not that abstraction enables transfer. Perception by the learner of similarity and difference across situations is the key to transfer. Through identifying similarities between it and other previously encountered situations, the learner identifies structure in a problem. The breadth of knowledge resources available to the learner influences their perception and this knowledge is cognitively bound to the situations in which it is developed.

Wagner (2006) used co-ordination class theory to explain a student's progress in learning the law of large numbers. He describes:

...the inseparability of the perception of structure in a problem from the knowledge of the principles needed to solve it. (p. 61)

The dynamic relationship between possession of knowledge resources and the perception of affordances in situations connects to Mason and Spence's idea of *knowing-to* and *knowing-about* (Mason & Spence, 1999). To solve a problem the learner must imagine being able to solve it, i.e. *know-to*. Yet the imagining is affected by *knowing-that* particular concepts are useful and *knowing-about* how to solve the problem. Co-ordination class theory and its appealing corollary, knowledge in pieces, reflect an actor-orientated perspective of learning that is discussed fully in section 2.4.

## 2.3 Views of Transfer

Recent literature contrasts traditional or classical views of transfer with actor-oriented or situated views. There is a polarization of views and methodologies in this work which neglects the consensus about some aspects of transfer and the fusion of methods used by many transfer researchers (Lobato, 2006; Lobato & Siebert, 2002). Lobato and Siebert (2002, p. 89) question the traditional view of transfer as "the application of knowledge learned in one situation to another". This definition is learner neutral in that it assumes situations are the same for all individuals. In contrast, actor-oriented theorists acknowledge the personal construction of knowledge by learners through identifying relations of similarity between different activities. Schliemann & Carraher, (2002, p. 244) stated:

It is now fairly widely accepted that specific contexts, far from being incidental, are essential to what is learned and thought.

The views about learning and transfer taken by researchers affect what they look for, what they do and therefore what they find. The next sections will examine both traditional and actor-orientated views and methodologies in detail with the aims of describing the state of the art on transfer and of establishing a modern view of how learning, and by inference, transfer occurs.

### 2.3.1 TYPES OF TRANSFER

The concepts of amount and distance of transfer are common in the literature. Amount refers to the gap between a learner's current knowledge and that required for transfer. Distance refers to the features of situations, mostly the complexity of the ideas and connections required for transfer. Types of transfer warrant discussion in framing the interpretation of learners' construction of concepts in this study. This raises issues about the predictability of transfer and the expectation of time that a learner may need to construct adequate concepts. Some theorists discuss types of transfer in terms of learner responses within a complex domain.

Authors use varied terms such as near and far, vertical or lateral (horizontal), and positive and negative to characterise the amount and distance of transfer (Wagner, 2003). In research studies these constructs have been uniformly difficult to operationalise (Mancy, 2010a). Putting a quantity on, or assigning a category to, the amount or distance of transfer required by a learner takes a precision of measurement for the initial and end state that is difficult to achieve.

Amount of transfer has a learner perspective in that it considers prior knowledge while distance describes the degree of non-congruence between situations. Perkins & Salomon (1994) contrasted low road and high road transfer. Low road transfer is between situations that appear similar in context to the learner whereas high road transfer is between situations that appear remote and alien. Perkins and Salomon discriminated between the thinking required for each type of transfer while accepting the educational significance of both. For low road transfer they advocated hugging techniques that involve controlled variation of situations around a common structure. For high road transfer they suggested bridging techniques that emphasise abstraction of properties across diverse situations. This perspective is similar to that of (Schwartz & Varma, In press) who use the terms similarity and dynamic transfer. Dynamic transfer involves the co-ordination of systems that appear dissimilar to the learner so it takes a long time. Similarly diSessa and Wagner (2005) differentiated three types of transfer related to the preparedness of the learner for successfully solving a problem:

Type A: Transfer for which the learner has well-prepared knowledge resources

Type B: Transfer for which the learner has sufficiently well-prepared knowledge resources that can be expected in reasonable time given learning support

Type C: Transfer for which the learner is relatively unprepared and where they use prior knowledge in new contexts (for them)

diSessa and Wagner suggested that many of the results from traditional research, in which transfer does not occur, happen because Type C transfer is required after minimal intervention.

Transfer types connect with the zone of proximal development (ZPD) (Vygotsky, 1962). Like Piaget, Vygotsky proposed temporally long stages of development so the zone was an area of significant conceptual growth rather than of learning specific knowledge and skills (Chaiklin, 2003). The ZPD is the distance between what the learner can achieve independently and that of which they were potentially capable

given access to an informed other person. Type B or high road transfer opportunities correspond to Vygotsky's idea of assisted learning so align with the ZPD. Type C transfer corresponds to unassisted learning. Opportunities for learners to independently transfer their knowledge to unfamiliar situations (Type C transfer) are in harmony with the Vygotsky's objective of the learner becoming self-generating.

Generally researchers agree that the greater the dissimilarity between situations the greater the difficulty and complexity for students in transfer. There is also acceptance that far transfer is educationally significant, takes a long time, and requires strong meta-cognitive self-direction from the learner. Whatever categories describe the amount of transfer required it is clear that there is a direct relationship expected between the amount and the difficulty of transfer for the learner, and therefore the likely time required for it to occur. The issue for this study is the degree to which it is possible to describe the amount of transfer for individual learners and locate the knowledge resources that facilitate transfer. Another consideration, though not the central focus of the research, is how learning as demonstrated by successful transfer, is facilitated by the learning environment, particularly through access to tasks and discourse (Anthony & Walshaw, 2007).

Lobato and Siebert (2002) discussed other types of transfer. They proposed between situation, within model, between representation, and situation to model types.

Examples of these types related to this study are:

- Between situations, e.g. ratio in a mixing colour context to a slope of ramp context;
- Within model, e.g. establishing the part-whole and part-part relationships in a ratio as fractional numbers;
- Between representations, e.g. connecting table and linear representations of rate or connecting decimal and fractional number notation;
- Situation to model, e.g. comparing scoring frequency of sports people using percentages or decimals.

Lobato and Siebert's work is useful as it provides a way to classify the transfer exhibited by learners in specific situations. Their taxonomy extends the conventional definition of transfer from being just about between-situation transfer to between models and representations. This is helpful in conjunction with consideration of conceptual field theory, to be discussed in Chapters Three and Four, in that models and representations arise that must be connected by learners in order to see similarities in and between contextually different situations. Lobato and Siebert's wider view of transfer raises issues about the balance and significance. Are all four types critical to the development of strong concepts in mathematics? Are some types more significant than other types? Do learners have personal preferences or dispositions for different types and how does this influence their conceptual growth?

The next sections contrast how researchers have defined transfer and the methods they have used to investigate it. Traditional is a generic term used to encompass a large number of studies prior to the mid-1990's. The prevailing focus was on the factors which influenced the success of transfer not on understanding how it occurred.

### **2.3.2 TRADITIONAL VIEWS OF TRANSFER**

Underpinning traditional views is a principle that the creation of an abstraction in one or many situations leads to its application in new situations. This abstraction is detached from the initial situations so the learner is free to apply it in a general way. This connects to broad theories of cognitive development that model processes for the creation of mathematical objects that can be used for the creation of more complex structures (Simpson, 2009).

Research reflecting the traditional view is usually associated with methodology that pre-ordained successful transfer through contrast of normative novice and expert performance. The role of the researcher is as observer and the studies typically involve a pre-assessment, a period of intervention or treatment, followed by a post-assessment. Larger scale experimental and control groups are used to evaluate the success of the intervention, method or procedure (treatment). Traditional studies consider learners' prior knowledge adequately factored by the stratification of samples using pre-assessment results. Subjects are recipients of the intervention or treatment and expected to apply their knowledge to a new situation. Schwartz, Bransford, and Sears (2005) referred to this methodology as Sequestered Problem Solving (SPS) in which the problems used in pre- and post-assessment differ only in surface features, meaning minor contextual or task variation rather than significant structural variation. Subjects have no opportunity to learn through feedback on their assessments. Transfer is constrained to given time points and measured by performance of anticipated behaviours.

Traditional research into transfer is almost uniform in reporting poor incidence of subjects applying their knowledge in new situations. Perkins and Salomon (1994) summarised abundant evidence that expected transfer often does not occur in classical studies. They noted that transfer can be positive where the learner notices similarity between situations and negative when the learner transfers inappropriately. The expectation yet failure of transfer is referred to as the transfer paradox meaning transfer does not occur in traditional studies but clearly must occur in other environments. The fact that humans learn presupposes that they transfer from one situation to another.

General failure of traditional studies to find transfer of learning gave weight to queries about the validity of the methodologies used in the traditional paradigm. Criticisms focused on the limitations in the measurement of transfer, on the short timeframe of studies, and the emphasis given to summative rather than formative data. These issues were addressed in considering the method adopted for transfer research.

### **2.3.3 SITUATED AND ACTOR-ORIENTED VIEWS OF TRANSFER**

Just as cognitive science perspectives influenced the methodologies employed by traditional researchers, and consequentially the data they gathered and results they found, the advent of constructivist learning theory influenced the researchers who developed situated and actor-oriented views of transfer. Simply put, situated views

consider contexts for application of a mathematical concept as influential to learners' engagement. Actor-oriented views go further in highlighting that the very perception of a situation is personal to each learner and is strongly influenced by their previous experience. The view of the learner as an active constructor of their own knowledge, mediated through environmental and social interaction, aligns with interpretive methods of inquiry that provide in depth commentary on the learning of individuals and the situations in which the learning occurs. Constructivist views are at odds with the traditional paradigm that, in its stereotypical form, depicts the learner as a passive participant of a treatment or intervention and that views learning as uniform, predictable, and easily measured.

The work of Lave (1988) strongly influenced the development of situated learning perspectives. Lave studied the everyday mathematical practices of people in supermarket shopping and cooking in weight-watching programmes. She found little evidence of transfer between school mathematics and the methods used by adults in everyday life situations. The people in her study showed high success rates in solving complex problems such as calculating best buys and food servings in context yet inability to solve parallel problems in classroom settings. Lave criticised the laboratory methodology of traditional research into transfer as too narrow in that it ignored the interaction between the individual and the situation. Other researchers sought to explain why different learners saw apparently isomorphic situations differently.

Actor-orientated views of transfer see the learner as playing an active role in the constructing their own knowledge through noticing similarity within and between authentic contexts. Royer, Mestre, and Dufresne (2005) suggested that similarity does not exist a priori between situations and must be perceived by the learner. They view context as an integral feature of a situation that affects learner engagement.

The role of learner perception *in situ* was central to the work of Greeno and his colleagues (Greeno, Smith, & Moore, 1993). They believed that for transfer to occur the learner must attune to affordances in the situation. Affordances are opportunities to act (Chick, 2007). For example, a student may see comparison of two-thirds and two-fifths as an opportunity to use proximity to one-half. Greeno *et al.* placed ownership of transfer in the hands of the learner, specifically in their ability to detect similarity between a new situation and those they have already experienced, and interpret the information attended to as opportunities to act. This view has congruence with that of Mason and Spence (1999) who contrasted the difference between *knowing-to* act in the moment and *knowing-about* the information required to act. Mason and Spence provide three reasons for knowledge remaining inert in a given situation, meta-processes that prohibit accessing knowledge, structural deficits (lack of *knowing-about*) and situatedness (relationship between the learner and the situation). Figure 1 summarises their connections between types of knowing. *Knowing-how* refers to being able to carry out a process as in knowing how to convert a fraction to a percentage. *Knowing-that* refers to understanding of concepts as in knowing that a fraction and its percentage describe the same amount. *Knowing-to* recognises affordance, the opportunity to act, as in *knowing-to* convert a percentage to a fraction to simplify a calculation.

Mason and Spence’s model that an individual can only act on affordances if they potentially see themselves acting resonates with experiences in the Early Numeracy Research Project (Clarke & Cheeseman, 2000). Interviewers found that students who believe they cannot solve a given problem often successfully do so when asked, “But if you could solve the problem what would you do?” (Personal communication with Doug Clarke, 16 October, 2000).

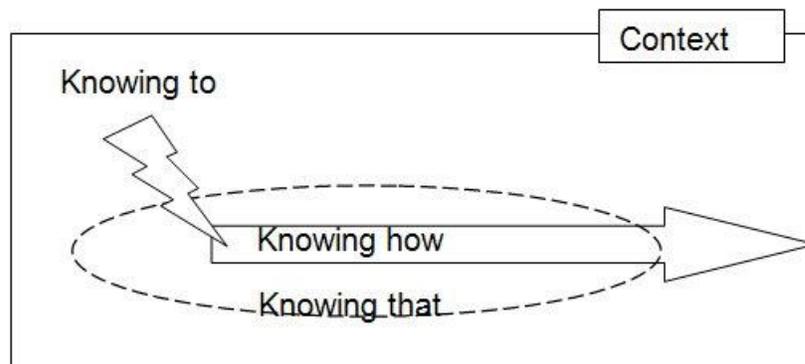


Figure 1: Connections between types of knowing (Mason & Spence, 1999)

The view of structure in situations as personally constructed resulted in some researchers considering a broader view of successful transfer. Lobato (1997) focused on transfer as the “personal construction of similarity” and found that finer grained analysis of transfer by learners revealed more than that expected using traditional research methodology. She added the significance of transfer as socially mediated through personal interaction and tools. Other situated theorists studied the role of social interaction between the learners, peers and teachers, and with the external environment in framing transfer (Engle, 2006).

Researchers in both the situated and actor-orientated genres usually adopt methodologies that involve clinical interviews with a small number of learners (Levrini & DiSessa, 2008; Lobato & Siebert, 2002; Wagner, 2006). The fundamental difference between these methods and those in the traditional genre is the focus on in-depth analysis of the process, of constructing new knowledge. This view assumes a broader view of evidence of transfer and assigns a greater significance to the role played by learners’ prior knowledge, attitudes and dispositions in new learning. Greeno, Smith, and Moore (1993, p. 100) wrote:

In the view of situated cognition, we need to characterise knowing, reasoning, understanding and so on as relations between cognitive agents and situations, and it is not meaningful to try to characterise what someone knows apart from situations in which the person engages in cognitive activity.

Like the traditional view of transfer, situated and actor-oriented views are not without paradox. A belief that knowledge is highly situated seems paradoxical to the observation that people conceive of and manipulate abstract ideas free of situations that lead to creation of the ideas. Situated learning theories are themselves examples

of abstractions, the properties of which people think upon without necessarily considering the seminal situations. If similarity of structure is so much determined by personal attendance, as actor-orientated perspectives suggest, then why do different people construct such similar views? The next section seeks to rationalise the apparent paradoxes in traditional and situated/actor-oriented views of transfer.

#### 2.3.4 MIDDLE GROUND

Debate over theory often results in polarized opinions that dismiss the possibility of middle ground. Such is the case with transfer. The paradoxes discussed previously suggest there is need of some compromise in the traditional and actor-orientated perspectives. Theorists generally accept that transfer happens because people learn. They also accept that people create concepts that are abstract ideas though there is disagreement about the process of abstraction. Traditional theorists view abstractions as applying to situations and actor-oriented theorists see abstractions as being created through and connected with situations.

Frey (2008) discussed the differences and similarities between the two views of transfer along four dimensions; experimental methodology, epistemology, role of subjects and definition of successful transfer. He argued that the classification of traditional transfer was too simplistic. Frey cited a range of historical studies that address the limitations of laboratory type experiments. These studies include classroom learning of children, examinations of the contribution of mistakes to learning, extended learning times, and examining the role of context.

I argue that it is more helpful to conceptualise traditional transfer, the body of literature from 1970 – 2000, as representative of a wide-ranging collection of frameworks, theoretical perspectives, experimental methodologies, epistemologies, models for subjects' involvements and measurements of successful transfer. (p.30)

Schwartz, Bransford, and Sears (2005) also called for a broadening of outlook towards research on transfer. They argue that traditional research transgresses mainly in the way it measures transfer. The researchers distinguish between transfer-out of situations (from one situation to another) and transfer-in situations (transfer in meeting the demands of a new situation). The focus on transfer out of situations frequently provides little evidence of success and makes people look “dumber” (Detterman, 1993).

Schwartz, Bransford, and Sears (2005) suggested the use of preparation for future learning (PFL) assessment to measure what people learn through transfer in situations. Their ideas suggest overcoming the paradox of little transfer in traditional studies with a redefinition of transfer and the way it is measured to include learners' partial connection of new knowledge in new situations, and dispositions and skills associated with creative thinking. They proposed a model for education aimed at the creation of flexibly transferred knowledge. While accepting the importance of efficiency in freeing cognitive resources for transfer they balanced this with a need for innovation. Their construct of an optimal adaptability corridor (see Figure 2) applies to both educational programmes and the hypothesised effect on learners in terms of their perception of mathematics as an activity. The significance of Schwartz,

Bransford, and Sears' theory is that transfer must be considered and measured not just in terms of transfer between situations that are structurally similar but also in terms of the conceptual connections formed, often partial, and the flexibility of thinking shown by learners in unfamiliar situations.

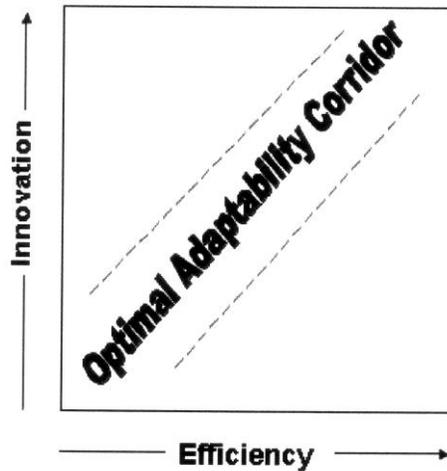


Figure 2: Optimal adaptability corridor (Schwartz, Bransford, & Sears, 2005)

The weight of theoretical opinion in the last decade leans towards situated or actor-orientated views of transfer. With that view go research methodologies based around in-depth analysis of how individual learners construct knowledge within a given domain. Traditional large-scale pre-test and post-test studies give powerful summative insight about large samples of learners with the objective of generalisation to populations. However, these studies shed little light on knowledge construction as a process. Therefore, this study adopts a design research methodology in an attempt to document the process of knowledge construction by individuals and the factors that may have led to its occurrence. The methodology does not imply acceptance or rejection of particular theories of learning. Design research develops theory through analysis of data in an iterative way in which the emerging theory and research design have dual influence (Baumgartner et al., 2003; Schoenfeld, 2006).

In the next section diSessa's broader theory of co-ordination classes is described (diSessa, 2008). Key sub-constructs in the development of rational number understanding (Kieren, 1980, 1988, 1993) and multiplication and division of whole numbers (Greer, 1992, 1994) fit the criteria for co-ordination classes. This makes co-ordination class theory useful as explanatory of learners' construction of knowledge in this study and their attempts to transfer in and across situations.

## 2.4 Co-ordination Class Theory

diSessa (2008) argued that debate about theories of conceptual development had reached an impasse that could only be resolved with agreement on the grain size of analysis. He saw traditional views inextricably connected to coherent views of cognition in which abstractions are the vehicle for transfer between situations. This contradicts diSessa's knowledge in pieces view of conceptual development in which large numbers of small elements must be co-ordinated to create a new concept. There is no detachment of the contextual connection of the knowledge elements in the modification of concepts. In fact, contextual knowledge elements are part of the resources available to the learner in meeting the situational demands. Abstraction is the result of transfer between situations, not the cause of it. diSessa's work suggests that the paradox of co-ordination class theory being an abstraction in itself is overcome by the fact that all theory of the in pieces genre is linked to the situations through which it is created. In pieces writing is full of exemplification of theory through case studies of individual learning stories.

diSessa (2008), and diSessa and Wagner (2005) highlighted problems of definition of the term concept in the literature. They described a co-ordination class as a particular type of concept. A well-developed co-ordination class enables the individual to project the concept onto a diverse range of contexts in the world. They did not claim co-ordination classes to be the only type of concept possible and offer phenomenological primitives (p-prims) as another type of concept. However, diSessa (2002) argued for the significance of co-ordination class as a ubiquitous type of concept. It is difficult to see any concepts other than those related to perceptually obvious phenomena, e.g. dropped objects fall, that do not require co-ordination of multiple pieces of knowledge.

Attention to small grain size of the knowledge elements is central to co-ordination class theory. diSessa argued that conceptual change can only be understood through attention to the sub-conceptual knowledge elements that are used by learners in the construction of co-ordination classes. There may be sharing of elements across classes. The interest of in pieces researchers is the organisation of knowledge elements in the creation of specific classes. Given an acceptance that concepts are mentally constructed it seems obvious that the learner personalises them. Emerging concepts seem big concepts to the learner at the time of construction and draw on little concepts that are already developed. However, little concepts now may have appeared to be big concepts when they were first constructed. A process of connecting multiple items of knowledge must be involved in the construction of big concepts. Co-ordination classes are big concepts for the learner at the time of construction. So learning of concepts has a connected web aspect to it as well as a hierarchical aspect as co-ordination classes are in turn co-ordinated to form more sophisticated concepts (Hiebert and Carpenter, 1992).

A corollary hypothesis is that the development of a co-ordination class is evolutionary rather than revolutionary as the learner attempts to co-ordinate knowledge elements productively and non-productively over extended periods. Contradictory applications and co-ordinations of knowledge elements may co-exist in

at any given time as the learner projects the concept onto different situations. Learner responses are not random. They reflect the prior knowledge of the learner, their attention to features of the situation and their cueing of knowledge elements perceived as relevant (diSessa, 1993). However, to an observer learners' strategies appear inconsistent and in pieces.

Cueing priority is the probability of the learner engaging a knowledge element or elements in a particular situation. Feedback to the learner, from social or phenomenological sources within the environment, establishes and reinforces a priority order in the activation of knowledge in particular situations. New knowledge that has no history in application naturally has a low cueing priority at first until its utility is established through application (Pratt & Noss, 2002). diSessa saw cueing priority as a better explanation of why learners change their preferences than the idea of perturbation provoking such change (Wheatley, 1992). diSessa (2008, p. 44) outlined the difficulty of conceptual change.

The ultimate problem is that multiple changes in the contextuality of multiple elements must all be co-ordinated to create a plausible successor to old ways of thinking.

Co-ordination class theory has considerable potential to inform learning and teaching since it is specific to particular knowledge domains and focuses on a smaller grain size of knowledge elements than broad theories. It is the day-to-day efforts of learners to co-ordinate old pieces of knowledge in response to new situations that of interest to teachers.

The inadequacy of old knowledge for new situations is prevalent in multiplicative thinking and proportional reasoning, the topic of this thesis. For example, learners inappropriately apply additive thinking to the side lengths of similar triangles under enlargement (Hart et al., 1981) and whole number size relations to numerators and denominators when ordering fractions (Pearn & Stephens, 2004a). If co-ordination class theory is applicable to this study evidence of situational inconsistency should emerge. Students' responses should be valid in terms of their views at the time. Co-ordination class theory should also provide insight into how learners successfully create new concepts through co-ordinating old knowledge.

#### **2.4.1 STRUCTURE OF CO-ORDINATION CLASSES**

The theory around the structure of co-ordination classes describes some commonalities of function and architecture while insisting that the nature of the knowledge elements and the process of co-ordination are specific to the concept or concepts under construction.

diSessa and Wagner (2005) provided several main features of co-ordination classes which they claimed as useful in interpreting the process of constructing concepts (of that type). They described the main intrinsic difficulties of creating a well-developed class as span and alignment. Span is the range of contexts that the learner recognises as applicable to the concept. Alignment is the obtaining of the same concept-appropriate information in different situations, even given the invocation of different knowledge elements.

Difficulties in learning co-ordination classes arise from inadequate span, in which the learner has insufficient knowledge resources to apply the concept across a wide range of situations, or to poor alignment where the learner is not able to determine the same concept details reliably across situations. This is similar to Skemp's model of present and goal states (Skemp, 1971). Inability of the learner to solve a particular problem is due to either an inadequate scheme with which to solve it or selection of an inappropriate scheme. An inadequate scheme may involve lack of significant pieces of knowledge or inability to navigate from present to goal state in a given situation. These difficulties correspond to lack of span and poor alignment respectively in co-ordination class theory. Poor alignment also corresponds to inappropriate scheme selection.

The relationship between transfer and span is reciprocal. Transfer between situations requires an increase in awareness of the range of situations to which a co-ordination class is applied, and in turn increases span or relevance for the class. Simpson (2009) described this relationship as a hand over hand growth in interpretive knowledge. The implication for this study is that the presence of strong co-ordination classes involves the learner in seeing similarity across variable situations. The corollary for instruction is that learners are encouraged to attend to the similarities and differences between situations in a way that focuses on the desired patterns and structures to the exclusion of other variables.

diSessa and Wagner (2005) also provided two features of co-ordination classes that relate to architecture, function and process. They gave two key functions, readout and inference, that are involved in the successful application of a concept to a given situation. Readout strategies involve attention by the learner to appropriate situation conditions. The causal or inferential net involves the learner in translating the readout information into action. To do so, the learner must map the readout data to the defining information of the concept in order to act on it. This seems similar to Skemp's (1979) idea of appropriate scheme selection. diSessa (2002) hypothesised that many problems of transfer can be attributed to dysfunction in the inferential net. In Skemp's terms this seems similar to navigation between present and goal state and implies the importance of scaffolding in instruction when learners have difficulty.

Two processes are involved in the construction of a co-ordination class, incorporation and displacement. Both processes involve the use of prior knowledge. Incorporation is the employment of prior knowledge for the partial construction of a new co-ordination class and displacement is the displacement, not dismissal, of prior knowledge where it is not applicable to the new concept. This classification of processes acknowledges that prior knowledge is significant in the construction of new co-ordination classes and that this knowledge can both assist and inhibit this construction.

Co-ordination class theory predicts both assistance and inhibition from old knowledge in the construction of new concepts. It raises the issue of how a learner recognises the usefulness of old knowledge or adapts old knowledge to meet new demands. There is considerable potential in this study for phenomenological primitive (p-prims), such as "taking some away leaves less" to be items of old

knowledge in need of adaptation for multiplicative thinking and proportional reasoning to develop.

#### 2.4.2 PHENOMENOLOGICAL PRIMITIVES OR PERCEPTS

While concepts are conceived ideas that are imposed on real world objects there are also ideas that occur naturally to people through their actions on objects. Imagine a collection of some kind. More objects are added to the collection. Therefore the collection has more. If a person likes the objects this is a positive outcome. If they dislike the objects, the outcome is negative. “Putting more in creates a bigger collection” may be referred to as a percept (Tall, et al., 2000) or a p-prim (diSessa, 1993).

diSessa identified p-prims in his research on the understanding of forces by undergraduate students. P-prims exhibit the following characteristics:

- Small and plentiful;
- Natural or data driven (experiential) not deduced;
- Feeling obvious and natural, therefore not a source of cognitive conflict;
- Working by recognition, may be used in some situations but not in others;
- Not articulated or used to explain anything.

Physics examples include “things go in the direction you push them” (force as a mover), “equal efforts cancel out” (dynamic balance), and “out of balance systems return to equilibrium” (return to equilibrium). The significance of p-prims is the uses of these primitive knowledge elements in the construction of co-ordination classes. Since p-prims are natural, elicited directly from phenomenological experience, and do not invoke conflict they must be reorganised rather than replaced.

Of significance to this study is that p-prims may play a key role in the development of students’ misconceptions in multiplicative thinking and fractional numbers. “Multiplication makes bigger” and “division makes smaller” (Fischbein, 1999; Fischbein, Deri, Nello, & Marino, 1985) are over-generalisations in multiplicative thinking that may develop from p-prims such as, “adding more makes bigger” and “cutting up makes smaller parts”. Resnick and Singer (1993) identified proto-quantitative reasoning of five and six year olds as arising naturally from everyday experiences. They identified fittingness and direction of co-variation; for example, bigger people need bigger clothes, as examples of proto-quantitative reasoning. These ideas appear to be candidates for classification as p-prims. Resnick and Singer argued that instruction did not utilize the potential of proto-quantitative ideas. It is equally true that these same ideas have the potential, if not modified, to inhibit the development of more sophisticated ideas. McGowen and Tall (2010) used the term *met-befores* to describe prior knowledge that appears obvious to the learner. *Met-befores* are trusted in the same way as p-prims but are broader than knowledge resulting from phenomena. McGowen and Tall cited examples of *met-befores* interfering with learners development of concepts.

### 2.4.3 APPLICATIONS OF CO-ORDINATION CLASS THEORY TO MATHEMATICS

The origins of co-ordination class theory lie in physics education. The description of co-ordination classes fits very well with significant concepts in mathematics. For example, a concept of counting whole numbers of objects requires the co-ordination of a large number of knowledge elements. These elements include recall items such as number sequences and symbol to word associations. Other elements involve knowledge of invariance of process such as the order of counting objects or the spread of objects does not affect the final count.

Other researchers have applied diSessa's theory to the field of mathematics and statistics education (Izsak, 2005; Simpson, 2009; Wagner, 2003, 2006). Izsak's, Wagner's and Simpson's work involves case studies of individual students. In each study students participate in teaching interviews over extended periods of time. The researchers adopt a grounded theory perspective of adaptive interviews in which they elicit learner responses and build up theories from the data.

Wagner (2003, 2006) gave a detailed description of Maria's change in thinking about the law of large numbers. Maria showed susceptibility to variations in problem type, situational conditions and contexts. Her available knowledge influenced her perception of the situations. This supports a prediction of situational variability in early learning of a co-ordination class. Wagner documented how Maria gradually comes to perceive situations as similar in some way thereby creating a complex network of similar problems. Her history of solving problems cued certain knowledge resources as relevant and transferable across the situations. He concluded:

What we see in a particular situation is largely a function of what we expect to see, although our expectations are not necessarily conscious ones. As a consequence of this, seeing complex structures within a given situation depends on having complex expectations – not necessarily ready-made structures previously stored that are retrieved in their entirety, but complex associations of descriptive and explanatory knowledge resources that have proven to be mutually supportive in other circumstances.

(Wagner, 2006, p. 63)

Simpson (2009) described the development of a concept of negative numbers with two eight and nine-year-old learners. She found productive and unproductive use of experiential knowledge about the world in the construction of mathematical concepts, e.g. sand equals hot, colder so more clothes. As anticipated, new knowledge is cued more readily only once it acquires an established track record across situations. The conceptual maps in Simpson's study grew over time as learners made more connections between knowledge elements. She concluded that abstraction is not a necessary requirement for transfer between situations but through transfer, and articulation that is more precise, concepts become more abstract. This suggests that co-ordination class theory tells much about the process for creating concepts that later become objects for further thought. Simpson adds that concepts are continually evolving and never complete.

Izsak's work centred on fifth graders representations of arrays as discrete arrangements of dots and rectangular areas. He documented the learning trajectory of two students as they come to differentiate the representations, co-ordinate dimensions

and totals, and evaluate the appropriateness of each representation. Izsak identified the selection of measurement attributes, and the use and evaluation of representations as key knowledge elements that the students learned to co-ordinate as their understanding developed.

There are several implications of these applications of co-ordination theory to mathematics. Research that seeks to describe concept construction should involve case studies of individual students to document the process of construction at a knowledge element level. It should occur over a prolonged period to allow construction to occur. The role of researcher should be to illuminate learners' strategies for solving problems and to elicit knowledge through provoking the learner to think differently. Such is the nature of this study.

## 2.5 Implications of Transfer for Instruction

A classroom over the course of one year was the setting of this research. Alongside close examination of the process of concept construction by learners, there is need for some discussion about the features of instruction that are associated with the process. A review of the literature suggests attributes of learning environments that support learners to achieve transfer of learning.

Schwartz and Varma (in press) provided four elements of learning environments that facilitate innovation. These features are:

- **Distributed cognition** meaning that information is held by members of a group or through inscriptions to lessen individual memory load and allow access to multiple pieces of information simultaneously;
- **Generation of alternative interpretations** meaning that divergent perspectives from individuals are encouraged and supported;
- **Multiple candidate structures** meaning that judgment is suspended as many possible solution paths are explored;
- **Focal point for co-ordinating efforts** meaning that endeavour is structured towards a common goal.

Schwartz and Varma (in press) also reflected a common theme in transfer research when they highlighted the importance of multiple opportunities for learners to discern commonalities and differences across situations. diSessa (2002) also supported this view but added that the situations must be carefully chosen to elicit different kinds of concept projection. Many theorists point out the need to move students' attention from personal episodic description of what they do in situations to a focus on images of pattern and structure (Gray, Pitta, & Tall, 2000). Pattern refers to consistency or regularity, and structure refers to the organisation and relationships of elements that are described by the pattern (Mulligan & Vergnaud, 2006).

Mancy (2010b) applied the findings of generalisation research from psychology in advocating the use of examples, rules and prototypes as facilitators of transfer. She found little support in psychology for exclusively rule based models. Her research

within a computer environment indicates that multiple examples and rules work together in assisting learners to transfer. Schwartz and Varma (in press) reported superior transfer from students who were free to solve unfamiliar problems before being shown prototypical solutions compared to students who were taught using rule based methods or those who attempted the problems without later exposure to a solution strategy. This matches studies of Japanese teaching methodology that puts students in problem solving situations before exposing them to prototypical solutions presented by peers or their teacher (Stigler & Hiebert, 1999).

The work of Gary and Tall (2001) suggested an instructional focus on both anticipation of the result of a process and on using the process as an object for further thought. This connects to object theory as a long-run view of learning. Anticipation of the result may be considered as *condensation* in Sfard's terms and using the process as an object as *reification* (Sfard, 1991). The implication is that expecting anticipation and object use leads to students doing so. This powerful application of object theory warrants attention.

Gray and Tall (2001) saw the connection of physical embodiments and symbols as critical tools in the development of procepts, symbol forms that encompass process and object. Folding back to physical phenomena, or base objects, seems fundamental to verification of ideas in the process of conceptual change (Pirie & Kieren, 1994).

Our observations of human activity reveal that the 'encapsulated object' is not simply produced by 'encapsulation' or 'reification' of process into object, but is greatly enhanced by using the configuration of the base objects involved as a precursor of the sophisticated mental abstraction. (Gray & Tall, 2001, p. 71)

Fractions, ratios, decimals and percentages seem ideal candidates for being procepts. Of interest are the varying processes that need to be embodied by students in symbols and how embodiment or lack of it impacts on concept development. What role do symbols play in assisting transfer, if at all? As Gray and Tall suggest, do symbols give students tags that enable them to talk and think about concepts? Chapter Five discusses features of the instructional environment that were significant in facilitating transfer in this study.

## **2.6 Considering Both Theories of Learning and the Implications**

This chapter describes two main models for conceptual change and transfer. The power of models is not in their truth but in their utility in describing and explaining some phenomenon, in this case learning. Transfer must occur in learning and also is an indicator that learning has occurred.

Object theories take a long term, broad view of learning that begins with a process of acting on things, real or imagined. Anticipation allows learners to think about the process as an object of thought. This object of thought becomes a new thing to act on and so learning becomes more sophisticated.

If object theories are useful, they should explain concept change at a macro-level and possibly indicate disparity in progress between learners in the study. There should be

evidence of learners projecting existing concepts onto new situations that is not directly dependent on them seeing the new situation like old ones. Students should be able to think of former processes as objects. The derived instructional implications such as focusing student attention on anticipation of process and application of object should prove fruitful.

The weight of recent opinion favours actor-orientated perspectives on transfer that take a closer view of learning at a personal level that is idiosyncratic and variable in the short-term. Learner perception is dependent on old knowledge, attitudes and dispositions, and mediated through social interaction. Learners' responses to situations are variable not simply because the situations are different. Individuals structure situations through the lens of previous experience. So the same situation is assigned different structure by different learners.

Co-ordination class theory holds promise as a model of conceptual change. Concepts or sub-constructs such as fraction, ratio, rate, quotient, and operator fit the description of co-ordination classes. These concepts involve the co-ordination of multiple knowledge elements. They also share knowledge elements. Understanding the rational number field as a co-ordination class requires integrated understanding of the sub-constructs (Kieren, 1980, 1988, 1993). Transfer made in one situation that informs the development of a concept or sub-construct can bootstrap the development of another concept if connections are made.

For co-ordination class theory to be applicable in this research, the following observations need to hold:

- Attention to the grain size of knowledge elements is essential to understanding concept development;
- Deep learning takes considerable time dependent on the preparedness of learners' old knowledge for the new concept/s;
- Learners show a richness of existing ideas and the interaction between these ideas and their perception of situations contributes to variable learning paths;
- Learners hold competing models at any given time mainly due to the activation of p-prims;
- Meta-cognitive views influence the affordances learners see in situations and thereby transfer.

Object theory and co-ordination class theory are different models for describing conceptual development (di Sessa, 2008). Given their support in the literature it is likely that both models are informative. An investigation into the efficacy of both models requires in-depth analysis of how learners construct concepts in and between situations. In keeping with recent investigations of co-ordination class theory this research uses design research methodology and use case studies of individual learners as the format of data presentation.

## 2.7 Relevance of Ontology

This chapter proposes ideas about how learning occurs and how transfer facilitates and reflects that process. Ontology, the structure of knowledge domains, informs our understanding of knowledge that may be transferred. The following two chapters summarise the literature on multiplicative thinking and proportional reasoning. They provide the ontological basis for this thesis.

Chapters Three and Four provide detailed descriptions of hypothetical learning trajectories in concept development for the two domains of multiplicative thinking and proportional reasoning. Learning trajectories, as identified by research, allow for an analysis of the knowledge resources required for conceptual growth and the types of transfer that are necessary.

The literature also provides structure in each domain. Theorists provide multiple taxonomies of problem types, unit structures and contexts, and provide rich descriptions of students' conceptions and misconceptions. This detail informs the focus on co-ordination classes as an explanatory model in the following ways:

- Misconceptions alert us to problems in span and alignment related to learners' read-out strategies and inferential nets, and the possible influence of p-prims;
- Problem types and contexts provide information about the span of concepts and situations;
- Conceptual field theory offers frames for identifying similarities and differences that should be attended to across situations.

The thesis now considers the corpus of work on multiplicative thinking and proportional reasoning. Proportional thinking involves application of multiplicative thinking so there is considerable connection between the two domains. As well as informing the theoretical framework of the thesis this part of the literature review also guided the classroom instruction in this study. I developed a Hypothetical Learning Trajectories (HLT) for multiplication and division, and proportional reasoning. Development of programmes of instruction occurred in a responsive way, from evaluation of students' actual learning in relation to the HLT.

## CHAPTER THREE: MULTIPLICATIVE THINKING

### 3.1 What is Multiplicative Thinking?

To define multiplicative thinking as the ability to solve problems that involve multiplication and division is simplistic and erroneous. Students use a range of strategies including skip counting and repeated addition to solve multiplication and division problems (Eil, 2005; Jacob, 2001; Mulligan & Mitchelmore, 1997; Steffe, 1994). The nature of the thinking, not the problem, is the issue. Application of the properties of multiplication and division across diverse situations is multiplicative thinking.

Clark and Karmii (1996) differentiated additive from multiplicative thinking by considering the quantity relationships involved. Diagrammatically they showed additive thinking as:



Figure 3: Model of additive thinking for  $5 + 5 + 5 = 15$

Each number in the situation has an identical referent, for example with “Five lollies and five lollies and five lollies equals 15 lollies”. The numbers five and 15 are counts and the label attached to the numbers, lollies, is a referent. Numbers in equations disguise the fact that quantities are amounts of a referent unit. The ellipses in the diagram represent the explicitly nested number sequence (Steffe & Cobb, 1988). Counts simultaneously measure the cardinality of a set and are partitionable into other composite counts.

In contrast, Clark and Kamii (1996) represented multiplicative thinking using an example of Figure 4.

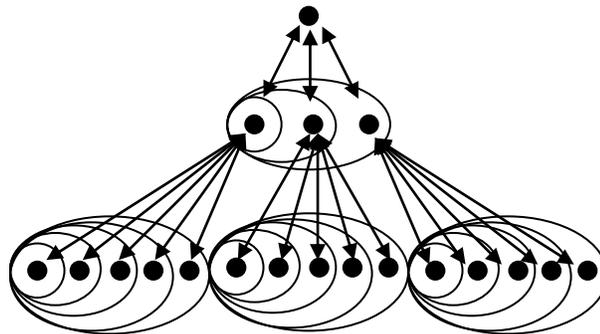


Figure 4: Model of multiplicative thinking for  $3 \times 5 = 15$

Two levels of abstraction are involved. Firstly, sets of five singletons became composite units. The counting of these composites is co-ordinated so they became

iterable units (Steffe, 1994) that are replicated a given number of times. Secondly, several inclusion relationships exist including potential partitioning of the sets of five, for example into sets of three and two, and partitioning of the number of sets, for example into two sets of five and one set of five.

Multiplicative thinking involves creation of the multiplicand, in this case the composite units of five, and the multiplier, the number of sets. In multiplication situations the numbers involved, three and five, relate to different referents, the number of sets and the objects in each set respectively. Greer (1992) suggested the different roles of the numbers as one reason for students' difficulty with learning multiplication.

The complexity of co-ordinating different types of number units or measures is a common theme in the literature. Steffe (1994) and Clark and Kamii (1996) portrayed multiplication as the efficient quantification of a set's cardinality through the co-ordination of iterative units. Davydov (1992) described multiplication as a transfer of count. He rejected established methods of teaching multiplication by repeated addition of equal sets, and saw multiplication as the measurement of some magnitude using a small measure exercised through use of a larger, intermediate measure.

To illustrate Davydov's point consider three containers, A, B, and C.

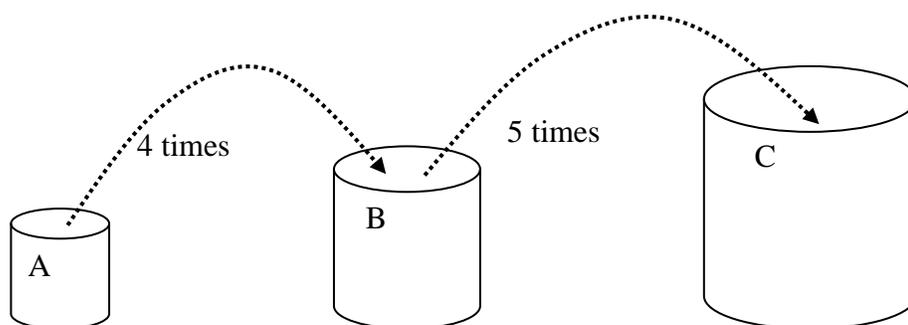


Figure 5: Model of Davydov's transfer of count

B acts as an intermediate measure. Four units of A fill B. Five units of B fill C. Finding the measure of C in A units occurs indirectly, using multiplication of  $4 \times 5 = 20$ , rather than directly through the counting of singleton units. According to Davydov, multiplication involved the creation of transitive relationships between measures. Boulet (1998) supported Davydov's concept of multiplication and demonstrated its consistency when applied to negative integers, rational and irrational numbers.

### 3.2 Properties of Multiplication and Division

Multiplicative thinking begins with the ability to construct both multiplier and multiplicand. Co-ordination of these two numbers into one product means that multiplication is a binary operation. Four properties apply to multiplication (Ell, 2005, p. 30).

Commutative Property	$p \times q = q \times p$
Associative Property	$p \times (q \times r) = (p \times q) \times r$
Distributive Property	$p \times (s + t) = (p \times s) + (p \times t)$
Inverse/reversibility	$p \times q = u$ so $u \div p = q$ and $u \div q = p$

The property of inverse means that division undoes multiplication and vice versa, i.e.  $p \times q \div q = p$ . The other properties do not hold for division.

Commutative Property	$u \div q \neq q \div u$
Associative Property	$u \div (q \div r) \neq (u \div q) \div r$
Distributive Property	$u \div (s + t) \neq (u \div s) + (u \div t)$
Inverse/reversibility	$u \div q = p$ so $p \times q = u$ and $q \times p = u$

Multiplicative thinking involves application of these properties of multiplication and division. Students apply these properties as *theorems in action* usually without explicit awareness or formal representation (Vergnaud, 1998). Expressing the operations and their properties algebraically disguises the complexity of the situations to which they are applied and the psychological complications for students in learning to apply them (Greer, 1992).

While the commutative, associative and distributive properties do not hold for division they are present in division strategies through the inverse relationship to multiplication. For example,  $24 \div 4 = 6$  so  $24 \div 6 = 4$  by the commutative property and  $24 \div 4 = (20 \div 4) + (4 \div 4)$  by the distributive property.

Two main research approaches to multiplicative thinking are prevalent in the literature. The conceptual field approach describes the mathematical structures involved, and the psychology approach describes students' misconceptions and the growth of their knowledge and understanding. Some studies bridge the two approaches.

### 3.3 Conceptual Field Theory

Research in the area of mathematical structure, known as conceptual field theory, focused on both the structure of the mathematics (Schwartz, 1988; Vergnaud, 1994), and on analysis of problem types (Fischbein, Deri, Nello & Marino, 1985; Greer, 1994). Work on multiplicative field theory was advanced by Vergnaud's classification of problems (Vergnaud, 1983, 1988, 1994, 1998).

Vergnaud defined the underlying structure of all problems for which multiplicative thinking was required. He regarded this as the "more fruitful approach to children's cognitive development" (Vergnaud, 1994, p. 41). His classification began with the structure of two or more measure-spaces,  $M_1, M_2, M_3, \dots$  etc., corresponding to the quantities and referents in multiplication or division problems. Classification of problem type was by both the relationship existing between the measure-spaces and the nature of the givens and unknowns. A summary of Vergnaud's classification is

given in Table 1. The fourth problem sub-type only appeared in Vergnaud (1994), the first three in his earlier papers (Vergnaud, 1983, 1988).

**Table 1: Vergnaud’s classification of multiplicative problem types**

<b>Problem Sub-type</b>	<b>Definition</b>
Isomorphism of measures (Simple proportion)	Direct proportional relationship between two measure-spaces, e.g. “Six oranges cost \$4.00. How much do 18 oranges cost?”
Product of measures (Concatenation of simple proportions)	Two measure-spaces mapped proportionally onto a third, e.g. “A sandpit is 6 metres wide and 8 metres long. What is the area of the sandpit?” or “Tyler has three pairs of shorts and four tops. How many different outfits can she make?”
Multiple proportion (Double proportion)	The required quantity-measure-space is proportional to two different and independent measure-spaces, e.g. “Ken has 12 calves to feed for 30 days. Each calf drinks two litres of milk per day. How much milk will the calves consume altogether in 30 days?”
Comparison of rates and ratios	The proportional relationships between two rates or ratios involving comparison of the measure-spaces, e.g. “Three boys share four pizzas and two girls share three pizzas. Who gets more pizza, a girl or a boy?”

The isomorphism of measures, product of measures, and comparison of rates and ratios types are most relevant to this study. Location of the unknown in the isomorphism of measure problems corresponds to common one-step multiplication or division situations. Consider the measure spaces, bags ( $M_1$ ) and lollies ( $M_2$ ), and the problem, “There are three bags. Each bag contains five lollies. How many lollies are there altogether?” There are two corresponding division problems arising from this context, quotative and partitive (Greer, 1994; Nesher, 1988).

Quotative division involves knowledge of the size of parts and requires finding the number of parts, e.g. “There are 15 lollies. Five lollies go in each bag. How many bags are there?” Partitive division involves knowledge of the number of parts and requires finding the size of the parts, e.g. “There are 15 lollies shared equally into three bags. How many lollies are in each bag?” Quotative division is sometimes referred to as division by measurement and partitive division as division by sharing.

Ratio tables represent multiplication and the two forms of division as scenarios in which the unknown is in a different cell (see Table 2). Vergnaud’s work on the multiplicative conceptual field highlighted that the numeric or algebraic expression of the multiplicative relationship  $3 \times 5 = 15$  or  $a \times b = c$  masked the complexity of the referents attached to the numbers 3, 5, and 15 or  $a$ ,  $b$  and  $c$  and the operational connections between multiplication and the two types of division.

**Table 2: Three one-step multiplication and division problems shown as ratio tables**

Bags( $M_1$ )	Lollies per bag( $M_2$ )	Bags( $M_1$ )	Lollies per bag( $M_2$ )	Bags( $M_1$ )	Lollies per bag( $M_2$ )
1	5	1	5	1	?
3	?	?	15	3	15
Multiplication		Quotative Division		Partitive Division	

Schwartz (1988) developed a classification system for multiplicative relationships based on the character of these referents. In addition and subtraction the quantities that are combined or separated have the same referent, be those quantities discrete (countable) or continuous (measurable). In this case, the referent is to one class or attribute of objects, lollies. Schwartz (1988, p. 41) labelled these number-referents as extensive quantities, and described addition and subtraction as “referent preserving compositions”.

Multiplication and division involved another type of number-referent relationship. Intensive quantities embody an operational relationship between two extensive quantities. For example, lollies-per-bag embodies a relationship between the number of lollies and number of bags, as kilometres-per-hour embodies a relationship between the number of kilometres and the number of hours. Lollies-per-bag and kilometres-per-hour are intensive quantities sometimes referred to as rates. Lollies, bags, kilometres and litres are extensive quantities. Table 3 shows Schwartz’s three types of multiplicative problem types. E refers to extensive quantities and I to intensive quantities.

**Table 3: Schwartz’s analysis of multiplicative problem type**

Type	Example
$I \times E = E'$	Three children (extensive quantity) have five lollies each (intensive quantity). How many lollies do they have altogether? (extensive quantity) $3 \text{ children (E)} \times 5 \text{ lollies-per-bag (I)} = 15 \text{ lollies (E')}$
$E \times E' = E''$	A parlour has four different flavours of icecream and three different toppings. How many different sundaes can you buy there? $4 \text{ flavours (E)} \times 3 \text{ toppings (E')} = 12 \text{ sundaes (E'')}$
$I \times I' = I''$	A heater uses 3 kilowatts of electricity per hour. Electricity costs 68 cents per kilowatt. What does it cost to run the heater each hour? $3 \text{ kilowatts per hour (I)} \times 68 \text{ cents per kilowatt (I')} = 204 \text{ cents per hour (I'')}$

Vergnaud and Schwartz’s taxonomies showed the increased complexity that multiplicative problems presented to students in comparison to additive problems through the variant nature of the measures involved. Behr, Harel, Post and Lesh (1994, p. 126) established two notational systems for describing the unit structures involved in number problems and a generic manipulative aid to represent the

structures. For example, the conception of five bags of three lollies as fifteen lollies in total was represented as:

Manipulative Aid	Unit Analysis
(OOO)(OOO)(OOO)(OOO)(OOO)	$1(3\text{-unit})+1(3\text{-unit})+1(3\text{-unit})+1(3\text{-unit})+1(3\text{-unit})$
((OOO)(OOO)(OOO)(OOO)(OOO))	$= 5(3\text{-unit})s$
(OOOOOOOOOOOOOOOO)	$= 1(15\text{-unit})$

The researchers undertook an exhaustive analysis of possible unit structures. This analysis had little impact on subsequent research and was criticized for failing to address the mathematics of learners and inadequately structuring more complex multiplicative tasks (Lamon, 2007). However, the work of Behr, Harel, Post, and Lesh illustrates the complexity of unit structures involved in multiplicative thinking and the critical importance of students being able to re-unitise quantities. Ironically re-unitising is revealed as a critical ability in Lamon’s research in proportional reasoning (Lamon, 2002).

### 3.4 Semantic Problem Types

Attention to the unit structures goes some way to explaining the complexity of multiplicative thinking but neglected the impact of context. The additional demands of semantic interpretation became a focus of research. Neshet (1988) described the means by which students mathematized the textual information contained in problems into operations. Interpretation of a word problem requires development of a text base, consisting of propositions, followed by creation of a situational model. Three semantically distinct types of problems were included in Neshet’s (1988, pp. 21-24) analysis.

**Table 4: Neshet’s three semantic problem types**

Problem Type	Characteristics
Mapping	Problems in which the multiplicand is mapped onto the multiplier, e.g. There are three bags (multiplier) with five lollies in each bag (multiplicand). How many lollies are there altogether? These problems can be modeled by equal additions.
Comparison	Problems involving application of a scalar referent is to a compared set or measurement, e.g. Jill has three times as many marbles (scalar) as Jack. Jack has five marbles (compared set or measure). How many marbles does Jill have?
Cartesian Product	Problems involving combination of two quantities to get a third, e.g. Tyler has three pairs of shorts and four tops. How many different outfits can she make?

Greer (1992, 1994) synthesised the taxonomies of Vergnaud, Schwartz and Neshet. In doing so, he differentiated multiplicative problem types by their measure-space, referent, and semantic structure and by the inclusion of discrete and continuous quantities (whole numbers, fractions, decimals). Greer (1994, p. 62) also questioned

the “degree of parsimony” in the previous classifications, and suggested that important psychological and pedagogical implications were not considered.

Schmidt and Weiser (1995, p. 56) supported Greer’s view and proposed a semantic analysis of one-step word problems “between the radical reduction to the arithmetic kernel on the one hand, and the possibly rich contexts of specific problem situations on the other”. Table 5 presents a synthesis of the taxonomies of Greer and Schmidt and Weiser. Double arrows indicate that the distinction between partitive and quotitive division is not applicable in those cases.

**Table 5: Analysis of multiplicative problem types**

Problem Class	Multiplication	Partitive Division (by the multiplier)	Quotative Division (by the multiplicand)
Equal Groups (discrete) or Measures (discrete with continuous)	There are three bags of lollies. Each bag contains five lollies. How many lollies are there altogether? You buy 3 kg of steak at a cost of \$8.60 per kg. How much do you pay?	15 lollies are shared equally into three bags. How many lollies go in each bag?  You pay \$25.80 for 3 kilograms of steak. What is the cost of the steak per kilogram?	There are 15 lollies. Five lollies are put into each bag. How many bags are filled?  You pay \$25.80 for some steak at \$8.60 per kilogram. How much steak do you get?
Rate (discrete or continuous)	Your car travels at a steady speed of 80 kilometres per hour for $2\frac{1}{4}$ hours. How far does the car go?	Your car travels 180 kilometres in $2\frac{1}{4}$ hours. What is its average speed?	Your car travels 180 kilometres at a steady speed of 80 kilometres per hour. How far does it travel?
Measure Scaling (discrete or continuous)	Six steps of Jack’s are the same length as one step of the Giant. How many Jack-steps measure the same length as five Giant steps?	Thirty Jack steps are the same length as five Giant steps. How many Jack steps measure the same as one Giant step?	Six steps of Jack’s are the same length as one step of the Giant. To measure the same length as 30 Jack steps, how many steps would the Giant take?
Multiplicative Comparison (discrete or continuous)	You have three times as many marbles as Mary. Mary has eight marbles. How many marbles do you have?	You have 24 marbles. That is three times as many as Mary has. How many lollies does Mary have?	You have 24 marbles. Mary has eight marbles. How many times more marbles do you have than Mary?
Multiplicative Change (discrete or continuous)	Your dog is eight times heavier than when it was a puppy. As a puppy it weighed 2 kilograms. How heavy is it now?	Your dog weighs 16 kilograms now. That is eight times heavier than when it was a puppy. How heavy was your dog as a puppy?	Your dog weighs 16 kilograms now. As a puppy, it weighed 2 kilograms. How much heavier is it now than then?
Rectangular Array (discrete) Rectangular Area (continuous)	A carpark has 8 rows of cars. There are 9 cars in each row. How many cars are in the park altogether? A rectangular berry patch is 4.6 metres long and 3.7 metres wide. What is the area of the patch in square metres?	There are 72 cars in the carpark. They are in eight equal rows. How many cars are in each row? A rectangular berry patch has an area of 17.02 square metres. It is 3.7 metres wide. How long is it?	There are 72 cars in the carpark. They are nine cars in each row. How many rows are there? A rectangular berry patch has an area of 17.02 square metres. It is 4.6 metres long. How wide is it?
Cartesian Product (discrete) Product of Measures (continuous)	There are four ways to travel from X to Y and three ways to travel from Y to Z. How many different ways can you travel from X to Z? A current of 3.4 amperes flows through a resistor of 4.6 ohms. What voltage is supplied?	There are 12 different way to travel from X to Z via Y. There are 3 ways to travel between Y and Z. How many ways can you travel from X to Y? A voltage of 15.64 volts is produced through a resistor of 4.6 ohms. What is the current, in amperes, flowing through the resistor?	There are 12 different way to travel from X to Z via Y. There are 4 ways to travel between X and Y. How many ways can you travel from Y to Z? A voltage of 15.64 volts is produced from a current of 3.4 amperes. What is the resistance, in ohms, of the resistor?

Some problem types offered by Greer, and Schmidt and Weiser were not mutually exclusive. For example, Greer (1992, p. 280) gave this problem as an example of product of measures.

“If a heater uses 3.3 kilowatts of electricity for 4.2 hours, how many kilowatt-hours is that?”

Extending Schwartz’s classification, this problem is of type  $I \times E = I'$  since 3.3 kilowatts per hour is an intensive measure. Therefore, it is possible to classify the problem as one of rate. Minor difficulties of mutual exclusivity aside, semantic problem type analysis further illustrates the complexity of situations to which multiplicative thinking is applied.

An important point of departure of Schmidt and Weiser’s classification from that of Greer was inclusion of a different problem type, described by Jacob (2001). Schmidt and Weiser (1995, p. 60) gave this problem;

“During his first year of life Otto the elephant trebles his weight at birth. In his second year he doubles his weight. Which multiple of his weight at birth has he got at the end of his second year of life?”

Separating problems of this type from those classified previously in this chapter is the absence of both quantities and referents. Solving these problems requires one to operate on the relationships between the quantities and referents without knowing them specifically. Collis (1975) referred to this as acceptance of “lack of closure”, a critical concept in the development of algebraic thinking.

### **3.5 Problem-Response Studies**

Other researchers investigated the difficulties children encounter in solving multiplication and division problems and suggested explanatory models (Bell, Greer, Grimison, & Mangan, 1989; Fischbein, Deri, Nello & Marino, 1985; Greer, 1994; Hart, et al., 1981). Hart *et al.* (1981), reporting on the strategies of 11 to 13 year olds, found that word problems presented as repeated addition are easier for students to solve than Cartesian product problems. Students frequently use addition-based strategies to solve multiplication problems, so difficulty with establishing the addends in Cartesian product situations explains the difference in difficulty.

Repeated addition and subtraction strategies are also common on division problems. Quotitive and partitive division appear to be of about equal difficulty but division as sharing proves easier than multiplication in some studies. Students find providing a word problem to match a symbolic expression difficult and many think reversed division expressions were of equal value, e.g.  $391 \div 23 = 23 \div 391$ . Hart proposed a three level progression in problem solving ability:

- Subtraction ;
- Recognition of multiplication and division;
- Construction of multiplication and division.

The percentage of students at level 3 increased from 40% for 11 year olds to 60% for 13 year olds suggesting that the development of multiplicative thinking took considerable time. Hart’s findings need consideration alongside recent research showing older students’ tendency to apply multiplicative thinking more readily than younger students even inappropriately in situations that are additive in nature (Van Dooren, De Bock, & Verschaffel, 2010).

Further enhancement of problem type analysis came from attention to other task variables. Fischbein *et al.* (1985) investigated why students changed their minds about the operation required in response to changes in the numerical conditions of the problem. In this research, students at grades 5, 7, and 9 were required to select the correct operation from a menu that modelled given multiplication and division word problems. Students tended to choose incorrect operations when decimals were involved, particularly as multipliers and divisors.

The researchers proposed that each arithmetic operation linked to a primitive intuitive model. For multiplication, the intuitive model was repeated addition, for partitive division it was sharing, and for quotative division it was repeated subtraction. These primitive intuitive models remained “rigidly attached to the concept long after the concept has acquired a formal status” (Fischbein et al., 1985, pp. 5-6) and mediated and constrained students’ selection of operations. Fischbein and colleagues hypothesized that primitive intuitive models accounted for students’ perception of constraints applying to multiplication and division problems. Their findings are consistent with the constructs of percepts (Tall, et al., 2000) and p-prims (diSessa, 1993) in that the students derive their beliefs from direct actions on physical objects. Harel and Behr (1994, p. 2) summarised these primitive constraints.

**Table 6: Harel and Behr’s summary of intuitive constraints**

Multiplicative situation	Intuitive constraints
Multiplication	1. Multiplier must be a whole number 2. Multiplication makes bigger
Partitive division	1. Divisor must be a whole number 2. Divisor must be smaller than dividend 3. Division makes smaller
Quotative division	1. Divisor must be smaller than dividend

Vergnaud (1988) referred to the underpinning mathematical relationships that students consider when making operational choices as *theorems in action*. In the language of co-ordination class theory students read out information from the situation and infer what mathematical knowledge may be useful in meeting the demands (diSessa & Wagner, 2005).

The hypothesis of intuitive primitive models proved overly simplistic. Bell, Greer, Grimson, and Mangan (1989, p. 447) found that choices of operation were more directed by the perceived ease of calculation and on perceptions about the “size-changing properties of operations” than on underlying conceptual structures. Harel,

Behr, Post, and Lesh (1994) found that the divisor less than the dividend constraint was not as robust as posited and found the involvement of other problem variables in strategy selection, including context, syntax, and rule violation.

These contradictory findings were also supported by De Corte and Verschaffel (1994) who required 10, 12 and 14 year old students to construct word problems for given numerical calculations. Other researchers detected a broader range of primitive models including repeated building up strategies for partitive division (Kouba, 1989; Mulligan & Mitchelmore, 1997). Sherwin and Fuson (2005, p. 354) rejected the hypothesis that strategies were determined by underlying conceptual structures asserting that changes were driven by students' access to "number-specific computational resources".

The debate crosses over the views about abstraction that were discussed in Chapter Two. Processes as objects theory supports a view of abstraction as detached and applied. This aligns with a view that primitive models explain students' strategy choices. Co-ordination class theory attributes some primitives to observed actions on physical objects. The theory also anticipates considerable variation in strategy choice influenced by students' available knowledge resources.

### **3.6 Models for Progression**

The difficulties encountered by students with multiplicative thinking mirror the historical development of mathematics (Sfard, 1991). This perspective is a common starting point for creating Hypothetical Learning Trajectories in teaching experiments (Gravemeijer, 2001). Significant changes in thinking about concepts that prove troublesome for students have been described as cognitive obstacles (Herscovics, 1989).

Historically multiplication and division have presented difficulties both in the remembering of basic facts and in calculation. This resulted in cultures finding ways to simplify the cognitive demands of the operations through the creation of algorithms and devices such as finger multiplication, Egyptian and Russian doubling methods, and the invention of logarithms (Seaquist, Padmanabhan, & Crowley, 2005).

Many researchers observed the problem solving behaviour of students and described progression in the sophistication of strategies (Anghileri, 1989; Cooney, Swanson, & Ladd, 1988; Kouba, 1989; Lefevre et al., 1996; Lemaire & Siegler, 1995; Mulligan & Mitchelmore, 1997; Steffe, 1994). Their work was predominantly restricted to students solving problems involving multiplication and division basic facts, i.e. up to  $10 \times 10 = 100$  and  $100 \div 10 = 10$ . Synthesis of this research shows that students progress through broad developmental phases of count-all to composite counting/repeated addition to known products and/or derived facts.

Sherin and Fuson (2005) cautioned that student strategy selection was extremely variable being heavily influenced by the values of the operands, classroom experiences particularly in regard to requisite knowledge, and contextual familiarity. Ell (2005), in her work with students on harder multiplication and division tasks,

reported similar strategy variability. The research into early progressions in multiplicative thinking is summarised in Table 7.

**Table 7: Developmental stages in early multiplication**

<b>Stage Researchers</b>	<b>Count-all</b>		<b>Composite counting Repeated addition</b>		<b>Known product Derived fact</b>	
(Kouba, 1989)	Direct representation (with physical objects)		Additive	Transitional counting	Recalled number facts	
(Anghileri, 1989)	Unary					
	Unitary counting	Rhythmic counting	Number pattern		Binary Known fact	
(Steffe, 1994)	Initial number sequence		Tacitly nested number sequence		Explicitly nested number sequence	
(Lemaire & Siegler, 1995)	Counting set of objects	Repeated addition			Retrieval	Rapid responses
(Lefevre, et al., 1996)			Repeated addition	Number series	Retrieval	Derived fact
(Mulligan & Mitchelmore, 1997)	Unitary counting (direct counting)	Repeated addition			Multiplicative calculation	
		Rhythmic counting	Repeated adding, additive doubling	Skip counting	Known multiplication fact	Derived multiplication fact
(Sherwin & Fuson, 2005)	Count-all		Additive calculation	Count-by	Learned product	Pattern-based, Hybrid

The congruence of findings suggests a generalised progression from unitary counting to forming additive composite units to co-ordination of both multiplier and multiplicand. There is some divergence of opinion at the highest stage of these frameworks. Some researchers promote knowledge of basic multiplication facts while others see a combination of knowledge and derivation as the arrival of multiplicative thinking. Sherin and Fuson (2005) believed that students' computational vocabulary merges into a rich, multiplicative structure for the whole numbers up to 81 and strategies become increasingly difficult to differentiate as the structure develops.

### **3.7 Connections to Counting or Historical Hegemony**

While mathematically multiplication is a binary operation, some researchers believe psychologically its emergence involves complex co-ordination of counting schemes (Steffe & Cobb, 1998). These schemes develop from co-ordination of unitary counting to the forming of composites to the co-ordination of iterable units. Nantais and Herscovics (1990) demonstrated that conservation of plurality, the development of a part-whole counting scheme, was the key determinant of readiness for multiplication.

Further support came from brain research that showed humans have an innate counting scheme but advanced mathematical abilities rest on symbolic notations,

words, and the arduous learning of computational algorithms (Dehaene, 1997; Feigenson, Dehaene, & Spelke, 2004). Number sense, as opposed to quantifying, involves the co-ordination of functions located in different modules within the brain. The congruence of findings on the early development of multiplicative thinking suggests that initially students view multiplicative situations not as unique but as a subset of those in which counting and additive thinking are applied.

In short, the abstraction of iterable units is based on counting by ones; on iterating the unit one (Steffe, 1994, p. 52).

Other researchers questioned whether the traditional counting to additive to multiplicative curriculum reflected real progression in student thinking or the teaching effect of established hegemony. Confrey and Maloney promoted splitting or equi-partitioning as an alternative mechanism for the growth of multiplicative thinking (Confrey & Maloney, 2010). Sophian and Madrid suggested emphasis of unit structures other than one in early mathematics learning (Sophian & Madrid, 2003). While it is debatable as to which situations best provide the genus for multiplicative thinking there is no doubt that the exercise of determining quantities ultimately requires reference to, and co-ordination of, singleton units.

### **3.8 Beyond Elementary Multiplication and Division**

The frameworks for the early development of multiplicative thinking fail to explain the transition from an understanding and knowledge of basic multiplication facts to applications of the properties of multiplication and division that are more complex. A gap exists between the research on simple multiplicative thinking and the significant volume of work on proportional reasoning (Lamon, 2006).

Siemon et al. (2006) developed a Hypothetical Learning Trajectory (HTL) for multiplicative thinking based on Rasch analysis of assessment items. Their sample of 3200 Australian students in year four to year eight provided strong evidence for the relative difficulty of tasks. Differentiation of the upper levels of the HTL showed increasing sophistication in applying and communicating the number properties of multiplication and division with whole numbers.

Tasks requiring quotitive division were more difficult than those involving partitive division and the researchers hypothesised a strong link between division of whole numbers and understanding of fractions, decimals and percentages. Development of the scales for Progressive Achievement Tests in New Zealand showed similar progression in the application of multiplicative part-whole strategies and the connection between multiplicative thinking and the emergence of fraction concepts (Darr, Neill, & Stephanou, 2006).

Jacob (2001) and Jacob and Willis (2003) proposed at least five broad stages for the development of multiplicative thinking, including two stages that extended those of the early frameworks:

1. One-to-One Counting;
2. Additive Composition;
3. Many-to-One Counting;
4. Multiplicative Relations;
5. Operating on the Operator.

Stages 1 – 3 of Jacob and Willis' framework corresponds approximately to the Count All, Composite Count and Known or Derived Facts stages of the earlier frameworks. Evidence of a student working at the Multiplicative Relations stage is their use of the commutative, distributive and inverse properties of multiplication and division. This suggests that students were able to form factors and use invariance properties, such as doubling and halving.

Operating on the Operator was a term used by Piaget and Inhelder (1962) to describe reasoning about operating on the result of an operation without having to reconstruct the operation itself. In this sense, the operations of multiplication and division became objects of thought rather than actions. Accepting operations as not finished was characteristic of formal operational thought and allowed access to qualitative reasoning about the effects of operations (Post, Behr, & Lesh, 1988). Jacob and Willis (2003) described this stage of multiplicative thinking as one in which students treated the numbers in a problem situation as variables.

Students at the Operating on the Operator stage transform quasi-variables (Fuji & Stephens, 2001) with recognition of variant or invariant effect (Harel & Behr, 1990). For example, if asked to provide dimensions of other cuboids that have the same volume as the cuboid in Figure 6 these students perform transformations on the factor dimensions rather than calculate the volume. Fuji and Stephens (2001) referred to this as relational thinking.

According to Zazkis and Campbell (1996) the ability to use operators as objects for further thought develops through generalisation from specific examples and involves a process stage where deductive arguments are supported by empirical verification. It is likely that Operating on the Operator thinking develops in transformation-specific ways rather than into a uniform encompassing stage. In other words, any encounter with multiplication and division, however simple, is seen or not seen by the students as opportunity to generalise about properties of the operations.

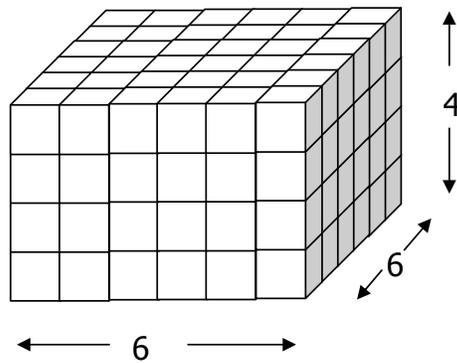


Figure 6: Cuboid dimension problem

### 3.9 Types of Division

Division involves a relationship between three quantities, a number of things to be divided (dividend), the number to divide by (divisor), and the result (quotient). The divisor can take two forms, the number of shares, in partitive division, or the quantity of the shares, in quotitive division. The choice of divisor dictates whether the quotient tells the quantity of each share, in partitive division, or the number of shares, in quotitive division.

The operation  $24 \div 6 = 4$  represents two different physical actions, sharing and measuring, as shown in Figure 7. In partitive division, the action of allocating a singleton to each part involves removing a set from the dividend that is equal in cardinality to the number of parts. Similarly, in quotitive division, removal of a set of given cardinality from the dividend is identical in effect to giving a singleton to that number of parts.

Thompson and Saldanha (2003, p. 30) say that students' appreciation that the same equation models both partitive and quotative division hinges on development of an anticipatory scheme for both sharing and forming equal sets.

When students understand the numerical equivalence of measuring and partitioning they understand that any measure of a quantity induces a partition of it and any partition of a quantity induces a measure of it.

It is not clear which division scheme, partitive or quotative, is the most intuitive to young students. Real life situations frequently demand sharing but school experiences, such solving  $24 \div 6 = \square$ , described as "How many sixes are in 24?", promote measurement. Young students prefer to use counting or additive build-up strategies for division (Mulligan, 1992) but make up sharing, rather than measuring, scenarios for given division equations (Silver, Shapiro, & Deutsch, 1993). Downton (2009) found little difference in the performance of eight and nine year old students on partitive and quotative division problems across a range of semantic types. Neuman (1999, p. 104) suggested, "Children think of both kinds (of division) as partitive, but deal with both kinds in a quotative way". Correa, Nunes, and Bryant

(1998) hypothesised a strong link between subtraction and the anticipatory scheme for quotative division.

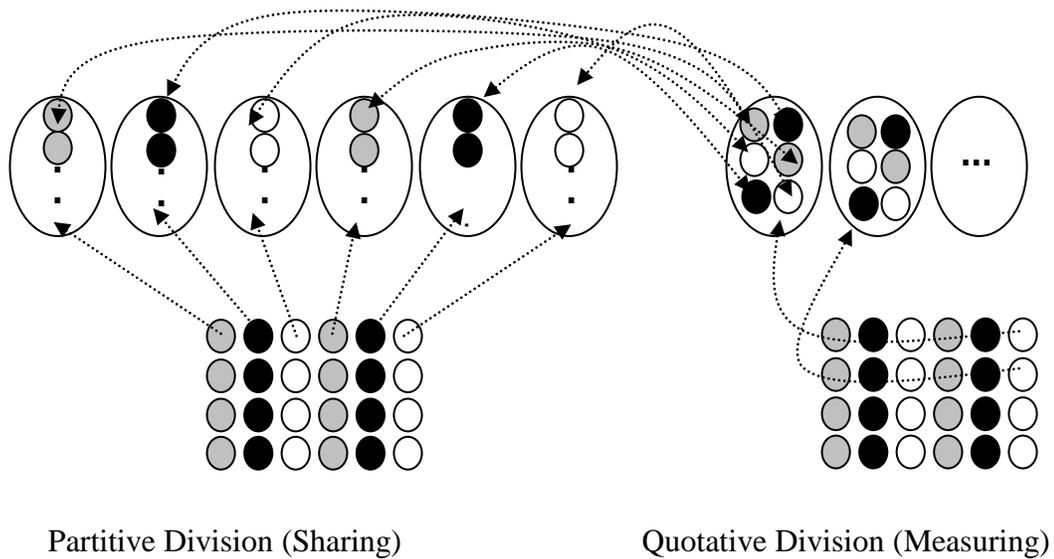


Figure 7: Links between partitive and quotative division

### 3.10 Difficulty of Division

The memorisation of division facts as a means to anticipate the results of division actions is problematic. Robinson, Arbutnott, Rose, McCarron, Globa, and Phonexay (2006) found that searching for a corresponding multiplication fact was the preferred strategy of Grade 7 students for solving division problems. They also noted that this multiplication reversal strategy was resistant to age effect (Robinson, Arbutnott, & Gibbons, 2002). Preference for the mediated retrieval strategy of reversing multiplication may lie in the difficulty of remembering division facts. This difficulty is due to learning history, division facts learned after addition, subtraction, and multiplication facts, and the interference with accurate recall caused by associative memory (Dehaene, 1997; Rickard, Healy, & Bourne, 1994). The inability to operate with and from division facts has clear implications for students' ability to find factors of given numbers, an important conceptual foundation for proportional reasoning (Kaput & West, 1994; Woodward, 2006; Young-Loveridge, 2006).

Research on students' ability to solve more difficult division problems is limited. The research points to considerable teaching effect on the success of students on complex division tasks. Anghileri (2001, p. 86) noted a reliance on standard algorithms among her sample of year 5 English students and found that only one-half of the algorithmic calculations resulted in correct answers. She suggested deferring the teaching of algorithms in favour of development of students' intuitive strategies through a process of "progressive schematisation".

Progressive schematisation is a principle of Realistic Mathematics Education in the Netherlands. It involves deliberate structuring of learning contexts with the intention of inviting students to re-invent key specific strategies and to abbreviate their strategies into more efficient forms (Beishuizen & Anghileri, 1998). Implicit in the approach is that students engage in reflective abstraction.

A comparative study of success with solving various computational problems showed that nine and ten year old Dutch students outperformed their English counterparts, particularly on division (Anghileri, Beishuizen, & van Putten, 2002). The recording of Dutch students was more structured allowing them to apply repeated “chunking” strategies successfully. In contrast, English students often unsuccessfully attempted a standard algorithm. Ruthven (1998) highlighted the importance of number fact and pattern knowledge and the refinement of students’ written recording strategies as critical to reliable division calculation. He reported English students’ tendency to invent convoluted, error-prone strategies, or to apply incorrect procedures to standard algorithms, thereby creating *malgorithms*.

Silver, Shapiro, and Deutsch (1993) found that students’ completed presentation of division algorithms masked the personal jottings they made in the process. These jottings frequently involved contributory counting, additive build-up, and multiplication calculations. The evidence suggests that the cognitive load involved in complex divisions is such that students need efficient ways to record their strategies. A written method that supports successive chunking, within a quotative division scheme, is easier for students to adopt successfully than a standard algorithm based on place value partitioning.

Anghileri, Beeishuizen, and van Putten (2002, p. 167) suggested this is because “Learning is most effective where written methods build upon pupil’s intuitive understanding in a progressive way”. Place value partitioning for division is difficult for students to interpret and apply, as are strategies involving partitioning of the dividend and/or divisor that leave the quotient invariant (Wright, 2003).

Studies into students’ ability to solve division problems with remainders revealed that interpretation of the remainder required a two-way mapping between the problem context and the mathematical calculation. Only 25% of American and Chinese students could interpret remainders in contextually correct ways despite their general algorithmic competency (Cai & Silver, 1995; Silver, et al., 1993). This result was hardly surprising given the difficulty of alternative representations of remainders from division, e.g.  $27 \div 4 = 6 \text{ r } 3$  or  $6 \frac{3}{4}$  or 6.75, the possibility of rounding answers, and the mapping of the possible interpretations to the demands of a given context.

### **3.11 Pedagogical Issues**

Traditional approaches to multiplication and division begin with the relationship between repeated addition and multiplication (Confrey & Smith, 1995; Davydov, 1992). This instructional approach reflects the counting and additive roots of students quantifying strategies for multiplication problems. However, students often cling to conceptions of multiplication as repeated addition, and division as sharing or repeated

subtraction (Hart, 1983) and this reliance is detrimental to their transfer of understanding to the variety of situations to which multiplication is applied, such as scaling and rates (Greer, 1992). Some researchers question the preordained developmental link between counting and multiplication (Confrey & Smith, 1994; Sophian & Madrid, 2003).

Alternative approaches to teaching multiplication emphasize the significance of unit size as well as count in determining quantity (Confrey & Maloney, 2010; Davydov, 1992; Sophian, 2004; Sophian & Madrid, 2003). The approaches explicitly deal with primitives associated with counting, i.e. higher count means more, and are multiplicative in the sense that students construct and co-ordinate the multiplicand (unit size) and multiplier (number of units).

Confrey and Smith (1994) and (1995, p. 42) advocated use of a splitting construct. They defined splitting as “an action of creating similar multiple versions of an original, an action which is often represented by a tree diagram”. This description applies to situations involving replication, similarity (scaling), and order of magnitude as well as partitioning. Essentially the approach is to begin with a unit (one) and act on it with successive  $n$ -splits, as opposed to beginning with composite units of singletons. A 3-year teaching experiment using splitting approaches resulted in significant learning gains, particularly for lesser prepared students (Confrey & Lachance, 2000).

Another important pedagogical issue considered by researchers was the relationship between students’ conceptual understanding and their computational proficiency. A consistent message was the significance of number fact knowledge balanced by a flexible approach to calculation strategies. Graveiimeijer and van Galen (2003) considered the memorisation of multiplication and division facts and the development of reliable algorithms for calculation to be essential. They criticised traditional presentation of preferred algorithms to students as non-negotiable procedures, favouring an approach that began with students’ attempts to solve contextual real problems. Guided re-invention of reliable algorithms arose from processing of students’ methods.

Fuson (2003) discussed the strong connection between conceptual and procedural aspects of learning and maintained that understanding was critical to accurate calculation. She promoted the use of accessible algorithms that allowed students to make successive approximations through partial products and quotients. Her recommendations were similar to those of Anghileri’s (2004) suggestion of using a chunking division algorithm.

### **3.12 Tools and Other Representations**

Many researchers highlighted the role of tools in facilitating conceptual and procedural knowledge of multiplication and division (Beishuizen & Anghileri, 1998; Cobb & Bowers, 1999; Lesh & Behr, 1987). Tools include words, symbols, diagrams, and physical representations of situations and mathematical concepts. Recording methods that allow for a diversity of students’ strategies, such as empty number lines

(Ruthven, 1998) and ratio tables (Fuson, 2003), result in more reliable calculation than formal uni-structured algorithms.

Researchers stressed the importance of students making connections between various representations in order to understand the transformations on quantities conveyed through words and symbols (Lesh & Behr, 1987; Levensen, Tirosh, & Tsamir, 2004). Two schools of thought exist about the attributes of physical or diagrammatic representations. One school attends to the properties of the representations themselves, sometime referred to as the epistemic fidelity of the representation in terms of its illustrative power. For example, Greer (1994) gave four criteria for the selection of powerful representations:

1. Applies to a wide range of problems (Generic);
2. Requires application of thought that is isomorphic to the conditions of the problems;
3. Allows for a progression in invented strategies from the naïve to the sophisticated;
4. Is compatible with other representations.

An alternative view is that representations only have illustrative power if learners grant it. Transparency of instructional devices lies not in the properties of the objects isomorphism to the target concept (Meira, 1998). Rather the learner makes connections between materials and concepts through use. Use of representations is mediated through sociocultural practices.

An example of the contrasting views about representations is the debate about use of arrays. From the epistemic fidelity perspective, arrays embody the binary nature of multiplication and connect to other mathematical constructs, like measurement of area and volume and Cartesian products. However, the multiplicative structure of arrays seems to be a later attribute attended to by students (Outhred & Mitchelmore, 2004) and depends strongly on their concepts about units of measurement and tessellation.

Transparency suggests that the process of abstracting multiplicative composition appears not to be a simple matter of imaging dynamic action on physical materials. Rather, understanding that an array embodies two factors involves classification by the student that a given situation can be modelled multiplicatively (Sullivan, Clarke, Cheeseman, & Mulligan, 2001). Students who can see the multiplicative structure of arrays appear to map pre-existing knowledge about iterative counting onto the spatial arrangement (Steffe & Cobb, 1998). Neuman (1999) found that students' internal cognitive construction of divisor and quotient in partitive division problems was manifest in external tools, such as diagrams, that resembled array-like structures. It appears that arrays are useful representations of the binary nature of multiplication,

and its invariance properties once students have the ability to construct factors (Fuson, 2003).

### **3.13 Summary and Hypothetical Learning Trajectory**

Multiplicative thinking is difficult for many students to learn and for teachers to teach. Most students enter school with informal knowledge that supports counting and additive schemes (Sophian & Madrid, 2003) and proto-quantitative ideas about proportional reasoning (Resnick & Singer, 1993). Students need to re-conceptualise their counting schemes to include multiplicative relationships between quantities as well as additive relationships (Clark & Kamii, 1996; Steffe, 1994).

Students often apply additive thinking in situations where multiplicative thinking is required and vice versa (Hart, 1983; Van Dooren, De Bock & Verschaffel, 2010). Teachers also apply the same primitive intuitive models for multiplication and division resulting in inappropriate modelling for students (Harel, Behr, Post & Lesh, 1994). Yet, multiplicative thinking remains foundational to the development of important mathematical concepts such as place value, proportional reasoning, and rate of change, measurement, and statistical sampling (Batturo, 1997; Mulligan & Watson, 1998; Siemon, Izard, Breed & Virgona, 2006).

Some theory about multiplicative thinking is about mathematical structure alone but not about how students interpret the mathematical structure. Researchers have talked at cross-purposes as one camp has described the structure of the mathematics through elegant models and the other camp has described the psychological complexity associated with how students interpret multiplicative situations. Few researchers have undertaken teaching experiments to connect students' thinking to the structure of mathematics (Confrey & Lachance, 2000).

Many things conspire to make multiplicative thinking difficult. Common usage of the words multiply and divide is suggestive of making bigger and making smaller (Greer, 1992). Multiplication and division are operations that apply to a wide variety of problem types in which the relationships between quantities and the referents of those quantities take on many different forms (Greer, 1992). Many of the problem types do not correspond to an equal groups view of multiplication yet the psychological origins of multiplicative thinking appear to be in the forming of additive units (Steffe, 1994). This led Jacob and Willis (2003, p. 460) to claim, "Multiplicative thinking cannot be generalised in any simple way from additive thinking".

Memorisation of basic multiplication and division facts and of place value structures places a burden on learners yet this knowledge is essential for efficient calculation (Woodward, 2006; Young-Loveridge, 2006). The human brain has no specific place to store all the knowledge in symbolic form and relies on verbal association to facilitate recall (Dehaene, 1997). The development of procedural and conceptual understanding of multiplication and division is a continually changing marriage in which new ideas become old objects for further thought (Anghileri, et al., 2002). Ell (2005, p. 49) summarised the nature of multiplicative thinking.

Rather than a static place to be arrived at, multiplicative thinking seems to be an ongoing process of change.

The challenge of this thesis was to describe the process of change, particularly as multiplicative thinking becomes the basis for operations on fractions, ratios and proportions. In an effort to do so a Hypothetical Learning Trajectory (HLT) was constructed for the growth of multiplication and division of whole numbers. The trajectory described key ideas in three phases. Given the age of the students in the study a phase of one-by-one counting was not included. Research quoted in each cell of the trajectory was exemplary and did not include all of the references discussed in this chapter.

**Table 8: Hypothetical Learning Trajectory for multiplication and division of whole numbers**

Whole Number Operations	Counting on and simple additive partitioning	Complex additive and emerging multiplicative	Complex multiplicative
Multiplication	Skip counting and repeated additive composition (Kouba, 1989)	Deriving multiplication facts using distributive property and doubling/halving (Mulligan & Mitchelmore, 1997)	Deriving multiplication answers using all properties of multiplication (Wright, 2003)
	Hybrids based on known facts and counting/adding (Sherwin & Fuson, 2005)	Known multiplication facts (Anghileri, 1989)	Relational thinking about multiplication (operating on the operator) (Jacob & Willis, 2003)
Division	Equal sharing using composite counts or additive build-up (Mulligan, 1992)	Division by multiplicative build-up (Robinson, et al., 2006)	Known division facts (Ruthven, 1998)
			Connection of partitive and quotative division (Thompson & Saldanha, 2003)
			Division by connecting multiplication properties and inverse (Siemon, et al., 2006)

## CHAPTER FOUR: PROPORTIONAL REASONING

### 4.1 What is Proportional Reasoning?

Some researchers view proportional reasoning from the perspective of a conceptual field theory, others view it in terms of how students learn and interact with the ideas. Both groups of researchers seek to increase understanding of proportional reasoning and many researchers work across the two perspectives. The literature reveals consistent divergence between mathematical structure and how learners perceive situations. Learners frequently fail to see the common proportional structure which applies to different situations.

Proportional reasoning involves a multiplicative comparison between quantities. So working with proportions involves multiplicative thinking. A quantity is a measurable attribute of an object though in some situations, often referred to as qualitative reasoning; it is possible to reason about quantities without carrying out any measurement. Measured quantities are composed of both numbers (counts) and referents (units) attached to them. For example, \$45.00 contains a number, forty-five, and a referent, dollar. Referents can be extensive, referring to an individual attribute of objects, such as mass or length, or intensive, referring to a relationship between attributes, such as density (mass across volume) or speed (distance over time).

The mathematical structure of a linear proportion is an equality between two terms of the form  $\frac{a}{b} = \frac{c}{d}$ , where  $a$ ,  $b$ ,  $c$ , and  $d$  are integers ( $b$  and  $d \neq 0$ ). The algebraic simplicity of this definition disguises many semantic and psychological dimensions of proportional reasoning as viewed from a learner's perspective. To see the equality relationship in two equivalent ratios, for example, requires recognising the constants of proportionality between and within the ratio pairs (Lamon, 2006). If  $a:b$  and  $c:d$  are equivalent then  $\frac{a}{c} = \frac{b}{d}$  and  $\frac{b}{a} = \frac{d}{c}$ .

$\frac{a}{b}$  and  $\frac{c}{d}$  appear in ratio contexts as relationship but are also rational numbers and so are members of an infinite quotient field (Kieren, 1993, p. 4). This means that there are an infinite number of names for the same rational number and an infinite number of numbers between any two rational numbers. For learners this marks a considerable cognitive shift from and co-ordination with the set of integers.

Implicit in the definition of a rational number is the theorem,  $a \div b = \frac{a}{b}$ , which will be called the *quotient theorem* in this thesis. While algebraically  $a \div b$  and  $\frac{a}{b}$  are alternate representations, in origin they are not.  $a \div b$  is a process of division, either partitive (sharing) or quotative (measuring), and  $\frac{a}{b}$  is a number with an ordinal and cardinal place between and with members of the infinite set of rational numbers. Theorems have their origins in actions on objects (Vergnaud, 1998). Partitive and quotative division are quite different operations in terms of the corresponding actions

on physical objects. The connection by learners of the two forms of division with fractions as quantities is unlikely to be a simple matter of transferring mathematical structure (Thompson & Saldanha, 2003).

As briefly mentioned above, the definition of a proportion applies to ratios and rates as well as to fractional numbers. From a structural perspective ratio in the Gaussian tradition is a binary relation between pairs of numbers, irrespective of the referents attached to the numbers (Lesh, Post, & Behr, 1988b). For learners the referents in a problem are not irrelevant variables but are viewed as situational conditions of the problem.

Consider the example below:

The instructions for cooking a chicken in the microwave specify 10 minutes on high power for every 500 grams of chicken.

How long should you cook a 2.5 kilogram chicken for?

This problem involves two binary relations. One relation is the conversion between measurement units, kilograms to grams, and the other is the relation between mass and cooking time. These relations can be represented in two ratio tables using Vergnaud’s classification of multiplicative structures (Vergnaud, 1983; 1988). Structurally both relations may be called isomorphism of measures.

**Table 9: Ratio tables for chicken cooking problem**

Kilograms	Grams	→	Mass (g)	Time (min.)
1	1000		500	10
2.5	2500		2500	?

Solving for the unknown cooking time is typical of missing value proportion problems that have three values for two rate or ratio pairs given and the fourth value sought. Two main types of multiplicative strategy exist, within or between the measure spaces (Lamon, 1993, 1994; Van Dooren, et al., 2010). Most of the research literature defines multiplicative comparisons that treat the measure spaces separately as either within, scalar, or extensive strategies (Lamon, 2006; Lesh, Post, & Behr, 1988a). Multiplicative strategies that relate the different measure spaces are defined as between, functional, or intensive. Table 10 exemplifies these different strategies applied to the chicken cooking problem.

**Table 10: Within and between strategies for the chicken cooking problem**

Mass (g)	Time(min.)	Mass (g)	Time (min.)
500	10	500	10
2500	?	2500	?

In this problem a within strategy resulting in  $10 \times 5 = 50$  minutes of cooking time is easier computationally than  $\frac{1}{50} \times 2500 = 50$  minutes. In the Gaussian tradition, two *rates* exist between mass and cooking time,  $\frac{1}{50}$  minutes per gram and 50 grams per minute. These rate numbers are reciprocals (Kaput & West, 1994). Location of the unknown within the ratio pairs determines which rate is applicable. For learners the location of the unknown and the choice of numbers and has a significant effect on problem difficulty (Lesh, Post & Behr, 1988a). Problems involving integral multipliers are considerably easier for learners than those involving fractional multipliers (S. Alatorre, 2002; S Alatorre & Figueras, 2004, 2005; Steinhorsdottir, 2005).

Use of the terms rate, ratio, within and between is not uniform across research. Karplus, Pulos, and Stage (1983) used between to refer to strategies connecting numbers in the same scientific systems (measure spaces). Within describes strategies across scientific systems. This thesis will use Lamon's (2006) definition of within and between strategies, described previously, rather than the Karplus *et al.* definition.

The ancient Greek tradition distinguished ratios and rates from the Gaussian definitions (Lesh, et al., 1988b). In the Greek tradition, a rate was a binary relation between two numbers in different measure spaces, and a ratio related numbers in the same measure spaces. So 240 kilometres in 3 hours is an example of a rate, 1 litre of concentrate to 4 litres of water is an example of a ratio. In this thesis, rates are distinguished from ratios, in the Greek tradition. While rates and ratios are isomorphic structurally, students often do not see them as such.

## 4.2 What is involved in Proportional Reasoning?

The common consensus is that proportional reasoning is essential to the understanding of mathematical and real-world concepts and is foundational to further study of mathematics. Proportional thinking underpins key mathematical ideas such as place value, percentages, decimals, measurement, rates of change, trigonometry, algebra, statistical inference and probability (Norton, 2005). It applies to a wide range of everyday contexts including enlarging documents, reading scale maps, duplicating recipes, calculating best deals, predicting outcomes and fair sharing (Dole, 2010). The understanding of many scientific concepts such as measure conversion, balance, pulleys and gears, consumption, density, and speed also depends on proportional reasoning.

Post, Lesh and Behr (1988a) and Lamon (2006) provided comprehensive lists of abilities demonstrative of proportional reasoning. These abilities are to:

1. Express the meanings of quantities and variables using the constant of proportionality in a variety of contexts. This involves the ability to mathematise a situation into a proportional reasoning model, carry out the transformations required to the quantities and referents, and map the mathematical solution to the original problem, with a critical awareness of the reasonableness of the solution. Use of the constant of proportionality assumes that both scalar and functional relationships within and between the rate pairs are accessible to the learner in integral and non-integral form.
2. Distinguish situations for which proportional reasoning is an appropriate model from those for which another mathematical model is required. An example of this is that under enlargement of an object by factor  $k$ , lengths change proportionally but areas change by a factor of  $k^2$  and volumes change by factor  $k^3$  (De Bock, Van Dooren, Janssens, & Verschaffel, 2002). Model appropriateness is known to be problematic for learners. Van Dooren, *et al.* (2010) described the age-related confusion of students between additive and multiplicative situations. Cramer, Post, and Currier (1993) documented the difficulty students had in distinguishing functional relationships of the form  $y = mx$  from those of  $y = mx + c$ .
3. Use and connect multiple representations. These representations include relations (functions), tables, graphs and diagrams. This also includes the connections between fractions, ratios/rates, decimals and percentages in which fractions and decimals come to be regarded as specific types of ratios (Lachance & Confrey, 2002). Students need to associate the slope of the graph of a proportional situation with the constant of proportionality and the intercept at the origin with the rate pair (0,0). They also need to recognize that the graph of an inverse proportion is a hyperbola.
4. Reason qualitatively as well as quantitatively about proportional situations. Qualitative thinking involves having a sense for the direction of change. Lamon (2007, p. 631) provided Table 11 for directional changes in a cookie sharing (quotient) context. These directions of change apply across proportional situations including changes to the numerators and denominators of rational numbers, including fractions. There is evidence that students' qualitative reasoning is not divorced from their quantitative reasoning in that the size of the numbers involved and the magnitude of change in the numbers have considerable impact on success, particularly in ambiguous change situations (Post, Behr & Lesh, 1988).

**Table 11: Directional changes in proportional reasoning situations**

Change in quantity <i>cookies per person</i>			
Change in number of cookies	Change in number of people		
	+	-	0
+	?	+	+
-	-	?	-
0	-	+	0

5. Use the language of proportionality. This requires students to have the mathematical register to express concepts and relationships using specialized and formal vocabulary. This vocabulary acts as both a tool of expression and a thinking tool through which complex ideas are developed (Vygotsky, 1962).

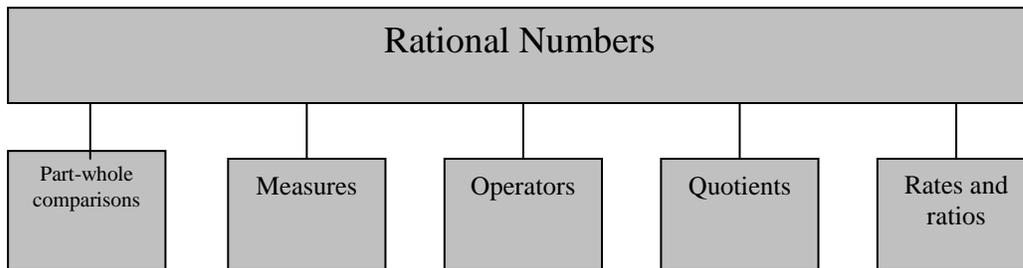
In summary Lamon (2006, p. 13) defined proportional reasoning as:

... supplying reasons in support of claims made about the structural relationships among four quantities, (say  $a, b, c, d$ ) in a context simultaneously involving covariance of quantities and invariance of ratios or products; this would consist of the ability to discern a multiplicative relationship between two quantities as well as the ability to extend the same relationship to other pairs of quantities.

Lesh, Post and Behr (1988b, p. 2) described proportional reasoning as “both the capstone of elementary arithmetic and the cornerstone of what is to follow”. They considered proportional reasoning as fundamental to the development of algebraic thinking in that it required students to engage in mathematical modelling in its simplest form, to work with statements of equivalence, to identify structural similarity across situations, and to work with variables in multiple forms, including as specific unknowns and fixed constants.

### 4.3 Kieren’s Sub-constructs for Rational Numbers as a Structure for Proportional Reasoning

Subsuming all proportional reasoning under the algebraic generic,  $\frac{a}{b} = \frac{c}{d}$ , neglects the multiple personalities of rational (fractional) numbers, the variety of situations to which they apply, and their subtle nuances and connections (Lamon, 2001). Kieren (1980; 1988; 1993) gave five sub-constructs for rational numbers shown in Figure 8. Kieren’s work was unusual in that he sought to explain observations of the thinking of learners in structural terms. The sub-constructs remained relevant to proportional reasoning research over a period of 30 years possibly because they provide a useful bridge between proportional reasoning structures and situations. In this thesis the sub-constructs were used to develop the instructional programme and to analyse students’ learning.



**Figure 8: Kieren's five constructs of rational number**

Kieren described a construct as a product of thinking that people tested against their experience using protocols. He did not suggest that the sub-constructs were independent of one another or complete but posited that they combined to create a generalized understanding of rational numbers. The sub-constructs build upon three intuitive constructs of partitioning, equivalencing quantitatively, and creating dividable units that were applied by very young students to solve fraction problems (Kieren, 1980; 1988; 1993; Resnick & Singer, 1993).

The literature is often ambiguous in describing the relationship between rational number and proportional reasoning. Understanding rational number requires the ability to reason proportionally, that is to work with multiplicative relationships within and between measure spaces. The great appeal of Kieren's model is that it makes explicit the core conceptual structures people use when applying rational number to situations. These situations invariably involve proportional reasoning. This thesis uses the terms rational number and fractions interchangeably which is a liberty. Fractions are typically defined as a subset of rational numbers,  $\frac{a}{b}$ , where  $a$  and  $b$  are positive integers,  $b \neq 0$  (Lamon, 2007). Students in this research worked with fractions.

### 4.3.1 PART-WHOLE

The part-whole sub-construct involves relating a part to a whole or vice versa. The whole was either continuous, as in an area or volume, or discrete, as in a countable set of objects.

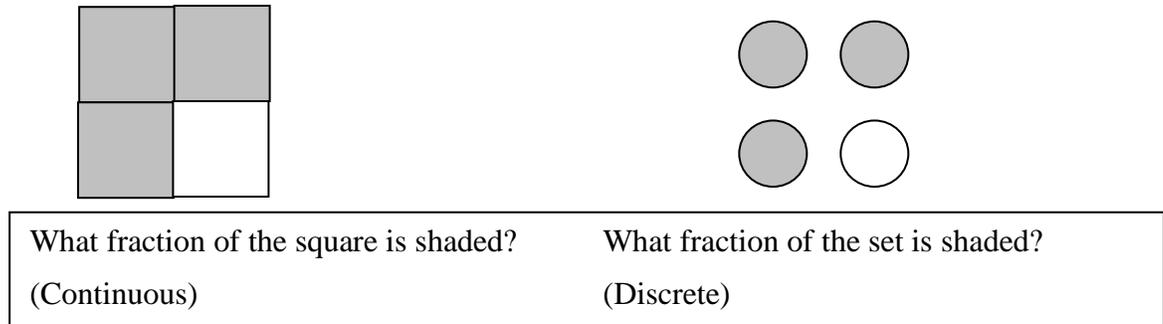


Figure 9: Continuous and discrete part-whole examples

### 4.3.2 MEASURES

The measures sub-construct describes a quantity in terms of another quantity that is set up as a unit of comparison. In the continuous context below, length A is measured using length B or B measured using A. The operators of comparison are reciprocals, if three of B measures A then one-third of A measures B. Similarly, the measure construct in discrete contexts involves using a referent set as a unit of comparison for another set. Later in his research, Kieren (1988) refined the sub-constructs by integrating part-whole as a special case of the measure sub-construct and further subsumed part-whole into the other constructs (Kieren, 1993).

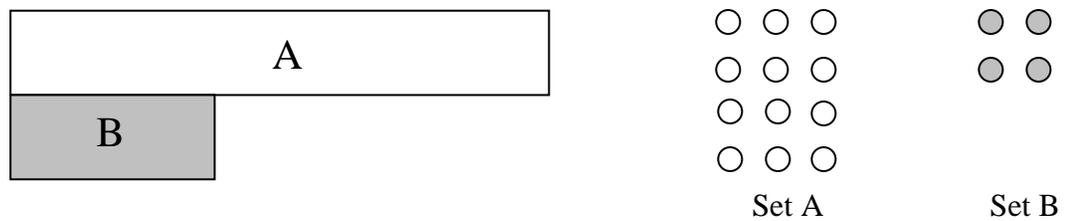


Figure 10: Measure sub-construct in continuous and discrete situations

The simplicity of the measures sub-construct belies its significance. Fractions as numbers are measures, so all operations, including ordering, addition, and subtraction, involve the measures sub-construct.

### 4.3.3 QUOTIENTS

The quotient sub-construct connects division with rational number through the *quotient theorem*,  $a \div b = \frac{a}{b}$ . Problems such as, “Three children share eight biscuits

equally. How much biscuit does each child get?” involve partitive division. Sharing problems require the connection of discrete and continuous representations that is potentially difficult for learners. The objects to be shared must be treated as discrete units of reference for “how much” yet still be partitionable in a continuous sense.

It is not clear from Kieren’s work where quotative division resides in the sub-constructs. Measurement division seems best classified under the measure sub-construct since a problem such as  $5 \div \frac{2}{3} = \square$  is usually interpreted as “How many measures of two-thirds measure five?” rather than as a rate, i.e.  $1 : \frac{2}{3}$  as  $\square : 5$ . In this thesis the quotient sub-construct refers to partitive division situations in which the quotient is a fraction of a referent whole or one. Measures are the sub-construct encompassing quotative division and fractions as quantities or numbers.

#### 4.3.4 OPERATORS

The operator sub-construct refers to situations in which a rational number maps a number or quantity onto another number or quantity (Kieren, 1980). Problems such as, “On the pirate ship Captain Crook gets two-thirds of the loot and the crew get one-third. There are 24 gold coins to share. How many coins does Captain Crook get?”  $\frac{2}{3} \times 24 = 16$  involves the operator sub-construct in that two-thirds operates on 24. In this sense rational operators act like exchange functions, for example, finding two-thirds of a number is equivalent to the function  $f(x) = \frac{2x}{3}$  (Kieren, 1980; Post, Lesh, Cramer, Harel, & Behr, 1993). Two orders for the operations exist. In the case of finding two-thirds of a number or quantity ( $q$ ) the possibilities are  $q \div 3 \times 2$ , and  $q \times 2 \div 3$  (Lamon, 1999).

Again the simplicity of the mathematics disguises potential complexities from a learning perspective. Embedded in the bi-partite nature of the symbols  $\frac{a}{b}$  is the duality of personality, number or operator (Lamon, 2006). This is a source of confusion for students that is explicitly addressed in Japan by differentiating operator and quantity fractions in instruction (Yoshida, 2004). The treatment of a fraction is dependent on context. For example, a student thinking of one-half as an operator may locate it at the centre of a number line when the position of the number one-half is required.

Applying a rational number as an operator involves either partitive or quotative division depending on the view of the rational number that is assumed from a context (Post, et al., 1993). As a number  $\frac{a}{b}$  is an extensive quantity made up of  $a$  iterations of the unit fraction  $\frac{1}{b}$ . This is a measure sub-construct idea that is critical to understanding a fraction as a number. Regarding  $\frac{a}{b}$  in this sense leads to partitive division to establish a  $\frac{1}{b}$  th share and then replicating it  $a$  times.

The part-whole relationship in a ratio  $a:b$  for measure  $a$  is  $\frac{a}{a+b}$ . This means  $a$  in every  $a + b$ . Consider the problem, “For every three cars Jill cleans Frank cleans two. Altogether they clean 30 cars. How many cars does Jill clean?” This problem

involves quotative division to find out how many lots of five cars total 30 cars. Multiplying the result of six by Jill's part in the ratio 3:2 gives 18 cars. Understanding the relationship between partitive and quotative division is critical to students' ability to operate with rational numbers (Thompson & Saldanha, 2003).

#### 4.3.5 RATIOS

Kieren's rates and ratios sub-construct follows the Greek tradition of defining a rate as a binary relation between quantity pairs with different measures, e.g. time and distance, and a ratio as a binary relation between quantity pairs with the same measures, e.g. litres of concentrate to litres of water. This distinction between rate and ratio becomes ambiguous when the measures are of the same attribute but involve different measurement units, e.g. 10 mL of weedkiller to 1L of water (Lesh, et al., 1988b).

Vergnaud's classification of multiplicative structures describes both ratios and direct rates as isomorphism of measures situations (Vergnaud, 1983; 1988; 1994). The connection between the ratio and the part-whole sub-constructs is that any ratio  $a:b$  can be seen as a unit whole made up of  $a + b$  parts. Viewing these parts as discrete, the part-whole relationships can be expressed as  $\frac{a}{a+b}$ , for the  $a$  measure, and  $\frac{b}{a+b}$ , for the  $b$  measure.

As an example, consider the ratio of 3 parts grey to 5 parts black (3:5).

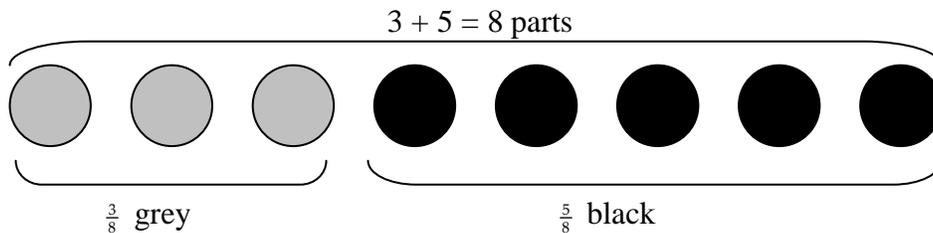


Figure 11: Connecting the ratio and part-whole sub-constructs

Other pairs with the same ratio, e.g. 6:10 and 9:15, share the same part-whole relationships but the numbers expressing the relationship are equivalent fractions. For nine parts grey to 15 parts black (9:15) the relationships are  $\frac{9}{24}$  parts grey and  $\frac{15}{24}$  parts black.  $\frac{3}{8}$  and  $\frac{9}{24}$  (grey), and  $\frac{5}{8}$  and  $\frac{15}{24}$  (black) are pairs of equivalent fractions.

Equivalent ratios provide a context in which the size of the whole or referent one changes but the part-whole relationship remains constant. The part-whole relationships in ratios are easier to interpret if the ratio is given as a frequency as opposed to a part to part ratio (Adjage & Pluvillage, 2007). Frequencies involve out of statements, for example, "Lisa got seven out of her 12 shots in". The part, seven, and the whole, 12, are explicit. In contrast, the part-whole relationship in the ratio 7:5 involves the inference that the whole is  $7 + 5 = 12$ .

Some situations labelled as ratios involve whole-whole comparison. For example, trigonometric ratios are the invariant relationships between corresponding sides of similar right-angled triangles. In the scaling situations below (Figure 12) the relationship of measures for the opposite and adjacent sides is the same for both triangles, i.e.  $3 \div 4 = 0.75$  and  $6 \div 8 = 0.75$ . This ratio is the tangent of the angle shown,  $36.87^\circ$ .

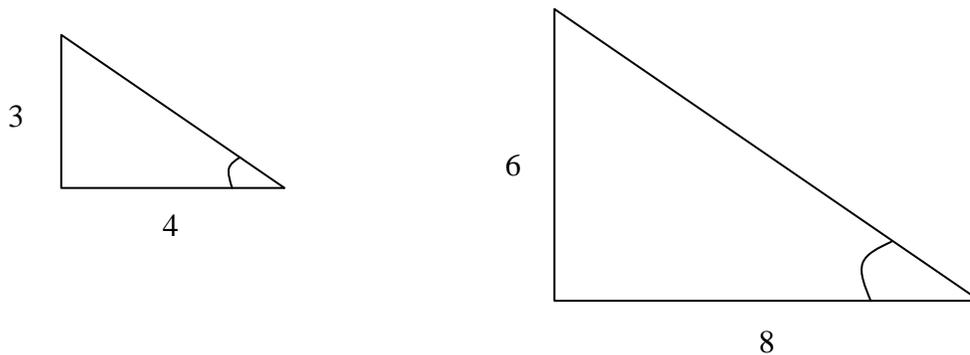


Figure 12: Ratios of sides for similar triangles

#### 4.3.6 RATES

Ratios and rates are structurally similar. In the Gaussian tradition, ratios were a special case of rates (Lamon, 2006). However, ratios and rates are different for learners in context. Rates do not have part-whole relationships, as the numbers have different referents and measure different attributes. Rates involve connection between separate wholes. Consider the rate \$15 buys ten pineapples. The pineapples and dollars remain independent sets and do not combine to form a whole made of 25 collected objects.



Figure 13: Image of the rate \$15 buys ten pineapples

A constant measure space to measure space relationship exists in equivalent rates through the creation of intensive measures, in this case dollars per pineapple or pineapples per dollar. Equally partitioning the rates creates equivalent rates, e.g. \$3 buys two pineapples. Replicating the rate gives equivalent rates, e.g. \$30 buys 20 pineapples. Equal partitioning and replication are multiplicative though learners

frequently use additive thinking inappropriately to find equivalent rates (Kaput & West, 1994; Van Dooren, et al., 2010).

#### 4.3.7 CONNECTIONS BETWEEN SUB-CONSTRUCTS

Proportional reasoning, as it pertains to rational numbers, involves understanding the connections between and subtleties of interpretation across Kieren's five sub-constructs. Appropriate connections involve discerning when a particular construct is relevant, for example applying measures rather than operators when dealing with fractions as numbers, and using sub-constructs flexibly to solve problems in context. Learners frequently employ different sub-constructs than those intended by task designers (Lamon, 2007; Mitchell & Horne, 2009). For example, to solve quotient problems students commonly use rate build-up. Instances of flexible use of sub-constructs are acts of transfer in the situations in which they occur.

The deep connectivity of the rational number sub-constructs made teaching only one interpretation impossible. The rational number personalities that I have discussed highlight different and essential characteristics of the rational numbers, but they are inextricably connected.  
Lamon (2007, p. 659)

While a large number of possible connections exist between sub-constructs some are more significant than others are. Part-whole is involved in the other four sub-constructs since they all require the establishment of a referent whole and the equal partitioning of that referent whole (Kieren, 1988; 1993). The ratio and rate, measure and operator sub-constructs connect in the relationships between the numbers in part-part ratio pairs, and between the numbers in rate and some ratio pairs that are whole-whole relationships. For example, with the ratio of 3:5, grey to black parts, the black part can be compared using the grey part and vice versa. This is a measurement idea. The black part is  $\frac{3}{5}$  of the grey part and the grey part is  $\frac{5}{3}$  of the black part. This is like finding the unknown operators which map three onto five and vice versa, i.e.  $\frac{5}{3} \times 3 = 5$  and  $\frac{3}{5} \times 5 = 3$ . These functional operators are constants of proportionality and apply to any pair with the same ratio or rate (Lamon, 1993). Similarly, scalars in ratio and rate situations are also operators.

The numbers in the same rate and ratio pairs form a set of ordered pairs, for example, (3,5), (6,10), (9, 15),... Graphing the relation on a number plane results in points representing the ordered pairs that are co-linear and the line on which they lie passes through the origin (see Figure 14). The slope of the line is given as a measure or number,  $\frac{5}{3}$ , the constant of proportionality between black and grey parts. In general, the slope of the graph for rate pairs equivalent to  $a:b$  is  $\frac{b}{a}$ .

The measures sub-construct also connects strongly with situations involving quotients and operators. The quotient theorem,  $a \div b = \frac{a}{b}$ , irrespective of its derivation from quotative or partitive division, involves measurement. For example, the portion from four parties sharing three objects equally has a referent measure. The mathematical convention is to define the share in terms of one object not the whole set of three

objects. The quotative division parallel is measurement of a length or set of three with a length or set of four.

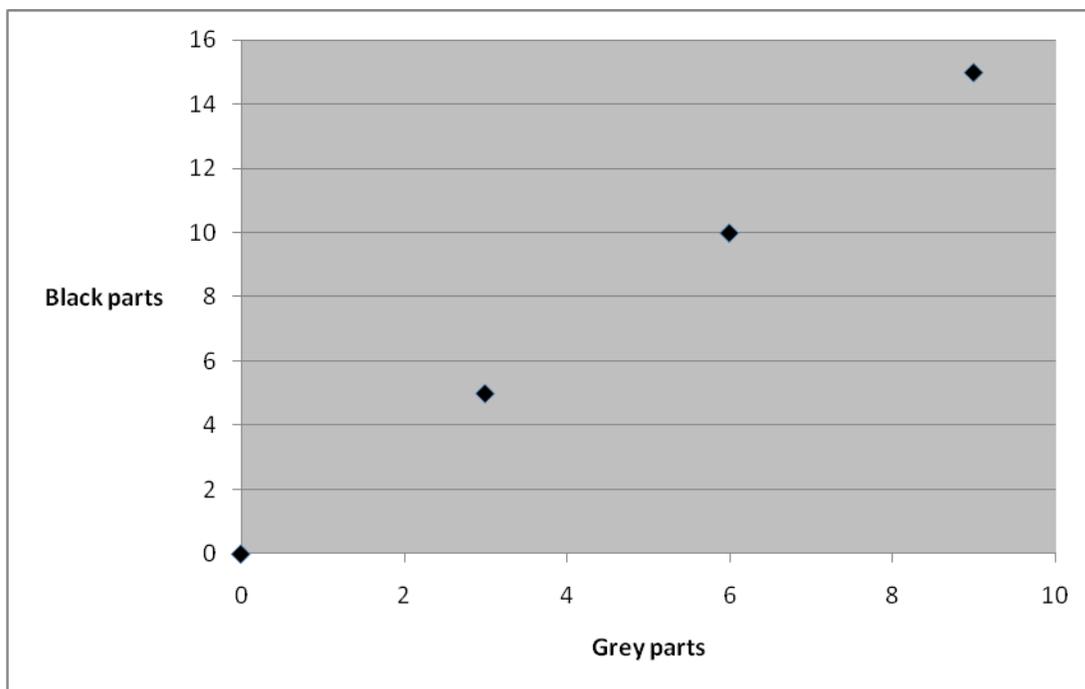


Figure 14 : Graph of equivalent ratio pairs for 3:5

Whenever a fraction acts as an operator on a discrete or a continuous quantity, that quantity is the referent one. In discrete situations, this can be confusing since a set of objects must be re-united as a new one. In finding  $\frac{a}{b}$  of the quantity part-whole thinking is used to establish a measure of  $\frac{1}{b}$ , that is iterated  $a$  times. While the order of the operation is arbitrary in principle, establishment of the unit fraction then iteration is more obvious from measurement perspective than iterating the quantity  $a$  times then finding  $\frac{1}{b}$  th of it.

Of interest in this thesis are the connections between sub-constructs used by learners in solving proportional reasoning problems in context. This will test the hypothesis that co-ordination of the sub-constructs contributes to a sound understanding of rational number that is manifest in successful transfer between situations.

#### 4.4 Complications

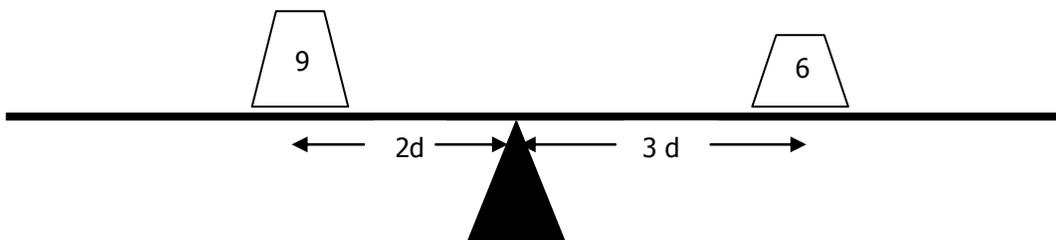
The preceding sections build a picture of the rational number field as complex. In simplest form, the mathematical structures require many connections and inferences. Each connection and inference is an act of transfer in itself. This section highlights other structural issues with rational numbers in which theorems established in some sub-constructs are inappropriate to other sub-constructs.

Rates and ratios, perform differently to fractions as numbers under operations, particularly addition and subtraction (Lesh, et al., 1988a). Adding fractions works as long as they are from the same unit whole and they share common denominators. The fractions combined are iterations of the same unit fraction so, in general  $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$ , e.g.  $\frac{5}{10} + \frac{9}{15} = \frac{15}{30} + \frac{18}{30} = \frac{33}{30}$ .

In contrast the result of combining ratios  $a:b$  and  $c:d$  is  $(a + c) : (b + d)$ . Combining ratios that do not share the same unit whole and viewing these ratios as part-whole relationships creates perplexing addition results, such as, “Mary gets five out of ten for one test and nine out of 15 for another test. Her combined mark is 14 out of 25. Does this mean  $\frac{5}{10} + \frac{9}{15} = \frac{14}{25}$ ?” Adding of numerators and denominators is a common strategy of learners (Crooks & Flockton, 2002; Flockton, Crooks, Smith, & Smith, 2006).

Combining rates also creates intuitively perplexing results. For example, travelling for half the time at 20 kilometres per hour and half the time at 30 kilometres per hour does not equate to an arithmetic average speed of 25 kilometres per hour. This result is given by the harmonic mean,  $h = \frac{2}{\frac{1}{20} + \frac{1}{30}} = 24$  km/h which is always less than arithmetic mean when the rates are different.

While direct linear proportions are expressed as  $\frac{a}{b} = \frac{c}{d}$ , inverse linear proportional contexts are expressed as  $ac = bd$ , where  $a, b, c,$  and  $d$  are integers. This corresponds to Vergnaud’s (1994) product of measures problem type within the multiplicative conceptual field. For example, the angular moment of an object on a balance is the product of its mass and its distance from the fulcrum (point of balance).



**Figure 15: Two different masses in balance**

In this situation, a mass of nine units is a distance of two units from the fulcrum. This balances a mass of six units that is three units of length from the fulcrum. This is an inverse proportional (product of measures) situation since  $9 \times 2 = 6 \times 3$ .

Inverse rate problems involve constant products, for example, “It takes six builders four weeks to build a house. How long will it take eight builders to construct the same house?” The product in this case,  $6 \times 4 = 24$ , is the unit rate of 24 days for one builder to construct the house. The problem involves solving the equation,  $6 \times 4 = 8 \times \square$ . The graphs of inverse proportional situations are hyperbolas.

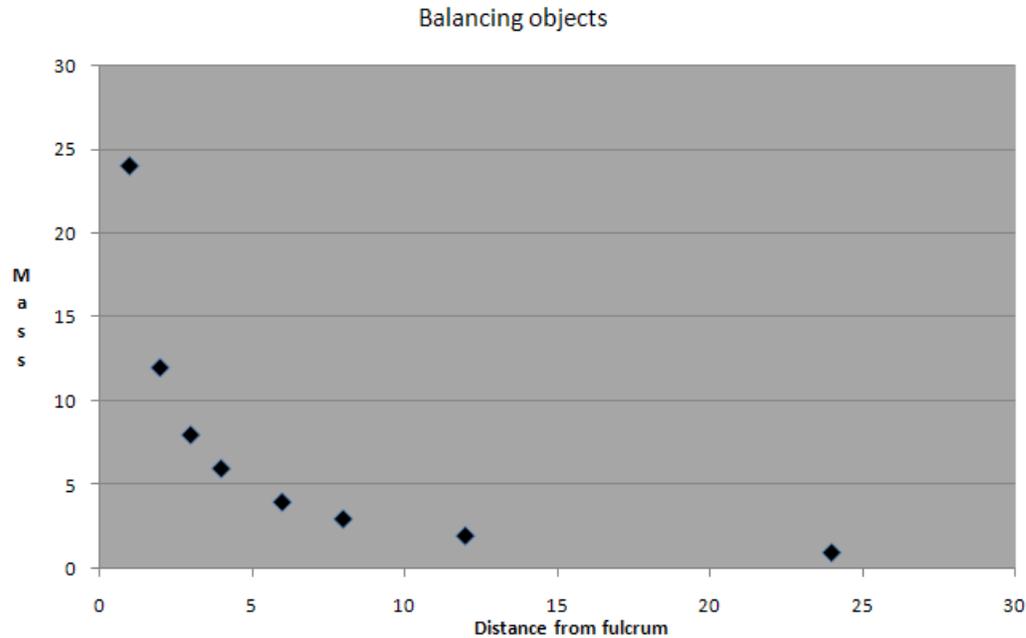


Figure 16: Graph of inverse proportional relationship

#### 4.5 Size Relations in Proportional Reasoning

Difficulties in considering different types of relationships between the numbers in ratio and rate pairs simultaneously may be fundamental to learner problems with ordering fractions (Gould, 2006; Pearn & Stephens, 2004b; Stafylidou & Vosniadou, 2004). Adjai & Pluvina (2007) described fractions as numbers that contain a proportional relationship. Changes to the numerator of a fraction work in an additive way, for example  $\frac{5}{9}$  is three-ninths greater than  $\frac{2}{9}$ . Changes to the denominator work in an inverse proportional way, for example,  $\frac{5}{9}$  is two-thirds of  $\frac{5}{6}$ .

Proportional reasoning situations often require determination of the size relation between two or more fractions, decimals, ratios or rates. For this reason Cramer and Post (1993) considered the ability to solve missing value problems to be an inadequate indicator of proportional reasoning. Ordering of rational numbers by size is more difficult than missing value problems as it involves an inferential judgment (Karplus, et al., 1983). The difficulty of the inference depends on the nature of the numbers and their position in the rational numbers, ratios or rates.

Given the situation of comparing two fractions,  $\frac{a}{b}$  and  $\frac{c}{b}$ , that share the same denominator the order relation is determined by which number,  $a$  or  $c$ , is larger since both fractions are iterations of the same unit fraction, e.g.  $\frac{4}{5} > \frac{3}{5}$  since  $4 > 3$ . The order relation for comparing  $\frac{a}{b}$  and  $\frac{a}{c}$  is determined by which number,  $b$  or  $c$ , is smallest since the smaller the denominator the greater the size of the iterated unit fraction.

In the absence of a convenient benchmark, ordering two fractions  $\frac{a}{b}$  and  $\frac{c}{d}$  takes consideration of both the numerators and denominators. Equating numerators or denominators through finding equivalent fractions makes comparison possible. For example,  $\frac{2}{3}$  and  $\frac{5}{8}$  might be compared by finding common numerators ( $\frac{10}{15}$  and  $\frac{10}{16}$ ), or denominators ( $\frac{16}{24}$  and  $\frac{15}{24}$ ). In the same way two ratios or rates with the same measurement units,  $a:b$  and  $c:d$ , can be compared by finding equivalent pairs in which  $m(a) = n(c)$  or  $m(b) = n(d)$ , where  $m$  and  $n$  are scalars (Norton, 2005), for example, 2:3 and 3:5 are changed to 6:9 and 6:10 or 10:15 and 9:15.

With comparison of fractions, ratios, and rates several inferences are significant for learners. Creation of equivalent forms is isomorphic with equal partitioning or replication that preserves the attribute involved, for example, a ratio situation may involve flavour or a rate situation may involve speed. With fractions, this conserves the measure or size of the number. Comparisons often involve opposite effect. For example,  $\frac{10}{15}$  is greater than  $\frac{10}{16}$  because sixteenths are smaller than fifteenths, and 6:9 is proportionally stronger in the first measure than 6:10.

#### **4.6 Students' Development of Proportional Reasoning**

Proportional reasoning is difficult for most learners. It develops late in students' schooling, if at all, and this development takes a considerable time (Tourniaire & Pulos, 1985). Hart (1981) found little difference in achievement on proportional reasoning tasks between 11 and 13 year old students, and Steefland (1984) estimated that only 50 per cent of late adolescents think proportionally. There is consensus that proportional reasoning does not develop through natural processes of enculturation and that it requires deliberate instruction (Lamon, 2006; Moss & Case, 1999; Resnick & Singer, 1993).

However, Alatorres (2004) showed that unschooled adults developed simple proportional schema in response to the demands of everyday life. A study of nurses' strategies for calculating drug dosage revealed that they developed their own reliable, scalar-based algorithms based on specialized knowledge of drug products and acceptance of invariance of concentration (Hoyles, Noss, & Stefano, 2001).

Students' development of proportional reasoning appears to be localized, at least initially, within the specific contexts in which it is encountered (Behr, Harel, Post, & Lesh, 1984). This suggests that a linear model of development is overly simplistic. Students exhibit variability of transfer between contexts and representations and proceed from situation-specific strategies to generalised concepts (Kaput & West, 1994). Karplus, Pulos, and Stage (1983) and Tourniaire and Pulos (1985) rejected the idea of a generic developmental sequence as pedagogically unhelpful and favoured exposing students to key aspects of proportional reasoning in a variety of situations, irrespective of their preparedness.

Hart (1988) suggested that proportional reasoning requires students to bring together conceptual understanding and computational resources. Memory load is strongly associated with students' ability to solve proportional reasoning problems (Noelting,

1980; Tourniaire, 1986; Tourniaire & Pulos, 1985). Mathematical knowledge acts not only as a limiting factor on students' capability to solve problems but also as an interpretive filter for making sense of empirical reality (Schwartz & Moore, 1998). A connection exists between students' computational resources, and the control systems that regulated their use (Lamon, 2006). These findings strongly support the value of co-ordination class theory as an explanatory model of conceptual growth.

The neo-Piagetian perspective (Case, 1992) suggested that the cognitive development of proportional reasoning, while domain specific, is associated with the development of central multiplicative structures (Lamon, 2006). The divergence of researcher opinion about progression in proportional reasoning may be the result of multiple factors that contrive to produce variability in students' performance across studies. It may also reflect process orientated views of learning strongly aligned to situated theory in contrast to long-term object views of learning.

#### **4.7 Significance of Task and Learner-related Variables**

A number of task variables impact on the relative difficulty of problems. These task variables combine with other situated factors, such as students' attitudes, beliefs, motivation, and learning preferences, to create variability of performance. Cramer, Post and Behr (1989) found that students who were able to cognitively restructure tasks were more capable at proportional reasoning than students who could not. Cognitive restructuring involves breaking component tasks away from the structure of presentation, providing a structure to a task that lacked one, or imposing a different structure on a problem than that initially provided. Tourniaire (1986) referred to such students as being field independent in contrast to field dependent.

Field dependent learners, who accepted and worked within externally provided structures, did not perform as well on proportional reasoning tasks as field independent learners, who applied their own structures flexibly. A significant feature of proportional reasoning is attending to key information within a situation, determining the nature of the relationships between elements of information and deciding how to act (Tourniaire & Pulos, 1985). Cramer et al. (1989) found the requirement for field independence to solve proportional reasoning tasks in three main ways; breaking up a structure into its parts, providing a structure where none is given, and imposing a different structure on a task than the structure given in the task organization. Tourniaire and Pulos (1985) found a positive correlation between general intelligence and proportional reasoning performance.

The link between problem solving ability and the availability of computational resources was substantiated by Kaput and West (1994) who found that students fell back to more primitive strategies when the problem conditions did not meet the knowledge they had available to implement their most sophisticated strategy. Simon (1979) referred to students providing a good enough solution in these circumstances as *satisficing*. Proportional reasoning requires the simultaneous processing of up to six variables in working memory (M-space). Proficiency with proportional reasoning is associated with working memory capacity (Tourniaire & Pulos, 1985). Kaput and

West (1994) also found that unsuccessful attempts at some proportional reasoning problems affected students' attitudes towards subsequent tasks, and often resulted in the use of less sophisticated strategies.

Task variables have a pronounced effect on problem difficulty. Students are more capable of solving problems where the context is familiar to them. For example, Allatorres and Figueras (2004), and Kaput & West (1994) found students were more proficient with speed and pricing problems than with other rate contexts such as scaling (Allain, 2000). Post, Cramer, Behr, Lesh, and Harel (1993) found that students were real-world dependent and struggled with hypothetical contexts such as interpreting speed as an inverse rate, e.g. hours per kilometre. Geometrically rich proportional reasoning tasks are difficult for students due to the added cognitive load associated with the contexts (Lo & Watanabe, 1997). The availability of manipulatives, and other representations, results in greater success on tasks (Tourniaire & Pulos, 1985).

The friendliness of numbers, particularly the ease of applying a scalar relationship, has a significant impact on problem difficulty (Misailidou & Williams, 2003). The easiest problems are those in which a given rate pair maps onto the target pair using an integer multiplier. For example, given 2:3, finding  $\square:15$  involves multiplying both numbers by five. Functional or between relationships tend to be harder for students to recognize and use than scalar relationships, due to the intensive quantities involved (Karplus, Pulos & Stage, 1983). Students often adopt additive strategies where non-integer multipliers are involved within or between measures (Cramer & Post, 1993) and where numbers in the problems were proximally close in size (Kaput & West, 1994). Provision of calculators makes no difference to success in these situations (Post, Lesh, Cramer, Harel & Behr, 1993).

The location of the unknown in missing value problems also changes the difficulty level of tasks (Post, Behr, Lesh, & Wachsmuth, 1986). Problems in which finding the unknown requires division, such as 16:24 as 4: $\square$ , are more difficult than those requiring multiplication, such as 2:3 as  $\square$ :24. Inverse operation problems, in general, tend to be more difficult. Ease of finding a unit rate also has a pronounced effect on task difficulty (Cramer, et al., 1993). For example, 8:16 as 3: $\square$  is easier than 8:12 as 6: $\square$ , because division by eight in the first task gives a unit rate of 1:2. Division by eight in the second task produces a non-integer unit rate 1:1  $\frac{1}{2}$  and division by four produces a non-unit rate 2:3. Use of unit rate strategies is strongly associated with successful proportional reasoning in many studies (Ben-Chaim, Key, Fitzgerald, Benedetto, & Miller, 1998; Kaput & West, 1994; Lo, 2004).

Successful interpretation of proportional reasoning tasks requires acceptance of homogeneity of distribution (Behr, Harel, Post, & Lesh, 1992; Kaput & West, 1994). This means that there is uniform diffusion of the given rate pair throughout the population of interest. It also means overcoming experiential obstacles to accept that specific rate pairs constitute a generalized statement about all rate pairs in the given situation. Many examples in real life counter this belief. Buying a greater quantity of something often reduces the price per unit. Running speed decreases over longer distances. Classifying a situation as proportional involves suspending natural

variation in favour of uniform rate. Task variables that assist students to accept the homogeneity principle include “for every” statements, containment of variables in the problem context, e.g. objects in packets, liquids in litres (Kaput & West, 1994), and discrete objects as opposed to continuous mixtures.

Alatorres and Figueras (2005) found an order of proportional reasoning context difficulty. Situations involving discrete objects were easier than continuous mixture situations. Probability tasks were extremely difficult for learners. The homogeneity principle of the proportional model is in cognitive conflict with the natural variability that occurs in sampling (Shaughnessy, 2003).

## **4.8 Development of Proportional Reasoning**

In spite of the variability in students’ performance associated with situations, task variables, and the complex yet subtle personalities of rational numbers, there is some evidence that a broad progression in understanding occurs. Stages of long-term progression are in harmony with object views of learning.

Piaget and Inhelder (1958) believed that proportional reasoning developed at the formal operations stage and that this occurred in early adolescence. The association of proportional reasoning with the advent of abstract thought is that it requires developing and working with classifications and propositions devoid of access to the actual objects involved. Reversibility, the use of inverse operations and reciprocity, class inclusion, equivalence, co-variation, combinatorics, and the use of logical operators are required in proportional reasoning. Subsequent research casts doubt on the adolescent emergence hypothesis. Younger students engage in simple proportional reasoning (Resnick & Singer, 1993) and many individuals never develop the formal operational thinking required (Lamon, 2006). Other researchers support Piaget and Inhelder’s assertions about the links between proportion reasoning and formal, logical thought (Kieren, 1980; Noelting, 1980).

### **4.8.1 INTUITIVE SCHEMES, P-PRIMS OR PERCEPTS**

Some research describes the nature of young learners’ intuitive schemes about number and quantity which are consistent with the concept of p-prims, percepts and met-befores discussed in Chapter Two. Resnick and Singer (1993) suggested how to build on intuitive knowledge to develop proportional reasoning. They described intuitive knowledge as that which develops naturally through students’ everyday life experience and is situation-specific rather than generalized. The researchers described two types of “proto-ratio” reasoning observed in young students, fittingness (match for size) and co-variation by direction of changes in relative size. According to Resnick and Singer, the inclusion of numbers, as adjectives, in tasks often leads to students abandoning their pro-quantitative schema.

Other intuitive reasoning capabilities also seem to exist in young learners. Schemes for whole number verbal counting and global comparison of quantity develop early and translate into global proportional comparison schemes and numerical structure for splitting and doubling (Confrey & Maloney, 2010; Moss & Case, 1999). Students

develop intuitive schemes for partitioning, recognizing equivalence, and creating divisible units (Kieren, 1988). These schemes involve the co-ordinated use of imagery, in the forms of physical, visual and numeric pattern, thought tools, like counting and matching, and informal use of language. Whole number and fractional number schemes appear to develop separately at first then merge at about 11-12 years of age (Moss & Case, 1999). It is uncertain to what extent the conceptual difficulties with fractions experienced by young learners are the result of an innate disposition for discrete counting that results in whole number bias, the cognitive demands of relating symbols to quantities or instruction that privileges discrete thinking (Ni & Zhou, 2005).

Kaput and West (1994) warned that intuitive whole number schemes often interfered with the development of proportional reasoning and rational number understanding. This caution is supported by research into students' misconceptions about fractions (Pearn & Stephens, 2004b), ordering decimals (Steinle & Stacey, 2004), and the inappropriate use of additive strategies in ratio comparison problems (Lo & Watanabe, 1997). Students who encounter difficulty with fractions or proportions look to their whole number schema for support (Behr, Wachsmuth, Post, & Lesh, 1984; Hart, et al., 1981; Lamon, 2006). There is not uniform support for the role of whole number thinking as an inhibitor of proportional thinking. Olive (1999) and Steffe (2003) believe that whole number schemes are reorganised by students to meet the demands of fractions rather than replaced by them, a view in harmony with co-ordination class theory. Steffe and Olive view the development of part-whole thinking with whole numbers as foundational to understanding of fractions.

#### **4.8.2 CENTRAL MULTIPLICATIVE STRUCTURES**

Hypothetical learning trajectories offer generic long-term developmental progressions. The trajectories go beyond the short-term variation in student responses due to situational and task variables to examine the generalized intuitive and formalized structures that underpin students' strategies. Lamon (2007, p.652) suggested that seven central multiplicative structures exist, that help students to develop anticipatory schemes for the prediction of physical actions, irrespective of context (see Figure 17). These structures derive from research on the development of multiplicative thinking and on Kieren's sub-constructs (Kieren, 1980, 1988, 1993). Structures are connected and both enable and underpin students thinking about rational number. For example, measurement, fitting one object into another, is done using a part of the attribute of interest (unitising), reasoning with that unit (norming), and using units of different sizes dependent on the precision required (reasoning up and down).

A brief description of each multiplicative structure follows:

- Measurement involves considering the size of numbers with reference to one or another number, e.g. comparing the size of  $\frac{2}{5}$  and  $\frac{4}{9}$  by proximity to one-half;

- Quantities and co-variation involves understanding what characteristics vary and remain invariant as two quantities (counts and referents) change together, e.g. comparing the growth of two people's heights over time;
- Relative thinking involves comparing two or more quantities in respect of a given attribute, e.g. comparing ratios of blue and yellow paint by darkness of green;
- Unitising involves creating units appropriate to a situation and norming involves thinking with those units, e.g. comparing  $\frac{2}{3}$  and  $\frac{3}{4}$  by using twelfths (equivalent fractions);
- Sharing and comparing involves considering the results of equal sharing and comparing the relative shares in different scenarios, e.g. Who gets more coca cola, two boys sharing a 600mL bottle or five girls sharing a 2L bottle?;
- Reasoning up and down involves relating quantities through multiplicative scaling, e.g.  $\frac{2}{3}$  of  $\square = 6$  so  $\frac{1}{3}$  of  $\square = 3$  so  $\square = 9$ ;
- Rational number interpretations involve relating the various ways rational numbers can be represented (percentages, decimals, ratios, etc.) and the connections between the representations; e.g. percentages used to compare frequencies like 43 out of 60 and 56 out of 75.

Lamon's (2002; 2005, 2007) four-year-long teaching experiment provided strong evidence for the significance of these multiplicative structures forming an integrated web of ideas that are the basis of multiplicative and proportional thinking. Her work supports the validity of both co-ordination class theory and object theory as useful models for the development of proportional reasoning. The process of development is lengthy and is characterised by considerable situational variation. Given appropriate instruction over time, learners develop a co-ordinated structure of rational number ideas. The co-ordinated structure occurs from students recognising similarity and difference in situations and contains a connected set of mathematical objects that are available for application to new situations.

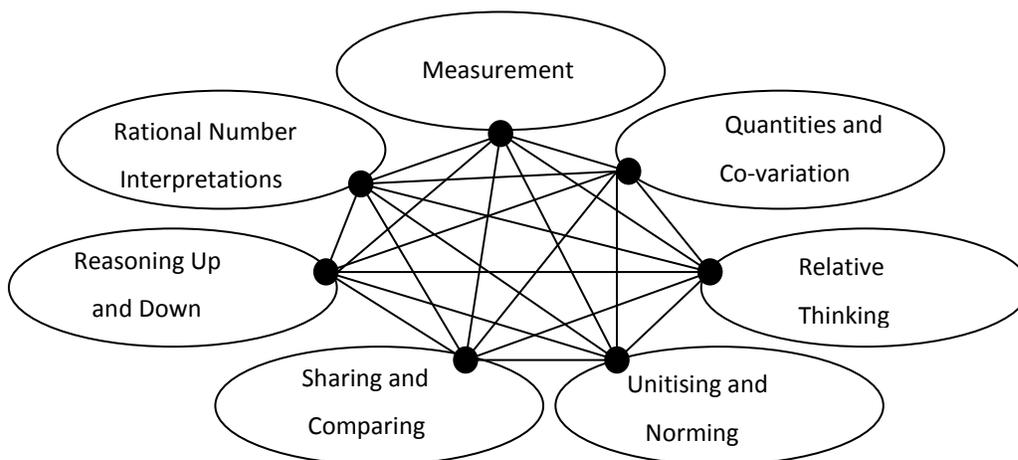


Figure 17: Lamon's seven central multiplicative structures

### 4.8.3 DEVELOPMENTAL PHASES

A synergy of studies into learners' solutions to proportional reasoning problems strongly supports a view that the development of proportional thinking is both localised in context but generic in nature (Case, 1992). Most studies used in Table 12 involve students answering missing value or comparison problems in ratio and rate problems, in both discrete and continuous contexts. There is also commonality in the phases of progression in students' responses to fractional number and decimal comparison problems.

The consistency of these studies suggests that students' thinking progresses through a sequence of broad phases. Naïve attempts at proportional reasoning problems are characterised by either incongruent attempts or focus on individual numbers or totals within the rate pairs. A phase of additive build-up strategies follows that involves repeated replication of a rate pair until a matching term occurs in the target rate pair. Abbreviation of additive build-up strategies occurs through use of multiplicative scalars. Multiplicative strategies involve treatment of the rate pair as a composite unit.

This multiplicative build-up phase is also associated with use of unit rate approaches in problems with favourable numbers (Hart, et al., 1981). Kaput and West (1994) described the unit rate approach as essentially additive in the sense that multiplication and division are treated as replicative procedures rather than as functional operators. Students frequently get confused as to which unit rate to use,  $1:\frac{b}{a}$  or  $\frac{a}{b}:1$ , when solving rate problems in unfamiliar contexts (Post, Behr & Lesh, 1988). There is common agreement across the studies that the ability to select critically from within and between relationships, particularly with non-integer operators, indicates sound proportional reasoning. This suggests abilities to identify structural similarity in proportional reasoning problems across situations (Lesh, et al., 1988a) and to apply quotative division in the "x referent A's for every y referent B's" sense.

All of the studies reported student use of incorrect additive-based strategies. These strategies involve subtractive difference relations, e.g.  $3:5 = 12:14$  (Alatorre & Figueras, 2002, 2003, 2004). Additive build-up is resistant to change and is frequently applied when repeated iteration of the initial rate onto the target rate leaves remainders or when the unit rate is not easily found (Christou & Philippou, 2002; Lo & Watanabe, 1997). Students often ignore easy scalar or functional relationships in favour of additive build-up though the incidence of additive thinking reduces with the presence of referents (Karplus, et al., 1983).

**Table 12: Generic phases in the development of proportional reasoning about rates (and ratios)**

Researchers	Non-proportional thinking	Inappropriate centrations	Additive build-up	Multiplicative build-up	Proportional reasoning
(Hart, 1981)	No coherent attempts	No rate applied	Additive build-up, halving/doubling	Rate used (scalar)	Non-integer ratios
(Tournaire & Pulos, 1985)	Guessing	Centration on conditions	Additive build-up, recognize constant differences	Integer relationships	Non-integer relationships
(Noelting, 1980)		Intuitive (lower)	Intuitive (middle), Order relations, e.g. 2:3 < 4:3	Creation of equivalence classes (concrete stages)	Formal (any ratio comparison), invariance of scalar and function relation
(Alatorre, 2002; Alatorre & Figueras, 2004, 2005)		Centrations on antecedents, conditions or totals	Order and grouping relations (mapping)	Multiplicative relations	Rate or ratio comparison
(Steinthorsdottir, 2005)	Incorrect additive thinking		Additive indivisible unit	Multiplicative indivisible unit	Divisible unit within and between relationships
(Ben-Chaim, et al., 1998)	Affective response	Focus on conditions	Additive build-up	Unit rate	Within and between relationships
(Kaput & West, 1994)			Co-ordinated build-up or down	Abbreviated build-up or unit rate	Within and between relationships
(Pantazzi & Pitta-Pantazi, 2006)			Interiorisation	Condensation	Reification
(Lo, 2004)			Build-up	Abbreviated build-up or unit rate	
Lamon (2006)	Random operation	Single unit focus	Build-up	Composite unit	Reversibility

#### 4.8.4 PARALLELS TO UNDERSTANDING OF FRACTIONS AND DECIMALS

Parallels exist between the performance of students on ratio and rate problems and their understanding of fractions and decimals. Learners over-generalise whole number thinking to the ordering of fractions by either focusing on the numerators or denominators, or by gap thinking where the distance from one is measured by the difference between numerator and denominator (Gould, 2006; Pearn & Stephens, 2004b; Streefland, 1993). Hart (1981) found that students tended to treat numerators and denominators as separate numbers rather than consider the relationship between them, e.g.  $\frac{2}{3} + \frac{3}{4} = \frac{5}{7}$ .

This thinking is consistent with centration on the antecedents and conditions, and subtractive relations seen in learners' thinking about ratios and rates. Difficulties with problems where iterating initial rates do not map by integral scalar onto the target rate parallel confusion with re-unitising decimal and whole number remainders in division problems (Confrey & Lachance, 2002).

Steeffland (1993) described whole number over-generalisation in proportional reasoning contexts as the presence of n-distractors. It took students in his study two years to become resistant to applying whole number thinking to fractions as they progressed through phases of absence of (cognitive) conflict, change in response to conflict, spontaneous refutation, freedom from n-distractors, and resistance to n-distractors phases. Post, Behr, Lesh, and Wachsmuth (1986) documented a growth path in students' understanding of order relations with fractions. Growth proceeded through understanding unit fractions, iterations of fractions, non-unit fractions (same denominators), and non-unit-fractions (different denominators). These phases closely aligned to conceptual growth with ratios and rates in which learners progressed from centration on individual numbers in rate pairs to understanding the scalar and functional relationships.

Competence with ordering decimals involves a combination of understanding equivalent fractions and multiplicative place value (Steinle & Stacey, 2004). Learners are susceptible to treating decimals as whole numbers in ordering tasks resulting in longer is larger strategies or relying on naïve views of decimal place value resulting in longer is smaller strategies (Roche & Clarke, 2006). Multiplicative thinking is required to apply equivalence with fractions in both discrete and continuous contexts (Pearn & Stephens, 2005). LaChance and Confrey (2002) suggested a similar connection with percentages.

Fractional numbers require acceptance of the homogeneity principle essential to understanding ratios and rates. Real life contexts frequently involve sharing in which the resulting parts are not equal (Hunting, 1999). This is contrary to the mathematical convention of fractions as iterations of equally sized parts. Ordering fractions also involves acceptance of a universal unit of comparison (Yoshida & Sawano, 2002). Hart (1981) reported that students had difficulty identifying the unit of reference in operations problems with fractional numbers, e.g. "What is the whole in  $\frac{1}{6}$  of  $\frac{3}{4}$ ?"

#### **4.8.5 WHY DO STUDENTS REASON INAPPROPRIATELY WITH PROPORTIONS?**

It is no surprise that learners find proportional reasoning difficult, given its complexity. The difficulties associated with learning to reason proportionally are a function of many factors including contextual and task variables, the multiple and connected personalities of rational numbers, demands on working memory and flexibility of thought and the connection of multiple representations. Complexity in the decision making associated with proportional reasoning requires development of the pre-frontal lobes which are the decision-making part of the human brain (Kwon, Lawson, Chung, & Kim, 2000). Learners need to be able to attend to the relevant information in situations, to distance themselves from their physical actions to

construct mental objects, to shift between quotative and partitive division, to curtail their fondness for additive build-up, and to relate quotients as fractions (Lo & Watanabe, 1997). The development of central multiplicative structures constrains or enables proportional thinking.

There is agreement among some researchers that learners' ability to anticipate actions on physical phenomena and apply these anticipations as objects of thought is essential to the development of proportional reasoning. Schwartz and Moore (1998, pp. 511-512) state:

The sketch of quantitative mental models suggests an explanation for the development of proportional reasoning that does not assume a developmental increase in working memory or the growth of fundamentally new conceptual structure... Mathematical tools help simplify this empirical complexity so that it can be understood within basic model structures.

Once high level proportional reasoning is accessible to students there is a tendency for them to apply it indiscriminately to all situations involving co-variation. De Bock, Verschaffel, and Janssens (2002) described this as students succumbing to the illusion of linearity. Freudenthal (1983, p. 267) described linearity as a suggestive property that was easily over-generalised and resistant to change. Historically linear thinking results in errors by capable mathematicians in situations as varied as geometric enlargement, speed and probability. For example, Aristotle claimed that an object that is ten times heavier than another object reaches the ground ten times faster than the other object when dropped simultaneously.

Reasons given for students' inappropriate use of proportional reasoning include use of intuitive reasoning (hunches), poor geometric knowledge, particularly about the areas and volume of non-regular shapes, and inadequate habits about how to approach word problems (De Bock, van Dooren, Janssens & Verschaffel, 2002; Modestou, Gagatsis, & Pitta-Pantazzi, 2004). Intuitions are synonymous with p-prims and percepts in that they are schema that appear to the possessor to be self-evident, have intrinsic certainty without need of justification, are coercive, applied globally rather than logically and formally, and are extrapolative across contexts (Fischbein, 1999). Students' tendency to locate the numbers in word problems and operate on those numbers without adequately considering the appropriate model to use is well documented (Van Dooren, De Bock, Hessles, Janssens, & Verschaffel, 2005). Steefland (1984) highlighted the importance of counter examples in helping students to re-think their linear dependence. Fischbein (1999) stressed that intuitive reasoning, whether correct or not, should be subject to proof because that reflected the nature of mathematical thinking and served to demonstrate to students that intuitively correct models often had limited generalisation.

#### **4.9 Pedagogical Considerations**

Representations and contexts play pivotal roles in the learning and teaching of proportional reasoning. Transfer between representations and contexts demonstrates generalised understanding of rational numbers (Moss & Case, 1999). Minimisation of the number of representations used lessens the perceptual distractors that interfere

with mathematisation (Behr, Wachsmuth, Post & Lesh, 1984). For example, Moss and Case (1999) used cylinders of water as the central representation in a teaching experiment that used percentages as benchmarks for the development of concepts about decimals. A linear representation for the teaching of decimals has also proved successful (Steinle & Stacey, 2004). Steefland (1984) used ratio tables and double number lines. Double number lines proved powerful representations for teacher trainees (Lo, 2004).

Figures 18 and 19 show use of these representations to model the problem, “A shirt usually costs \$48.00 but you get 25% off the price. How much do you pay?”

Percentage	100	50	25	75
Price	48	24	12	36

Figure 18: Ratio table

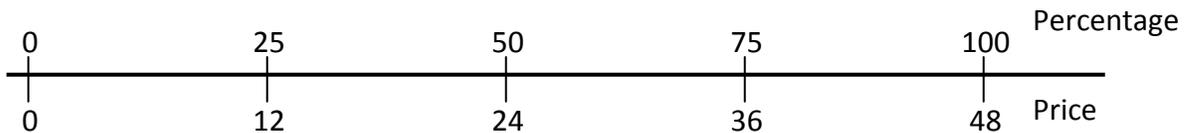


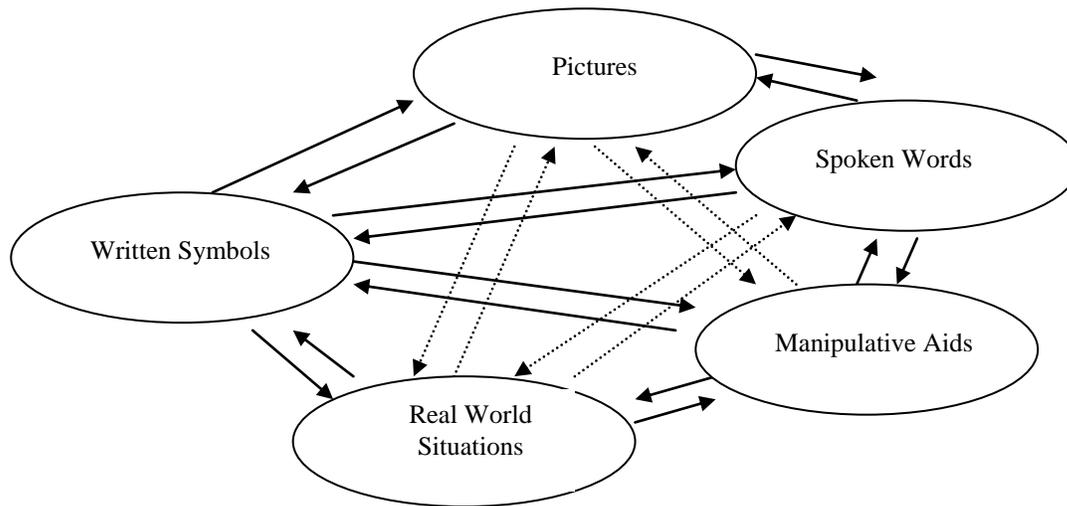
Figure 19: Double number line

Graphical representation of rate pairs assists students to discriminate between proportional and non-proportional situations (Cramer, et al., 1993). However, not all proportional reasoning problems are easy to represent with concrete materials. Many rate problems involve intensive quantities as measures for attributes such as colour density, strength of taste, and crowdedness. The quantities have referents that are much less tangible than extensive quantities (Lamon, 2006).

The literature uniformly views representations as vehicles for the development of abstract conceptual ideas not ends in themselves. Moss and Case (1999) provided four principles for the selection and use of representations:

- i. Develop concepts over procedures;
- ii. Use continuous as opposed to discrete;
- iii. Promote splitting as a natural form of calculation;
- iv. Connect equivalent forms of representation.

Post, Cramer, Behr, and Lesh (1993) supported the need to link the structural relationships between representations and the problem conditions, or related concepts, and the operations and transformations performed in solving problems. Lesh (1979) provided this translation model (Figure 20).



**Figure 20: Translation model for representations (Lesh, 1979)**

Authentic contexts, those familiar to students through their everyday worlds, are important to the development of proportional reasoning. Steefland (1993) and Lamon (2001) capitalised on students' sharing experiences as vehicles for developing understanding of the part-whole, measure, and quotient sub-constructs. Steefland (1993) used pricing as a context for developing the operator sub-construct. Ilany, Keret, and Ben-Chaim (2004) used a problem-solving approach with trainee teachers based on ratios, rates, and scaling. Lo and Watanabe (1997) promoted measurement and geometry as providing rich contexts for investigation. Lamon (2006) also highlighted the need for students to understand the proportional nature of unit creation and subdivision in measurement. While use of familiar contexts appears appropriate for initial instruction, the ability of students to apply proportional reasoning to non-familiar contexts also seems significant as an indicator of generalised proportional reasoning.

There is strong support for development of conceptual knowledge, before formalised work with symbols and algorithms (Cramer & Post, 1993; Hart, et al., 1981; Lamon, 2006). For example, in any equality of proportion situations finding a missing value in the relationships  $\frac{a}{b} = \frac{c}{d}$  can be found using the relationship  $ad = bc$ . The resulting cross-multiplication algorithm is an algebraic derivation that has no connection with the quantities involved in proportional reasoning problems (Lesh, et al., 1988a). Reliance on algorithms without understanding often makes easy problems hard and students often lack self-monitoring behaviours to check the reasonableness of their answers (Hart, et al., 1981). Algorithms masked students lack of understanding, particularly of the constant of proportionality (Lamon, 2006).

Unfortunately we sometimes confuse efficiency and meaning, and by default, even with the best of intentions, we introduce a concept in the most efficient but least meaningful manner. The standard algorithm for proportionality -  $\frac{a}{b} = \frac{c}{x}$ , a, b, c given, find x - is one of those areas. The standard solution procedure is to cross multiply and solve for x. That is,  $ax = cb$ , or  $x = cb/a$ . The algorithm in and of itself is a mechanical process devoid of meaning in the real world context.

Post, Lesh, & Behr (1988, p. 4)

In reporting the results of a four-year teaching experiment Lamon (2006) noted that students taught using a relational, rather than rule-based pedagogy, showed no noticeable difference in problem solving ability between situations involving integer and non-integer relationships. Given the lengthy development of proportional reasoning, delaying the teaching of procedures in favour of teaching for conceptual understanding constitutes a significant challenge to assessment regimes that demand incremental steps in competence (Lamon, 2006).

Research also considers the importance of requisite knowledge and sequences and emphases for the teaching of proportional reasoning. Lo and Watanabe (1997) and Thompson and Saldanha (2003) stressed the importance of multiplication and division with whole numbers as foundation for understanding of rational number. Students use chunking and building-up strategies to avoid multiplicative comparison. In particular, the ability to find multiples and divisors, to understand the conceptual connection between partitive and quotative division, and to have multiplicative understanding of place value are critical. Post, Lesh, Cramer, Harel, and Behr (1993) promoted re-unitising as foundational. This connected with Lamon's (1993) idea of unitising and norming, as a central multiplicative structure. Pedagogy aimed at developing re-unitising encourages students to see sets and continuous objects in multiple ways, for example four dozen cans as eight six-packs or 24 twenty-four packs, half a circle as two quarters of a circle or a quarter of two circles.

There is widespread support for use of all five of Kieran's sub-constructs in teaching rational number and the explicit emphasis on relationships between these sub-constructs (Kieren, 1980; Lamon, 2006; Post, et al., 1993). It is less clear as to the relative weighting and sequencing to be given, or certainty about whether understanding all of the sub-constructs is necessary or sufficient for the generalisation of rational numbers as an infinite quotient field (Kieren, 1988). Lamon (2006) suggested the measure sub-construct was the most easily connected to the other sub-constructs, and that operators and quotients were less powerful than measures, ratios, and part-whole sub-constructs. This was contradictory to the work of Steefland (1993) who used quotients and operators extensively. Kieren (1980) highlighted the importance of rational numbers as operators to students' ability to solve proportional reasoning problems with non-integer relationships. He suggested an across sub-construct progression through the following phases:

- i. Primitive subdivision by equal parts (Mack, 1993);
- ii. Constructing (and norming with) units;
- iii. Using inverses;
- iv. Creating ordered pairs;
- v. Simultaneous comparison;
- vi. Partitioning to establish equivalence classes.

Karplus, Pulos, and Stage (1983) supported Kieren's approach of concentrating on the relationships between variables and focusing students on classifying comparisons as equal (additive) difference or equal (multiplicative) ratio.

Research focuses on how best to progress students from using additive build-up and unit rate strategies to multiplicative scalar and function based strategies. Some researchers suggest posing problems where additive methods are cumbersome and ineffective or where the unit rate method is inefficient (Christou & Philippou, 2002). For example,  $2:5 = 20:$  provides easy multiplicative access by a factor of ten and  $3:4 = 15:$  is more easily solved by a scalar of five than by calculating the unit rate of  $1:1\frac{1}{3}$ . Hart (1981) suggested using the gross distortions in similarity with the application of additive thinking as a means to create cognitive conflict. For example, consider if learners suggest that a  $47\text{cm} \times 50\text{cm}$  rectangle is closer to a square than a  $76\text{cm} \times 80\text{cm}$  rectangle because of less (additive) difference, i.e.  $50 - 47 = 3$  versus  $80 - 76 = 4$ . The teacher uses a  $2\text{cm} \times 5\text{cm}$  rectangle, also with an additive difference of 3cm, as a counter-example.

There is little discussion in the research about alignment of instruction to meet the learning preferences of students. Field-dependent students require a higher degree of teacher intervention, than field-independent students (Cramer, et al., 1989). Field independence is a personal trait related to preference for working with a given structure and adapting it while field dependence is a preference for working within a given structure. A significant feature of proportional reasoning is attending to key information within a situation, determining the nature of the relationships between elements of information and deciding how to act (Tourniaire & Pulos, 1985). Learning preferences, along with other pedagogical issues relevant to the teaching experiment, will be discussed in Chapter Five of this thesis as background to the case studies of individual learners.

#### 4.10 Summary and Hypothetical Learning Trajectory

The following hypothetical learning trajectory shown in Table 13 is created from the research on the development of proportional reasoning combined with speculation about other key ideas. The trajectory uses Kieren's sub-constructs of rational number as a domain structure subsuming part-whole into the other four sub-constructs of measure, quotient, operator, and ratios and rates. Part-whole refers to both the equi-partitioning of a whole into a given number of parts and the reconstruction of a whole given a fractional part. Part-whole thinking is used in all four sub-constructs (Kieren, 1993). For example, in quotient situations involving  $a$  objects shared among  $b$  parties objects are partitioned and recombined to name the equal share.

Naming of the phases, Unit Splitting and Replication to Fractional comparison is indicative of a progression from additive-based whole number thinking to multiplicative-based thinking about rational numbers. The description in each cell refers to both conceptual understanding and procedural fluency. For example, a student understanding equivalence as a multiplicative relation demonstrates that a relationship exists between the relative size of measurement units and how many of them fit into the same target space. Eight-twelfths fit into the same space as two-thirds since twelfths are one-quarter the size of thirds. This understanding is also demonstrated by procedural fluency in computing equivalent fractions.

Alignment of the concepts in each cell is speculative and does not infer a precise sequence of development in that learners will be synchronised by phase across all sub-constructs at given points in time. Sources within this literature review justify the inclusion of ideas and are referenced where appropriate. These references are a sample of those which could be cited from this literature review.

The HLT tables for multiplication and division of whole numbers and for proportional reasoning informed the teaching design in an iterative way. The sub-constructs provided the focus for instruction over periods of one or two weeks. Also, for some extended periods instruction was aimed at decimals and percentages. For these units the HLT was used to check for broad application of these numbers to different situations and problem types. Long term planning is described later in Section 5.11.3.

Students' progression in relation to the HLT also informed instruction. Judgments were made frequently about the understanding of individual students and groups of students. Instructional time spent on particular sub-constructs or number representations was responsive to students' observed understanding. At times instruction changed focus because students progressed rapidly and at other times because progress appeared to have stalled. In the later situations I looked across the sub-constructs for useful connections that might advance the students' thinking.

**Table 13: Hypothetical learning trajectory for proportional reasoning**

Sub-construct /Phase	Unit Splitting and Replication	Unit Co-ordination	Fractional Equivalence	Fractional Comparison	
Measure	Halving based splits	Repeated splitting preserving equivalence (Confrey & Maloney, 2010)	Equivalence as multiplicative relation (Olive, 1999)	Order of fractions (Karplus, et al., 1983; Pearn & Stephens, 2004b)	
	More equal parts so smaller parts	Numerator as iterations – denominator as iterated (Post, et al., 1993)	Equivalent fractions as same number (Yoshida & Sawano, 2002)	Difference between fractions	
	Physical comparison of equivalent measure	Size of improper fractions in relation to whole numbers	Combining and separating fractions with related denominators (Pirie & Kieren, 1994)	Combining and separating fractions (Pirie & Kieren, 1994)	
			Measurement of one with non-unit fractions (Steffe, 2003)	Measurement of fraction with fraction (Steffe, 2003)	
Operator	Unit fraction related to amount	Non-unit fraction of amount as iterations of unit fraction	Non-unit fraction multiplicatively (variable unknown) (Post, et al., 1993)	Properties of multiplication and division with fractions	
			Operator and operated affect size of result	Fraction as unknown operator (Lamon, 1993)	
Quotient	Practical equal sharing by halving (restricted shares) (Hunting, 1999)	Practical equal sharing (open shares)	Quotient theorem, $a \div b = \frac{a}{b}$ . (Lamon, 1994; 2001)	Comparison of shares using equivalence	
			Equal and unequal shares	Remainders as fractions (Confrey & Lachance, 2000)	
Rate/Ratio	Practical duplication of rates	Equivalent rates by replication (Norton, 2005)	Conversion to unit rate (Ben-Chaim, et al., 1998)	Comparison of rates (measures unchanged) (Kaput & West, 1994)	
			Equivalent rates by multiplication and division (Noelting, 1980)	Graphs of linear rates (Post, et al., 1988)	
				Comparison of rates (measures changed)	
	Part-whole relationship in ratios		Equivalent ratios by multiplication and division	Inverse rates (Vergnaud, 1983; 1988; 1994)	
	Practical duplication of ratios	Equivalent ratios by replication (Lamon, 2007; Norton, 2005)	Part:part	Whole:whole	Comparison of frequencies (Adjage & Pluvillage, 2007)

## CHAPTER FIVE: METHODOLOGY

### 5.1 Research Aims

Chapter Five outlines the methodological underpinnings of the research. It discusses the research question and explains the rationale for the research design.

There are four foundational considerations in any research programme. They are the paradigmatic assumptions, the nature of the phenomena of interest, the questions asked about the phenomena, and the research method used (Geertz, Geertz, & Rosen, 1979). These considerations are inter-dependent. Defining one influences the others.

This study aims to answer the following research question with two parts:

**How do students develop their multiplicative thinking and proportional reasoning?**

- a. **Does a Hypothetical Learning Trajectory (HLT) reflect the actual learning trajectory of students?**
- b. **What model of conceptual learning, object theory or co-ordination class theory, best represents the growth of multiplicative thinking and proportional reasoning, and the transfer of knowledge within and between situations?**

Multiplicative thinking and proportional reasoning have possessive connotations. That is, the student has and exhibits something called thinking or reasoning. Central to this study was the process by which students acquired concepts. The research-informed Hypothetical Learning Trajectory (HLT) given in Chapters Three and Four provided a path of concept acquisition. Individual case studies considered the match between individual student's hypothetical and actual learning trajectories.

In developing case studies, I also looked for two features in the data that were of potential interest in explaining the process of concept development. Firstly, I looked for evidence in support of process to object theory (Sfard, 1991, 1998) to show that actions on physical phenomena became anticipated then encapsulated as objects of thought. Secondly, I looked for evidence in support of co-ordination class theory (diSessa & Wagner, 2005) to show that the construction of co-ordination classes was explanatory to students' transfer within and between situations.

### 5.2 Choice of Methodology

The research question, "How do students develop their multiplicative thinking and proportional reasoning?" involved in-depth examination of the process of concept construction in order to compare the HLT and the actual learning trajectories of individual students. The contrasting of object and in pieces views of learning also required insights into how students solved problems in new situations, the knowledge elements they used in doing so and processes of co-ordinating the elements that they

used. Interpretive methodologies that involve gathering and analysing large amounts of data about a small number of subjects suited the objectives of this study.

...good research is a matter not of finding the one best method but carefully framing that question most important to the investigator and the field and then identifying a disciplined way in which to inquire into it that will enliven both the scholar and his or her community.

(Schulman, 1997, p. 4)

Research methodology must align to the nature and object of the study. The research question had student knowledge and conceptual change as the phenomena of study. The setting of the research was in a classroom over a prolonged period. This setting lent itself to a qualitative design for several reasons:

- The variables involved in students' learning and conceptual change were complex and inter-related;
- The HLT and models of conceptual development and transfer were conjectures that were tested in relation to the data;
- Instructional decisions were based on data about students' conceptual growth as mapped by the HLT so the conjectures influenced the phenomena;
- The design of the learning and teaching environment was iterative and therefore could not be precisely defined and controlled from start to finish of the study;
- My role with the teacher and students in the study was inevitably interactive therefore strongly influencing the phenomena.

Both the objects and setting suggested the appropriateness of design research methodology which includes a modern interpretation of teaching experiments. The data was presented as case studies so required analysis of the knowledge and cognitive changes of several individual students. These aspects involved a fusion of cognitive psychology and situated learning theory (Steffe, Thompson, & von Glasersfeld, 2000).

The two parts of the research question both focus on student cognition. Traditionally cognitive psychology approaches (CP) study qualitative differences in student cognition and regard social interaction as a variable in the development of students' knowledge. Of interest to CP researchers are the mental processing functions that account for the relationship between stimulus and response. CP adopt mainly, though not exclusively, quantitative methodology. However recent studies of students' learning in mathematical domains often adopt a design research approach (Simpson, 2009; Steffe, 1988, 2003; Wagner, 2003). These studies represent a shift in research emphasis from snapshot views of learning to descriptions of the learning process over time and the development of explanatory theory. For design research the focus is on enhancing the cognitive constructions of individual students with tentative claims about generalisation. Apart from the specific characteristics of the situation being investigated, the sample of students in a multiple case study is usually insufficient in terms of size and representativeness to warrant inferences about the population of all students.

The research focus on the nature of conceptual change also lent itself readily to a situated learning (SL) perspective. The focus for SL is the interactions between individuals within a group and is particular to that situation. Situated learning approaches study the interaction between individuals in a social group, together with the tools they use. Underpinned by social constructivist theories that see learning as socially constructed, SL describes and explains qualitative differences in students' participation (Cobb & Bowers, 1999).

### **5.3 Teaching Experiments as Design Research**

#### **5.3.1 BACKGROUND**

Recent literature frequently classifies teaching experiments under the generic heading of design research, sometimes referred to as design experiments. Design research is an emerging category of research that involves cycles of design, enactment, analysis and redesign of explanatory theory (Baugartner, Bell, Brophy, Hoadley, Hsi & Joseph, 2003; Schoenfeld, 2006). This constitutes a redefinition of traditional teaching experiment methodology in that the initial conjecture is subject to change in a reflexive way on the basis of evidence. Typifying this view diSessa and Cobb (2004, p. 80) described teaching experiments as "...iterative, situated, and theory-based attempts simultaneously to understand and improve educational processes".

This modern interpretation of teaching experiment methodology neglects its history. Teaching experiment methodology was first used in USSR as early as 1900 and more recently in the Netherlands in the 1970s (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; van den Heuvel-Panhuizen, 2000). Usually these studies began with hypotheses about teaching methods, student thinking, or curriculum materials that were tested in the naturalistic setting of classrooms or in controlled situations with individual students. Early teaching experiments usually adopted quantitative data analysis in considering the effect of the treatment on low, middle and high achievers.

Design research gained popularity through the later 1990s because it addressed the limitations of methodologies from other fields, particularly those of the positivist tradition, in terms of providing finer grained analysis of the interaction of factors associated with students' learning (Brown, 1992). Design research based teaching experiments fill a void that emerged from consideration of constructivist learning theory (Steffe, et al., 2000). Interest in teaching experiments also links to international comparisons of instruction, particularly lesson study in Japan (Collins, Joseph, & Bielaczyc, 2004; Isoda, Stephens, Ohara & Miyakawa, 2007). Collaborative polishing of lesson scripts by teams of teachers in Japan is associated with high achievement in mathematics at a system level.

Within modern design research a teaching experiment describes events relevant to a particular situation. The setting is either one or more classrooms over the duration of a sequence of teaching episodes or restricted teaching situation with a small number of students, sometimes an individual student. Four key elements are involved; the

agent (teacher), the witness (researcher), the community of learners (students) and the method for recording events for retrospective analysis (data) (Steffe, Thompson & von Glasersfeld, 2000). The data analysis of a teaching experiment provides a chronological narrative with a focus on the actors in the study and their interpretation of events.

In keeping with the situated nature of design research, a growing body of teaching experiment literature further pushes the boundaries of the researcher as passive observer. Some researchers assume the role of teacher and researcher, working in partnership with the usual class teacher. In some studies the researcher becomes the teacher (Ball, 2000). Simon (1995, p. 121) wrote:

When the researcher/theorist assumes the role of the teacher in a research project he is uniquely positioned to study in a direct way the interaction of his theory and practice.

This study, as a teaching experiment, involved a collaborative relationship between researcher, teacher and students. I was both teacher and researcher so was an integral participant in the learning environment. Adopting a dual role offered both possibilities and constraints. As teacher, I was directly positioned to influence instruction and use my knowledge of research and pedagogy in doing so. The presence of two teachers in the classroom offered students enhanced learning opportunities in comparison to their normal classroom situation. Conversely, my involvement in day-to-day instruction limited the collection of data that may have been useful in analysing the social interactions. For example, the duration of the study and logistics made videoing of classroom interactions impractical.

### **5.3.2 CRITIQUE OF DESIGN RESEARCH**

Design research focuses on issues considered problematic and that suggests a need for beneficial change. In that sense, the methodology reflects a political agenda for the betterment of education. Researchers and curriculum designers identify multiplicative thinking and proportional thinking as critical abilities that have significant impact on students' further learning in mathematics and on their prospects as adult citizens (Dole, 2010). It has been estimated that only half of the adult population of USA reason proportionally (Lamon, 2005, 2007).

The greater good feature of design research was also described by Cobb, Confrey, diSessa, Lehrer and Schauble (2003, p. 2). These researchers described design experiments as having five cross-cutting features:

1. Theory (or conjecture) driven with the setting as a paradigm case;
2. Interventionist in vision;
3. Prospective and reflective;
4. Iterative in design;
5. Specific, located in particular practice.

According to Cobb *et al.*'s work the objective of teaching experiments as design research is the creation of models to describe, explain and predict teacher and student

action in the learning situation (Lesh & Kelly, 2000). Studies begin with a conjecture. The conjecture is localised to a specific area of mathematics or to a pedagogical approach (Gravemeijer, 2001). It reflects sound searches of relevant research literature together with speculative beliefs about approaches that lead to better outcomes for students.

The initial conjecture in this study was that a hypothetical learning trajectory (HLT) mapped the growth path in learning for multiplicative thinking and proportional reasoning and offered a useful structure for sequencing instruction. Detailed discussion of the HLT is presented later in this chapter.

Therefore, the initial conjecture had both mathematical and pedagogical dimensions. Teaching experiments involve a dialectical interaction between the conjecture and the dimensions of instruction, curriculum, methods, teacher and student roles, and assessment methods (Confrey & Lachance, 2000). A teaching experiment using design research methodology involves testing the assumptions implicit in the conjecture against the data.

The second conjecture of this research was that both object theory and co-ordination class theory offered insights into students' learning of concepts and thereby their ability to transfer within and between situations. This conjecture emerged from the data which showed considerable variation in students' responses that required explanation.

### **5.3.3 HYPOTHETICAL VERSUS ACTUAL LEARNING TRAJECTORIES**

The evolving cycle of data analysis and adjustment of instruction is a common feature of teaching experiments. Through the mathematics teaching cycle the detail of the Hypothetical Learning Trajectory (HLT) was continually modified in response to analysis of the classroom events (see Figure 22). In this design research, the HLT was a significant component of the theory developed during the course of the study.

Simon (1995) originally used the construct of a Hypothetical Learning Trajectory. His seminal paper cast the classroom teacher as a researcher, hypothesising a conceptual growth path for students and aligning activities to promote growth along the path. Modification of activities, learning goals and the hypothesised conceptual growth path occurred through reflection on students' learning.

HLT construction became topical for researchers who used the knowledge base about student learning in particular domains to inform their hypothesised growth paths. Three heuristics apply in the creation of the HLT for students. The first heuristic requires guides re-invention through progressive mathematising that often mirrors the historical development of the mathematical concept/s (see Figure 22). The second heuristic concerns didactical phenomenology, the search for model-eliciting activities that promote the students' conceptual development. The third heuristic focuses on emerging models that assist students to mathematise the problematic situations initially but later lead to the models being the objects of mathematisation (Gravemeijer, 2001).

An HLT does not attempt to anticipate the learning journey of each student in a particular group. Rather it describes a collective journey of the group as a community of learners. Of central importance are the evolving mathematical practices including those governing enquiry and explanation (Yackel & Cobb, 1996). Embedded in the learning situation are symbols, representations, and metaphors. These tools are integral to the social learning process, not variables separate from it. Tools mediate the actions that lead to construction of mathematical ideas and provide the means to validate the ideas.

Development of HLTs involves a distinction between student mathematics and the mathematics of students (Steffe, et al., 2000). Students' realities, their cognitive constructions, are the inspiration for models to explain how and why the events occur. Studies use multiple sources of data. The words and actions of students and their efforts to solve problems are the evidence of constructions.

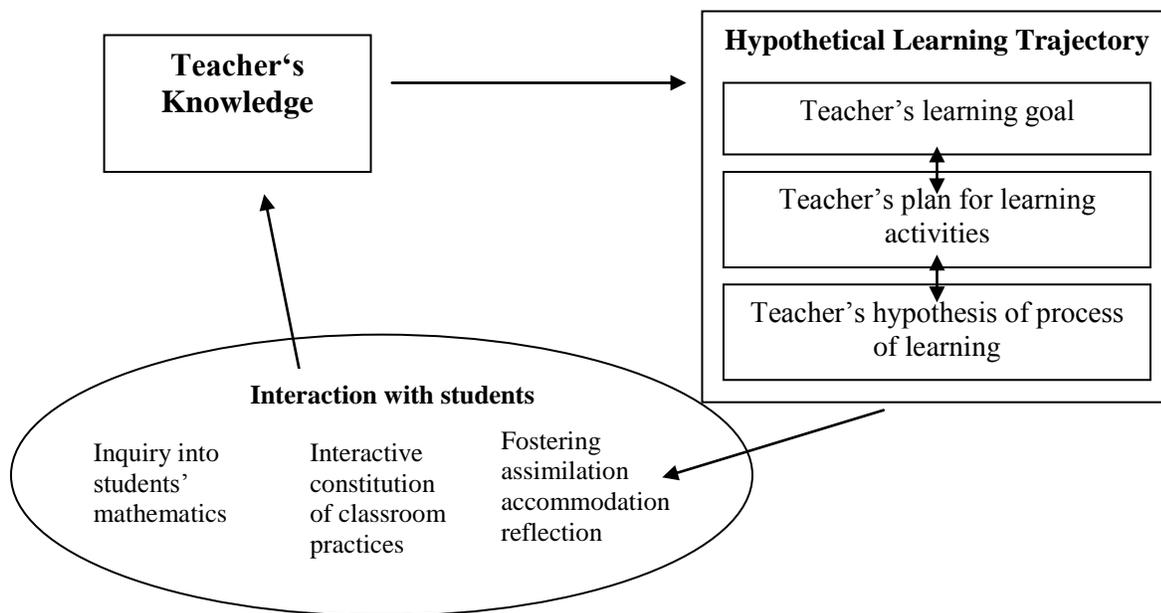


Figure 21: Mathematics Teaching Cycle (Gravemeijer, 2001, p. 151)

Recent developments in HLT research reflect attempts to bridge the hypothetical and actual learning trajectories of individual students. Researchers use the term learning trajectories. The absence of hypothetical is claim to the actual learning that takes place. For example, Steffe (2003) described the learning trajectory in fraction schemes of two grade five students over a teaching experiment spanning one month. Confrey and Maloney (2010) provided a progress report on their large-scale study into students' responses to equi-partitioning tasks.

They proposed an early iteration of a learning trajectory for equi-partitioning and describe a learning trajectory, in general, as:

...a researcher-conjectured, empirically supported description of the ordered network of constructs a student encounters through instruction (i.e. activities, tasks, tools, forms of interaction and methods of evaluation)...

Confrey and Maloney (2010, p. 1)

Simon's original perspective on trajectories also changed over time as he stated that for a teacher (or researcher) to promote conceptual change they must consider current state and goal state (Simon, Tzur, Heinz, & Kinzel, 2004).

A worthy focus of a teaching experiment research is analysis of the difference between a hypothesised trajectory and the learning that actually occurs. A common criticism of HLTs is that variation between students, and in the responses of individual students to different situations, makes generic, linear growth paths unrealistic and restrictive (Lesh & Yoon, 2004; Watson & Mason, 2006).

Some original design research involved analysis at a macro-level by treating students as a collective group. This thesis considers the difference between the hypothesised and actual learning trajectories for six individual students. The data is presented as in-depth case studies. The analysis is fine grained and scrutinises the HLT at the level of its appropriateness to the learning of each student. The advantage of linking case studies of individual learners with teaching experiment design is that it provides a view of the learning process in a research informed teaching environment. The cycles of evaluating students' cognition in relation to the HLT and the consequent adjustment of instruction potentially increases the probability of conceptual change.

#### **5.3.4 CASE STUDIES**

A case study is usually an in-depth investigation of a bounded real-life system over time (Bassey, 1999; Yin, 2006). However, the research reported in this thesis did not use the grounded theory approach normally associated with case study methodology (Glaser & Strauss, 1967). In grounded methodology theories about the system under investigation are developed through relating multiple sources of data. Design research begins with a conjecture or conjectures already established.

Case studies of individual students are used to present the data in Chapters Six to Ten (see Figure 22). Both quantitative and qualitative data are used. The main strength of case studies is that they provide rich descriptions of situations as they naturally occur. The data may establish cause and effect relationships. Use of multiple sources of data makes cases studies useful in describing the interplay of factors and variables within complex situations, particularly social situations.

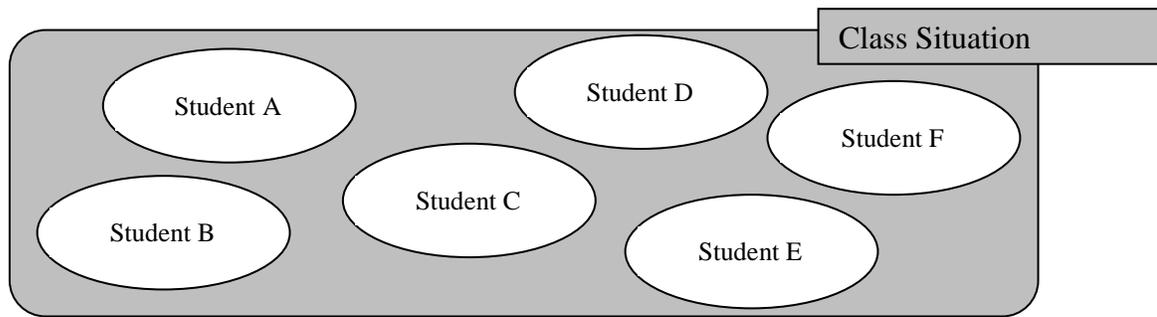


Figure 22: Case studies within a case study

The strengths are balanced by weaknesses. Narrow focus to one system limits generalisation to other systems (Bassey, 1999). Case studies are difficult to cross-check by other researchers and the selection of evidence is open to bias especially given the closeness of researcher to the researched phenomena in some studies.

#### 5.4 Standards of Evidence

Several issues surround the use of design research methodology and its appropriateness as a legitimate form of inquiry. Criticisms of teaching experiments relate to validity and reliability. The criticism lead to attempts to establish standards for evidence (Kelly, 2004; Dede, 2004). Schoenfeld (2002, p. 467) provided an organisational frame for judging the quality of studies (see Figure 23). Generality, or scope, refers to the range of situations to which the claims apply to, trustworthiness to reliability of the claims, and importance to the significance of the claims to improving overall educational outcomes.

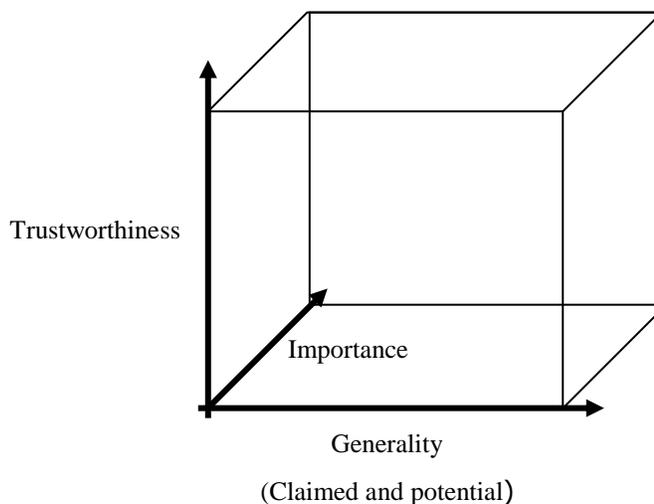


Figure 23: Three important dimensions along which studies can be characterised.

Design research methodologies are vulnerable to criticism against all three dimensions. Generalisability is an issue for all educational studies. Small scale studies are particularly susceptible to inappropriate claims that extend beyond the reach of the study, given the small numbers of actors involved and the specific aspects of the situation. However, it is every educational researcher's desire for their research to be generalisable, irrespective of the nature of their research (Schulman, 1997).

Trustworthiness relates to argumentation and warrant (Kelly, 2004). Argumentation concerns the logical process by which a methodology derives findings from the data and warrant refers to how to differentiate sound claims from unsound claims (Confrey & Kazak, 2006). Given that teaching experiment studies accept the multitude of independent and dependent variables existing *in situ* and manipulate the material, context, and scope of the intervention, it is easier to establish relationships between variables than direct causality. While the "association does not prove causality" mantra equally applies to quantitative research, teaching experiments also lack the equivalent of a statistical measure of effect size to validate the strength of claims.

The final dimension of Schoenfeld's (2002) model, significance, concerns the importance of the phenomena to the broader educational community. Dede (2004, p. 107) commented that with many teaching experiments, "This elephantine effort resulted in the birth of mouse-like insights." This criticism suggests giving more thought to the initial conjecture and to the data collected to support or reject it. My involvement in the research, the setting up of initial conjectures, the iterative and interventionist nature of the design, and the necessity for selection of evidence all raise questions of potential bias. Standards for the selection of evidence remain a major challenge in this study, as they have been for design research methodology in general.

Transferability is also a corollary of the significance dimension and is also a product of the specific nature of situations. Dede (2004, p. 114) argued that teaching experiments conducted in favourable circumstances fail to address issues of usability, scalability and sustainability across the breadth of educational settings.

## **5.5 Generative power**

Design research begins with a conjecture as a starting point for theory construction. This study began with the conjecture of the Hypothetical Learning Trajectory (HLT), derived from research in the domain/s concerned, and retrospectively considered the value of object and co-ordination theories of conceptual growth and transfer. This feature corresponds with the first dimension of Schoenfeld's model related to the ability to generalise. diSessa and Cobb (2004, p. 84) stated that quality studies produce "ontological innovations", defined constructs that provided insights into both how to observe and what to observe for. These constructs apply to other settings through a generalised theory; the situations of the specific design research do not. It is

problematic that diSessa and Cobb shift the point of argument from the findings of studies to theories that emerge from the findings of those studies. This does not resolve the argument of quantitative research advocates that single, small scale teaching experiments provide insufficient evidence for extrapolating the findings to a population. Two points in defence of diSessa and Cobb's position are relevant.

Firstly, the conjecture that teaching experiments investigate and theorise about derives from an extensive review of the literature. The literature reflects a range of methodologies and research designs. It provides a body of data in support of the conjecture, thereby adding to the weight of evidence. Secondly, criticisms of teaching experiments that they produce huge bodies of data (Dede, 2004) are at odds with the idea that the data is insufficient for generalisation.

A fairer position is that teaching experiments produce deep insights into the occurrence of phenomena across a small number of subjects as opposed to relatively shallow insights derived from quantitative data on a large number of subjects. It is to the deep insights and the development of explanatory constructs that diSessa and Cobb refer. A HLT, and object and co-ordination class theories are examples of such constructs (diSessa, 2008). A third line of defence of the generalisable theory position is that theory developed in one small scale study is often shown to be useful in other studies. This is true of co-ordination class theory.

DiSessa and Cobb (2004) provide criteria for judging the quality of constructs. Constructs are ideas that constitute a theory inferred from a teaching experiment. The criteria are:

- Descriptive power – Does the theory capture the key constructs to describe the phenomena and in doing so guide what to look for in observing them?
- Explanatory power - Does the theory explain how and why the phenomena occurred?
- Scope – How broad is the range of phenomena to which the theory applied?
- Predictive power – How good is the theory at predicting events involving the phenomena?
- Rigour and specificity – How well defined are the constructs in the theory, and how well described are the relationships between them?
- Falsibility – Can the theory be proven wrong?

The criteria put heavy demands on theory that are far from trivial. Given the inherent complexity of teaching experiments the creation of succinct constructs that are descriptive and explanatory requires an extensive synthesis of the data. Scope and predictive power are balancing imperatives. The opportunity is for generalisability but the constraint is the scope of phenomena to which the theory is applicable. There is also a relationship between specificity and falsibility. Tighter description of constructs and their relationships makes them more readily operationalised but also increases the opportunity for disproving. Disproving can involve one aspect of a construct but have the effect of eroding the whole theory.

## 5.6 Trustworthiness

Trustworthiness concerns the issues of validity and reliability. Qualitative methodology literature readily acknowledges that researchers' beliefs, knowledge and interests affect what they study and how they study it. All selection of primary data is theory-laden in that the researcher deliberately chooses to collect it based on its suitability for a purpose that is predetermined (Hall, 2000). The effect of evidence selection in terms of the results of research is well known, particularly in the field of classroom-based research. For example, in the TIMSS video study (Clarke & Mesti, 2003; Jacobs, Kawanaka, & Stigler, 1999) coding of classroom interactions both revealed and masked phenomena of interest.

Measurement of the dependent and independent variables usually present the greatest challenge in teaching experiments (Cobb, et al., 2003). By selecting measures, researchers define constructs of interest. Measures need to be feasible to administer, yet be able to capture the phenomena reliably. The act of measuring frequently acts on the phenomena that it seeks to measure (Lesh, 2002).

Partially in response to the criticisms about generalisability, selection of evidence, and warrant and argumentation, advocates of teaching experiments sought to establish standards for the gathering and processing of data (Cobb, et al., 2003; Confrey & Lachance, 2000; Steffe, et al., 2000).

The first standard for effective research concerns the rigour of data collection. The archiving of data allows for potential scrutiny by others in the research community. This implies that the data constitutes an authentic and representative record of events reflecting the experiences of the participants in the study. Researchers acknowledge the tension between the need for economisation of data collection, and the potential for biased selection of evidence. Justification for the preference of some data over other data needs to be transparent (Cohen, et al., 2000).

Congruence is the second standard for valid and reliable evidence. Multiple voices and sources of evidence triangulate to support interpretations of events and of the perceptions of the actors. Potential sources of evidence include personal journals, field notes, videoed interviews or lessons, assessment responses, samples of work, body language, and analysis of discourse. A democratic relationship needs to exist between researcher and those researched for this to occur. Multiple observances suggest stability of phenomena (Steffe, et al., 2000). Triangulation between different data sources and different voices suggests coherent interpretation or divergent interpretation of the actors (Anderson & Herr, 1999). Other workers suggest that multiple interpretations be built into research designs. For example, Lesh and Kelly (2000) advocated the use of multi-tiered designs involving students, teachers and researchers, over three phases, in which explanatory models were successively developed.

The third standard concerns ethics, the integrity of reporting. Sufficiently detailed description of the study enables accurate replication, albeit in non-identical circumstances. Transparent reporting of researcher experiences, beliefs, and values, and the relevant characteristics of the setting, acknowledgement of potential error,

including all relevant factors contributing to variance, and the rationale for data selection, are important standards. Any claims made, particularly regarding generalisability, are commensurate with the strength of the data available to support them. Significance reflects the quality of the data, not the quantity. The onus is on the researcher to describe their research adequately in an open and honest manner (Schoenfeld, 2002; Schulman, 1997).

## **5.7 Justification of Design**

The research question of this study was “How do students develop their multiplicative thinking and proportional reasoning?” The breadth of the question offered considerable choice in methodology. Testing of a Hypothetical Learning Trajectory (HLT) lent itself to a teaching experiment, in keeping with modern classifications that include such studies under design research methodology (Molina, Castro & Castro, 2007). The HLT was the conjecture under scrutiny (Cobb, Confrey, diSessa, Lehrer & Schauble, 2003). The classroom teaching was interventionist in that it sought to improve students’ understanding, iterative in that evidence of students’ thinking was used to modify instruction and the HLT, and prospective and reflective in that the teacher and I explored and reflected on concepts and situations perceived by us as accessible to students. In keeping with Simon’s (1995) original intention in defining the learning trajectory construct, the study was located in the particular practice of classroom teaching in which the HLT was used as a basis for planning instruction and simultaneously open to modification.

Rather than opting for a linear description of conceptual growth I opted for a construct approach. Kieren’s (1980, 1988, 1993) sub-constructs of rational number were used as the primary organisational frame. The main reason for this choice was that Kieren’s sub-constructs provide central conceptual structures in rational number and have a history of long-term scholarly scrutiny. The sub-constructs and therefore the design of the HLT offered a broad curriculum for the students that allowed for a considerable range of contexts. The data analysis considered other constructs such as decimals and percentages, graphs and probability as required.

While the study was a teaching experiment the research question was focused on the conceptual growth of students rather than on pedagogical issues in regard to teaching multiplicative thinking and proportional reasoning. Mapping conceptual growth required fine-grained records of learning for individual students over a prolonged period of time as a way to evaluate the efficacy of the HLT.

The second focus of the research was on object and co-ordination class theories of conceptual growth, and the implications for transfer of learning. This focus emerged during the course of the study. The individuality and variation observed in students’ learning prompted a retrospective evaluation of the data in terms of match to object and co-ordination class theories. So the consideration of these theories resulted from the teaching experiment rather than informed it as a conjecture “put in harm’s way”.

Literature on conceptual change suggests that a long period is needed to observe the process of significant conceptual change. The two models of conceptual change,

process to objects and co-ordination classes, support different perspectives but equally long-term views. Processes as objects require observations of anticipation of actions on physical entities and that anticipated actions become objects of further thought. Co-ordination class theory demands attention to the impact of different situations in terms of span and alignment, and the preferential cueing and co-ordination of fine-grained knowledge elements. The long-term nature of the research and in-depth consideration of all the work of the case study students, including in-depth teaching interviews, meant that the data was appropriate for testing the HLT and considering models of conceptual change.

Therefore the use of design research and of case studies to present the data was appropriate to answer the research question and the two issues of interest. Buell, Stone, Naeger, Bruce and Schatz (2011) used this methodology to consider how best to implement a learning resource in science. They classify their study as a *design-oriented case study* in which they describe the use of iterative cycles of implementation within a single classroom.

## **5.8 Class Situation**

The study took place in one class of a medium sized middle school. The school was located in a large town with a population of about 18,000 people in the North Island of New Zealand. It catered for students in years seven to 10. Most students at the school were aged 10 to 14 years. The school roll was about 500 students.

Students at the school came from two main geographical backgrounds. Some lived in the country and travelled to the school by bus each day. Most lived in the town and travelled to school by walking, car or bicycle. The decile rating of the school was eight. This rating reflected the socio-economic status of families in the community as gleaned from census data. So overall students came from well-resourced homes and enjoyed positive parental support. The school was selected due to its close proximity, 22 kilometres, to my base. Teachers at the school had just completed two years of professional development with me in 2006.

The class in which the research took place was a digital class. This meant that each student had access to a computer whenever needed. The students elected to be members of the class and their parents paid an additional fee to cover the cost of technology. Several students were subsidised into the class by the school management as it was felt that a digital learning environment was beneficial for them.

## **5.9 The Actors**

### **5.9.1 THE RESEARCHER**

I was an experienced mathematics educator. I had seven years experience as a classroom teacher in primary schools, and then worked as a mathematics education lecturer at a Teachers' College for two years. I had a further 18 years experience working as a mathematics adviser working in schools in New Zealand. This role

involved providing professional development for teachers and the modelling of teaching in classrooms.

In addition to advisory work I was extensively involved in writing resources for teachers and students, in directing professional development projects, leading the New Zealand Numeracy Development Projects and co-ordinating the writing of the New Zealand Curriculum for the mathematics and statistics learning area.

My Masters degree involved research into the learning and teaching of algebra and I had presented research papers on algebraic and multiplicative thinking to several conferences. As one of the authors of the New Zealand Number Framework, I was interested in progressions in student thinking, particularly in multiplication and division and proportions and ratios. These domains needed clearer elaboration in the research literature and had potential in informing further development of the Number Framework.

### **5.9.2 THE TEACHER**

The class teacher was in her fourth year of teaching. She had specific interest in information communication technology and how it could be used in educational settings. In the two years of professional development the school was involved in I had taught and observed in the teacher's class on seven occasions. She was selected because of her willingness to participate in the study, her positive attitude to mathematics teaching and her openness to new pedagogical approaches.

Her management of student behaviour was sound and she taught in a collaborative, interactive way. Group instruction was common and students' ideas were valued and acted on. Students were encouraged to make their own learning decisions and work independently. While mathematics was not a particular area of curriculum strength for the teacher, the experiment was seen by her as an opportunity to learn.

### **5.9.3 ROLES**

In this study the roles of teacher and researcher were combined. While the class teacher continued to play an instructional role she also provided observational feedback to me. In the lengthy times between teaching sequences the teacher taught her class independent of interaction with me. I had a dual role. I taught the majority of lessons during the teaching weeks of the study and gathered data on student learning.

The availability of two teachers was significant for several reasons. It enhanced the learning opportunities for students afforded by the study through small group instruction. This met an ethical requirement for care of student well-being. Secondly the collaborative relationship between teacher and researcher allowed for reflective discussion that enhanced the lessons provided for students. The complexity of ideas taught during the year necessitated that I adopt a teaching role. So, in this research, as teacher and researcher, I was an integral part of the classroom culture and environment.

#### **5.9.4 STUDENTS**

The students who participated in the study were all classified as year eight in 2007. At the beginning of the year there were 26 students in the class, 16 boys and 10 girls. Their ages ranged from 10 years 11 months to 13 years one month at the beginning of the school year. During the course of the year six students left the class and were replaced by four other students, two boys and two girls.

The students ranged considerably in academic achievement. In mathematics the range in achievement at the beginning of the year was from stanine one to eight as measured by Progress and Achievement Tests (PAT) (Darr, Neill & Stephanou, 2006). Parental permission was sought and gained for all students in the class to participate in the study. This permission included ethical consent to use the data for the purposes of educational research and wider dissemination of the findings through academic publication.

#### **5.9.5 CASE STUDY STUDENTS**

At the beginning of the year nine of the 26 class students were identified as potential candidates for case studies. The selection was based on getting a diverse sample by gender, ethnicity and initial academic achievement in mathematics. Extensive data were collected on each student.

From the group of nine students six were chosen for the writing of case studies within this thesis. This further refining of the case study group was made due to limits in the capacity of the thesis to describe the learning of all nine students. Two students were non-identical twins. Their case studies offered interesting comparisons so they were selected for data analysis. One student in the group was not included as she returned to Korea at the end of Term Three (September, 2007). Two other students provided similar gender and achievement dimensions to other students who were chosen.

The students whose case studies were documented in this thesis were (gender, ethnicity, achievement):

Simon (male, Maori, high)

Rachel (female, European, high)

Ben (male, European, high)

Odette (female, Maori, medium)

Jason (male, European, medium)

Linda (female, European, low)

#### **5.10 Phases of the Teaching Experiment**

I undertook to co-operatively teach the class and gather data during four phases during the 2007 school year. In the original plan these phases were to include four weeks in each of the four terms. Ill-health limited me to two weeks during Term Two

but all other commitments were met. I also taught geometry to the class for two extra days in Term Two.

The phases of the teaching experiment were as follows:

Term One: 26 February – 22 March

Term Two: 7 May – 18 May, 6 – 7 June

Term Three: 16 July – 10 August

Term Four: 22 October – 16 November

At the end of each lesson all student bookwork was viewed to gain information about student learning. At times individual students met me to discuss conceptual difficulties or their solutions. The teacher and researcher discussed students' progress during the interval and broadly planned instruction for the following day. I returned to my office to prepare resources and plans for instruction on the following day. In this way the teaching experiment was iterative and reactive to the observed needs of the students.

Between the phases of the teaching experiment the teacher taught the class free of involvement from me. During that time I continued my usual work as a mathematics adviser.

## **5.11 Sources of Data**

### **5.11.1 TEACHING DIARY**

At times during lessons and at the end of processing student bookwork I recorded my observations of the day in a teaching diary. All passages were dated. This diary provided an on-going anecdotal record during the course of the teaching experiment. The purpose of the diary was to capture events and observations in as timely a manner as possible to their occurrence. The diary included comments on:

- Perceptions of students' conceptual obstacles and misconceptions
- Strategies used by students to solve problems including representations used
- Pedagogical issues related to representations, group dynamics, instructional directions
- Notes from the teacher
- Mini-interviews with students
- Results of one-off assessments

The handwritten diary was put in electronic form to capture the data. This made the diary available for searching.

### 5.11.2 INTERVIEWS

Interviews of the nine students selected for the case study group happened on the following dates:

- 21 February (initial)
- 26 March (end of Term One)
- 16 August (end of Term Three)
- 19 November (end of Term Four)

I sought evidence of students' learning in multiplicative thinking and proportional reasoning through use of designed interview questions. There was hope that the interviews might capture the construction of new knowledge. I developed different sets of questions for the high, middle, and low achieving students for each interview (see Appendices 1-2 for examples). The questions usually contained a mixture of problems that ranged across constructs of interest and reflected a balance of situations that were similar and dissimilar to those encountered by the students during recent instruction.

Prior to the interview students were given a photocopy of the interview questions and given time to answer the questions in a private location. Students were encouraged to record their working on the photocopy, though sometimes individuals chose to record elsewhere or mask their recording.

I interviewed students individually and asked them to explain their solutions to the problems. The interview space was usually an office and was free of interruption or distraction. I videoed each interview and created a CD-ROM to preserve the data. The student's individual record contained a transcript of each interview.

Given the dual role of the interviewer, as teacher and researcher, a *teaching interview* protocol was used (Hershkowitz, Schwarz & Dreyfus, 2001). In this protocol, the interviewer's behaviour was deliberately interactive as opposed to the detached behaviour associated with traditional clinical protocols. The interviewer offered assurance at times, sought explanations of ideas, offered alternative ideas and pointed out problematic or conflicting explanations. Although students had prior access to the problems the protocol also used a think-aloud method (Brenner, 2006) in that students explained their strategies aloud and often created new ones in response to questioning.

### 5.11.3 BOOKWORK AND WORKSHEETS

To preserve the data I photocopied all bookwork produced by the case study group students. This written work often included worksheets provided by the teacher and me. Other work came from responses to texts and electronic learning objects available through the Internet. Numbering of the photocopied pages enabled access. At the end of each week, I wrote anecdotal notes from analysis of the bookwork and added these notes to the individual record of each student. The notes included descriptions of the mathematical task involved, the student's response to the task, and interpretive comments about the significance of the work in terms of the development of key sub-constructs.

### 5.11.4 STANDARDISED TESTS

I used two norm-referenced standardised tests to gather global data about the mathematical achievement of all the class students. The school created an *Assessment for Teaching and Learning (asTTle)* mathematics test for its own assessment purposes (Hattie et al., 2004). This test sampled questions in number knowledge and operations, and algebra strands from levels three to five of the New Zealand Mathematics Curriculum (Ministry of Education, 1992). Levels three to five spanned a range of average achievement expected by students aged nine years to 15 years. The same test was used at the beginning of the year (1 March) and the end of the year (7 November). Students received the tests in paper form. The class teacher marked all of the tests. *asTTle* tests returned a judgment on the best fit curriculum level for each domain of the test and the correctness or incorrectness of the students' responses to each item.

*Progressive Achievement Tests (PAT): Mathematics* (Darr, et al., 2006) were also used to obtain norm-referenced achievement data on all the class students. The teacher administered the tests at the beginning of the year (16 February) and at the end of the year (7 December). In line with PAT administration guidelines, students received different test booklets (Booklets 3 – 5), dependent on their perceived achievement level at the time. Different booklets improved the engagement of students in the test.

Processing of the student answer sheets used an online marking service that provided detailed reports on each student. Each report returned data against a common (PATM) scale and provided a stanine for each student in relation to norms for year eight students. The report also specified which items were answered correctly and answered incorrectly.

### 5.11.5 OTHER ASSESSMENTS

At the beginning of the year, on 20 February, all students in the class were interviewed with the *Global Strategy Stage interview (GloSS)*, Form H (Ministry of Education, 2007). Both the teacher and I carried out the interviews. The interview required students to solve number problems mentally and explain their solution strategies. The problems ranged in difficulty across three domains; addition and

subtraction, multiplication and division, and proportions and ratios. Students' answers were classified by the strategy stages of *The New Zealand Number Framework* (Ministry of Education, 2008). A sample of knowledge items from The Numeracy Project Assessment, *NumPA* (Ministry of Education, 2006) was also used. These items covered key knowledge of fractions, place value and basic facts.

All students in the class attempted a pencil and paper test on 16 July, at the beginning of Term Three. The test was in three forms, one for each achievement group in the class. Group A's test (low achieving group) covered a variety of addition, subtraction, multiplication and division word problems and three questions that addressed fractions as measures, operators and frequencies. Group B's test (middle achieving group) contained items about fractions as measures and operators, fraction to decimal conversions, calculating percentages, relational thinking about multiplication of whole numbers, division and measurement.

The test for Group C (high achieving group) had items about fraction to decimal conversions, comparison of fractions as operators, ratio, graphing of a measurement relationship, and comparing the price of clothing with percentage discounts.

#### **5.11.6 TEACHING PLANS**

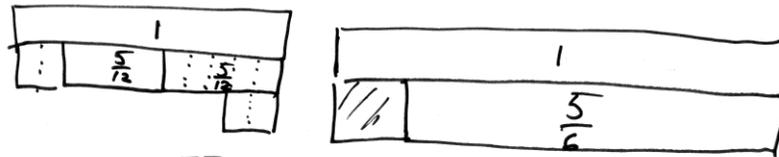
For each teaching week the teacher and I developed comprehensive instructional plans. The plans included whole class activities, the learning outcomes for each group lesson, required resources, book references and independent activities. I attached copies of worksheets developed for each group to each plan and archived them electronically to preserve the data and allow searching. An example of a weekly plan is included as Appendix 3. Powerpoint presentations used for whole class introductory lessons and problem solving lessons were also archived. A small number of video clips were used to introduce concepts. These clips were also archived.

#### **5.11.7 MODELLING BOOKS**

Each term I used a new modelling book for each teaching group. This was an A2 sized scrapbook drawn on with black felt pen. It acted like a small whiteboard. Each book captured the symbolic expressions and equations, diagrams and pictures, and words and stories from group lessons. The teacher or I did most of the recording but occasionally students recorded their strategies in the modelling books.

The modelling books provided information about the tools used by the teacher and me in the course of lessons. Together with the weekly plans the inscriptions gave some insight into how the ideas were developed during lessons and the strategies students used.

$$1 \div \left(\frac{5}{6}\right) = \frac{6}{5} \quad 1\frac{1}{5} \quad 1.2$$

$$1 \div \left(\frac{5}{12}\right) = \frac{12}{5} \quad 2\frac{2}{5}$$


$$1 \div \frac{57}{99} = \frac{99}{57}$$

Figure 24: Example of recording in the modelling books

## 5.12 Classroom Programme

The following section describes key characteristics of the classroom programme. Understanding the events that contributed to changes in student thinking is important background information for the reader. However, the research question was about conceptual change of individual students as a window to transfer of learning and to learning trajectories in multiplicative and proportional reasoning. For this reason, this section only provides a summary of events rather than an in-depth analysis.

### 5.12.1 TYPICAL LESSON

On most days the lesson format involved three main phases:

1. Whole class focus (approximately 15 – 20 minutes)
2. Achievement-based group instruction and independent work (approximately 45 minutes)
3. Whole class sharing of problem solving from group work (approximately 5 – 10 minutes)

Lessons frequently went for longer than the allotted 80 minutes as teachers and students became immersed in the activities. On some days, the above lesson format was abandoned to allow for learning as a whole class. For example, on every second Tuesday the timetable required teaching of mathematics in the afternoon. On these days, the class frequently spent an hour solving mathematics problems in small co-operative groups or worked on an investigation together, such as finding relationships in a dataset using computer software.

### 5.12.2 GROUPING OF STUDENTS

The teacher and I used a variety of methods for grouping students aligned to the purpose of the instruction. We used whole class grouping in several ways:

- Common focus supported by mixed achievement, co-operative grouping

Whole class topics during the year were percentages, figurative numbers, patterns and algebra, integers, powers and exponents, probability and data analysis.

- Common focus with differentiated task

Differentiation of task difficulty enabled all students, irrespective of level of achievement, to access common tasks. Students often selected the level of difficulty they felt comfortable with. This applied particularly to tasks where number knowledge was developed. For example, students selected one of the three levels of difficulty for The Estimation Game (Figure 25) or worked sequentially from easiest to hardest in identifying the function for given graphs (Figure 26).

Problem	Estimate	Actual	Difference
200 - 87			
92 - 46			
134 - 89			
401 - 197			
1005 - 296			
Problem	Estimate	Actual	Difference
26 x 12			
68 x 5			
24 x 33			
37 x 43			
15 x 75			
Problem	Estimate	Actual	Difference
16 out of 26			
34 out of 53			
7 out of 19			
44 out of 79			
88 out of 198			

Figure 25: Differentiated tasks for The Estimation Game

- Whole class sharing of strategies

At the end of each lesson, students attempted problems related to their instruction for that day and invited to record how they solved the problems on the whiteboard. Students from lower achieving groups frequently attempted problems set for the higher achieving students.

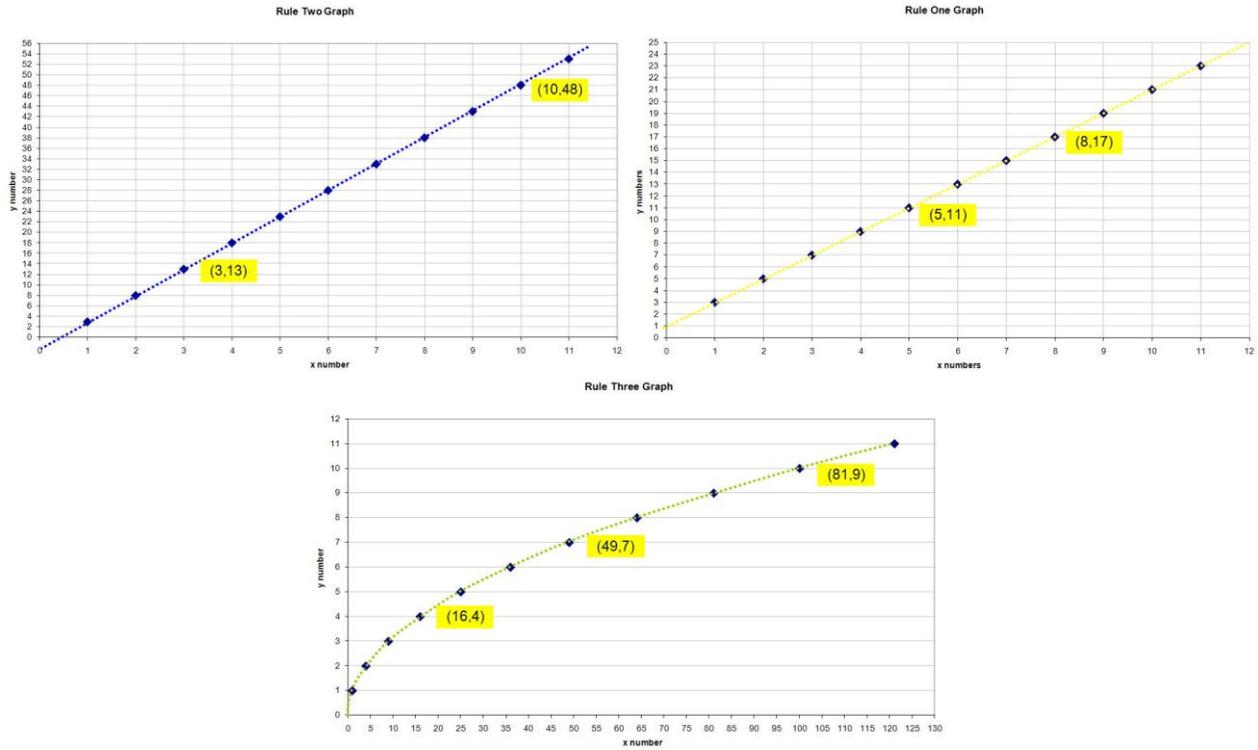


Figure 26: Differentiated graph to function tasks

- Co-operative problem solving

Occasionally we divided the whole class into mixed achievement teams, usually based on social compatibility, to compete in problem solving challenges (Figure 27).

1. What solid does this net fold up to make?

How many different nets for a cube can you find?  
Use the squared paper to record your nets.

Warning: This will not be considered to be a different net.

Figure 27: Example of a whole class problem solving task

Achievement based groups were established for the purpose of providing targeted instruction at appropriate levels of difficulty. These groupings changed considerably throughout the year. Students moved between groups, often at their request, as they felt the challenge was either too little or too great.

Initially three groups were created from assessment data. With two teachers available this allowed for all groups to receive instruction each day. By the end of term the number of students in the high achieving group had increased to the point that the group was split, on the basis of collaborative compatibility. So the content taught to the students in each achievement based group varied. For example, the lowest group began the year working on strategies for calculating with whole numbers while the highest group worked on fractions as measures. Judgment on appropriate content for each group was made by the teacher and me by relating students' responses to tasks against the types of knowledge and corresponding phase/s of the HLT. So the delivered curriculum was different for different students in instructional group situations.

Independent individual work was also a feature of the classroom programme when students were not involved with group instruction. There was expectation that students attempted problems without support initially but students were also invited to discuss difficulties they were having with their classmates and ask for teacher support if needed.

### **5.12.3 INTENDED LEARNING OUTCOMES**

Planning for instruction was responsive on a daily basis. Viewing of all student work at the end of the lesson allowed formative assessment as the basis for planning the next lesson. Consideration of the assessment data with reference to the HLT informed the intended learning outcomes. Sometimes there was revision of topics in situations where students' lack of retention seemed to affect their learning of new concepts adversely. Sometimes there was an assumption of conceptual knowledge from observations so no teaching of the knowledge happened.

An overview of the outcomes is given in Table 14 and is broken down by term and instructional group.

**Table 14: Intended learning outcomes by term and group**

<b>Term One</b>	<b>26 February to 15 March</b>
Whole Class	Basic facts for multiplication and division, estimation, percentages, square, triangular, rectangular and prime numbers, relations
Group 1	Strategies for adding and subtracting whole numbers, deriving from known multiplication facts
Group 2	Strategies for multiplication and division of whole numbers, multiplicative place value, multiplying and dividing by powers of ten
Group 3	Equivalent fractions, ordering fractions, frequency (out of) problems, probability, ratios, fractions as quotients, operations with decimals

<b>Term Two</b>	<b>7 May to 18 May</b>
Whole Class	Integers, multiplication and division basic facts
Group 1	Fractions as partitions of the whole, improper fractions, multiplication facts, finding fractions of sets
Group 2	Equivalent fractions, ordering fractions, adding and subtracting fractions
Group 3	Decimals and percentages as numbers, percentages as operators and frequencies, ordering decimals, recurring decimals, scaling

<b>Term Three</b>	<b>16 July to 10 August</b>
Whole Class	Exponents and powers, patterns in multiplication facts, patterns and algebra, probability, finding rules for relations, finding rules for graphs, problem solving strategies
Group 1	Strategies for subtraction of whole numbers, strategies for multiplication and division of whole numbers
Group 2	Fractions to decimal conversions, properties of multiplication connected to division of whole numbers, ordering decimals, adding and subtracting decimals, quotients as fractions
Group 3	Estimating calculations with decimals, division by fractions, fractions as unknown operators, strategies to solve rate problems, inverse rates, graphing proportional relationships, decimal conversions

Term Four	23 October to 16 November
Whole Class	Mathematical games, data analysis, probability
Group 1	Equivalent fractions, decimal fractions, fractions as operators on whole numbers
Group 2	Ratios, frequencies, percentages as proportions, percentages as whole to whole comparisons, fractions as operators, ratios as percentages, comparing ratios
Group 3	Sampling distributions of a population, finding possible outcomes, scaling of similar triangles, trigonometric ratios in similar right-angled triangles and from a unit triangle, finding unknown lengths using sine, cosine, tangent, Pythagoras theorem, bearings
Group 4	

#### 5.12.4 INSTRUCTIONAL APPROACH

Classroom environments are complex social situations and are therefore difficult to describe. Classifications of instructional situations frequently create dichotomies that fail to adequately capture the interconnectedness of significant variables (Clarke, 2005). While acknowledging the inherent problems, it is worthwhile describing some features of the instructional approach. These features provide information about factors that potentially affected the conceptual growth of students and their transfer of learning across situations.

#### Contextual problem solving

All instruction began with students' attempts to solve a problem. I chose or created problems with an expectation that students would engage in the construction of mathematical ideas through exploration. Choice of contexts reflected the interests of the students and key mathematical ideas. Sport and food were common themes and we used the names of students from the class in scenarios.



5. Alana has two-thirds of a tank of petrol left. It takes four-ninths of a tank to drive to and from Tauranga. How many return trips can she make?



5. The pack contained 825 breakfast crunchies. Stephen ate 456 of them in one sitting. How many crunchies were left in the packet.

Figure 28: Examples of contextual problems

Controlled variation of context encouraged students to see common mathematical structure in different situations. This usually involved fixing a situation and changing the task variables such as number size. For example, I introduced ratio using the situation of mixing paint or cocktails. Students gained some familiarity with solving problems in the first context before the introduction of new contexts. In the teaching of ratio, contexts included raising chickens, planting seedlings, scoring in sports, and characteristics such as eye colour or handedness.

### **Physical actions on materials leading to anticipated actions to ideas as objects for thought**

Physical materials or visual representations were used as models for the problem solving attempts of students. Often problems involved actions on physical objects. For example, I introduced trigonometric ratios through drawing large unit triangles on the carpet with chalk. Equivalent fraction instruction began with folding and cutting of paper strips and led to use of plastic fraction strip manipulatives for adding and subtracting fractions. The students used unifix cubes to build models of powers of three and represent ratios as part-whole relationships.

Students were encouraged to image the results of actions on physical objects (Pirie & Kieren, 1994). Techniques for imaging included masking the objects or asking students to predict the outcome. The aim of instruction was generalisation, so the physical actions were an opportunity to create several examples of a principle in order to describe that principle. For example, equal sharing of circles in a pizza context led students to anticipate the result of using the quotient theorem and represent it algebraically. Presence of physical objects allowed for validation of conjectures.



**Figure 29: Trigonometric ratios introduced by drawing unit triangles**

### Student generated solutions mediated through representational tools

I used representational tools to model student solutions in connection with the use of physical materials and diagrams. Symbols for operations, fractions, decimals, percentages and ratios were common as was the use of tables, graphs, equations and double number lines. I offered representations as ways to record and support thinking rather than as formalised algorithms.

Pattern recognition was encouraged by recording in structured formats that usually provide three examples. Care was taken to promote recursive thinking through sequencing where it was appropriate. I promoted relational thinking through disruption to sequential patterns. At times, it made sense to combine the two methods. In the example below, I used sequencing to highlight the multiplicative structure of powers of two, but disrupted the sequence to promote attention to the properties of exponents in multiplication.

$$\begin{array}{cccccccc} 2^1 & 2^2 & 2^3 & 2^4 & 2^5 & 2^6 & 2^7 & 2^8 \\ 2 & 4 & 8 & 16 & 32 & 64 & 128 & 256 \end{array}$$

$$2^2 \times 2^3 = 2^5$$

$$4 \times 8 = 32$$

$$2^3 \times 2^4 = 2^7$$

$$8 \times 16 = 128$$

$$2^2 \times 2^6 = 2^8$$

$$4 \times 64 = 256$$

Figure 30: Example of sequenced and non-sequenced recording

## Attention to student constructions and cueing preferences

The learning conversations in instructional groups and individual conferences provided formative data that helped to transform students' ideas. Written work sometimes included contrasting views or conceptual questions aimed at revealing student thinking.

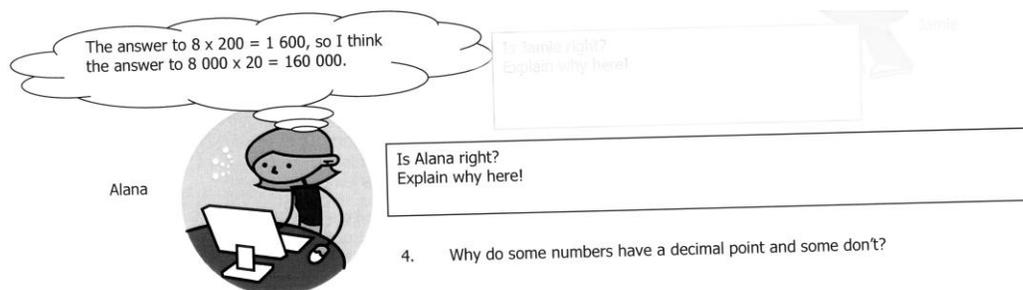


Figure 31: Revealing conceptual ideas through written tasks

Students frequently over-generalised and provided partially correct thinking. For example, Olaf thought  $\frac{7}{16}$  and 40% were the same number because both numbers were one fraction below one-half,  $\frac{1}{16}$  and  $\frac{1}{10}$  respectively (7 May). Incorrect ideas were addressed by contrasting the idea with the outcome of physical manipulation of materials, by connecting the given idea to another idea the student already believed, e.g.  $\frac{1}{4} = 0.4$  (incorrect) with  $\frac{1}{2} = 0.5$  (believed), or appealing to pattern.

I deliberately created situations to modify students' over-generalisations and partially correct ideas. In Term Four, Group Two students did not attend to the part-whole relationships in comparing ratios of blue and yellow paint. The students often used additive comparison of parts inappropriately. I used a spreadsheet to show the effect of additive difference as the whole changed in ratios (see Figure 32).

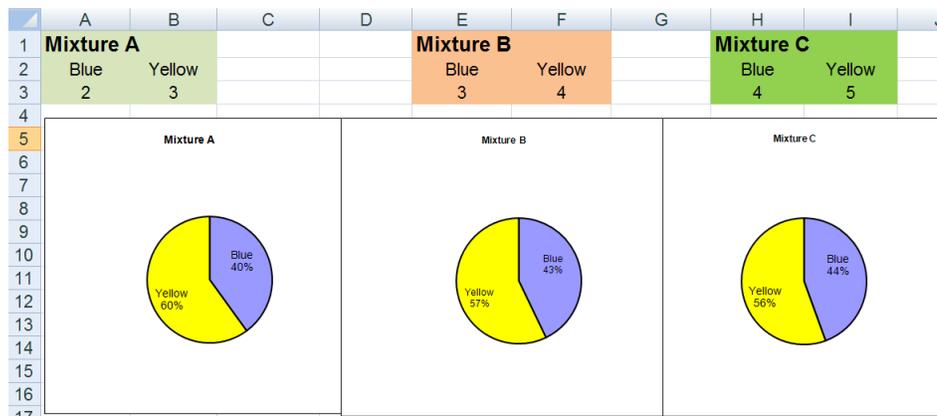


Figure 32: Spreadsheet design for comparing ratios

Alteration of task variables served to privilege some problem solving strategies over others. Careful selection of numbers in problems helped address students' cueing preferences. For example, in Figure 33,  $81 \div 3 = \square$  was used with students to introduce the use of proportional thinking as an alternative way to solve some division problems. The students' strategies at that point relied on standard place value partitioning.

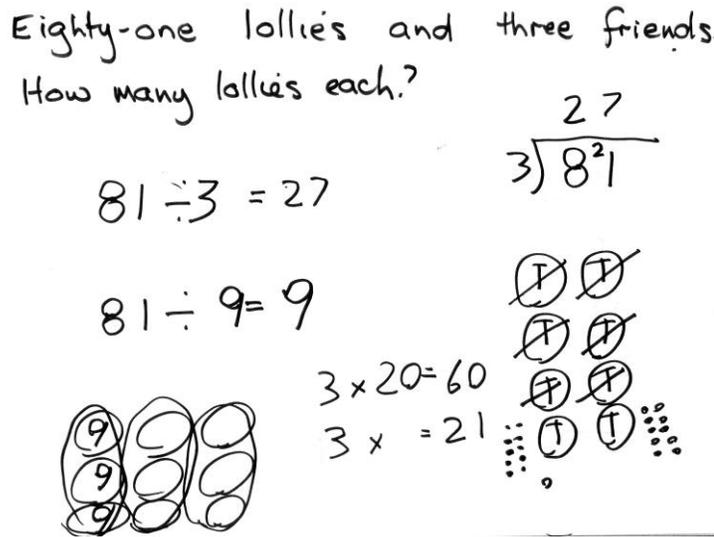


Figure 33: Use of task variables to address cueing preferences

## 5.13 Standards of Evidence

### 5.13.1 DATA PROCESSING AND SELECTION

I considered all the relevant available data in making judgments. The conjecture that initially framed searching of the data was the Hypothetical Learning Trajectory. The major data sources were the teaching diary, student work samples, and the weekly plans. I categorised all data by the sub-constructs of the HLT, and compared and contrasted data from different sources. The grey coding in the Learning Trajectory Tables (see Appendix 4) acknowledged disparity between sources to indicate that evidence of scheme and knowledge possession was often inconclusive. Examples of written work, teaching diary notes and interview transcripts appear in the case studies of students in this thesis.

In most instances, choice of these samples was representative of a wider body of data though some selection was for clarity and/or interest. I selected some data items because they represented events that I considered significant in the development of a student's thinking. Typically, these examples involved resolution of cognitive conflict or reversion of knowledge thought to be secure previously.

The common occurrence of situational variation in student thinking prompted analysis of models of conceptual growth. The conjectures of object and co-ordination class theory arose from a review of the literature on transfer (Chapter Two) and provided a secondary analysis. Some explanation of situational variation was the aim of that analysis.

### **5.13.2 INDIVIDUAL STUDENT CASE NOTES**

I compiled a cumulative electronic file for each of the nine students in the case study group for analysis. The file was organised temporally and each item of data dated for access. It consisted of data from tests, interview transcripts, relevant sections of the teaching diary, work samples and notes compiled on each student at the end of teaching weeks. The electronic format allowed for easy searching of the data and the temporal layout facilitated co-ordination of the data sources.

### **5.13.3 LEARNING TRAJECTORY TABLE**

From the literature search given in Chapters Three and Four I created a Learning Trajectory Table (Appendix 4). The table was a composite of the stages of development proposed by researchers for multiplicative thinking and proportional reasoning. It consisted of four main phases hypothesised as temporally related in terms of students' development of sub-constructs.

The table brought together a complex collection of different knowledge forms considered important in the development of multiplication and division of whole numbers and rational number. These forms included factual knowledge, e.g. basic multiplication facts, theorem knowledge that anticipated processes, e.g. non-unit fractions as iterations of unit fractions, and knowledge of procedures, e.g. combining and separating fractions with unlike denominators.

A table was developed for each student in the case study group at four time points; beginning of the year, end of Term One (March), end of Term Three (October) and end of year (December). Data from interviews, tests, observations recorded in the teaching diary and collected work samples was co-ordinated to make judgments about students' achievement for each knowledge element. I triangulated between data sources as much as possible in an effort to enhance the trustworthiness of my judgments. At times triangulation was not possible given the breadth of knowledge in the table and the restricted opportunities for assessment. At all times I undertook to represent students' achievement fairly given the available evidence.

At given time points each cell was shaded white, grey or black. White indicated that insufficient data was available to make a judgment; grey indicated that the data showed the student demonstrated understanding and proficiency on occasions and black indicated strong evidence that the student had demonstrated secure understanding and proficiency across a variety of situations. The colour shading represented conceptual growth clearly but had associated limitations.

White shading meant that no conclusion could be drawn but visually created an impression that knowledge was not possessed. Black shading involved a judgment that knowledge possession was secure given naturally limited data items. Given hypothesised situational variation in knowledge possession and activation from co-ordination class theory (diSessa, 2008), any claim to certainty about scheme and knowledge possession was speculative. I made assumptions that secure knowledge possession at one phase meant that contributory knowledge from previous phases was also secure. While it was sensible to make these assumptions, some were speculative. Grey shading represented a state of partial application of knowledge. In some cases a range of knowledge was involved so one shade of grey did not accurately represent the improvements in possession of some knowledge items over time.

#### 5.13.4 LEARNING TRAJECTORY MAPS

Data from the Learning Trajectory Tables for each student at each time point were further synthesised into a Learning Trajectory Map. Concentric hexagons represented phases of the Learning Trajectory Table with progress to a later phase represented by movement to an outer hexagon. The corners of each hexagon mapped to a phase in one of the sub-constructs, multiplication and division of whole numbers, measures, quotients, operators, rates and ratios (see Figure 34). White, grey, and black circles represented students' achievement translated from their Learning Trajectory Table. Lines connecting the circles gave the profile shape, but did not indicate the state of connection between the sub-constructs.

Figure 35 shows an example of how data from the learning trajectory table was represented as dots on the learning trajectory map. The figure shows both representations for Rachel at the end of Term One. In the multiplication and division of whole numbers sub-construct Rachel derived multiplication answers using the properties of multiplication and demonstrated partial consistency with applying other types of knowledge. This partial alignment was represented as a grey dot in the Multiplication and Division corner of the third largest hexagon. This location aligns with the Complex Multiplicative phase of the learning trajectory table. Rachel demonstrated consistent alignment at the Fraction Equivalence phase for the Rates sub-construct as represented by black shading in the table. This consistency is represented on the learning map as a black dot in the Rates corner of third largest hexagon which corresponds to the third phase of the learning trajectory table, Fraction Equivalence. Similarly, Rachel's partial consistency at the Fraction Comparison phase for the measures sub-construct is represented by a grey dot on the fourth largest hexagon.

Several acknowledgements of the limitations of the graphing of progress in this way require mention. There is considerable complexity and breadth to the knowledge contained in the learning trajectory framework. It was neither possible, nor desirable, to assess all of these ideas at given points in time. Some assumptions were made about the location of circles where data was not available based on the data from previous time points. Single dots represent application of multiple scheme and knowledge types so some conceptual growth did not show as darker shading or

outward movement between time points. Some students improved their knowledge in part of a sub-construct but this did not alter the overall judgment for the whole sub-construct.

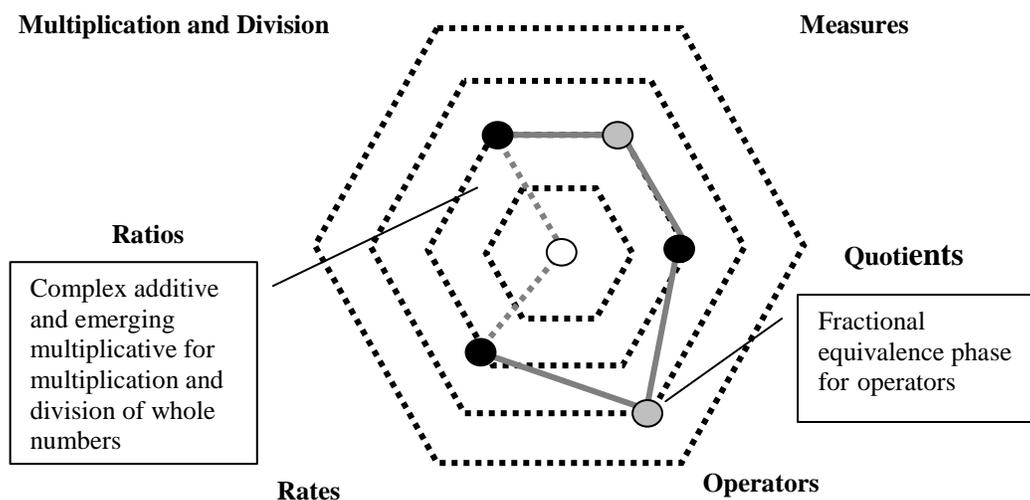
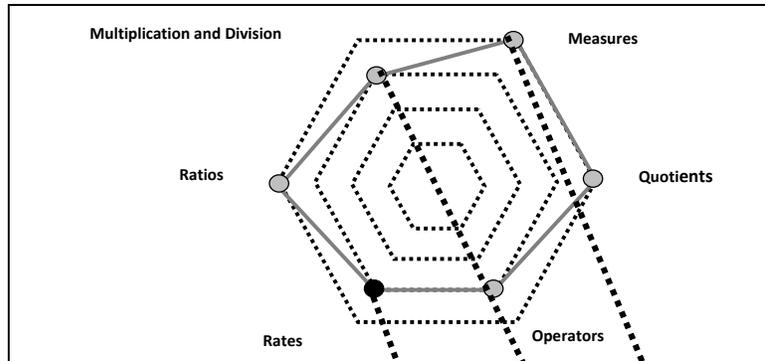


Figure 34: Example of Learning Trajectory Map

No movement of circles outwards occurred unless supported by data. This means that some improvements in span and alignment may have gone undetected at the time they first occurred. A circle for a given construct was only located at an earlier phase if evidence existed of reversion. In other words, some reversion in ideas may have gone undetected due to unavailability of data.

This map provides a concise if simplistic view of conceptual growth in order to examine improvements in span and alignment. Span refers to the breadth of situations to which a student applies their knowledge. Outward movement of circles indicates improvement in span. Alignment refers to the consistency with which a student applies the knowledge in solving problems. A white circle indicates the absence of evidence that the student has possession of the knowledge required. Progression from grey (partial possession) to black circles (consistent possession) indicates improvement in alignment.



Whole Number Operations	Counting on and simple additive partitioning	Complex additive and emerging multiplicative	Complex multiplicative	
Multiplication	Skip counting and repeated additive composition Hybrids based on known facts and counting/adding	Deriving multiplication facts using distributive property and doubling/halving Known multiplication facts	Deriving multiplication answers using all properties of multiplication Relational thinking about multiplication (operating of the operator)	
Division	Equal sharing using composite counts or additive build-up	Division by multiplicative build-up	Known division facts Connection of partitive and quotitive division Division by connecting multiplication properties and inverse	
Construct/Phase	Unit splitting and replication	Unit co-ordination	Fractional equivalence	Fractional comparison
Part-Whole	Halving based splits More equal parts so smaller parts	Repeated splitting preserving equivalence Numerator as iterations - Denominator as iterated	Equivalence as multiplicative relation Equivalent fractions as same number	Order of fractions Difference between fractions
Measure	Physical comparison of equivalent measure	Size of improper fractions in relation to whole numbers	Combining and separating fractions with related denominators Measurement of one with non-unit fractions	Combining and separating fractions Measurement of fraction with fraction
Operator	Unit fraction related to amount	Non-unit fraction of amount as iterations of unit fraction	Non-unit fraction multiplicatively (variable unknown) Operator and operated affect size of result	Properties of multiplication and division with fractions Fraction as unknown operator
Quotient	Practical equal sharing by halving (restricted shares)	Practical equal sharing (open shares)	Quotient theorem, $a \div b = \dots$ Equal and unequal shares	Comparison of shares using equivalence Remainders as fractions
Rate/Ratio	Practical duplication of rates	Equivalent rates by replication	Conversion to unit rate Equivalent rates by multiplication and division	Comparison of rates (measures unchanged) Graphs of linear rates Comparison of rates (measures changed) Inverse rates Comparison of ratios
	Part-whole relationship in ratios Practical duplication of ratios	Equivalent ratios by replication	Equivalent ratios by multiplication and division Part:part Whole:whole	

Rebecca – End of term one Learning Trajectory Map

Figure 35: Relationship of Learning Trajectory Table and Map

## 5.14 Ethics

In keeping with the proposal submitted to the Ethics Committee of the granting University, I gained written permission from both students and their caregivers prior to participation in the study. I obtained written permission from the class teacher and the principal of the school. The permissions granted to me rights to use the written and video evidence for the purpose of the study and for further use in research papers. The teacher and any student had the right to withdraw from the study at any time.

I undertook to preserve the anonymity of the school, teacher and students in the writing of the thesis and any further papers. In this thesis, details of the school are non-specific making identification difficult and “the teacher” is used to protect the identity of the teacher.

I considered potential harm to participants in the study. Joint planning time between the teacher and I was minimised as much as possible and the teacher’s role as classroom leader was respected at all times. The teacher appeared to appreciate my presence in her classroom. She wrote at the end of the experiment:

I have thoroughly enjoyed the experience and feel that my math teaching has been strengthened greatly through observing you on a variety of levels. (10 December)

The presence of two teachers in the room for mathematics instruction enhanced learning opportunities for students. I took up the normal role of classroom teacher both educationally and pastorally and became involved in other class events when appropriate. I made every effort to fit in with the normal routine of the class. The context for study respected the personal identities, interests and cultures of students in the class. Review of all student work between class lessons provided opportunities for the iterative review of instruction. I held Individual conferences with students when learning needs became apparent and used an assessment regime consistent with normal classroom practice. The case study students appeared to enjoy the opportunities for interviews and sometimes requested to see a video playback of proceedings. I openly shared the results of assessments with the students.

## CHAPTER SIX: CASE STUDY OF BEN

### 6.1 Beginnings

#### 6.1.1 PERSONAL CHARACTERISTICS

Ben was a male of European ethnicity aged 11 years old one month at the start of the school year. Therefore, he was younger than most students in the class by a full year. He struggled to interact socially with the older boys and had just one close friend who left part-way through the year. At that point Ben gravitated to other able but immature boys in the class.

Occasionally he was arrogant and aloof towards his fellow classmates in an effort to appear confident. Other students did not always appreciate his considerable wit. This made him sometimes unpopular and the subject of exclusion. However, he was usually keen to join in class events and participate in sports. Ben listed his interests as reading fantasy books, collecting dragon models, and playing on the computer.

Ben showed variable application to his studies. When motivated he applied himself to the point of frustration at times and was very intent on getting answers correct. At other times, he did not apply himself. He showed flexibility of thinking at times and rigid adherence to routines at others. Ben was very able at all aspects of his schoolwork and was a willing participant in discussions. He entered the New South Wales mathematics competition and narrowly missed gaining a merit certificate.

#### 6.1.2 INITIAL TESTS

Norm-referenced tests of mathematics used at the beginning of the year confirmed that Ben was achieving at a level well above average for his class. On an AsTTle test he was judged to be working at Level 4P (4 Proficient). This meant that he was achieving competently at level four of the mathematics curriculum, achieved on average by students during years eight and nine (Ben was in year eight).

His PAT result placed him in Stanine 7 which put him in the top 23 percent of students for his class level. He got all but three number knowledge and operations questions correct. Ben's correctly answered questions covered knowledge of ordering decimals and simple fractions, whole number place value, integers, and many examples of whole number calculation, including simple rates.

He was unable to answer questions that involved a complex rate, a percentage of an amount, and an example of estimation. In algebra he was able to predict further members of number and spatial patterns but had difficulty interpreting time and distance graphs, and identifying which equation solved a given problem.

#### 6.1.3 INITIAL GLOSS INTERVIEW

Ben also performed at an above average level during his initial GloSS interview (20 February). He correctly solved contextual problems that required the mental

calculations  $89 + \square = 143$  and  $5.33 - 2.9 = \square$ . In both cases he rounded the number to be subtracted and compensated. For example, he rounded 2.9 to 3, worked out  $5.33 - 3.0 = 2.33$  then compensated to find 2.43.

Ben was not as strong in the multiplication and division domain. He answered the following question:

The builder has 280 posts. She needs eight posts to build a pen.

How many pens can she build?

Ben attempted to carry out a written algorithm in his head which resulted in an incorrect answer of 32.5 pens due to miscalculating  $280 - 240 = 20$ . In the previous multiplication question he was able to calculate how many doughnuts were in nine packets of 13 doughnuts by leveraging off his knowledge of  $12 \times 9 = 108$ , i.e.  $9 \times 13 = 108 + 9$ .

In a big “Canterbury Colours” winegum packet there are 24 reds and 16 blacks.

A smaller packet with same mix has a total of 10 winegums.

How many black winegums are in that packet?

For the problem above Ben coordinated the 24:16 ratio to get a whole of 40 winegums. He realised that this total was four times the small packet so concluded that the small packet must contain four black winegums because “four times four is sixteen.”

Ben used mental calculation strategies for addition and subtraction of whole numbers and decimals but relied on imaged algorithms for division. He had a preference for rounding and compensating to solve addition and subtraction problems with whole numbers and decimals. He understood the distributive property as it applied to both multiplication and division of whole numbers. His strategy with the Canterbury Colours ratio problem showed that he co-ordinated the part-whole relationships and understood how to replicate and reduce ratios using a common factor.

Ben’s knowledge was also strong. He knew all his basic facts for addition, subtraction, multiplication and division and appeared to know the place value of decimals, e.g. How many hundredths are in all of 2.097? He recognised simple equivalent fractions, converted percentages to decimals and correctly ordered a set of three fractions;  $\frac{1}{2}$ ,  $\frac{2}{3}$ , and  $\frac{7}{10}$ . His only error was  $45 \div 6 = 7.3$  rather than 7.5, in which he recorded the remainder, three, as a decimal.

#### **6.1.4 INITIAL NUMBER INTERVIEW**

The initial number interview (21 February) consisted of six questions designed to assess Ben’s knowledge of the rational number sub-constructs. A parallel interview was used at the end of Term One.

In the first question two boys delivered pamphlets in a ratio of 2:3. Ben was asked what fraction of the pay each boy should get.

B: Kyle's getting two thirds because he only delivers two pamphlets for every three that Joe delivers.

I: Oh, so he should get two-thirds of whatever Joe gets.

B: Yeah.

I: So if they got 100 dollars how much should each person get then?

B: (Long pause) If they got paid 100 dollars between both of them?

I: Yes.

B: Okay...

I: You can scribble down things if you want to (handing over the pen).

B: Let me think...Joe would get \$33 while Kyle gets \$66, I think.

I: You could check to see if \$33 is two thirds of \$66.

B: No \$33 is not two thirds of \$66...

I: Now, there's a challenge to split \$100 so Joe gets two thirds of what Kyle gets.

B: (Pause) \$22...\$44 is two thirds of \$66.

I: And what do 44 and 66 add up to?

B: 110 (disgusted).

Ben focused on a part to part relationship within the ratio. This made it difficult for him to partition the whole pay of \$100. The transcript showed that Ben had considerable knowledge about fractions as operators, e.g.  $\frac{2}{3}$  of 66 is 44.

In a measures context Ben was asked to find a new road sign between one-half and two thirds. He wrote  $\frac{3}{5}$ . The interview showed that Ben used knowledge of one-half as a benchmark and mental images of three-fifths and two-thirds.

B: Because  $\frac{3}{5}$  is bigger than  $\frac{1}{2}$  and smaller than  $\frac{2}{3}$ .

I: And why did you settle on  $\frac{3}{5}$ ? Was there a reason?

B: Well  $\frac{3}{4}$  was bigger than  $\frac{1}{2}$  and  $\frac{2}{3}$  so I couldn't do that and  $\frac{1}{4}$  is smaller than both of them.

I: So quarters are off the menu.

B: Yeah, okay so no quarters, I tried thirds but  $\frac{1}{3}$  is smaller than  $\frac{1}{2}$  so I couldn't do that.

I: So you settled on fifths but how do you know that  $\frac{3}{5}$  is bigger than  $\frac{1}{2}$  but smaller than  $\frac{2}{3}$ ?

B: Because five is an odd number and it is not divisible by two so it would be bigger because  $\frac{1}{2}$  can kinda go into  $\frac{3}{5}$  (meaning that  $\frac{3}{5}$  must be greater than  $\frac{1}{2}$ ).

I: I think I know what you are saying but how do you know it is smaller than  $\frac{2}{3}$ ?

B: I did a little graph in my head and saw that  $\frac{2}{3}$  was bigger.

I: But what did you see in your head, pies or something like that?

B: Yeah, pies. I saw these pizzas, kind of.

Given Ben's knowledge of equivalent fractions the transcript was surprising. He went on a sequential journey through increasing denominators until a fraction met the criteria. There was subtle equivalence thinking in his confidence that three-fifths was greater than one-half. Relying on visual images of two-thirds and three-fifths produced a fortuitous result given the closeness of these fractions.

Ben solved a rate problem in the same test by finding the unit rate and scaling, i.e. 48 cakes in 16 hours equated to three cakes every hour so 27 cakes in 9 hours. He did so by working out "How many sixteens go into 48?" and knew that this meant "that every hour he can bake three cakes."

A measurement question required Ben to find out how many 375 mL glasses could be filled with nine litres of juice. He recorded the following algorithm (Figure 36).

The image shows a handwritten long division problem. On the left, the number 375 is divided by 9. The quotient is 41, and the remainder is 6. The numbers are written in a cursive, handwritten style. To the right of the division, the text "41 glasses" is written in the same cursive style.

Figure 36: Ben used an algorithm for division (21 February)

His transposition of dividend and divisor caused no apparent contradiction for Ben in the interview. His attendance was to reasonableness of answer size rather than whether his calculation matched the conditions of the problem.

B: I just did a little divisiony thing and she had 6 millilitres left over

I: Or six somethings left over, so 41 glasses, okay. Forty-one is quite a lot.

B: Yes, but nine litres!

Ben's thinking about a percentage problem involving calves ( $\frac{9}{30} = \square\%$ ) revealed weaknesses in his knowledge of percentages and decimals. His initial answer of 33% was correct but uncertainty about the decimal remainder caused Ben to change his strategy. His recorded answer of twenty-seven was probably the result of nine multiplied by three. Despite uncertainty about how to calculate the answer, Ben proposed a likely possibility as a way of sufficing.

B: I divided nine into 30 but I couldn't do it.

I: What do you mean you couldn't do it?

B: The remainder was point something.

I: Can you show me? Can you write it down somehow?

B: I just did it in my head, nine divided by 30, and I tried to round off the percentage and I got 27.

I: Maybe something to do with nine threes?

B: Yeah, I came up with 33 but I didn't know what to do with the decimal point.

The final question compared the shares of five boys with three pizzas to three girls with two pizzas. Ben used a rate strategy based on cutting all the pizzas into fifths. He compared 15 pieces shared between three boys with ten pieces shared between three girls and concluded that the girls got more.

I: So the boys here got how many fifths each?

B: Three.

I: And the girls...?

B: They will get three pieces each and there will be one left over.

This indicated Ben's preference for using rates particularly when the quantities were whole numbers of units, 15 fifths and 10 fifths respectively.

### 6.1.5 INITIAL SUMMARY

Ben's knowledge resources in number were considerable. He had sound knowledge of basic facts, whole number place value, and mental strategies for addition and subtraction. With division problems he tended to reverse the dividend and divisor and rely on imaged algorithms. He appealed to reasonableness of answer size rather than checking to see that his measurement of "what with which" related to the problem conditions. Ben was unsure about remainders that resulted from division and did not have reliable connections between fractions and percentages.

Ben had a strong preference for applying rate base solutions and used multiplication and division to replicate or reduce rates and ratios. He understood the part-part relationships in ratios but was inconsistent in applying the part-whole relationships. Ben appeared to understand that fractional operators were important in connecting numbers in rate and ratio pairs. In measure situations he did not apply his knowledge of equivalent fractions and he had a process view of sharing situations that involved creating equivalent pieces so he could apply rate thinking.

In general, Ben has excellent preparedness for developing proportional reasoning. There was evidence of partial establishment of co-ordination classes for operators, rates and ratios but for measures and quotients there was much to be constructed.

## 6.2 Progression in Whole Number Operations

### 6.2.1 DEVELOPMENT OF MENTAL STRATEGIES

Some whole class work during the year allowed Ben to rehearse and develop his knowledge of operations with whole numbers, particularly his mental strategies and basic fact knowledge. He became very fluent at identifying common factors in whole numbers though games and other activities. For example, in the estimation game Ben had a short time to estimate the product of two whole numbers. His estimates got progressively closer to the actual answer over the period of six days. Ben used strategies such as doubling and halving (see  $5 \times 74 = 37 \times 10$ ) and rounding with compensation (see  $7 \times 97 = 7 \times 100 - 21$ ). Strategies he used for estimation rapidly became calculation techniques.

Problem	Estimate	Actual	Difference
$8 \times 68$	544	544	0
$89 \times 6$	534	534	0
$5 \times 74$	370	370	0
$63 \times 9$	567	567	0
$7 \times 97$	679	679	0
Total differences			0

Problem	Estimate	Actual	Difference
$26 \times 16$	416	416	0
$86 \times 5$	430	430	0
$18 \times 34$	612	612	0
$28 \times 22$	610	616	6
$14 \times 38$	536	532	4
Total differences			10

Figure 37: Ben used a variety of estimation techniques for multiplication (2 March)

Ben was highly motivated by practice sheets that presented multiplication facts in a variety of ways. This activity seemed to benefit him in connecting representations such as fractions as quotients and division algorithms that he previously found troublesome (see Figure 38).

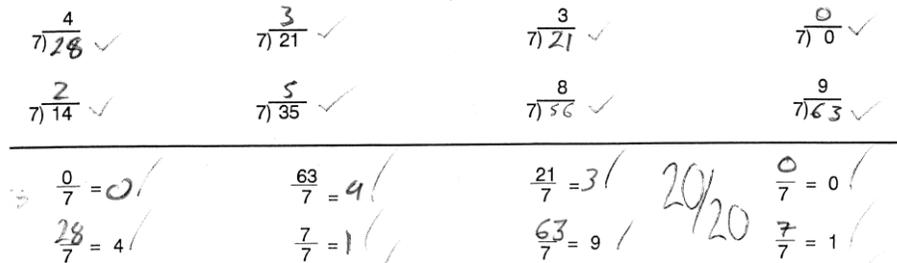


Figure 38: Ben practised multiplication facts in a variety of representations (23-27 July)

Ben also completed grids of multiplication and division facts rapidly where the unknowns required application of inverse operations, e.g.  $7 \times \square = 63$  or  $\square \times 6 = 18$ .

The aim of some activities was to encourage students to operate as comfortably with fractional or decimal factors as they did with whole numbers. In the triangular puzzles (see Figure 39) Ben found three factors for the number in the middle. One of the three factors was missing at first. His answers showed developing comfort with using decimals in multiplicative situations. In norm-referenced tests at the end of the year Ben answered all but one item about whole numbers correctly. These items included calculating powers of whole numbers, estimation of addition, subtraction and multiplication answers, and order of operations.

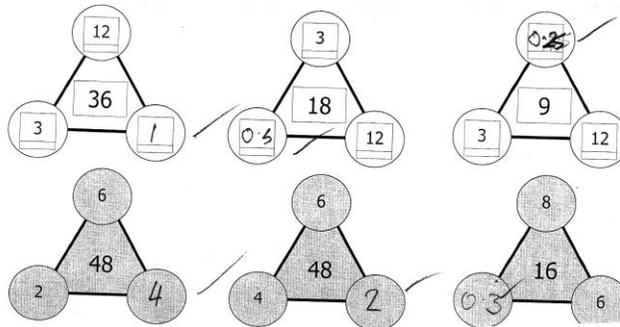


Figure 39: Ben solved missing factor problems using decimals (12 - 16 November)

## 6.2.2 RELATIONSHIPS IN NUMBER PAIRS

Some instruction involved finding relationships within and between square, rectangular and triangular numbers. Ben understood these families of numbers and could detect if a number was prime or non-prime. He also found square roots of numbers.

With function machine activities using computer spreadsheets Ben developed considerable expertise in finding the relations governing sets of ordered pairs, including square and triangular numbers.

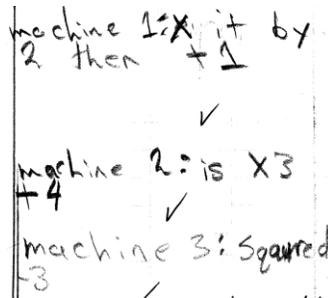


Figure 40: Ben found the relations governing ordered pairs (23 March)

Ben also applied whole number operations in a series of pattern and relationship activities. He solved linear equations with the support of strip diagrams and found unknown values for both dependent and independent variables in matchstick patterns (see Figure 41). Ben applied multiplicative thinking rather than additive recursive thinking to these problems. He seemed to transfer the undoing nature of inverse operations to these situations. He also found rules for relations including simple powers, quadratics and proportions, e.g.  $n \div 4 + 2$ . Though not explicitly taught the conventions of algebraic symbolism Ben readily invented expressions to represent the relations he found.

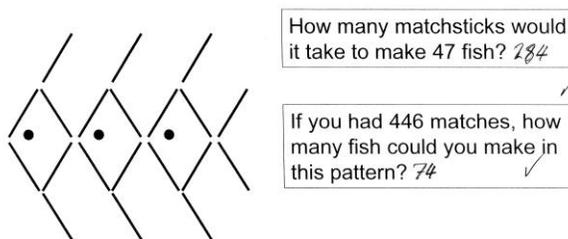


Figure 41: Ben used multiplicative thinking to find missing values for a linear relation (1 July)

## 6.3 Development of Rational Number Sub-constructs

### 6.3.1 PART-WHOLE AND MEASURES

Considerable instruction in Term One was devoted to learning about fractions as measures. At first, Ben struggled to understand that equivalent fractions were the same size in relation to a fixed referent one. He understood that the size of a fraction depended on the relative size of both numerator and denominator and used multiplicative thinking to estimate the proximity of given fractions to benchmarks

such as zero, one-half and one. In Figure 42 Ben decided which benchmark the fraction  $\frac{45}{98}$  was closest to and which fractions lay half-way between  $\frac{2}{5}$  and  $\frac{1}{3}$ , and  $\frac{3}{4}$  and  $\frac{2}{3}$ . Note that he expressed the half-way fractions in non-conventional form.

Alice and Bob are  
 because if you  
 double 45 you  
 only get 90 but your  
 total number is  
 98 which is 8 more  
 than 90. So if  
 you add more  
 4 more to 45 and  
 you will have  
 exactly 2

What fraction is half way between  $\frac{2}{5}$  and  $\frac{1}{3}$ ?  $\frac{5.5}{15}$

What fraction is half way between  $\frac{3}{4}$  and  $\frac{2}{3}$ ?  $\frac{8.5}{12}$

Figure 42: Ben compared a fraction to benchmarks and found a fraction between two fractions (28 February and 1 March)

In the second week, Ben became more proficient at relating splits of unit fractions back to the referent one. His knowledge of basic multiplication facts helped him in freeing up memory resources so Ben could generalise similarity across the splitting situations he encountered.

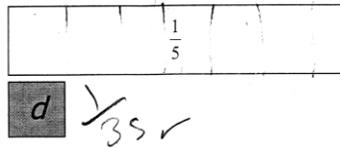


Figure 43: Ben co-ordinated the split of a unit fraction with a referent one (5 March)

Ben used his emerging fluency with equivalence to build number lines with fractions. Fraction strip materials were available and Ben used these materials to establish the relative position of simple fractions. While there were occasional slips (see placement of  $\frac{5}{8}$  in Figure 44) there were also signs of co-ordination of numerator and denominator in Ben's judgment of size relations.



Figure 44: Ben created a number line for fractional numbers (7 March)

In his end of Term One interview Ben decided that two-fifths was between one-quarter and one-third. As in his initial interview Ben relied on a combination of visual image and simple equivalence knowledge. The class work on equivalent fractions, particularly with linear ordering, had no impact on his cueing preference. Note the inference that two-quarters (one-half) are bigger than one-third so two-fifths are smaller than one-third since fifths are smaller than quarters.

I: How did you decide that two-fifths is smaller than one-third?

B: I drew it up in my head and one-third is slightly bigger than two-fifths but I couldn't use two-quarters, and a fifth is smaller than a quarter, and if I put in two-sixths that's the same as one-third.

By early in Term Three Ben was introduced to division by a fraction as measurement using length based plastic manipulatives, e.g.  $\frac{5}{6} \div \frac{2}{3} = \square$  as "How many two-thirds measure five-sixths?" He recognised the pattern of digits from three examples, i.e.  $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$ , and applied this to a range of problems (see Figure 45). While it was not clear that Ben fully understood the measurement relations and units that produced the symbolic pattern, he did appreciate its simplicity and the structure of problems to which it applied.

Jay has three and a half blocks of chocolate left from Easter.  
Each day he eats two-fifths of a block.  
How many days of chocolate does Jay have left?

$$8 \frac{3}{4}$$

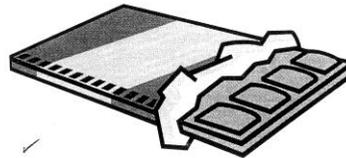


Figure 45: Ben divided by a fraction (6 August)

In his end of Term Three interview (16 August) Ben solved a structurally similar problem. In a car travel context he worked out  $1 \frac{1}{2} \div \frac{2}{5} = \square$ . Ben felt that rounding was appropriate in the context.

B: One tank is five fifths and half a tank is about three-fifths (leads to one and a half as eight-fifths), two and a half fifths. So four trips takes two, four, six, eight-fifths... four trips.

Strong use of equivalent fractions and rate based strategies characterized his approach to measures problems.

### 6.3.2 QUOTIENTS

In Term One Ben's strategy for solving quotient problems was consistent. Given any situation  $a$  objects shared between  $b$  parties, he divided each object into  $b$ ths and shared the pieces equally. His answer was always  $\frac{a}{ab}$ . For example, in Figure 46 he shared eight sausages between five dogs.

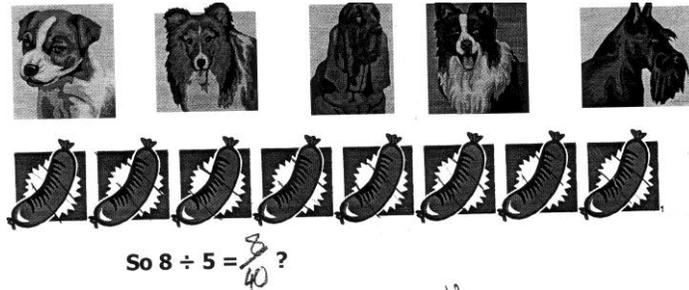


Figure 46: Ben's partitioned each whole by the quotient (18 March)

At this point Ben had an anticipatory scheme for finding fair shares that involved dividing the objects in the required number of pieces. In his end of Term One interview (25 March) Ben compared the shares of three boys with two pizzas with five girls with three pizzas. He wrote "boy(s) get  $\frac{3}{6}$  more." Ben drew lines that cut each pizza into sixths.

I: Why did you choose sixths?

B: Because that was the easiest number that they both go into (meaning two and three).

I: So you can divide 5 into 6 nice and easily.

B: No, not as easily, but it was pretty easy...um...So the boys each got three pieces, ah...four pieces.

I: So how would you write that? How much pizza did they each get?

B: They got four sixes each.

I: And what about the girls?

B: The girls each got one sixth of each pizza with a whole piece left and so the girls got five sixes which one six left over. So the girls got three sixes and there were three sixes left over.

I: So what did you do with the extras?

B: I changed it to a decimal which was...now what was the decimal?...point six? I don't remember what the decimal was but the boys got more pizza.

I: But you can't tell how much more.

B: No.

In this transcript, Ben confused sixths and sixes in his speech but not conceptually. He still cued iterative sharing first but now named the shares with reference to a single pizza, e.g. four-sixths of a pizza. The anticipatory scheme offered by the quotient theorem,  $a \div b = \frac{a}{b}$ , was not used.

In Term Three quotients occurred incidentally through fraction to decimal and percentage conversions and through basic multiplication practice. Ben's approach to sharing problems changed during Term Three. In his end of term interview (16 August) Ben applied the quotient theorem to determine the shares of two boys with five pizzas and three girls with eight pizzas respectively. He used equivalent fractions to compare the shares of  $2\frac{1}{2}$  pizzas with  $2\frac{2}{3}$  pizzas and wrote "Girls get  $\frac{1}{6}$  more."

Greater fluency with equivalence in measurement and ratio/frequency had a complementary effect on Ben's co-ordination class for quotients.

Ben confirmed the sound establishment of his co-ordination class for quotients in his interview at the end of Term Four (19 November). He compared the equal shares for five girls with three pizzas to eight boys with five pizzas. Ben used the quotient theorem to get shares of three-fifths and five-eighths of a pizza respectively.

B: They don't have a common factor that I can divide them down to.

I: So what can you use to compare them?

Ben used his fingers to track progress through the multiples of eight. He got 48 in error instead of 40 ( $5 \times 8 = 40$ ) and recorded  $\frac{27}{48}$  incorrectly using  $8 \times 3 = 27$ . Soon he decided that 40 was the lowest common multiple and recorded  $\frac{24}{40}$  and  $\frac{25}{40}$  to correctly compare shares.

I: Could you have changed them to percentages or decimals?

B: I sure could have. Three-fifths is 60%. One-eighth would be 12 and a bit percent...12.5 (dividing 100 by 8).

This vignette illustrated the complexity of, and cognitive load associated with, coordinating the knowledge elements required to solve comparison problems with quotients. By this time the quotient rule was a piece of knowledge for Ben that allowed him to anticipate the result of sharing without doing so. In practice and recall

situations Ben provided ample evidence that he knew his basic multiplication and division facts fluently. Yet having dedicated himself to a solution strategy, and faced with the consequent cognitive load, Ben reverted to a more primitive way to find the lowest common factor of five and eight. A disposition of mindfulness pervaded Ben's approach that caused him to reject his initial answers and resulted in him getting a correct solution.

### 6.3.3 OPERATORS

Through his work on ratios, Ben recognised the operator between two whole numbers. For example, in his end of Term One interview he knew for the ratio 3:4 that the first measure was three-quarters of the second (26 March).

In Term Three Ben used a fraction as an operator in a discrete context that required thinking of a 24-pack of cans as a referent one (17 July). He re-unitised smaller packs in terms of the referent whole, e.g. two-thirds of 24 is 16 cans,  $16 \div 6 = 2\frac{2}{3}$  so  $2\frac{2}{3}$  six-packs can be made from two-thirds of a 24 pack. This marked a change in Ben's thinking about remainders in that previously he either treated them as whole numbers or confused them with decimals, e.g. remainder of 4 meant 0.4.

Ben quickly detected the symbolic pattern in fraction multiplication when the concept was introduced using an array based model. He understood that multiplication meant "of", e.g.  $\frac{2}{3} \times \frac{3}{4} = \square$  means two-thirds of three-quarters (26 July). Ben transferred his knowledge of fraction multiplication to checking decimal multiplication problems, e.g.  $0.8 \times 0.2 = 0.16$  because  $\frac{8}{10} \times \frac{2}{10} = \frac{16}{100}$ .

Later in Term Three, the instructional focus shifted to finding unknown operators to connect two whole numbers multiplicatively. Previously Ben had shown the ability to use common factors to reduce and scale up rate pairs. In group instruction Ben recognised how a reciprocal relationship existed between the operators that mapped  $a$  onto  $b$  ( $\frac{b}{a}$ ) and  $b$  onto  $a$  ( $\frac{a}{b}$ ). However, his attendance was to patterns in the symbols rather than to the unit structure that underpinned why the pattern occurred (teaching diary, 26 July). In practice he correctly answered all the examples, e.g.  $7 \times \frac{9}{7} = 9$  and  $9 \times \frac{7}{9} = 7$ .

In his final interview at the end of Term Four (19 November) Ben compared the remaining distance to travel of two families. One family had completed two-thirds of a 140 kilometre journey while the other family had completed four-twelfths of a 96 kilometre journey. Ben found the distance travelled by each family using multiplication and division. He did not notice any equivalence relation between two-thirds and four-twelfths to simplify the problem.

### 6.3.4 RATES AND RATIOS

In comparing ratios in a fruit cocktail context (12 March) Ben understood that flavour was conserved as a ratio was replicated. This fitted with his preference for rate

thinking shown in the initial interviews. However, when given three examples of ratio comparison his answers were inconsistent. Ben appeared to attend to additive difference in incorrectly comparing 4 mango:5 orange with 5 mango:7 orange yet attended to proportionality in correctly comparing 5 passionfruit:6 orange with 7 passionfruit:9 orange.

Ben's difficulty in applying the part-whole relationships in ratios appeared again in his end of Term One interview (25 March). Given a pamphlet delivering scenario with a ratio of 4:3 Ben decided what fraction of the total pay each deliverer should get. He wrote "Rhandal should get paid three-quarters of the money Jay gets," identifying a part-part relationship. While he was able to readout key information in the problem Ben was unable to infer that the part-whole relationships were required to solve it.

I: Why do you say the Rhandal should get paid  $\frac{3}{4}$  of the money Jay gets?

B: Because Jay delivers four and Rhandal delivers three pamphlets in the time that Jay delivers four... so if Jay gets paid \$100 then Rhandal will get \$75.

I: But that's not how it's going to happen. Suppose you had a \$100 to pay both of them, what fraction of it do you think Rhandal should get?

B: He should get less than a half but more than a quarter. About two-fifths.

The same interview showed that Ben's preference for finding a unit rate remained intact. To solve the problem 18 cakes in 12 hours is the same rate as  $\square$  cakes in 8 hours Ben created a ratio table. He erased the table prior to the interview so the image below was traced.

12 cakes  
1 = 1 1/2  
2 = 3  
6 = 9  
8 = 12  
10 = 15  
12 = 18

Figure 47: Ben created a ratio table (25 March)

Ben viewed rate and ratio problems as different. Ratios cued part-part relationships strongly while his use of unit rate was the only sign of him applying measure-to-measure relationships in rates.

Early in Term Three Ben solved problems with inverse rates. He appeared to understand the isomorphism of product structure in the problems and apply it with no apparent difficulty. In a mural painting context he realised that if two students took

12 hours to paint a mural then three students could paint it in eight hours ( $2 \times 12 = 3 \times 8$ ). Ben saw a similarity of structure between painting murals, conserving area of rectangles and balance situations. He explained why five equal weights on position seven would not balance four weights on position nine.

No because  $5 \times 7 = 35$  and  $9 \times 4 = 36$ . 36 and 35 dont match so they wouldn't balance ✓

Figure 48: Ben used product to test if a balance scenario would work (23 July)

The teaching diary of 19 July commented on students' perceptions of balance problems and the unpredictability of students seeing relationships.

Ben has found the relationship while Jessie has not. (Why?)

In the same time period Ben's previously strong knowledge of rates was susceptible to contextual change. The teaching diary of 23 July recorded how Ben and another boy believed that a running rate of 12 laps in eight minutes was equivalent to a rate of ten laps in six minutes. Something in this scenario provoked attendance to additive difference.

In his end of Term Three interview Ben solved a problem about the price of apples.

At Jordan's fruit shop he charges \$8.00 for 12 apples. If he keeps the price of apples the same, how much should Jordan charge for 21 apples?

Ben recorded the following algorithm:

$$\begin{array}{r} 0.66 \times \\ 21 \\ \hline 13.86 \end{array}$$

Figure 49: Ben used an algorithm to solve a rate problem (16 August)

He correctly identified the operator between 12 and eight was 0.66, or two-thirds, and used it appropriately to multiply 21 (number of apples) by to get the correct money amount (\$13.86). Ben was unable to recognise how his rounding affected the accuracy of his answer. This was the first noticed instance in which Ben's knowledge of finding unknown fractional operators transferred to his strategies with rates.

B: I divided eight by twelve which is six and six recurring so I did sixty-six times twenty-one which is 13.86.

I: That's a weird number of apples isn't it?

B: It's not the apples. It's the amount of money he would get for twenty-one apples.

I: Oh, so it's thirteen dollars and eighty-six cents. Is that exactly right?

B: No, it's a little bit off, maybe.

I: What would make it a little bit off?

B: One of my equations going wrong.

In Term Four Ben's teaching group worked on trigonometry. Ben understood that steepness could be represented by the ratio of two sides of a triangle (rise:run). He recognised that for three similar right-angled triangles, with internal angles of  $90^\circ$ ,  $41^\circ$  and  $49^\circ$ , dividing the opposite side by the adjacent side gave answers that were "always equal (to) 0.7" (10 November). He appeared to generalise the constant ratio property of similar triangles from a few examples.

Ben worked with another student Zane to measure the height of trees in the playground. He described their method below.

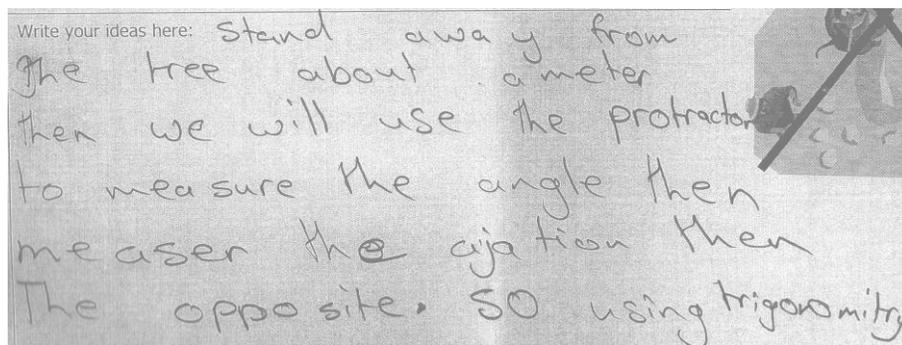


Figure 50: Ben described his method for finding the height of trees using trigonometry (5 November)

A teacher led conversation with Ben and Zane revealed that they were confident with using trigonometric ratios but had difficulty using inverse operations to find unknown measures.

T: How do you work this out?

B: Use cosine (0.87).

T: Why cosine?

B: Because you have the adjacent side.

T: But what side do you want?

Z: The opposite.

B: We should use tangent (recognising that tangent is the ratio of opposite and adjacent sides).

Ben writes  $\tan 28^\circ = h/22\text{m}$ ,  $0.53 = h/22\text{m}$ .

Z: You go  $22 \div 0.53$  (Uses a calculator and gets 41m).

T: Can that be right?

B: No, it must be less than 22m (recognising that the angle is less than  $45^\circ$ ).

Z: Oh, multiply by 22 (Uses a calculator to get 11.66m). Yes that's right.

Ben expressed the unknown side length in an equation showing that he co-ordinated knowledge about conservation of ratio, knowledge of trigonometric ratios and fraction as quotients. He used a triangle with internal angles of  $90^\circ$ ,  $45^\circ$ , and  $45^\circ$  as a benchmark that allowed him to make judgments about the reasonableness of calculation results. This was common to many students in his group.

In his final interview (19 November) Ben attempted a similar problem. He realised that the adjacent side length (34 metres) and that the tangent of  $35^\circ$  (0.7) were significant pieces of information. His first attempt was to calculate  $34 \div 0.7$  using a calculator.

B: That's not right.

I: Why not?

B: Because this angle is 35 degrees so this side (pointing to opposite) is shorter than 34. I made a mistake there (keys in  $34 \times 0.7 = 23.8$  and checks by calculating  $23.8 \div 34 = 0.70$ ). I'm still not sure.

I: Doesn't that match up with here? (pointing to table).

B: Yes it does but I only got the measurement of the small triangle.

I: Does it make sense when you look at the triangle?

B: It could be but I doubt it. (Ben is confused by the drawing which is not to scale and maps 17, half of adjacent side, onto opposite side which appears less than 17).

Ben looked for multiple sources for validation. Where these multiple sources provided different outcomes, he was uncertain. There was little doubt that a triangle drawn to scale would have confirmed his operation. For some reason he did not trust that the ratio of sides for the unit triangle remained invariant under enlargement to the

target triangle. This episode suggested that reasonableness of measure was a stronger source of validation for Ben than the structure of equivalent ratios and that he relied on the confluence of multiple information sources for validation.

The following problem from the same interview confirmed that Ben still preferred unit rate strategies. Firstly, he recorded an algorithm for  $100 \div 15$  (see Figure 51).

5. Fifteen pineapples cost ten dollars. At the same rate how much would you pay for six pineapples?

$\$3.96$   $\frac{0}{15} \overline{)100}$   
 $1 = 66.6\text{¢}$



Figure 51: Ben solved a rate problem (19 November)

I: What are you trying to find out?

B: How many cents, how much money it is for one pineapple.

I: What do you think, is it more or less than one dollar?

B: Less than a dollar.

I: Can you use this 15 to 10 as a way to break it down further?

B: Two-thirds.

I: What's two-thirds of a dollar?

B: 66 cents... (calculates  $6 \times 0.66$  mentally as  $\$3.96$  then checks on calculator. Ben is happy with his answer and checks the rate by calculating  $15 \times 0.66 = 9.9$ ).

My questioning prompted Ben's attention to the possibility of using equivalence. This indicated that while he had both unit rate and equivalent fraction strategies available his cueing preference remained intact during the year. Ben's ability to solve rate problems had broadened in that time to assimilate his increased comfort with decimals.

## 6.4 Probability

Some teaching time in Term Four was devoted to probability. Ben recognised the need for a sample to be representative of the population. For example, he commented on forecasting the elections for student council based on a sample of 30 girls and 100

year seven students respectively. He wrote “Kayla only asked 30 girls and Josh only asked year 7’s” (26 October).

Drawing counters from a bag with replacement simulated the sampling of fish in a lake. Ben recorded his results in a table and seemed to appreciate that there was uncertainty in predictions of the fish population from his sampling. He wrote, “There are more orange and green fish than the other colour fish,” avoiding any exact prediction of proportion. Ben felt that the only exact way to count the fish was to remove them all from the lake “tag them, count them and let them go” (25 October).

With situations in which he perceived there was a deterministic theoretical model Ben used percentages to represent probabilities. For example, in one scenario the students created justice rooms in which the fate of a defendant was either reward with a bag of gold or death by a tiger. The number of doorways that led to each outcome determined the probability of either reward or death. In independent, one-stage event situations Ben determined the correct percentage but in dependent, multi-stage situations he was unable to do so.

Tiger	Gold	Gold
█	█	█
x		

Ben wrote 66.6% chance of gold

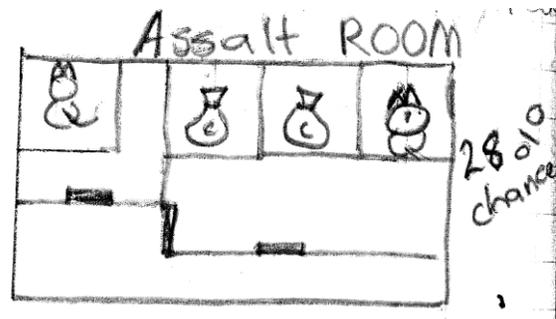


Figure 52: Ben uses percentages to represent probabilities (30 October)

## 6.5 Graphs

Ben's first use of graphs to represent relations occurred in Term Three during work on algebra. He graphed the quadratic pattern created by a relation between steps on a staircase pattern and the total number of matches needed to make the pattern.

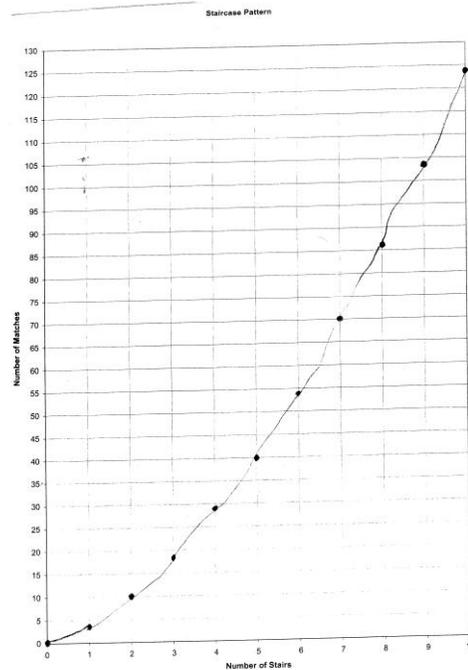
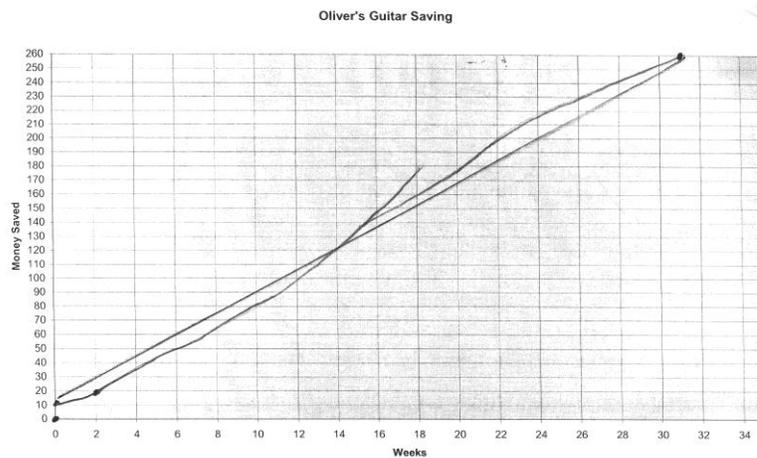


Figure 53: Ben drew a graph to show a quadratic relation (26 July)

Presented with a graph showing three rate pairs for the price of oranges Ben had mixed success at finding other rate pairs by interpolation and extrapolation. He was able to read specific rate pairs from the graph and solved a problem that required combining two displayed rate pairs, i.e. 25 oranges: \$15  $\oplus$  10 oranges: \$6 = 35 oranges: \$21.

Another attempt to graph a linear relation showed that Ben understood that a linear graph represented a constant rate. Figure 54 was his solution to the problem, “Oliver has \$12 saved up. He wants to buy an electric guitar that costs \$260. If he saves \$8 every week how long will it take before he can buy the guitar?” The horizontal scale being in units of two confused Ben. It was likely that he solved the problem arithmetically at first then graphed the relation.



**Figure 54: Ben graphed a linear relation (27 July)**

In his interview at the end of Term Three (16 August) Ben compared the running speed of three people, Jessie, Odette and Ben. A data point on a single number plane represented each person's speed. The axes represented time in minutes and number of laps. Ben recorded, "Odette because she ran more laps in less time."

Ben compared his rate with Odette's rate but neglected Jessie. To compare Odette with Jessie he estimated a unit rate of minutes per lap for each person. There was no attendance to slope as a representation of speed.

B: Odette is faster because I ran 6 laps in 12 minutes and she ran 8 laps in 11 minutes.

I: So she's definitely faster than you. Is she faster than Jessie?

B: Ah, no...um...9 minutes. Jessie would actually be faster. Odette ran one lap every 1 minute and thirty or forty and her laps (Jessie) only took about 1 minute twenty or something.

Late in Term Four there was a possibility that Ben's classwork with similar triangles in trigonometry influenced his consideration of slope as a rate of change. In his final interview in Term Four (19 November) Ben was asked who of three people had the best hourly rate (see Figure 55). I drew the lines.

The interview showed that Ben recognised that co-variation required attendance to both measures simultaneously. Firstly, he used a calculator to find Rebecca (\$13.30 per hour) and Ben's (\$11.25 per hour) unit pay rate. He attended to rotation as an attribute rather than slope and was most comfortable comparing rates numerically or when one measure was equalised on the graph.

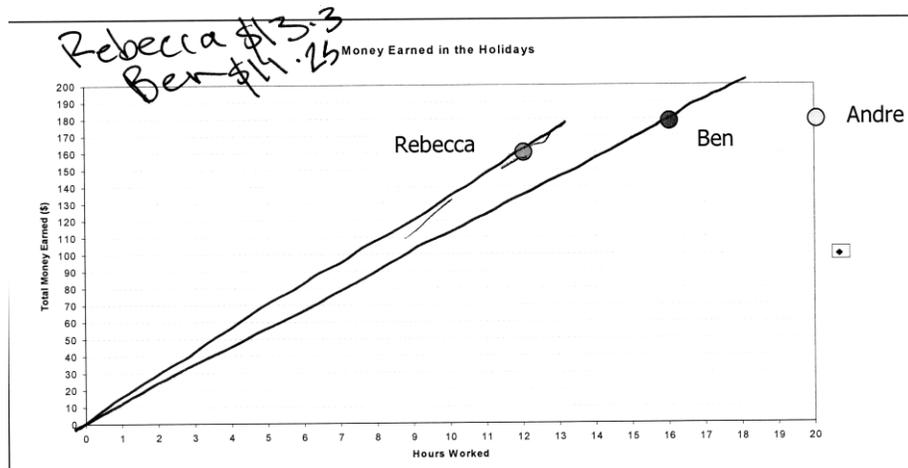


Figure 55: Ben compared the pay rates of three people shown on a graph (19 November)

I: I thought you might notice this (drawing slope lines). What do you notice?

B: This line's more curved (less slope).

I: Is that what you want to say?

B: This line is more pushed down towards the hours and this one is pushed more towards the amount earned (as a rotation).

I: What does that mean as it gets pushed more round to here?

B: It means they work less but get paid more.

I: But what if she (Rebecca) worked the same amount as Andre?

B: She'd be up here (pointing to a point on the slope line where the hours were equated).

I: So what is the slope telling you?

B: How much money they earned because it goes over the tens (pointing to axis) and how many hours they worked (pointing to the y-axis).

Ben's understanding of slope on a graph was emerging. He recognised that linear graphs represented constant rates and that the line represented both measures simultaneously. However, he could not use slope as a method of comparison unless one of the measures was equalised. He did not transfer his knowledge that ratios of sides of right-angled triangles conserve under enlargement to interpreting the slope of graphs.

## 6.6 Decimals

In week four of Term One instruction for Ben's teaching group focused on decimals using representations such as decimats, lengths of paper and circular regions. Ben's knowledge of decimal place value to two places appeared strong and he was able to add and subtract decimals, as he had done in his initial assessment. However, when calculations involved decimals to three places Ben's strategies unravelled. Comparing Rhandal's long jump of 5.768 metres with the world record of 8.25 metres he wrote 2.48 with no working. Given his previous preference for rounding, Ben probably rounded 5.768 to 5.77 and added on until he reached 8.25.

Division with remainders caused Ben difficulty initially. In one activity he was asked to complete triangles relating quotients (e.g.  $17 \div 8$ ) with fractions (e.g.  $\frac{17}{8}$ ) and decimals (e.g. 2.125). Ben used a calculator to complete all but one triangle correctly. He may have used trial and error on this task but the speed of Ben's work suggested that he was developing an appreciation of the denominators and divisors that related to certain decimals.

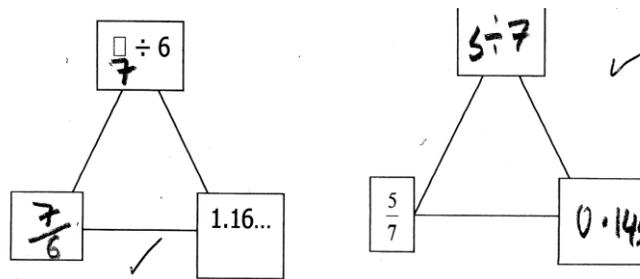


Figure 56: Ben completed fraction to quotient to decimal triangles (18 March)

Ben also used a computer-learning object to find fractions that matched given decimals and percentages. By altering the numerator and denominator, Ben was able to find fractions equivalent to 0.3737 and 1.8679.

His success at fraction to decimal conversion remained erratic. In a digit placing activity Ben correctly answered  $\frac{2}{8} = 0.25$  but also wrote  $\frac{3}{7} = 0.45$ , instead of  $\frac{3}{4} = 0.75$ , and  $\frac{4}{9} = 2.25$ , instead of  $\frac{9}{4} = 2.25$ . At times he confused denominators with decimals, e.g.  $\frac{1}{5} = 0.5$  (22 March). In one lesson decimals as quotients were modelled by cutting decimats. In Figure 57 Ben drew a model for seven whole decimats shared between five people (A, B, C, D, and E) and wrote  $7 \div 5 = 1.20$ . The redundant zero indicated a potential that he thought of decimals like whole numbers. His work with decimats also indicated other confusions for Ben. He was content to write  $1 \div 4 = 0.4$  and  $3 \div 4 = 0.75$  underneath each other. Divisor as decimal thinking sat comfortably alongside his materials-based confidence in three divided by four with no sign of cognitive conflict.

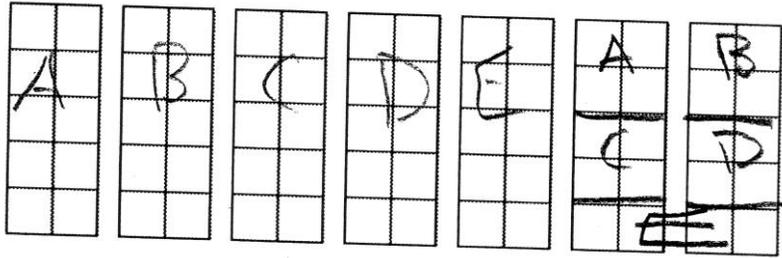


Figure 57: Ben used decimats to model a quotient, i.e.  $7 \div 5 = 1.2$  (14 March)

In his end of Term One interview (25 March) Ben confirmed money as his priority model for validating operations on decimals. Hoping Ben would rethink his answer that two-fifths was smaller than one-third the interviewer called on decimal conversion knowledge.

I: Okay so sixths wouldn't work. I'm just wondering if that's actually true. What's two-fifths as a decimal?

B: (long pause) Ahhh...ten...no point two five.

I: So you think it's point two five.

B: No, is it point two like money?

I: (writes  $\frac{2}{5} = \frac{?}{10}$ ).

B: That's four-tenths.

I: What's the decimal for four-tenths?

B: Point three five...no forty.

In the same interview Ben solved a measurement problem that required the operation,  $9L \div 0.45L = \square$ . He used a build-up strategy that also showed his treatment of two-place decimals as whole numbers.

B: I worked out 0.45 six times and came up with the answer 2.70 and I times-ed that by three which gave me the answer eighty point ten. Then I added another 0.45 litres of it, that so the answer was 19 people with a 35...no...it would be 20 people... (pause) no it's twenty people.

Ben's apparent expertise in decimal place value as suggested by the standardised tests at the beginning of the year was based on relating problems back to money. In particular he applied an "out of one hundred" preference. That explained his difficulty with converting tenths to decimal form, the use of redundant zeros and failure of his strategies for addition and subtraction when three place decimals were involved.

Early in Term Two Ben located the position of fractions, decimals and percentages correctly on the same number line, e.g.  $\frac{3}{8}$ , 0.38,  $\frac{6}{5}$ , 1.3. I introduced recurring decimals to provoke the students to consider decimals with more than two places. Ben spotted the pattern of recurring digits in decimals equivalent to sevenths, e.g.  $\frac{2}{7} = 0.285714285714286$  ( $0.\overline{285714}$ ).

By Term Three Ben possessed a broad repertoire of known fraction to decimal conversions. These included some decimals to three or more places and recurring decimals such as  $\frac{3}{8} = 0.375$ ,  $\frac{653}{1000} = 0.653$  and  $\frac{1}{9} = 0.\overline{1}$ . For  $\frac{46537}{10000}$  he wrote 0.6537 possibly believing that decimal referred only to the digits to the right of the decimal point.

He understood how a basic multiplication fact could be scaled up or down by powers of ten to calculate or estimate decimal multiplication problems. On 1 August he used  $7 \times 4 = 28$  to calculate  $70 \times 0.4 = 28$  and  $0.7 \times 400 = 280$ . Ben seemed highly attuned to symbolic pattern and used patterns extensively where he perceived strategic advantage.

In his end of Term Three interview (16 August) Ben chose \$12.72 as the best estimate of the cost of 1.470 kilograms of steak at \$8.60 per kilogram. In Figure 58 he eliminated the other answers as too high or too low. In norm-referenced standardised tests at the end of the year Ben answered all but two items related to decimals correctly. He correctly answered questions on decimal place value, adding decimals, converting fractions to decimals and using rounding to estimate the answer to decimal multiplication. His incorrect answers related to the question "How many integers are between  $\sqrt{15}$  and  $\sqrt{63}$ ?" and expression of a number in scientific notation. Success on both items depended on co-ordinating many knowledge elements outside of the decimal domain that were not taught to the students in Ben's class.

a \$127.19      b \$12.72      c \$1.27      d \$0.13

Explain your choice. it can't be a because it only \$8.60 per kilo and he only buys 1.479 kilos c is less money than 8.60 so the right answer is B

Figure 58: Ben selected the best estimate for the cost of steak using elimination (16 August)

## 6.7 Percentages

Early in Term One I gave a series of four mini-lessons to the whole class. A Slavonic abacus served as a representation of percentages as fractions out of one hundred. I aimed to establish a vocabulary of fraction-to-percentage benchmarks such as one-quarter as twenty-five percent. In a test Ben answered all three questions correctly. His ready acceptance of percentages matched his “out of 100” preference with decimals.

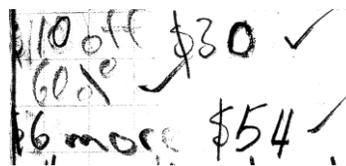


Figure 59: Ben answered three percentage problems correctly (9 March)

Once equivalence with fixed referent ones was established in Term One instruction for Ben’s teaching group progressed to comparing frequencies. Ben transferred his equivalence thinking from measure situations to frequencies. Asked to compare the data from two shooters in a sporting context (8 March) he converted each part-whole relationship to a percentage. For example, in comparing “Jordan gets 15 out of 24 first serves in, Ben gets 12 out of 20 in”, he used rounding and multiplicative mapping onto 100 to calculate each frequency as a percentage. Ben rounded  $\frac{15}{24}$  to  $\frac{16}{25}$  so he could scale up to 100 by multiplying both numerator and denominator by four.

He adjusted numerator and denominator by adding or subtracting the same whole number with several shooting comparisons, believing conservation of the fraction. Ben transferred his estimation preferences with multiplication, failing to appreciate how additive compensation changed the fraction. He recognised the potential use of percentages as equivalent fractions to compare frequencies.

Ben also had difficulty co-ordinating his knowledge of fractions in comparing other frequencies. In the interview with me below (9 March) he had to compare  $\frac{18}{26}$  and  $\frac{18}{27}$ . Ben’s first reaction was to attend to the greatest denominator. His reaction to my comment suggested he understood the subtle change to a frequency when the base number reduced.

T: What about 14 out of 21 (Andre) or 18 out of 26 (Jamie)?

B: 14 out of 21 is two-thirds. That’s 33.3%

T: But isn’t one-third equal to 33.3 %?

B: 66.6% (correcting).

T: What's the problem with 18 out of 26?

B: There's no common factor.

T: But 18 out of 27 is close.

B: That has a factor of 9 so two-thirds or 66.6%.

T: Which is bigger – 18 out of 26 or 18 out of 27?

B: 18 out of 27.

T: The bad news is that 18 out of 26 is bigger – why?

B: Oh yes. Jamie only took 26 shots, less than 27.

In his end of Term One interview Ben opted for a reasonable estimate of a percent rather than providing an exact answer. The problem involved calculating  $\frac{27}{45}$  as a percentage. Ben used fifty percent as a benchmark but confused his measure spaces when adjusting the estimate. This was surprising given his strength with solving rate problems using common factors in Term One.

I: You have 58%. Where did that come from?

B: I knew that about 22.5 was 50% and so I did that and then I figured out that there was about another 5% so I tried to figure out what 5% would be if it was out of 45. That's about 8 so I figured out that it's about 55 percent.

Further practice of frequency problems in Term Two resulted in Ben becoming more consistent in getting correct answers. Percentage strips provided a physical model for validating calculations. Ben used the materials effectively to calculate frequencies as percentages and percentages as operators.

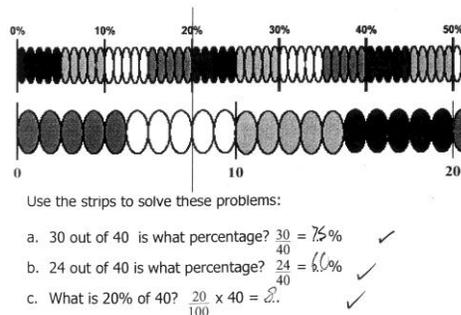


Figure 60: Ben used percentage strips for frequency and operator problems (9 May)

Ben preferred to express frequencies as fractions, simplify the fractions by division, and find equivalent percentages. Ben also knew how to use a percentage as an operator, e.g. 30% of \$320 = \$96.

The image shows two handwritten mathematical expressions. The first expression is  $\frac{15}{50} = \frac{3}{10} = 30\%$ , with a checkmark below it. The second expression is  $\frac{24}{36} = \frac{4}{6} = \frac{2}{3} = 33\frac{1}{3}\% = 66.\bar{6}\%$ .

Figure 61: Ben simplified fractions to convert them to percentages (9 March)

During a week of instruction in Term Two Ben worked on calculating percentage discounts, using benchmarks such as 50%, 10%, and 5% for calculating frequencies, and solving problems, in which the unknown was in varied places, e.g.  $\square\%$  of 66 = 38.

By the last week in Term Three Ben operated on percentage problems with greater flexibility than observed previously. He was comfortable with getting decimal percentage answers and used sophisticated rounding methods to estimate. He knew that a percentage frequency could be found using division and multiplying the decimal answer by 100, and recognised that dividing 100 by the base of the frequency gave a unit rate (Teaching diary, 7 August, 2010). Having calculated  $\frac{11}{28} = 39.2\%$  (Rhandal's shooting percentage) using a calculator, Ben answered the question, "Is 40% of 28 closer to 5, 10, 15 or 20? Explain how you used Rhandal's percentage to work that out?" Ben wrote:

Closer to 10 because 40 is 2.5 of 100%, 28 is  $2.5 \approx 11$  (meaning  $2.5 \times 11 \approx 28$ ).

Despite emerging competence, Ben's methods remained unreliable. He gave an answer of 38.8% to the problem, "Kayla buys a skirt. It normally costs \$35 but she gets it for \$21. What discount does she get?" No working was given. Ben could not programme a spreadsheet to find the capital return of \$600 after five years of compound interest of 11%.

In his end of Term Three interview (16 August) Ben tried to find 80 percent of 35 shots in a netball context. He wrote  $80\% = 15$ .

B: I understood it. I just couldn't get 80 percent of 35 shots. I got it into two and a bit... (meaning 35 goes into 100 two and a bit times).

I: Oh, I see. You mean how many thirty-fives into 100?

B: Yes.

I: Do you know another name for 80%?

B: Eight-tenths.

I: or...

B: One quarter...er...three quarters.

I: Write some of this down.

B:(Records  $80\% = 8/10$ , double counts in lots of 20%, 1-20%, 2-40%, 3-60%, 4-80%)  
No, it's four-fifths.

I: Does that help?

B: A bit.

I: How many shots did she take?

B: 35.

I: And she got four-fifths in. What would one fifth be?

B: Five.

I: Five fives are twenty-five?

B: Six...no seven (division build-up, not facts).

In this episode, the cognitive load of co-ordinating the multiple knowledge items required to solve the problem crowded Ben's thinking. Most of the knowledge existed independently, though the equivalence of eight-tenths and four-fifths needed reconstruction. Lack of fluency with knowledge co-ordination played a significant part in Ben's inability to solve the problem even with support from the interviewer.

In his end of year interview (19 November) Ben solved a contextual problem of the form "18 out of  $\square$  equals 67%." This problem involved co-ordination of the part-whole relationship and did not give ready access to equivalence relations.

Ben worked out  $67 \div 18 = 3.7222...$  on his calculator.

B: Each word is worth 3.7%. Eighteen words is 67% so 19 words is 70.7%...

I: That's going to take a long time.

B: Well 10 words is 37%. Three 37% is a bit too much so there must be nine words, that's 27 words.

I: Where did the nine come from?

B: Well, I got rid of one of the ten lots of 3.7% ( $9 \times 3.7$  is closer to one third of 100).

I: Does that check out with the 67%?

B: Yes, that's right.

I: Why are you sure?

B: Because I've done lots of these problems before.

There was certitude in Ben's final response that paid homage to his past experience with a variety of percentage problems. He was in no doubt that his answer was correct. The multiple knowledge elements Ben co-ordinated in his solution were diverse. He recognised that 67:18 was the rate of percentage to words and that division gave a unit rate. Ben knew that a multiplicative and additive build-up strategy would map 3.7 onto 100. Subtle knowledge about size relations was involved in the inferential connections he made. Ten questions equated to 37%, 37% was more than one-third of 100 so nine questions was a better estimate of one-third.

Despite the sophistication of his strategy he did not attend to useful knowledge elements. For example, dividing 100 by 3.7 gave the number of questions. Sixty-seven percent was very close to two-thirds. Ben's strategies were not predictable from the problem conditions. There was no ubiquitous abstraction that guaranteed successful transfer from one situation to the next, only situations in which sometimes he co-ordinated the required knowledge made relevant by his own choice of solution path.

## 6.8 Summary

### 6.8.1 PROCESSES TO OBJECTS

Ben was a high-achieving student for his class level and age. At the beginning of the year he had strong strategies for whole number operations and a sound knowledge base, including basic facts and place value. Ben applied his understandings of the properties of multiplication and division with whole numbers in object-like ways to proportional reasoning problems. For example, Ben used common factors extensively to simplify ratios, identify equivalent fractions and identify unknown fractional operators.

There was plentiful evidence to suggest that Ben created and applied abstractions that were object-like across multiple contexts. He took two terms to develop control of the quotient theorem but having done so he applied it to a variety of situations and tasks such as expressing fractions as decimals. Ben applied understanding of equivalent

fractions as representing the same quantity to division by fractions and finding part-part relationships in rates and ratios.

However, the path to theorem status was full of potholes. Contextual variation was associated with variable responses. Usually proficient at solving rate problems using multiplication and division, he opted for additive difference in a speed context. He appeared to accept conservation of side length ratios in similar triangles yet questioned it in finding the height of a tree. Ben struggled to connect the slope of a graph of ordered pairs with constant rate.

Increased sophistication required modification rather than replacement of existing ideas and preferences. Development of the quotient theorem was an excellent example. At first, Ben used partitioning of the objects and equal sharing in quotient situations. He treated the total number of objects as the referent in quantifying the shares, e.g. two pizzas shared between three people gave two-sixths of a share. It took experience with multiple situations over a prolonged time for Ben to create an anticipated process that became object-like.

Sometimes he trusted theorems and applied them inappropriately. Working on equivalent fractions, he tried rounding strategies at first in an effort to make calculations easier, unaware his rounding changed the value of the fractions. Application of anticipated processes as objects of thought was unreliable at times and susceptible to cognitive load. For example, his knowledge of common factors grew strongly in terms one and two, and he increased his register of fraction-decimal-percentage facts. Yet late in the year he used finger counting to find a common multiple so he could compare  $\frac{3}{5}$  and  $\frac{5}{8}$  in a pizza sharing context. Knowledge of a theorem was no guarantee of Ben's use of it in any particular situation.

### 6.8.2 CO-ORDINATION CLASSES

Co-ordination class theory anticipates situational variation in problem solving due to the demands of readout and inference. This was certainly true in Ben's case. The knowledge co-ordination demands of complex problems resulted in knowledge that was fluent in one setting being difficult to access in other settings. Ben chose strategies at times that were efficient and solutions appeared easy. At other times his choice of strategy sent him down tortuous paths where the load of knowledge retrieval and integration was too much.

A noticeable feature of Ben's progress was the durability of preferences. Conversion to unit rate cued highly throughout the year. This was true even when he had readily accessible strategies based on equivalent fractions and was able to co-ordinate non-unit rates. Rounding and compensating dominated his approach to calculation. In reasoning about fractions this preference led to some incorrect strategies at first, e.g.  $\frac{15}{24} = \frac{16}{25}$ . By the end of the year, Ben had adjusted his rounding and compensation strategies to accommodate his improved knowledge about fractions. His preferences did not change. Ben got better at exercising them.

Preferences influenced the ease with which Ben accepted new ideas. His preference for viewing decimals as "out of one hundred" made simple percentage problems

accessible. Ben treated percentages as natural extensions of whole number operators and took considerable time to regard them simultaneously as proportions. With ratios his first attention was to the part-part relationships. This preference prepared him for finding unknown fraction operators connecting rate pairs. The same preferences that helped him learn some ideas inhibited his adoption of other ideas. He struggled to integrate his “out of 100” preference with the demands of decimals with more than three places and to work with part-whole relationships in frequency situations.

The simultaneous co-existence of incompatible knowledge was evident with Ben. For example, his tendency to think of whole number remainders as decimals, e.g.  $\frac{1}{4} = 0.4$ , took considerable time to amend. He knew 0.25 represented one-quarter yet gave 0.4 as the result of one divided by four. Trust in a knowledge element was instrumental in the development of co-ordination classes. Ben relied on multiple sources of evidence in coming to trust a result. For example, he relied on his trust in money to find that eight-tenths was equivalent to four-fifths through double counting. In the situation where the scale of the diagram was at odds with his prediction of tree height using tangent he did not trust his answer.

Different situations afforded different ways to trust in results. Given a calculator to solve problems with decimals he trusted the reasonableness of the answer size rather than his reasoning about the structure of the operation. There were several instances of new situations provoking Ben to mistrust knowledge that had previously appeared sound. On other occasions, Ben placed undue trust in ideas that were false. For example, he thought  $\frac{18}{27}$  was greater than  $\frac{18}{26}$  in comparing frequencies. His openness to questioning, and a desire to find pattern, seemed to be key personal dispositions that enabled Ben to make progress.

A co-ordination class was never fully developed. Ben continually modified his knowledge. At times the modification seemed progressive and other times regressive. Ben shared knowledge between his co-ordination classes. Knowledge of fraction equivalence in part-whole and measure situations was used by Ben to identify equivalent part-part relationships in ratios and to determine the difference of shares in quotients. He used unknown fractional operators to identify measure-measure relationships with rates. The issue was not the completeness of a co-ordination class but Ben’s trust in it and its sufficiency for the purpose. Ben’s strategic choice affected the ease with which knowledge elements were co-ordinated.

### **6.8.3 SPAN AND ALIGNMENT**

Ben began the year with good knowledge and strategies for solving multiplication and division problems with whole numbers. With the exception of quotient problems Ben’s map showed his knowledge of the other fraction sub-constructs was consistent with the hypothesised learning trajectory. He was able to apply multiplicative thinking to a broad range of situations including finding equivalent fractions, using non-unit fractions as operators and scaling rates and ratios. His alignment, reliable execution of strategies, was variable across these sub-constructs.

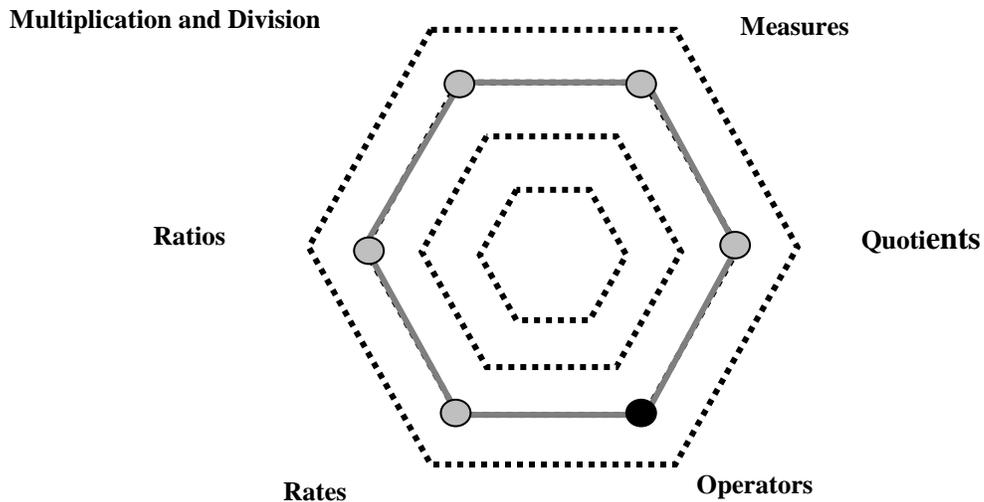


Figure 62: Learning Trajectory Map: Ben beginning of Term One

Changes to the map for the end of Term One captured Ben’s development of common factor knowledge and his enhanced strategies for multiplication and division of whole numbers, in particular his use of the associative property when partitioning factors multiplicatively (see Figure 63). Ben also improved his understanding of equivalent fractions as the same measure and recognised the multiplicative operators between parts in ratios, i.e. for the ratio  $a:b$  the operators are  $\frac{a}{b}$  and  $\frac{b}{a}$ . While still struggling with using a common referent in quotient problems he recognised situations when equal shares occurred.

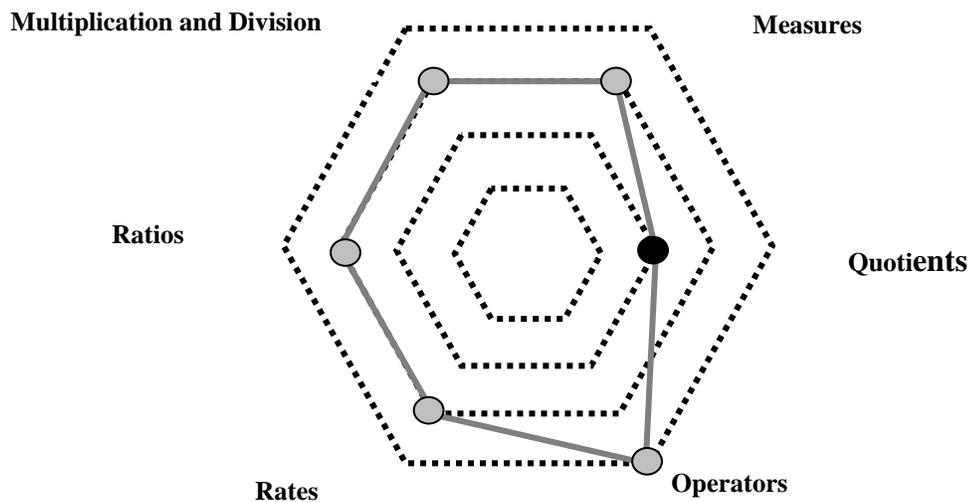


Figure 63: Learning Trajectory Map: Ben end of Term One

During terms two and three Ben consolidated his mental strategies for multiplication and division of whole numbers to include the operations as inverses and connect partitive and quotative division. This transferred to his reliability in finding equivalent fractions and applying equivalence to ordering fractions. It is likely that Ben could add and subtract fractions at this point but no data was available to support this conjecture. Ben also knew the quotient theorem and used his knowledge of equivalent measures to compare shares. There was also evidence that he was able to solve inverse rate problems on most occasions.

The predominance of grey circles in the map for the end of Term Three indicated that Ben broadened the range of situations to which he applied proportional reasoning but his alignment was not consistent (see Figure 64). The sub-constructs were in a state of flux.

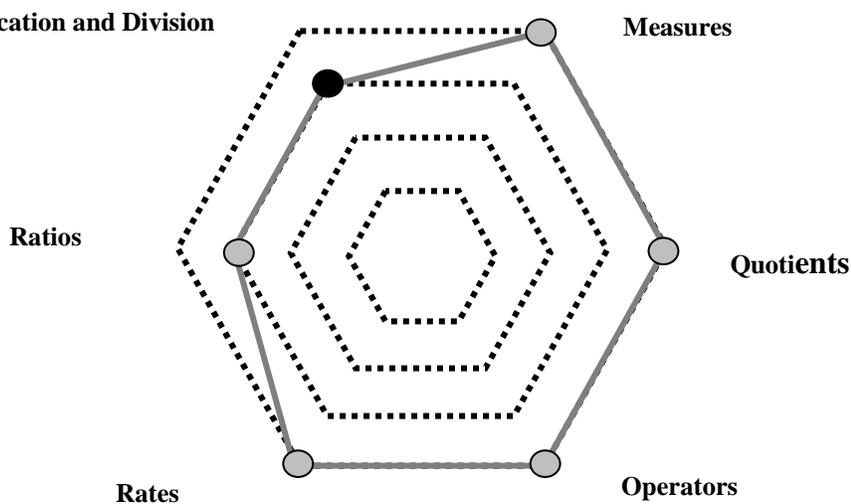


Figure 64: Learning Trajectory Map: Ben end of Term Three

The map for the end of Term Four suggested that Ben's progress during the term was limited (see Figure 65). Alignment only improved significantly in his measurement with fractional units (quotative division). The ceiling effect of the map limited any representation of increase in span. Ben's only demonstrable development was his coordination of equivalent fractions as measures to compare frequencies reliably. It appeared that Ben's knowledge of the sub-constructs was in a process of integration as his strategies were often characterised by a flexible connection of knowledge across the sub-constructs.

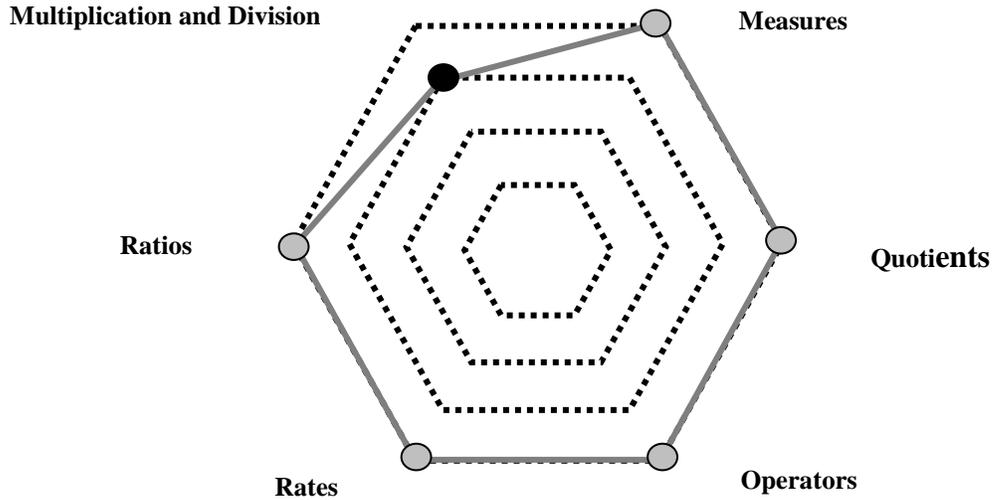


Figure 65: Learning Trajectory Map: Ben end of Term Four

A simultaneous view of Ben's maps over the course of the year showed a progressive improvement in span, the situations to which he could apply multiplicative thinking and proportional reasoning (see Figure 66). Alignment, reliability of strategy, at one phase of the trajectory sometimes preceded progress to a higher phase in the same construct but consistent alignment was not a necessary condition for improvements in span. By the end of the year Ben's constructs were not totally reliable but there were strong indications that he was co-ordinating his knowledge across the sub-constructs. This suggested that for high achieving students like Ben cognitive development in a complex domain was characterised by progressive increases in span supported by better alignment over long periods.

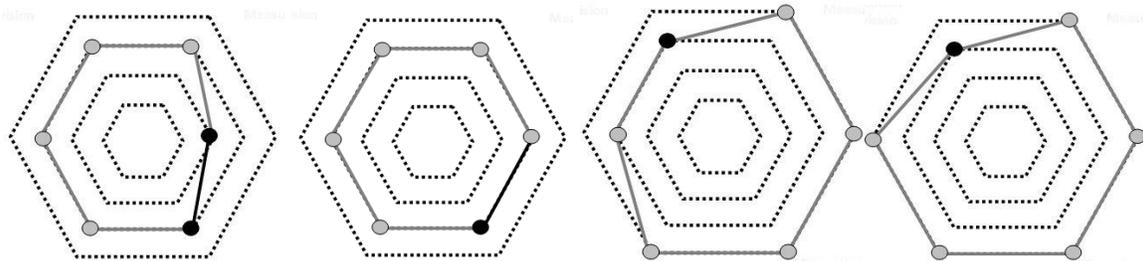


Figure 66: Simultaneous view of Ben's Learning Trajectory Maps

## CHAPTER SEVEN: CASE STUDY OF RACHEL

### 7.1 Beginnings

#### 7.1.1 PERSONAL CHARACTERISTICS

At the beginning of the school year Rachel was exactly 12 years old. Her parents both worked in professional occupations and she identified herself as a European New Zealander. Rachel was a representative basketball player and enjoyed all forms of sport and exercise, especially rowing. She was a willing participant in other cultural activities as well and gave shopping as a favourite pastime. Rachel was highly motivated towards her schoolwork and applied herself consistently to classroom tasks. While quiet and thoughtful she participated openly in group discussions. She had a small selected group of friends, predominantly made up of similarly mature and academically able girls. Mathematics was Rachel's favourite subject at school. She enjoyed solving problems and was persistent and innovative in doing so.

#### 7.1.2 INITIAL TESTS

Rachel showed above average achievement for her age on the tests of mathematics at the beginning of the year. She was accredited an overall level of 4A (Level 4 Achieved) on the asTTle test sat on March 1. Her performance on number operation tasks was slightly superior to her performance on number knowledge tasks.

Her PAT assessment (February) returned a stanine of seven which positioned her in the top 23 percent of students for her class level. She answered all but one number task correctly. Her correct responses involved whole number place value, calculations with whole numbers including division with a single digit divisor, simple percentage conversion and replication of rate pairs, finding fractions and percentages of amounts, and the ordering and addition of decimals. Her only incorrect answer related to the ordering of the decimals, 5.6, 5.22, 5.315, and 5.08. Rachel's main areas of weakness were in geometry and statistics.

#### 7.1.3 INITIAL GLOSS INTERVIEW

Her initial GloSS interview (20 February) confirmed that Rachel's mathematics achievement was above average. Her knowledge with whole numbers was fluent and extensive. This knowledge included most basic facts for addition, subtraction, multiplication and division, and nested place value, e.g. 723 hundreds in 72 345. She readily identified simple equivalent fractions in a sorting task and applied nested place value to decimals. Rachel was unable to connect decimals with percentages or order a set of non-fractions with different denominators. This result coupled with the AsTTle test suggested that Rachel had a strong understanding of whole numbers and an emerging understanding of fractions, decimals and percentages.

Her responses to mental arithmetic problems in context revealed a preference for imaging written algorithms except in situations for which she had no established

procedure. She correctly solved a problem involving length of electrical cable using a decomposition based algorithm for  $5.33 - 2.90 = \square$ . She also imaged an algorithm to solve a division problem involving building animal pens from posts, i.e.  $280 \div 8 = 35$ . The Canterbury wine gum problem involved finding an equivalent ratio for 24:16. By identifying four as a common factor mentally, Rachel reduced the ratio to 6:4.

In summary, the interview revealed that Rachel determined the operations required to solve number problems with whole numbers and used her strength with written algorithms to calculate answers correctly. She only resorted to mental calculation when no readily accessible algorithm was available.

#### 7.1.4 INITIAL NUMBER INTERVIEW

Rachel's first number interview occurred early in the school year (21 February). She was able to represent the ratio of "two for every three" as part-whole fractions,  $\frac{2}{5}$  and  $\frac{3}{5}$  in a pamphlet run context. In a road journey context she had to find a new sign to go between one-half and two-thirds. Her first response was three-quarters. This may have been due to attending to the pattern in numerators, 1, 2, 3, and denominators 2, 3, 4. The following interview segment showed that Rachel did not see this question as affording an opportunity to apply equivalence. She also appeared to interpret questioning by the interviewer as an indication that her answer was incorrect.

I: Can you explain why three-quarters is between one-half and two-thirds?

R: Because two-thirds is over three-fourths and one-half is less than three-quarters.

I: What tells you that two-thirds is more than three-quarters?

R: Oh...it isn't...I don't know.

I: How would you decide if it's bigger?

R: Oh...it's not right.

I: What makes you say it's not right?

R: Because I don't think two-thirds is as big as a fourth (meaning three-quarters)

I: You're not sure? Normally would you measure it or draw a picture of it?

R: Yeah.

I: So there is no way that you can just look at them and tell that one's bigger than the other?

R: No.

Rachel's response was consistent with her inability to order fractions during her GloSS interview in that both tasks afforded the use of equivalent fractions. However, Rachel's answer to a rate problem was consistent with the capability she showed on the standardized tests. The question read, "Hamish can bake 48 cakes in 16 hours. How many cakes can he bake in 9 hours?" Her recording showed that Rachel found the unit rate by multiplication and scaled the unit rate by a factor of nine. Her interview confirmed this strategy.

The image shows handwritten mathematical work. On the left, there is a multiplication problem: 16 multiplied by 3, with a horizontal line under the 16 and the result 48 written below. To the right of this, it says "27 cakes". Below that, there is the equation  $3 \times 9 = 27$ , where the 27 has a diagonal line through it. At the bottom, it says "3 cakes per hour".

Figure 67: Rachel applied a unit rate (21 February)

Rachel also found how many glasses of 375 millilitres fill from nine litres of juice. She correctly wrote an answer of 24. Erased but still visible was a written algorithm for  $1000 - 375 = 625$ . Rachel explained her strategy.

R: I worked out how many glasses were in 1000, and there was 9000 litres, so I times how many there was in nine, aww... what?

I: So you said first, "How many of those will go into 1000?" How did you work that out?

R: I thought about three and with all of the left overs. I added them all and did it again.

I: I see, you found out how many 375s fitted into all the leftovers.

The strategy showed ability to connect a sequence of operations that solved the problem. Rachel's explanation did not match her strategy as three measures of 375 millilitres exceed one litre. However, both her response and written recording correlate to the use of some form of repeated addition and subtraction strategy. Rachel's access to knowledge resources about whole number subtraction and addition, coupled with knowing that 1000 millilitres make one litre enabled her to develop a correct solution albeit by cumbersome means.

Question five of the interview was about percentages, "Remi has 30 calves. Nine of the calves are Friesians. What percentage of the calves is Friesians?" Rachel explained her lack of response.

R: I couldn't do that one.

I: What was the problem? What did you find hard?

R: I just don't get percentage.

I: You've never come across it before?

R: I have but I never get it right.

I: So do you know what percentage means?

R: I think it's like out of a hundred.

Rachel's reply indicated an emotive override of any attempt at solving the problem. In Mason's (1999) terms, she could not imagine herself in the position of solving the problem. Her past experiences resulted in a reaction that prohibited any affordance for action despite the availability of knowledge about percentage as out of one hundred.

The final problem of the interview involved comparing the equal shares of five boys with three pizzas compared to three girls with two pizzas. Figure 68 shows her recording. The interview showed that Rachel's strategy was potentially fruitful and involved considerable thinking about equivalent fractions. Co-ordination of the knowledge resources proved difficult given the complexity of the solution strategy.

$$\begin{array}{l} 5/10 + 1/10 = 6/10 \\ 3/5 \end{array} \qquad \begin{array}{r} 3/6 \\ 2/6 \\ \hline 5/6 \end{array}$$

Figure 68: Rachel's recording for pizza sharing problem (21 February)

R: I gave every boy half a pizza and then I cut up the half that was left, and that was one-tenth and one-tenth plus five-tenths is six-tenths and then I simplified that to three-fifths.

I: And what happened here with the girls?

R: I gave them all half a pizza again and with the half that was left I divided it into three so that was one-third so that's one sixth of the whole thing, so one-sixth plus three-sixths equals... (puzzled)... oh... it's five sixths, I don't know.

I: So what you mean is, half for you, half for you, and half for you (pointing). You chop up the half left over into what you call sixths. How many sixths is a half?

R: Three.

I: So they got three-sixths and one-sixth which is ...?

R: Four-sixths.

I: Do you know another name for four-sixths?

R: One-third.

I: And you had a bit of a job figuring out who got the most.

R: The girls.

I: What tells you that?

R: I don't know...the girls' pieces just looked like they would be bigger.

The intervention of the interviewer in co-ordinating knowledge resources for Rachel helped uncover thinking that was not obvious from her recording. As with problem two she did not see that the comparison of shares afforded the opportunity to use equivalence.

### **7.1.5 INITIAL SUMMARY**

Rachel began the school year with good understanding of whole number place value. Her use of imaged algorithms masked the detection of properties for multiplication and division of whole numbers. The interview responses suggested that Rachel solved whole number division problems with known multiplication facts, trial and improvement sequences of multiplication algorithms or repeated addition.

Rachel showed the ability to work with rate pairs using common factors and recognised the part-whole relationships in ratios. Context appeared to cause her no difficulty with these problem types. Some of her approaches to problems with fractions were sensitive to variation of problem type and conditions. She applied equivalence in some situations but not in others and she had considerable difficulty with co-ordination of fraction knowledge, possibly due to limits on working memory. Previous experience with percentages cued an emotional block.

Given Rachel's availability of knowledge resources at the beginning of the year, she had good preparedness for creating adequate co-ordination classes for fractions, ratios, rates, operators and quotients.

## 7.2 Progression in Whole Number Operations

While fractions and decimals dominated the focus of Rachel's learning group there were many differentiated whole class activities that supported the development of whole number operations. Rachel's progress with whole numbers during the year involved the development of two main areas, application of the properties of multiplication and division in the form of flexible mental strategies, and the identification of relations in number pairs. I considered that these areas were significant to the development of proportional reasoning.

### 7.2.1 DEVELOPMENT OF MENTAL STRATEGIES

During Term One Rachel mastered recall of all her basic multiplication facts and became proficient at finding common factors for given pairs of whole numbers. She departed from her preferred written algorithms and used mental computation based on the variance and invariance properties of multiplication. For example, the estimation game gave her limited time to find a best estimate for calculations. Rachel became proficient at estimating calculations as complex as two-digit multiplication. In the examples below she applied complex mental strategies to get exact answers to the problems rather than estimate, e.g.  $86 \times 5 = 43 \times 10$  and  $28 \times 22 = 28 \times 20 + 28 \times 2$ . These strategies were shared with the whole class during the follow-up to each game.

Problem	Estimate	Actual	Difference
$26 \times 16$	416	416	0
$86 \times 5$	430	430	0
$18 \times 34$	612	612	0
$28 \times 22$	616	616	0
$14 \times 38$	532	532	0
Total differences			0

Figure 69: Rachel used multiplication strategies during the estimation game (23 March)

In Term Three basic multiplication fact practice involved finding missing values in grids. This required inverse thinking of the forms,  $a \times \square = c$  or  $c \div \square = a$ . Rachel showed fluency and flexibility in completing these grids and with practice sheets that showed multiplication facts in multiple representational forms like  $7 \times \square = 63$  and  $\frac{63}{\square} = 7$ .

She also showed proficiency in applying division in both partitive and quotative situations. For example, in a statistics lesson Rachel represented frequencies out of 22 using a pie chart. She calculated  $360 \div 22 = 16.36$  to find degrees per person. This marked a significant departure from her preference for build-up methods to solve division problems.

Given the strong teaching focus on proportional reasoning in Term Four there was little additional evidence of growth in Rachel's understanding of whole numbers. In her final PAT and AsTTle assessments Rachel correctly solved all but two questions involving whole numbers correctly. The incorrect items involved calculating  $309 - (90 + 89 + 76 + 88)$  and  $8 \times 16$  given  $8 \times 16 = 64$  respectively. She answered both items correctly at the beginning of the year and had adequate knowledge resources available at the end of the year to answer them. Rachel correctly answered questions involving estimation of two digit multiplication, exponents, and order of operations.

By the end of the year Rachel had a broader range of calculation strategies with whole numbers than she had at the beginning. She still used written algorithms but also applied a broad range of mental strategies for multiplication and division of whole numbers.

### 7.2.2 RELATIONSHIPS IN NUMBER PAIRS

In Term One, I used a computer spreadsheet environment to encourage functional thinking. Students received sets of ordered pairs and had to find a rule for each relation. After two weeks of this activity Rachel identified linear relationships such as "x 9 - 6" and sometimes identified more complex relationships such as  $n$  to the  $n$ th triangular number.

In a lesson on internal angles of polygons Rachel noticed that the relationship between the number of sides and the sum of the internal angles of polygons was linear. In a mini-interview she said, "You add 180 (degrees) for each extra side" (17 May). It was not clear what Rachel attended to in reaching her generalisation, number pattern or the geometric principle that adding another side added another internal triangle to the polygon. The class discussed both ideas during the lesson. While Rachel's rule involved recursive (one term to the next) thinking it was clear that Rachel sought to find relationships in spatial, as well as number, contexts.

During whole class work on algebra in Term Three Rachel used strip models effectively to solve linear equations such as  $7t - 16 = 25$ . Rachel wrote expressions such as  $n \times n + 1$  to represent the relation in sets of ordered pairs presented using a spreadsheet and wrote equations such as  $n \times (3 + n) + 2 = (1 + n) \times (2 + n)$  to represent patterns of equations.

In a solar system context, Rachel estimated the magnitude of Pluto's solar year by focusing on the relationship between the other planets and their solar years. She was able to find linear relationships in geometric patterns, and extended this to relationships in quadratic relationships. In Figure 70 she noticed that the total number of matchsticks was  $n(2n + 2)$  where  $n$  equalled diamonds across.

By the end of the year, Rachel was a proficient relationship seeker. She enhanced her register of possible relationships in number pairs from the start of the year. She readily found additive and multiplicative rules as well as simple quadratics, and powers.

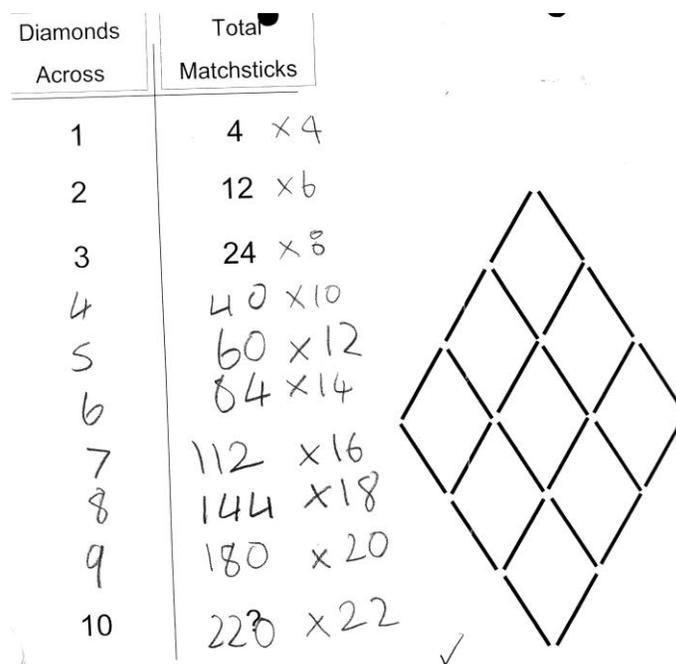


Figure 70 Rachel extended a quadratic pattern (25 July)

## 7.3 Development of Rational Number Sub-constructs

### 7.3.1 PART-WHOLE AND MEASURES

The first week of instruction for Rachel's group focused on understanding equivalent fractions as the same measure of a fixed one. Rachel seemed to accept this idea in practical situations such as placing equivalent fractions on the same paper strip number line. Some interference in the idea of equivalent fractions as the naming the same amount came from whole number primitives such as larger number therefore larger amount. For example, when asked to justify her answer to  $\frac{2}{5} + \frac{6}{10} = \frac{10}{10}$  she replied, "I think it is equal because if I make two-fifths into a bigger fraction, it's in tenths and four-tenths plus six-tenths equals ten-tenths."

By the end of week two Rachel had secure procedures for finding equivalent fractions as measures. She adopted a multiplicative algorithm for finding common denominators, e.g.  $\frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$ , and for finding equivalent fractions, e.g.  $\frac{3}{4} \times \frac{5}{5} = \frac{15}{20}$ .

Influence in her development of these algorithms came from working with her close friend Jessie who relied on them extensively. Finding common denominators was her preferred method of comparing two fractions with unlike denominators. On March 1, Rachel found the difference between one-sixth and one-seventh:

It is  $\frac{1}{42}$  and I worked it out by making a number that both 6 & 7 can divided in & I got 42.

Rachel used common factors to reduce fractions to simpler forms using common factors, e.g.  $\frac{8}{20} = \frac{2}{5}$  (March 7) and could name splits of unit fractions without reference to a unit whole, e.g.  $\frac{1}{3}$  of  $\frac{1}{13} = \frac{1}{39}$  (March 7). She had difficulty naming the fraction mid-way between two fractions tending to concentrate on the denominator that was required rather than the total fraction. Her difficulties appeared to be due to co-ordination of all of the knowledge elements involved rather than in possession of the elements. She consistently placed equivalent fractions at the same place on a number line and attended to scale in ordering fractions.

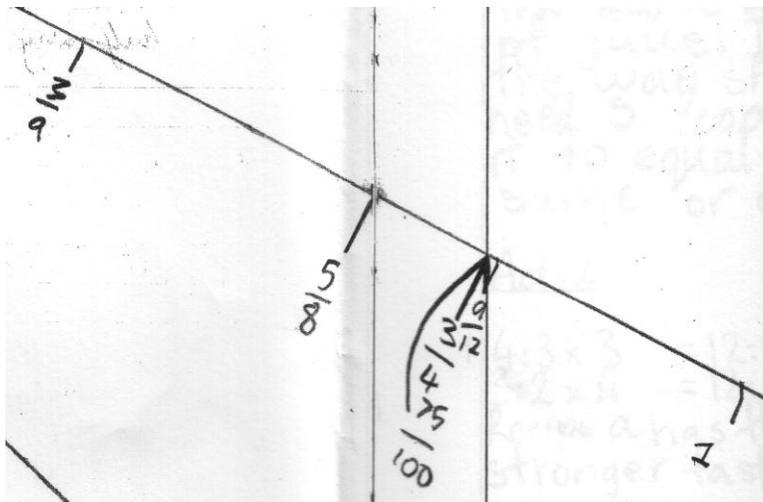


Figure 71: Rachel ordered fractions on a number line (10 March)

While instruction involved experiences with physical materials such as folding paper strips and manipulating plastic fraction strips, Rachel saw these as illustrative and preferred to operate with symbolic representations. She was able to validate her results through reference to some mental image of the concepts, not through continued access to the physical materials.

During Term One Rachel came to see equivalent fractions as the same measure of a fixed one. In her end of Term One interview (26 March) she showed improved co-ordination of the knowledge elements needed to name the fraction mid-way between two fractions.

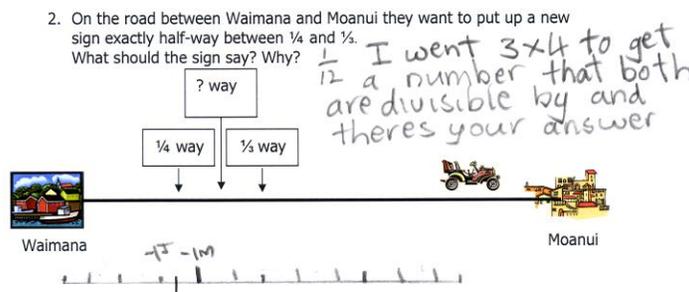


Figure 72: Rachel found a fraction between one-quarter and one-third (26 March)

The following interview showed that Rachel had a stronger understanding of the relationship between partitioning and equivalence than she had at the beginning of the year. However, she still found co-ordination of all the required knowledge elements difficult. Naming of the mid-way fraction required co-ordinating the size of the partition and the expression of the complete unit as a single fraction.

R: I drew up a number line because I didn't get it, then I got it. You go  $3 \times 4$  is 12 which both fractions fit into. They go into a number line so the number in the middle is one twenty-fourth.

I: One twenty-fourth. I see you've done it here. You've got... this is a quarter and this is a third and you're finding the one that is half way in-between here.

R: Yeah.

I: And you're saying this bit here is one-twenty-fourth. But how much is this altogether?

R: Each piece is one-twelfth.

I: And you're saying this little mark here – that is supposed to be half way?

R: Is supposed to be one twenty-fourth.

I: Yes, but how many twenty-fourths is it in all of this?

R: Seven.

I: How did you get that?

R: In one twelfth there's two, and there's three twelfths, so that's six and one more.

I: So the sign would read seven twenty-fourths.

In Term Three Rachel applied the measure sub-construct to discrete situations. She reunited parts of 24 can packs of cola as fractions in their simplest form, e.g. 18 cans is three-quarters of a 24 can pack. In context she was also able to measure one fractional unit with another, e.g. five-eighths of 24 cans is two and a half six packs. Her use of equations reflected a measurement (quotative) view of division with fractions and showed the flexibility she had developed with reuniting in discrete situations. In these situations Rachel's co-ordination classes for ratio, measure and operator appeared to be further co-ordinated into a type of super co-ordination class for proportions, albeit in an early form.

$$c \left| \begin{array}{l} 5 \\ 1-8 \\ 4 \end{array} \right. = \frac{2}{8} = \frac{2 \cancel{2}}{1 \cancel{4}} \quad 2 \frac{1}{2} \text{ packs}$$

Figure 73: Rachel found how many six packs could be made from five-eighths of a twenty-four pack (17 July)

Her competence in discrete situations transferred to continuous situations. She coordinated division as measuring with her concept of equivalent fractions to solve fraction division problems. In purely symbolic form, she created an algorithm gleaned from attending to pattern and adopted equivalent fraction based strategies to solve problems. Note her conversion of three-quarters and one-sixth to  $\frac{18}{24}$  and  $\frac{4}{24}$  respectively in Figure 74.

Andre has three-quarters of a loaf of Mollenberg bread left. He uses one sixth of a loaf each time he makes a toasted sandwich. How many tasted sandwiches can he make with the bread he has?

$$\frac{5}{4} = \frac{18}{24} \quad \frac{1}{6} = \frac{4}{24} \quad \frac{18}{24} \div \frac{4}{24} = \frac{4}{24} \times \frac{1}{2} = 1$$

Figure 74: Rachel used equivalent fractions to solve a division problem (16 August)

Rachel saw similarity between discrete and continuous situations. In discrete situations the individual items that make up the set are distracters. She regarded the whole set as the referent one and worked with it as though it was continuous. Rachel also treated a fraction of a referent one as a new referent, and measured it with another fraction of the original one. Her capacity to re-unitise and norm in terms of multiple referent ones was a key feature of Rachel's strategies at this time.

In her end of Term Three interview (16 August) Rachel solved  $1\frac{1}{2} \div \frac{2}{5} = \square$  to find out how many trips a car could make on one and a half tanks of petrol. She drew Figure 75 to solve the problem.



Figure 75: Rachel drew a strip diagram to solve a fraction division problem (16 August)

Her correct answer of  $3\frac{3}{4}$  trips was not obtained through observable use of equivalent fractions. This may have indicated flexibility of method and sufficing in the situation

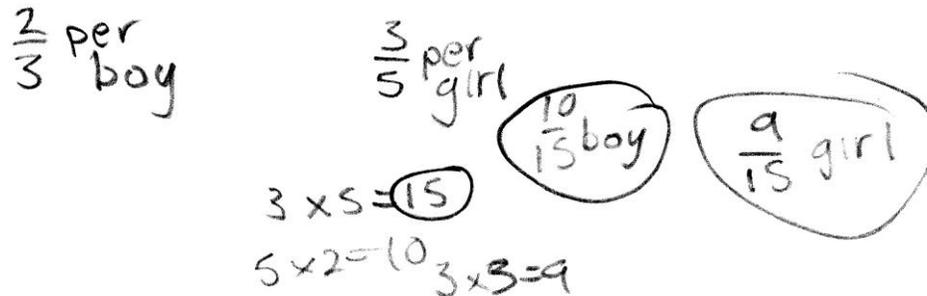
as opposed to a lack of transfer. However, coupled with her non-use of equivalence in a quotient problem from the same interview it also indicated that knowledge of equivalence had slipped in fluency and/or in cueing priority.

Several observations describe the development of Rachel's knowledge of the part-whole and measure constructs. Her fluency with multiplication and division of whole numbers enabled her development of an equivalent fraction scheme. Reliability of calculation helped Rachel see similarity across situations. Reliability worked in tandem with attendance to the structure of splitting in making generalisation possible.

Flexible reunitising of quantities was an integral part of Rachel's reasoning. As the situations became more complex she had to conceive of multiple referent ones simultaneously. This came from a base of perceiving a single referent one flexibly. Finally, Rachel's possession of an equivalence concept for fractions as measures did not mean that she applied it consistently to relevant situations. It took considerable time before she consistently applied the equivalence of fractions as numbers to situations involving quotients, ratios and operators.

### 7.3.2 QUOTIENT

By the end of Term One Rachel had abandoned her previous methods of finding equal shares by repeated partitioning. She used the quotient theorem,  $a \div b = \frac{a}{b}$ , in the context of sharing pizzas among boys and girls. Significantly, she co-ordinated the equal shares in a quotient situation with fractions as measures by naming the shares as equivalent fractions to enable comparison (see Figure 76).



**Figure 76: Rachel used the quotient theorem and equivalent fractions to compare shares (26 March)**

By this time Rachel was connecting her constructs for measures and quotients. This indicated that these classes developed sufficiently to enable transfer. In her interview at the end of Term Three (16 August) Rachel compared the shares for two boys with five pizzas with three girls with eight pizzas. She readily applied the quotient theorem to get shares of  $2\frac{1}{2}$  pizzas (boys) and  $2\frac{2}{3}$  pizzas (girls). Her paper answer stated that boys got more pizza though she changed this answer at interview. She was unable to give the difference between  $2\frac{2}{3}$  and  $2\frac{1}{2}$  other than to say that "it could be one tenth or something like that". This episode showed how tenuous successful transfer was.

That which transferred successfully in a similar situation five months previously transferred only in part on this occasion.

In her final interview Rachel successfully found the difference in shares between five girls with three pizzas and eight boys with five pizzas. She calculated the difference using equivalent fractions and decimals (see Figure 77). In discussion with me, she was unable to give the difference between 0.625 and 0.6 as a fraction (twenty-five thousandths). Use of equivalent fractions returned as a way for Rachel to compare shares. The return was as confounding as its absence in the Term Three interview.

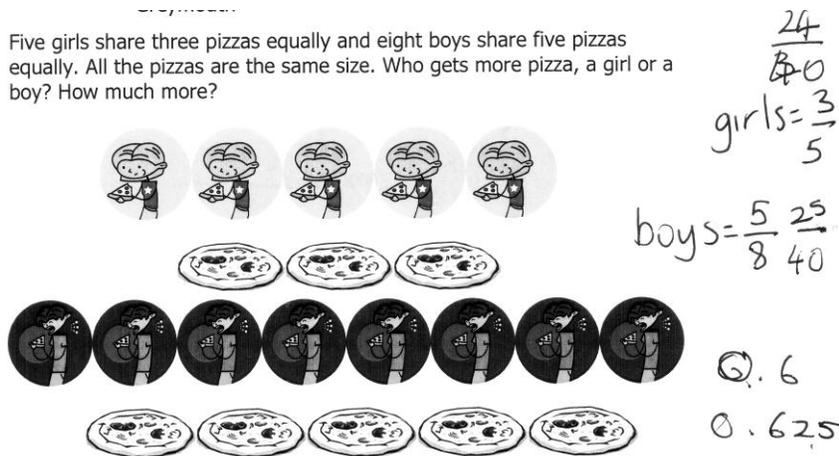


Figure 77: Rachel compared shares in a pizza context using fractions and decimals (19 November)

### 7.3.3 OPERATORS

No explicit instruction on fractions as operators occurred in terms one and two. A question in the pencil and paper test at the beginning of Term Two showed that Rachel had good control over finding a fraction of a whole number amount. The problem required multiple calculation steps and attendance to multiple pieces of information. She correctly concluded that all three families had the same distance to travel (see Figure 78).

By the second week of Term Three Rachel was able to multiply fractions and see the operation as finding a fraction of another fraction. She quickly understood an array-based model for the operation and connected fraction multiplication to multiplication of decimals, e.g.  $0.6 \times 0.32 = 0.192$  because  $\frac{6}{10} \times \frac{32}{100} = \frac{192}{1000}$ . Rachel applied this knowledge to finding the maximum volume of an open cuboid that made from a 26cm by 26cm square of paper. She used a spreadsheet to find the dimensions to one decimal place (4.3 x 17.4 x 17.4).

2. Which of these three rally drivers has the **furthest** distance, in kilometres, left to go on their race?

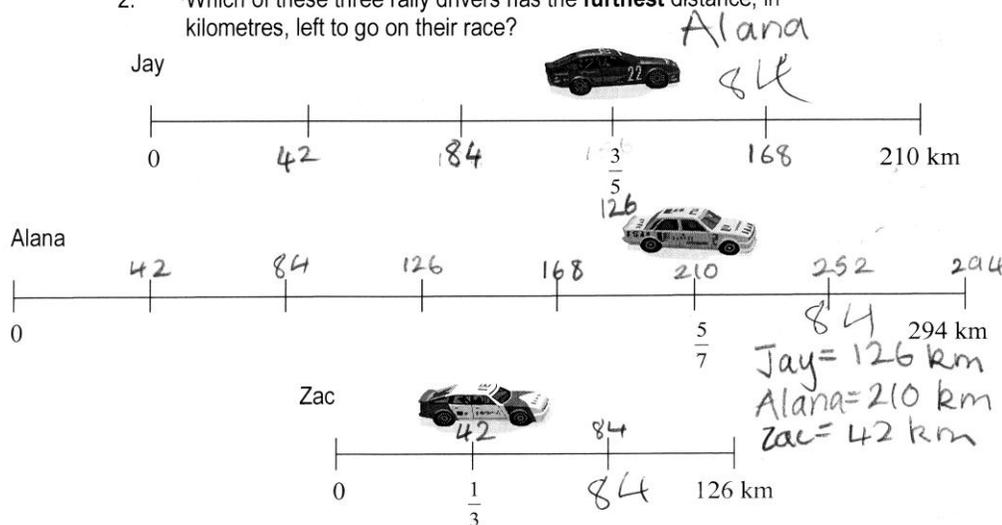


Figure 78: Rachel calculated fractions of whole numbers (16 July)

In Term Three instruction for Rachel's teaching group focused on the multiplicative operator that connected values in rate pairs, e.g.  $\square \times 12 = 18$ . Rachel used common factors to find fractional operators in simple rate situations. For example, to compare Jordan who rode 18 laps with Alana who rode 12 laps she used three as a common factor to conclude "Jordan rode  $\frac{6}{4}$  times as many laps as Alana" and "Alana rode  $\frac{4}{6}$  times as many laps as Jordan." Her understanding of multiplicative operators was very susceptible to change of context. In a concrete making context that involved a 24:9:15 ratio of gravel to cement to water Rachel wrote that the gravel was "2 x bigger & 6 tonne" than the cement. In this situation she possibly accessed knowledge of other linear relations that were learned in algebraic settings. In rate situations she applied multiplicative operators but in this ratio situation she applied a combination of multiplicative and additive thinking.

	12	24	36	48	60	72	10	100	1000
$\frac{6}{3}$	<del>24</del>	<del>48</del>	<del>72</del>	96	120	144	20	200	2000
$\frac{5}{2}$	<del>30</del>	<del>60</del>	<del>90</del>	120	150	180	25	250	2500
$\frac{3}{5}$	<del>18</del>	<del>36</del>	<del>54</del>	<del>72</del>	36	54	6	60	600
		14.4		28.8					

Figure 79: Rachel practised fractions as operators using grids (5-9 November)

In her final interview at the end of Term Four (19 November) Rachel compared two-thirds of 210 kilometres with four-twelfths of 96 kilometres in a context that involved finding the remainders of two journeys. She calculated both answers mentally and

had no need to attend to equivalence relationships between two-thirds and four-twelfths as operators. The intention of the item was to provoke use of doubling and halving thereby revealing whether students believed that the properties of multiplication applied to fractions as operators. Rachel's strategy satisfied to solve the problem so she had no need to apply the properties.

### 7.3.4 RATES AND RATIOS

In her initial assessments Rachel knew the part-whole fractions within a ratio. However she struggled to co-ordinate her knowledge of equivalence in fixed one situations with ratio situations in which the referent whole changed. For example, when asked to compare the accuracy of two shooters she responded, "I think Jessie is the better shot 'cause her number of shots (goals) is closer to her number of shots than Simon." In this situation she read out (attended to) appropriate information but she inferred that additive difference was a more appropriate model than multiplicative comparison. In sporting contexts, such as goal shooting, the associate primitives of more goals better player and less misses better player seemed particularly attractive to her.

By the end of week three Rachel was using different forms of comparison in frequency and ratio contexts. She understood the conservation of flavour in equivalent ratios of fruit juice mixtures. She transferred her knowledge of scaling equivalent rates to ratios.

$$\begin{array}{l} 5:4 \times 7 = 35:28 \leftarrow C \\ 7:5 \times 5 = 35:25 \leftarrow D \\ \text{Recipe C has the} \\ \text{strongest taste} \checkmark \end{array}$$

Figure 80: Rachel used multiplication of ratios to compare the taste of two fruit cocktails (March 12 )

However, in frequency situations she used equivalent fractions. The problem in Figure 81 was to compare the first serve accuracy of two tennis players, Jordan with 15 out of 24 serves in and Ben with 12 out of 20 serves in. The difference in strategies showed that Rachel did not perceive similarity of structure in these two situations, ratio and frequency. She appeared to have all the required knowledge elements available to apply equivalent fractions to both contexts. Initial testing showed that she knew the part-whole fractions in ratios but did not co-ordinate this knowledge with her use of equivalence in measurement situations with fixed ones. Lack of transfer was not due to the complexity associated with variable referent wholes, as this applied to both contexts. Semantic triggers such as "out of" may have helped Rachel to connect fractions to frequencies more readily than in ratio situations

in which the “out of” structure needed to be inferred from the part-to-part information.

$$\begin{array}{l}
 \text{Jordan} = 15/24 = 5/8 \\
 \text{Ben} = 12/20 = 3/5 \\
 \frac{5}{8} \times \frac{5}{5} = \frac{25}{40} \quad \text{Jordan} \\
 \frac{3}{5} \times \frac{8}{8} = \frac{24}{40} \quad \checkmark
 \end{array}$$

Figure 81: Rachel used equivalent fractions to compare serving frequencies in tennis (March 11)

At the end of Term One interview Rachel showed maintenance of her knowledge of part-whole relationships in ratios. Given a ratio of 4:3 in a newspaper delivery context she identified  $\frac{4}{7}$  and  $\frac{3}{7}$  as the part-whole fractions. She demonstrated improved flexibility with rates through use of common factors and equivalent non-unit rates, i.e. 18 cakes in 12 hours as six cakes in four hours, therefore 12 cakes in eight hours.

During Term Two, students in Rachel’s group considered how the properties of shapes changed under enlargement. Her initial thoughts were that side lengths grew additively, e.g. a triangle with sides of 3:4:5 was similar to a triangle with sides of 6:7:8. After drawing triangles to test her conjecture, Rachel readily accepted that multiplicative scalars applied to the side lengths of similar shapes.

Rachel showed co-ordination of the part-whole relationships in ratios and between ratios and operators in the following question from the initial Term Three interview (16 July).

Josh and Jessica pick apples. In the day Josh picks 48 cases and Jessica picks 32 cases. They get paid \$250 altogether. How much of the money should each person get?

She used a common factor of four to reduce 48:32 to 12:8 then 3:2 and calculated three-fifths of \$250 and two-fifths of \$250.

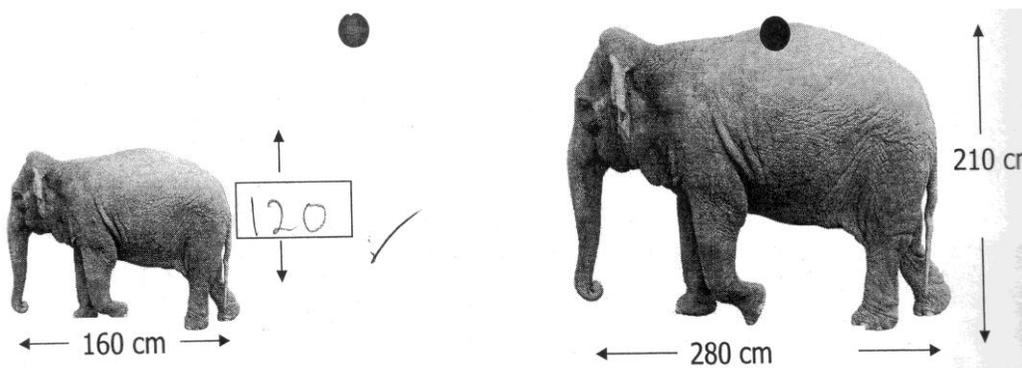
$$\begin{array}{l}
 \text{Jessica} = \frac{2}{5} \\
 \text{Josh} = \frac{3}{5} \\
 \begin{array}{r}
 12 \\
 4 \overline{) 48} \\
 \underline{40} \\
 8 \\
 4 \overline{) 32} \\
 \underline{32} \\
 0
 \end{array}
 \end{array}$$

$$\begin{array}{l}
 250 \div 5 = 50 \\
 \text{Jessica } 50 \times 2 = 100 \\
 \text{Josh } 50 \times 3 = 150
 \end{array}$$

Figure 82: Rachel co-ordinated ratios and operators (16 July)

In solving the problem it appeared that her previous learning resulted in a cueing priority for both ratios and operators that allowed Rachel to successfully co-ordinate across sub-constructs. This connection did not occur easily with the enlargement problems in Term Two. Rachel saw 48:32 as the same ratio as 3:2 and saw  $\frac{3}{5}$  and  $\frac{2}{5}$  as the two part-whole relationships. She then used these fractions as operators on \$250 to obtain the required ratio of \$150:\$100.

The proportionality of side lengths from similar shapes was reintroduced in Term Three. Rachel treated the matching lengths as rate pairs and applied common factor reasoning to establish scalars.



**Figure 83: Rachel applied rate thinking to scaling figures (8 August )**

Transfer of ratio to similarity of shape was not an obvious choice for Rachel. This was hardly surprising given that the structure of whole-to-whole comparison, as in side length to side length, was different to the part-part ratios with which she was most familiar. Rachel perceived ratios in enlargement and discrete mixture situations as different. Similarity existed in the multiplicative relationship between the parts and wholes compared. Rachel appeared to have sufficient multiplicative knowledge resources available to notice that similarity of structure.

Some instruction in Term Three was devoted to inverse rate problems (isomorphism of product). A painting context gave the information that two students could paint a mural in 12 hours. Rachel could not find the unit rate, i.e. one student paints one-twenty-fourth of the mural in one hour, and incorrectly stated that three students would take nine hours. The final question asked the time that 18 students would take to paint 12 murals. Her table indicated that Rachel recognised the constant product property but could not make use of it in the context (see Figure 84). The presence of three measures in the situation, students, hours and murals, was outside Rachel's experience with rate situations.

No. kids	hours	No. kids
8	3	18
2	12	
3	8	
16	1:50	
1	24	No

Figure 84: Rachel used a table to calculate inverse rate values in a painting context (9 July)

Rachel's teaching group encountered several other inverse rate scenarios and representations, including the use of a number balance. For example, Rachel drew a number plane to graph the inverse relationship between the number of lollies per bag and the number of bags that gave 144 lollies in total. She also drew tables to see which combinations of weights and arm locations (distances from the fulcrum) balanced and did not balance. Students were encouraged to notice similarity across the constant product situations and compare the characteristics of those situations to their idea of direct rate (isomorphism of measure). In her interview at the end of Term Three (16 August) Rachel correctly solved a balance problem. A boxer pup weighing 16 kilograms was at three measures from the fulcrum. Rachel found the weight of a cat positioned at eight measures from the fulcrum needed for the scales to balance. She identified the situation as appropriate for equality of product.

R: Because  $16 \times 3$  is 48 and the other one has to equal 48. So  $6 \times 8$  is 48 so he must be six kilograms.

Work in Term Four on similar triangles, as a lead into trigonometry, showed Rachel recognised that fractional operators connected ratio and rate pairs. In simple scaling situations she often used common factors to identify scale factors.

Rachel's absence from school for six days made it difficult for her to construct a strong concept of ratio in similar triangle situations. She measured the angles in diagrams rather than applied trigonometric ratios to mask her lack of knowledge. After two lessons Rachel was able to scale a unit triangle to find unknown side lengths but did not appreciate that trigonometric ratios were the same for similar right angled triangles of any size, e.g.  $\text{opposite} \div \text{hypotenuse}$  gave a common ratio (sine) for all similar right-angled triangles. Obtaining different answers for different similar triangles using a ruler and calculator did not support her identification of common ratios (see Figure 85).

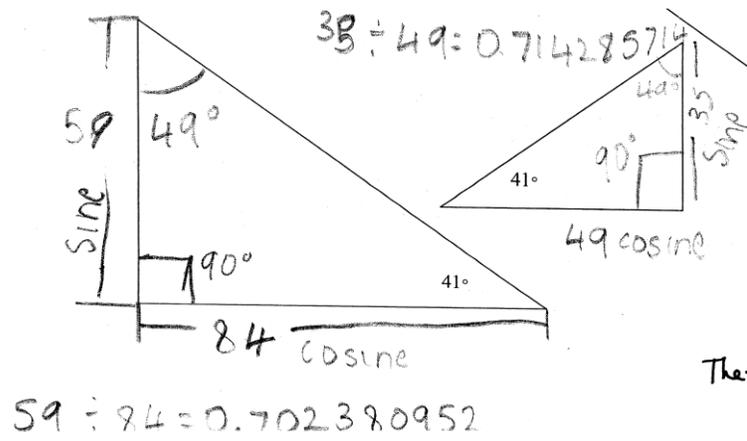


Figure 85: Rachel obtained different calculator answers for common ratios of similar triangles (26 October)

Rachel partially overcame the effect of her absence through her relationship with Aleisha. Both students worked together on using trigonometry to find the heights of the tallest trees in the playground. Their explanation of method In Figure 86 showed they were able to apply tangent as a ratio to calculate the height of a tree accurately.

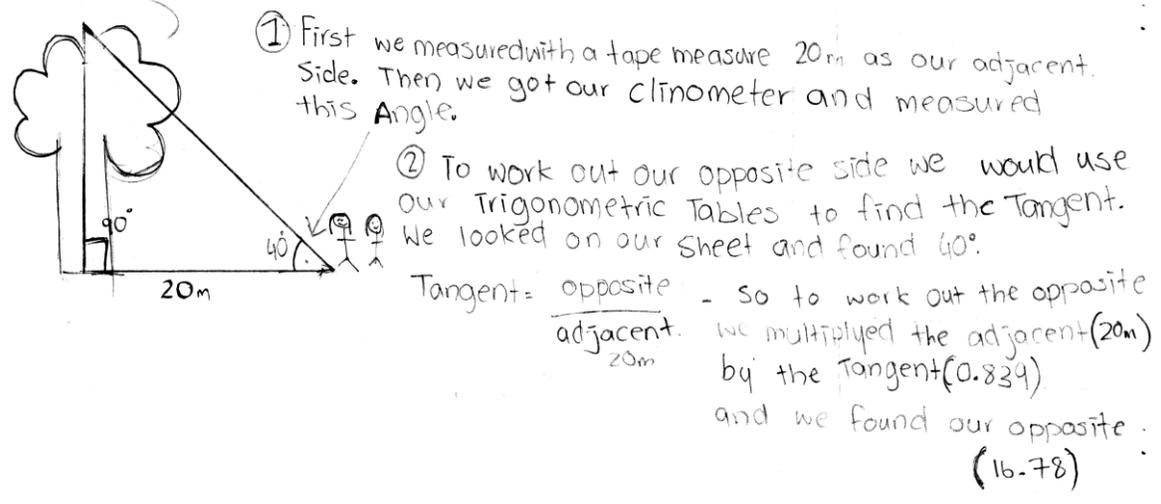
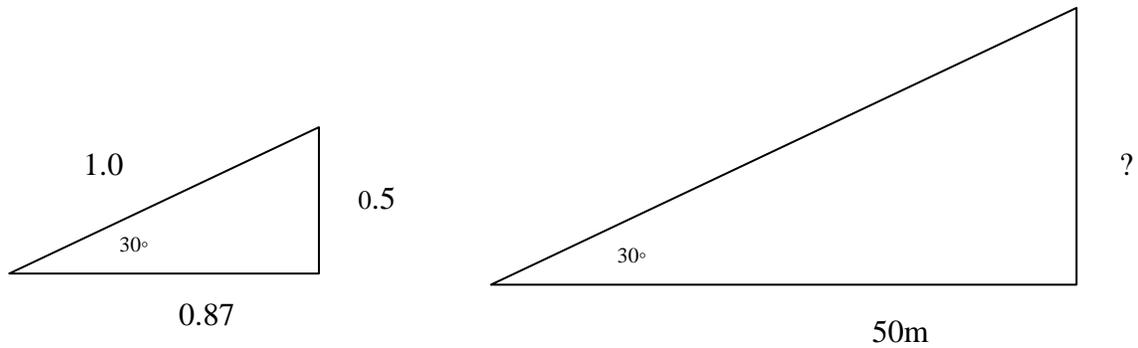


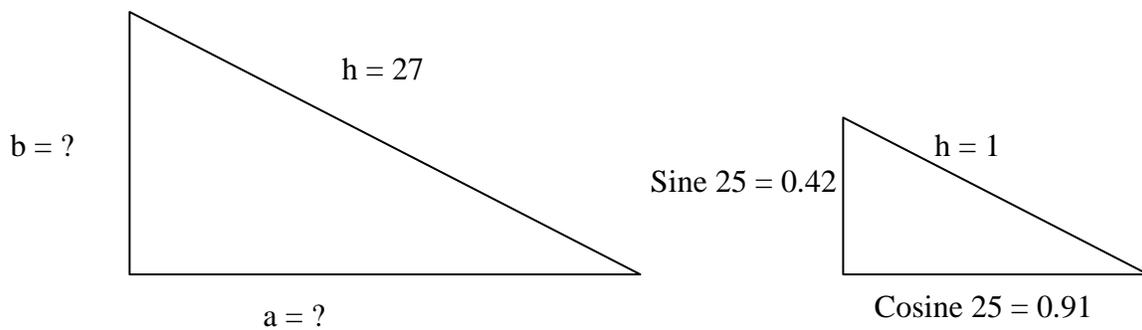
Figure 86: Rachel and Aleisha described their method for finding the height of a tree (25 October)

With less friendly numbers division by a decimal less than one required a transformation in her thinking about scale factors. I asked the students in her group to compare a unit triangle with a similar triangle as given in Figure 87.



**Figure 87: Similar triangle problem one involving scaling by a non-integer**

Finding a uniform method to find the length of an unknown side caused ongoing uncertainty especially in situations where the numbers involved were decimals. Rachel experimented with various operations. For example, given Figure 88 and asked to find the length of the opposite side she tried both  $27 \div 0.42$  and  $27 \div 0.91$  to find an answer that seemed sensible.



**Figure 88: Similar triangle problem two involving scaling by a non-integer**

It was through solving many similar examples with an appeal to reasonableness that enabled Rachel to see what operation worked in these situations. Primitives such as multiplication makes bigger and division makes smaller did not completely account for why the requirement of difficult calculation reduced her attendance to similarity in the properties of the triangles under scaling. Confidence in her ability to estimate the length of unknown sides gave Rachel a way to check which calculation worked. She simply satisfied in meeting the demands of the situation.

In her final interview Rachel did not attempt the trigonometry question in Figure 89 as she could not recall which ratio to use. The recording is that of the interviewer.

The tree needs to be felled because it is dead on the inside. From the measurements given work out the height of the tree.

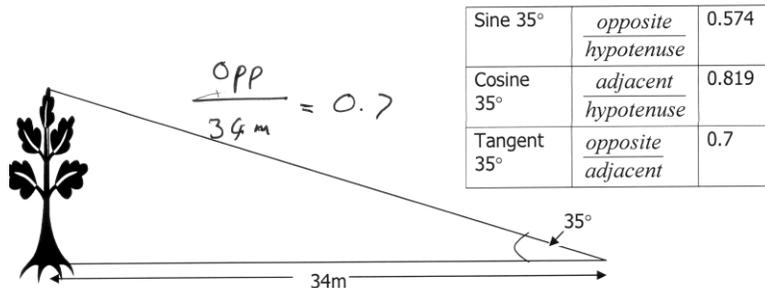


Figure 89: Tree height problem from Rachel's final interview

I helped Rachel identify which sides of the diagram were adjacent and opposite to the angle of 35 degrees and pointed out that the table meant that the opposite side length over the adjacent side length was 0.7.

I: What would you do there?

R: Multiply 34 by point seven.

Failure to recall the labelling of sides relative to an angle and her inability to read out the required information from the table prevented Rachel from starting on the problem. Once those barriers were removed, she co-ordinated her knowledge of fractions as quotients and operators sufficiently to obtain a correct solution.

In her final interview a rate problem involving pineapples and dollars proved trivial. Rachel calculated the unit rate of 1.5 pineapples per dollar and found the missing multiplicand in  $\square \times 1.5 = 6$  to find the correct answer.

Several observations are relevant about Rachel's learning of ratios and rates. At the beginning of the year she had knowledge of the part-whole relationships in ratios and was very fluent at using multiplication to replicate rates. Rachel worked with ratios in frequency situations more easily than in part-part situations such as discrete mixtures or whole-whole situations such as scaling triangles. This was possibly due to their close analogy to an "out of" view of fractions. Rachel perceived scaling situations initially as dissimilar to mixture situations but she came to view them as similar in the sense that multiplicative relationships applied. Whole-whole ratios sometimes required the finding of non-integral unknown operators. If unfriendly numbers existed in ratio pairs, particularly decimals, Rachel trialled calculations with a view that reasonableness confirmed validity.

## 7.4 Probability

Probability provided another context for part-whole reasoning about fractions during Term Four. In a game involving the product of two dice Rachel speculated that chance could be described using fractions. The teaching diary noted:

They (students) do not possess structured ways to find all of the outcomes yet so their fractions are speculative estimations. Most have a naïve idea of sample size as a predictor. “More is better” prevails but they do not recognise that larger samples minimise “proportional variation” from theoretical predictions. (29 October)

I used statistical sampling as a vehicle for developing part-whole thinking in probabilistic contexts. One situation involved use of a bag of coloured counters as a model for fish in a lake. At first Rachel saw little point in sampling counters when it was possible to count all of them by emptying the bag. She recorded her samples using in a table containing invented symbols for the different fish types. In one exercise she was told to tag a sample of ten fish (counters) with replacement into the lake (bag). She selected a new sample of ten fish to find out how many were tagged in order to estimate the fish population of the lake. Rachel wrote:

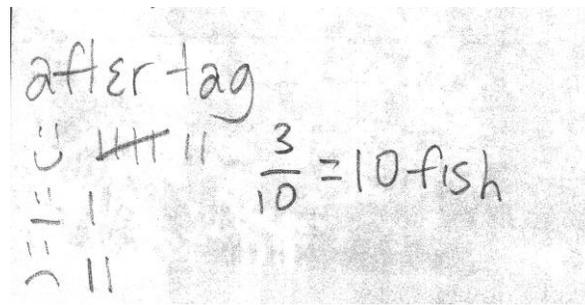


Figure 90: Rachel used a tagged sample to estimate the total population (25 October)

She concluded, “After our samples there should be 30 people on the island (fish in the lake)” Rachel recognised that if three-tenths of the population was ten fish then three times that number, 30 fish, was a good estimate of the total population. Overtime she came to appreciate that probability required an acceptance of variation that was at odds with her deterministic preferences.

## 7.5 Graphs

There was no explicit teaching of scatterplots in terms one and two to represent co-variation. The initial pencil and paper test at the beginning of Term Three contained an item in which students gave an estimate for the handspan of a 165 cm tall person from a dataset represented with a scatterplot. Rachel gave 17.5cm as her answer. This showed she could read data from scatterplots and her deterministic view with little acknowledgement of variation. Rachel attended to the vertical scale (height) and selected the middle of data points with that height to read off her estimate on the horizontal scale (handspan).

In whole class algebra instruction she learned to identify relations from number plane graphs showing several data points. Rachel was able to write expressions to represent relations.

$$\begin{array}{l} \text{Graph 1} = x^2 + 1 \\ \text{Graph 2} = x^{\frac{5}{2}} \\ \text{Graph 3} = n^{0.3} \\ \text{Graph 4} = n^2 - n \end{array}$$

Figure 91: Rachel wrote expressions for relations shown on spreadsheet graphs (6-10 August)

The algebra instruction and rate problems gave Rachel multiple opportunities to connect graphs with the sets of ordered pairs they represented. In rate situations, she attended to both measures in answering questions related to graphs. Given three points on the following graph (Figure 92) she drew a straight line to find several rate pairs.

Asked if she could buy 35 oranges for \$20 she answered, “No because if the grid carried on it would cost more than \$20” (23 July). Her answer reflected a focus on the line as a predictive tool. There was no evidence that she saw the slope as a representation of unit rate. Rate situations were a subset of the multiple situations in which she had encountered linear graphs. Rachel’s noticing of similarity across the situations was that constant differences produced linear graphs.

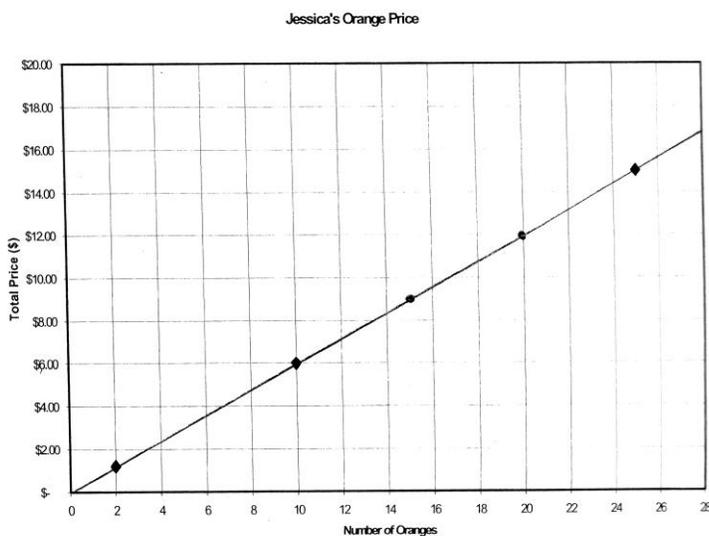


Figure 92: Rachel used a straight line to extrapolate and interpolate missing rate pairs (24 July)

Confirmation that Rachel did not see slope as representing rate of change came from her interview at the end of Term Three (16 August). She was given three points on a number plane that represented the time in minutes and the number of laps of three runners. Asked who of the runners was fastest Rachel used a line connecting the origin with the point representing her favoured runner. The line was used as a way to equalise one measure rather than as a representation of rate.

R: I think that Jessie is fastest. I drew a straight line and matched the lines up (meaning for equivalent time).

I: What leads you to that conclusion?

R: If I took a ruler (uses a remote as a straightedge) and drew the line to here (matching time) Odette would be slightly lower.

Instruction on statistics provided Rachel with further opportunities to connect slope of a graph with rate of change. One situation involved using Tinkerplots™ to create a graph of the world record for the men's high jump over time (1896 – 2000). Rachel drew a trend line through the data points and noticed the achievement of relatively small improvements over recent times.

In her final interview (19 November) Rachel compared the pay rates of three people shown as ordered pairs on a number plane (see Figure 93). At first, she eliminated Andre as a possibility for the highest paid.

R: I know it's not Andre because Ben is higher than him (equal money less hours).

Rachel used a calculator to find the unit rate for Rachel and Ben and recorded these numbers in the top left of the frame. The interviewer then drew a line of slope from the origin to Rachel's data point as shown in Figure 93.

I: I thought you might do this (drawing the slope line)...What does that tell you, when you draw a line?

R: Like when you get to there it shows you (pointing at the point) the amount of money they have got when they have got the same amount of hours as you.

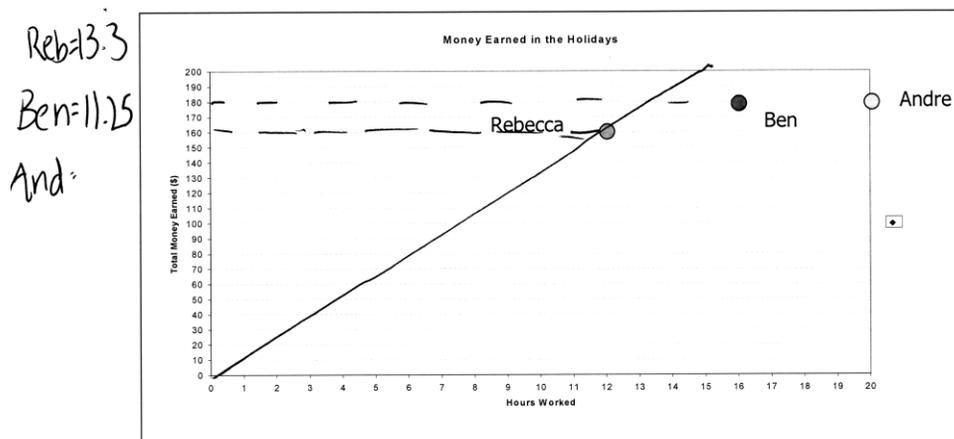


Figure 93: Rachel solved a pay rate problem using unit rates

As previously, Rachel saw the line as a way of equalising one measure to compare the other measure across the two rates. She did not see the slope as representing rate of change, a derived measure from co-variation of time and money. There were many potential obstacles to Rachel's transfer in this situation including non-acceptance of the homogeneity of rate, attendance to unit change as additive difference through other linear relations and her long established priority cueing for finding unit rate using given rate pairs. Rates presented different difficulties in comparison situations than ratios. Rachel used equivalent fractions to represent the part-whole relationships in ratios. Rates held no parallel structure for her.

## 7.6 Decimals

Term One, weeks three and four, involved explicit instruction about the place value structure of decimals and the addition and subtraction of decimals. The physical models used to develop the ideas included, an area model called the decimat, one metre long paper strips and circular regions. Rachel preferred to use a written algorithm to add and subtract decimals and quickly applied the relative size of the place values involved including renaming, e.g. 2.4 as 24 tenths. At first she was able to connect quotients to fractions, e.g.  $3 \div 5 = \frac{3}{5}$ , in converting fractions to decimals, e.g.  $\frac{3}{5} = 0.6$ , but was unable to work in reverse, e.g.  $0.6 = \frac{3}{5}$ .

Rachel's ability to consider decimal place value in flexible ways took considerable time to develop during the later part of Term One. She struggled to express decimals as fractions and quotients, and expressed whole number remainders from division as decimals, e.g.  $5 \div 4 = 1.1$ . Consistency of symbolic pattern seemed to be a strong validation mechanism for Rachel as she sought to co-ordinate her knowledge of decimals through multiple situations. In Figure 94 Rachel used the decimat model to anticipate the decimal for five-eighths. Note that she used a division of one-tenth into quarters (bottom right) and symbolic pattern to confirm her prediction. She also made a recording slip in the sequence of equations, i.e.  $10 \div 4 = 2.25$  (2.5).

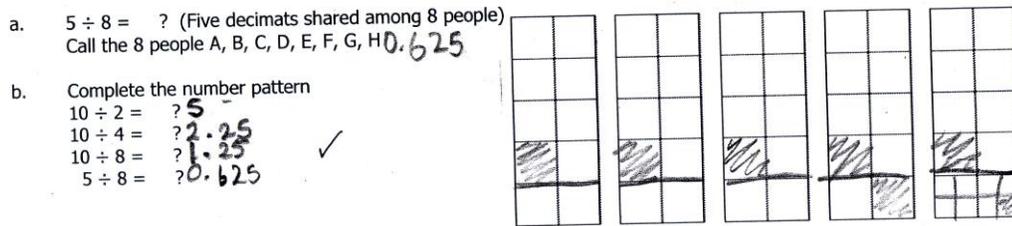


Figure 94: Rachel used a combination of decimat model and number pattern to predict the decimal for five-eighths (19 March)

At the end of Term One Rachel showed no change in her strategy for solving a decimal division problem, i.e. nine litres  $\div$  0.45 litres in a fruit juice context. Her method showed a preference for treating decimals as whole numbers of equal length, e.g. 0.45 as 0.450, and a collecting leftovers strategy. This indicated her developing knowledge of decimal place value had not transferred to quotative division situations.

Group work in Term Two gave Rachel more chances to make fraction to decimal connections. She was able to place decimals and fractions correctly on the same number line.

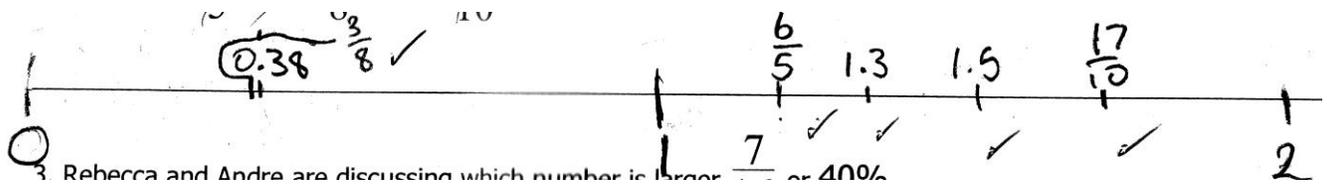


Figure 95: Rachel placed decimals and fractions on the same number line (7 May)

However connecting the recurring decimal for a unit fraction, such as one-seventh, with anticipating the decimal for a non-unit fraction, such as four-sevenths, proved difficult. Rachel understood why recurring decimals occurred but was unable to coordinate fraction to decimal conversions. The connection of knowledge elements, rather than the absence of knowledge, appeared to be at the heart of Rachel's inability to complete these tasks. Elements included non-unit fractions as iterations of unit fractions, the canon of place value, and decimal notation. Rachel's knowledge of the individual elements was sound.

T: Why do decimals recur sometimes?

R: Because there's always a bit left over. It may be one-tenth, it could be three-tenths, etc.

(15 May)

Expressing the remainders of whole number division as decimals, fractions or whole numbers was difficult for Rachel in Term Two. For example, she did not associate a calculator answer of 10.52 to  $263 \div 25$  with its other expressions of  $10 \frac{13}{25}$  and  $10 \text{ r}13$ .

In her pencil and paper assessment at the beginning of Term Three Rachel was unable to use knowledge that the decimal for three-eighths is 0.375 to write three-twenty-

fourths ( $\frac{3}{24}$ ) and nineteen-eighths ( $\frac{19}{8}$ ). Recognising that  $\frac{3}{24} = 0.125$  from  $\frac{3}{8} = 0.375$  involved multiple acts of knowledge co-ordination, such as  $\frac{3}{24}$  is one third of  $\frac{3}{8}$  therefore it is the same as  $\frac{1}{8}$  and  $375 \div 3 = 125$  so  $0.375 \div 3 = 0.125$ . At this time Rachel was unable to make these multiple co-ordinations even though many of the knowledge elements were well established singularly. She wrote,  $\frac{19}{8} = 2.$ , noticed that the decimal was greater than two, but did not connect the remainder of  $\frac{3}{8}$  with 0.375 to give a complete answer. This showed co-ordination in part, where Rachel used some of the knowledge elements appropriately but did not make all the necessary connections. During Term Three, Rachel showed progressive improvement in connecting fractions and decimals. Using decimat models in conjunction with a calculator Rachel connected fractions with quotients and expressed these numbers as decimals. She correctly established that  $\frac{10}{11} = 0.9090\dots$  given  $\frac{1}{11} = 0.0909\dots$ , using multiplication by ten. Her co-ordination of scaling by powers of ten and multiplication of fractions enabled her to connect a single multiplication answer with whole numbers, e.g.  $7 \times 4 = 28$ , to many decimal multiplication answers, e.g.  $0.7 \times 400 = 280$  and  $70 \times 0.04 = 2.8$ .

a.  $\frac{3}{2} = 1.5$  ✓    b.  $\frac{3}{4} = 0.75$  ✓    c.  $\frac{3}{8} = 0.375$  ✓    d.  $\frac{3}{16} = 0.1875$  ✓  
 e.  $\frac{1}{3} = 0.\dot{3}$  ✓    f.  $\frac{1}{9} = 0.\dot{1}$  ✓    g.  $\frac{1}{27} = 0.\overline{037}$  repeating    h.  $\frac{1}{81} = 0.\overline{012345679}$  repeating  
 i.  $\frac{6}{10} = 0.6$  ✓    j.  $\frac{65}{100} = 0.65$  ✓    k.  $\frac{653}{1000} = 0.653$  ✓    l.  $\frac{6537}{10000} = 0.6537$  ✓

Figure 96: Rachel converted common fractions to decimals (24 July)

Rachel's knowledge of the alternative ways to view remainders from division grew progressively through Term Three. In solving 47 divided by 11 to find the unknown side length of a rectangular field Rachel appeared to make a connection for herself. To express the remainder in the required decimal form she carried out a division algorithm then used her knowledge that three-quarters equals 0.75 to complete her answer.

$$\begin{array}{r} 4 \overline{)47} \text{ r}3 \\ \underline{44} \phantom{00} \\ 3 \phantom{00} \end{array}$$

$$4 \overline{)3} \frac{3}{4}$$
  
11.75 cm Work sheet

Figure 97: Rachel connected whole number, fraction and decimal remainders (23 July)

An attempt to provoke estimation of decimal multiplication in the end of Term Three interview (16 August) was uninformative. Rachel had the following scenario:

Rump steak costs \$8.60 per kilogram. Jay buys 1.479 kilograms. How much does he pay? (Choose one of these answers).

\$127.19    \$12.72            \$1.27            \$0.13

She identified \$12.72 as the only viable choice as both \$1.27 and \$0.13 were not possible “because these are less than \$8.60 which is one kilogram but this (\$127.19) is too much.” The lack of a viable alternative to \$12.72 limited the information it yielded about Rachel’s number sense with decimals.

In testing at the end of the year, with PAT and AsTTle tools, Rachel answered the majority of decimal questions correctly. She was able to identify the position of 5.16 and 0.4 on number lines, add two place decimals, order two place decimals, check decimal addition calculations using inverse operations, estimate decimal multiplication by rounding, and convert  $\frac{16}{20}$  to a decimal form. Her incorrect answer involved ordering decimals with a varied number of up to three places. She got the same item incorrect at the start of the year.

Rachel’s knowledge of decimals was at no point complete. It was only adequate for some problems. She drew on resources from her co-ordination classes for measures, quotients and operators in constructing more and more complex ideas. While ordering ragged decimals still caused her difficulty, her knowledge was sufficiently complete for her to think in sophisticated ways about multiplication and division with decimals.

## 7.7 Percentages

A series of whole class mini-lessons on percentages used a Slavonic abacus as a manipulative for demonstrating fraction to percentage conversions. The lessons developed benchmark fractions as percentages and applied them to simple percentage calculations. Rachel learned these ideas easily and on a short test solved three percentage calculation items correctly, e.g. 12.5% of \$72.

By the end of Term One her knowledge of common factors and equivalent fractions gave Rachel computational fluency with percentage problems about frequencies. Her response to expressing 27 out of 45 as a percentage showed that she now viewed percentages as equivalent fractions in these situations.

$$\begin{array}{l}
 27 \div 9 = 3 \\
 45 \div 9 = 5 \\
 100 \div 5 = 20 \\
 3 \times 20 = 60
 \end{array}$$

Figure 98: Rachel calculated a percentage in a frequency situation (26 March)

During Term Two some instruction was provided on using percentages as operators, e.g. 40% of \$36. Rachel relied heavily on using ten percent as a benchmark and building up to the required percentage. Her strategy showed a transfer between rate and operator sub-constructs and a tendency to view a percentage operator as an honorary whole number rather than as a fraction. In a conversation with me Rachel was surprised to learn that 12.5% of an amount gave the same answer as one-eighth of that amount (10 May). A short exposure to percentages as fractional operators in Term One was insufficient to embed this idea. By the end of Term Two Rachel seemed to accept treatment of percentage operators in the same way as whole numbers and as fractions. However, she seemed unable to co-ordinate the two views simultaneously. In Figure 99, she tried to calculate 12.5% of \$96. Rachel knew that 12.5% was equivalent to one-eighth but then calculated 80% using  $8 \times 10\% = 80\%$ .

She retained her ability to solve percentage as operator problems, using rate based strategies, into Term Three. In the initial pencil and paper test Rachel compared the price of two pairs of jeans, Jean City at a cost of \$119.95 less 35 percent and Denim Dungeon at a cost of \$79.95 less seven percent for cash. For both calculations Rachel rounded the retail price to a whole number of dollars and used the distributive property to find partitions of the required percentage, e.g. 7% as  $10\% = \$8$  so  $1\% = \$0.80$  so  $7\% = \$5.60$ . In doing so, she co-ordinated properties of whole number multiplication with unit and composite rates.

$$\begin{array}{l}
 12.5\% \text{ of } 96 \\
 12.5 = \frac{1}{8} \text{ of } 100 \\
 96 \div 10 = 9.6 \\
 \begin{array}{r}
 9.6 \\
 \times 8 \\
 \hline
 76.8
 \end{array}
 \end{array}$$

Figure 99: Rachel unsuccessfully treated a percentage operator as a whole number and as a fraction simultaneously (10 May)

Rachel knew the connection between decimals and percentages as equivalent forms. She commented on the following strategy, attributed to Ben, for finding ten-eighths as a percentage, “Ten out of eighteen is  $\frac{10}{18}$ . That’s the same as  $\frac{5}{9}$  and  $\frac{1}{9}$  is 0.11 recurring. What do I do now?” Rachel wrote: Ben = x by 5 then x by 100 & there is your answer. (17 July)

Her answer showed co-ordination of multiple knowledge pieces including five-ninths as five iterations of one-ninth, multiplication applied to recurring decimals, i.e.  $5 \times 0.11\dots = 0.55\dots$ , percentage as “out of one hundred” and multiplying a decimal by one hundred gave the equivalent percentage.

Rachel was reluctant to estimate answers to percentage problems. She tended to calculate exact answers first to compare the accuracy of different estimation methods. In response to problems with unfriendly numbers Rachel seemed in search of a single algorithm. The use of a calculator without being taught the key in procedure saw her experiment with possible algorithms. In one situation she had to work out the discount if Kayla paid \$21 for a skirt that normally cost \$35. Rachel recorded the

algorithms  $35 \overline{)2.851429}100$  and  $2.8571429 \times 21 = 60$  (8 August) to find out the percentage for 21 out of 35. Given her previous fondness for using common factors this represented a shift to her cueing preferences. I noted her preference for dividing 100 by the base number in a frequency in the teaching diary.

Andre, Ben, Rachel, Joshua, Zac and Jordan all recognised that  $100 \div 36$  gave the percentage equivalent to one unit (unit rate). (8 August)

Providing examples of percentage problems where the numbers did not share a common factor or the base number did not scale easily to 100 resulted in Rachel seeking a trusted procedure or algorithm. Reduction to a unit rate cued highly in her treatment of rate problems. Seeing percentage operator situations as rate situations had a consequential effect on her preference for solving such problems.

In her interview at the end of Term Three (16 August) Rebecca attempted an algorithm to find eighty percent of 35 shots (see Figure 100). Her calculation of  $100 \div 35 = 2r30$  did not provide the tidy scale factor she desired so she abandoned her working. Note transposition of 35 and 100 in the division equation.

3. Rebecca got 80% of her shots in during the last netball game. She took 35 shots.

How many goals did she get?

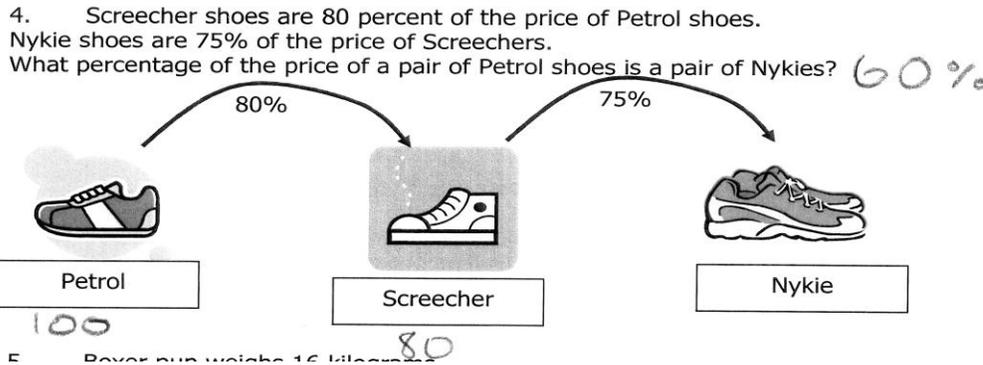
$$35 \div 100 = \text{blah}$$

$$\text{blah} \times 80 = \text{answer}$$

$$35 \overline{)100} \quad 2r30$$

Figure 100; Rachel applied an algorithm for solving percentage problems (16 August)

In the same interview she found a generalised percentage of a percentage. Rather than operate on the unknown as a generalised number Rachel assigned the price of a pair of *Petrol* shoes a nominal price of \$100. The problem did not require variable thinking so Rachel assigned an amount that made calculating the generalised percentage easy.



**Figure 101: Rachel assigned a fixed value to solve a generalised percentage problem (16 August)**

Rachel's final interview presented her with a percentage problem in which the base number was unknown. A calculator was available to her.

You sit a spelling test and you get 18 words right. The teacher gives you 67% correct. How many words are in the test?

I: How many words are in the test?

R: Eighteen and some more (joking). You could go  $67 \div 18 = 3.72$ .

I: What does that tell you?

R: What each percentage is (unit rate of percentage per word). Then you could go 100 divide by 3.72 (gets 26.88). So 26 questions.

I: You have some rounding there. Which is it closer to?

R: 27.

I: Does  $18/27 = 67\%$  check out? (pointing to written answer).

R: Yes, half of 27 is  $13 \frac{1}{2}$  which is 50%.

I: Look at those numbers (pointing to 18 and 27). Is there anything in those numbers that helps you?

R: They are both in the nines times table.

I: So can you simplify that?

R: Two-thirds.

I: And that checks with 67%.

R: Yes.

The dialogue revealed Rachel's ability to access multiple knowledge elements with support rather than independently. She was unfazed by the complexity of decimal answers the calculator produced and appeared to have understanding of the part and whole relationships she was calculating. The interviewing protocol provided considerable data about Rachel's accessible knowledge though she did not choose to use it independently.

PAT or AsTTle testing at the end of the year did not pose sufficiently complex questions to measure Rachel's achievement with percentages. She answered all the problems correctly including finding 15% of 300, converting 70 out of 200 to a percentage, and comparing 30% of \$7.00 with \$3.00.

## **7.8 Summary**

### **7.8.1 PROCESSES TO OBJECTS**

Many examples of anticipated processes are evident in Rachel's case study. Her drive to anticipate actions on objects without carrying out the actions was a feature of Rachel's mathematical disposition. For example, decimats were used as a physical model to represent decimals as quotients, e.g.  $3 \div 4 = 0.75$ . Rachel moved to symbolic forms to anticipate the results of actions after three examples.

She began the year with a strong preference for finding unit rate and scaling the unit rate to solve problems. Unit rate is the most fundamental of multiplicative relationships. There was obviously a time in Rachel's conceptual development when she carried out operations on physical objects or images of them to scale unit rates. Her entry data suggested that she now anticipated these physical actions by multiplication and she knew a range of facts that supported this anticipation.

The unit rate strategy is a theorem in action. There was no evidence that Rachel consciously drew on a formal algebraic notion of the theorem when she used it, she trusted that it worked in certain situations. Her anticipation of actions on rates became embellished to include scaling or reducing a given rate pair by a common factor to produce non-unit rates. This was one indicator that Rachel used unit rate as an object of thought.

During the year there were multiple instances in which Rachel applied unit and non-unit rate strategies to situations. She was not confident with percentages at the beginning of the year. She came to see percentage change situations as rates.

Problems such as 80% of 35 shots became opportunities to find other rate pairs by scaling, e.g. 10%:3.5 shots or  $(100 \div 35)\%: 1$  shot. Rachel applied rate thinking to measure problems such as  $1 \frac{1}{2} \div \frac{2}{5} = 3 \frac{3}{4}$  by building pairs, i.e. 1:  $\frac{2}{5}$ , 2:  $\frac{4}{5}$ , 3:  $1 \frac{1}{5}$  ...

Several other anticipated processes appeared to become objects for more advanced construction of concepts. She applied multiplication and division of whole numbers as inverse operations, the quotient theorem, and converting a fraction to an equivalent form by multiplying numerator and denominator by the same factor at different times. It was questionable to conclude the encapsulation of these processes as objects in a complete and tidy way from the data.

Equivalence of fractions was a fleeting notion, applied in some situations but not in others. For example, Rachel used equivalence to compare shares in quotient situations at the end of Term One, did not use it in Term Three with the same type of problem, and reinstated it in Term Four. She used multiplication and division as inverse operations fluently with whole numbers. Faced with calculations involving decimals in trigonometry Rachel abandoned structural thinking about inverse. She favoured establishing which operation gave a reasonable answer.

The evidence suggests that in Rachel's case processes became anticipated but encapsulation was a doubtful metaphor. For an anticipated process to be applied Rachel trusted in its efficacy and co-ordinated sufficient knowledge to make it work in a given situation. Embellishment of anticipated processes occurred by application. They were not complete and encapsulated, just sufficient and available.

### 7.8.2 CO-ORDINATION CLASSES

Rachel's case study supported the hypothesis that co-ordination class theory was explanatory to the development of proportional reasoning. Data at the start of the year showed Rachel to be a capable mathematician for her age. She had advanced multiplicative thinking with whole numbers that gave reason to anticipate ready progress on proportional reasoning. Rachel did progress but the process was lengthy and spasmodic.

Rachel's construction of measures, operators, quotients, and rates and ratios matched the development of co-ordination classes. This was also true of her work with graphs, probability, decimals and percentages. There was considerable variability in the way Rachel strategised across different situations. Examples of this variability included:

- Expressing quotients as decimals but being unable to express decimals as quotients, e.g.  $3 \div 5 = 0.6$
- Using equivalence in frequency situations but using scaling in ratio situations
- Finding unknown fractional operators in rate situations but not in ratio situations

- Applying multiplication and division as inverse operations with whole numbers but not with decimals
- Using multiplicative thinking to scale ratios and rates in some situations and additive thinking in others, e.g. enlargement of shapes

In this discussion the term situation is used in a variety of ways. Situations may be story contexts, associations between representations, and connections between and within the sub-constructs or models. While the ontology literature described many situations as structurally similar, e.g. isomorphism of measure, it was clear from Rachel's case study that similarity was in the eye of the beholder. Co-ordination class theory predicted the progressive increase in span of a concept, the breadth of situations to which the concept could be applied. For example, Rachel viewed frequency situations as different to ratio situations despite the part-part and variable whole similarity of the situations.

Frequency situations made the "out of" inference unnecessary but in ratio situations Rachel had to infer the part-whole relationships in order to apply equivalence of measure. Rachel knew the part-to-whole relationships of ratios in isolation, applied the relationships to find shares of a whole but did not apply them consistently in determining equivalent ratios. In enlargement situations, the ratios involved whole-to-whole comparison, such as the relationship between side lengths. Ontologically these ratios were isomorphism of measure situations, similar to frequencies and ratios, but Rachel did not perceive them as such initially. As predicted by co-ordination class theory it was her eye for similarity across many examples that led to Rachel applying multiplicative rather than additive thinking to whole-whole ratios.

Rachel's case study is shot-through with examples of successful and unsuccessful co-ordination of knowledge elements. There was evidence of phenomenological primitives such as more goals therefore better shooter that interfered with her construction of proportionality. Some knowledge elements such as more shares therefore smaller pieces and division makes smaller seemed to be derived primitives or *met-befores*. Once observed by focused attention they acquired p-prim like status. Rachel shared many elements in the construction of concepts as co-ordination classes, knowledge of common factors for example. Rachel borrowed knowledge across constructs in ways that indicated construction of a super co-ordination class for proportionality.

The structure of co-ordination classes was in the head of me, the researcher. It was not apparent that Rachel explicitly thought in terms of an evolving network of co-ordination classes. She possessed language and symbols to name concepts and talk about them, such as equations for operations and terms like "two-thirds per boy". Language and symbols appeared to play a key role for Rachel in identifying different situations as similar and in detecting patterns and relationships.

### 7.8.3 SPAN AND ALIGNMENT

I graphed Rachel's conceptual growth on the learning trajectory map for multiplicative thinking and proportional reasoning. Her initial map showed Rachel's relatively advanced concepts of measures, ratios and rates (see Figure 102). Rachel knew how to find equivalent fractions but no data existed to see if she transferred this knowledge to addition and subtraction or measurement with fractions. She relied on conversion to unit rate and scaled ratios by multiplication and division. Rachel used equal sharing strategies in quotient situations.

Knowledge of most basic multiplication and division facts supported her development of fraction sub-constructs. Rachel used the distributive property to derive unknown multiplication facts and solved division problems by multiplicative build-up.

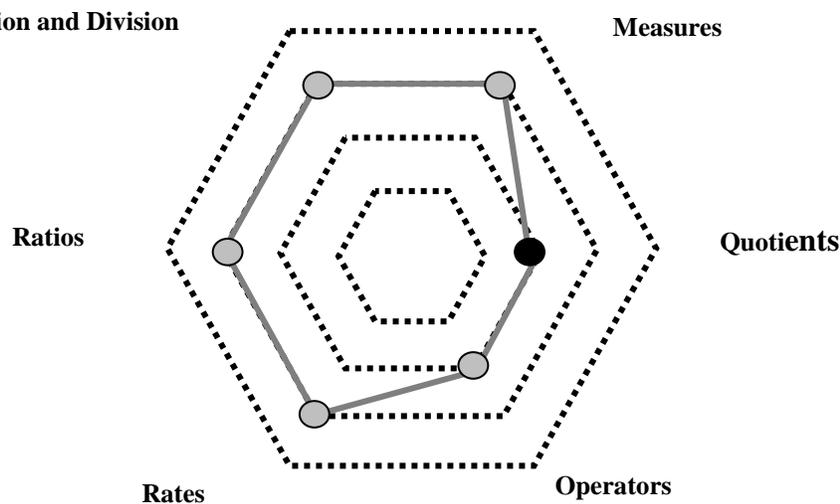


Figure 102: Learning Trajectory Map: Rachel beginning of Term One

Term One showed a progressive development across all the fraction sub-constructs. Rachel developed mental strategies for multiplication and division based on the properties of these operations. She learned to find common factors efficiently. Rachel's enhanced her knowledge of equivalent fractions by seeing them as equal measures and she applied equivalence to adding and subtracting fractions, ordering fractions, and comparing shares in quotient situations, frequencies and ratios.

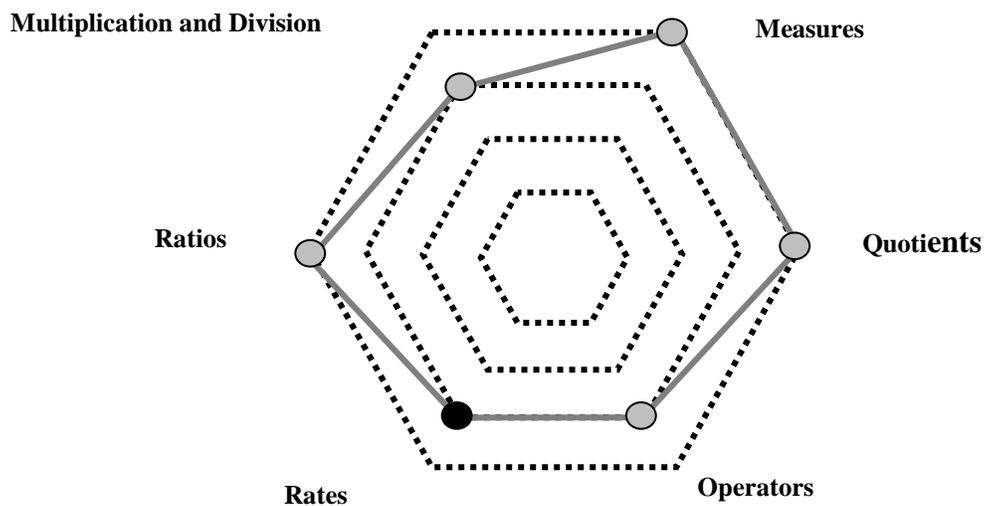


Figure 103: Learning Trajectory Map: Rachel end of Term One

Through terms two and three Rachel developed her mental strategies for division with whole numbers. She connected multiplication and division as inverse operations and partitive with quotative division. Rachel's solutions to problems involving the measure sub-construct showed strong alignment (reliability). She saw subtraction as difference between fractions and division as measurement with fractions. She identified and used the operator connecting members of rate and ratio pairs, and solved problems with inverse rates.

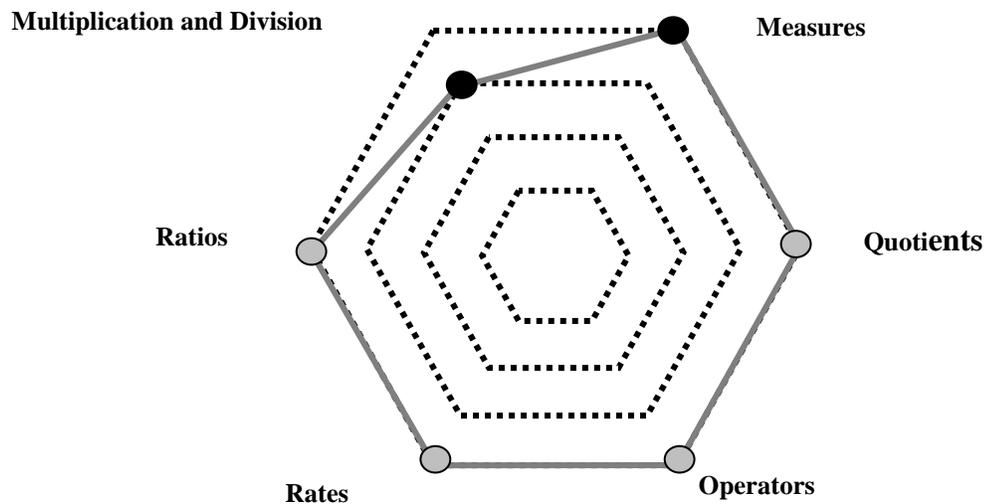


Figure 104: Learning Trajectory Map: Rachel end of Term Three

Rachel's progress in Term Four was adversely affected by her ill-health. The span of Rachel's proportional reasoning was at the limits of the learning trajectory map. Her strategies indicated that Rachel was integrating her knowledge across the fraction sub-constructs. Rachel's alignment in ratio situations became reliable but her strategies and knowledge in situations involving the other fraction sub-constructs were unreliable.

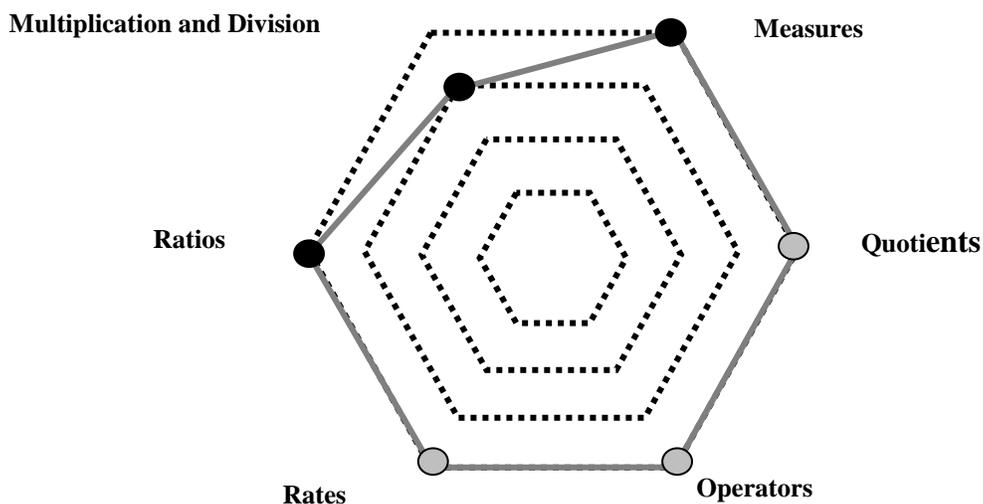


Figure 105: Learning Trajectory Map: Rachel end of Term Four

The simultaneous view of Rachel's maps showed considerable progress in multiplicative thinking and proportional reasoning during the year (see Figure 106). Her initial cognitive growth was in multiplicative strategies with whole numbers and the measures sub-construct. This reflected the teaching emphasis. Rachel connected measures to other constructs so her profile became very regular by the end of the year. While the span of situations to which she applied proportional reasoning developed, her strategies still lacked reliability (alignment) at that time. This observation was consistent with Rachel's co-ordination of her knowledge across constructs.

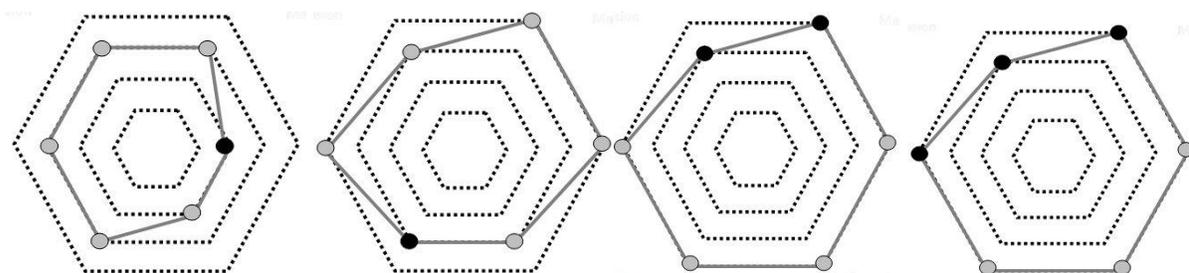


Figure 106: Simultaneous view of Rachel's Learning Trajectory Maps

## CHAPTER EIGHT: CASE STUDY OF ODETTE

### 8.1 Beginnings

#### 8.1.1 PERSONAL CHARACTERISTICS

At the beginning of the 2007 school year Odette was 12 years nine months old. She identified her ethnicity as Maori and appeared equally comfortable with her father's German ancestry. She readily recalled the rich experiences she had as a younger child when both of her parents undertook missionary work in Africa. Her twin brother, Simon, was also in the class. Both students were home-schooled by their parents until the age of eleven.

Odette had a wide circle of friends, mainly other girls, and appeared to be well liked by other students in the class. She approached her schoolwork with extreme dedication, participated freely in any discussions and completed all independent work with diligence. She was frequently a leader in instigating class events. According to her class teacher, Odette had particular strengths in language. She was an excellent sportsperson and was the reigning school tennis champion.

#### 8.1.2 INITIAL TESTS

Initial testing in February and early March showed that Odette was performing at an average level in mathematics for her age. Her AsTTle test result described her as working at level 3P with more strength in number operations than number knowledge. Her PAT result was stanine five based on her class level. This placed her at an average level. Analysis of her performance on PAT items suggested good knowledge of whole number place value, simple fractions and decimals to tenths. She demonstrated understanding of multiplication and division in a variety of situations, including an application of a simple rate and to quotative division. Her areas of weakness included extension of the decimal system to hundredths and thousandths, knowledge of integers, solving percentage problems, and finding and describing the relationships found in sequential patterns. Odette's performance on the geometry, measurement and statistics questions was consistent with average achievement. She displayed relative strength in spatial visualisation, including symmetry, and in probability.

#### 8.1.3 INITIAL GLOSS INTERVIEW

The GloSS interview (20 February) revealed that Odette had a strong repertoire of basic facts for addition with some minor gaps in her recall of subtraction and multiplication basic facts. She relied heavily on visual images to order unit fractions, worked out the number of tens in 239 using her knowledge that ten tens equal 100 and located 6.9 on a number line. Odette was unable to progress further in the place value knowledge questions. This confirmed the limitations of her decimal knowledge

but contradicted the apparent strength in whole number place value described in the PAT result.

Her mental strategies for multi-digit subtraction were limited to visualising algorithms though she showed greater flexibility in using multiplication, mainly though use of doubling and halving strategies, e.g. 4:52 so 8:104 by doubling. This was consistent with her strong performance on rate problems in the initial AsTTle and PAT results. Odette was unable to find three-fifths of 30 centimetres but easily established that one-eighth of 16 was two using basic facts knowledge. This further exposed her difficulties with using non-unit fractions in operator contexts and her preference for doubling and halving.

#### **8.1.4 INITIAL NUMBER INTERVIEW**

Her first interview further exemplified Odette's fondness for repeated halving (21 February) when she partitioned graphics of confectionary worms into quarters and fifths. Her partitioning indicated that Odette understood the size relation of partitions, that is, the greater the number of parts the smaller the parts. Her anticipation of the result of repeated halving was restricted to three iterations. This suggested Odette possessed an emerging multiplication-based anticipatory scheme for equal partitioning in continuous contexts.

O: I would cut it here which is sort of the middle, then I would chop in the middle of that and the middle of that (halving halves).

I: If you cut your quarters in half, how many pieces would you have?

O: (Pause) Eight (uncertainly).

I: And they would be called?

O: Eighths.

I: And if you chopped the eighths in half how many pieces would you have?

O: Twelve (unsure).

In attempting to cut a worm into fifths Odette created an iterative unit that mapped onto the whole five times.

O: There and there and there and there (indicating places to cut).

I: You seem to be using your quarters to help you decide where to make that first cut. Is that right, when you had to make that first cut what told you?

O: I just kinda know.

I: So is a fifth bigger or smaller than a quarter? Which one is smaller – a fifth or a quarter?

O: A fifth.

I: Why?

O: Because I have to cut up the pieces into smaller bits than before. I have to cut that into five bits and that into four bits.

Odette's strength in halving and her preference for visual images when dealing with fractions contributed to her fluent answers to other partitioning task in the interview. Presented with a fraction strip model, I asked her, "How many sixths are the same as one half?"

O: Three because there are six in the whole bar and it would have to be a half of six.

Her lack of understanding of non-unit fractions as operators showed in her efforts to find three-quarters of 24, in sharing jellybeans onto a birthday cake. She focused on the numerator, possibly because it was the first recognisable whole number in the instructions.

O: (finger tracking) Would you get...(pause)...(looks stuck).

I: What are you trying to do to work it out?

O: I'm trying to divide twenty-four by three.

I: Okay.

O: That would equal eight, and then...I'm not sure.

The final question in her initial interview required Odette to work out the greater share, three boys sharing two pizzas or four girls sharing three pizzas. Odette demonstrated an equal partitioning scheme by drawing appropriate divisions on the pizzas in the graphic. She used a unit fraction as an iterative unit in describing the girls' share as three-quarters that appeared at odds with her previous engagement with three-quarters as an operator. She based her answer, that girls got more than boys did, on a visual comparison rather than on any concept of two-thirds and three-quarters as numbers reflecting a quantity. She did not possess an anticipatory process for determining the share in these quotient situations.

O: They'll get the same.

I: Can you show me how much pizza each person will get.

O: (Cutting pizza into thirds which are approximately equal) There'd be three pieces.

I: And what do you call those pieces?

O: One-thirds (continues cutting) and each person gets two pieces

I: Yes, two-thirds.

O: (Cutting pizzas in quarters and attempting to share – does not deal – tries grouping the pieces). They'll, each girl, will get two and a half pieces.

I: Can you please show me one girl's share?

O: (Starts dealing). They'll get one from this pizza, one from this pizza and ... hang on.

I: So they'll get one piece from each pizza, how much will they get altogether?

O: Three...three quarters.

I: So these guys get two-thirds and the girls...?

O: They'll get more.

### **8.1.5 SUMMARY OF ENTRY LEVEL**

At the beginning of this study, Odette showed some competence with whole number place value but limited understanding of fractions, decimals and percentages. Her approach to whole number calculation tended to involve algorithms, both visualised and recorded, but she created meaning-driven solutions in situations where she had no known procedure. She showed evidence of some multiplicative thinking with whole numbers but this linked closely to repeated doubling.

Odette showed strong preference for visual images when dealing with fractions. She did not co-ordinate her understanding of three-quarters as a continuous quantity with three-quarters as an operator.

## **8.2 Progression in Whole Number Operations**

Odette showed strong growth in her mental strategies with whole numbers over the first four teaching weeks (anecdotal record 22 March). She used the empty number

line as a representation to enhance her addition and subtraction strategies and successfully applied the distributive property for addition through standard place value and compensation based strategies. She learned her missing basic multiplication facts within the first week of instruction. Odette transferred use of the distributive property to multiplication problems. However, she often combined multiplication with additive thinking.

$0. 48 \times 5 = 240$   
 $I. 5 \times 40 = 200$  then  $8 \times 5 = 40$   
 $\$ 200 + 40 = 240$   
 $II. 50 \times 5 = 250$  then  $-10 = 240$   
 $Alana's answer$

**Figure 107: Odette using the distributive property (2 March)**

$12 \times 4 = 48$   
 $13 \times 4 = 52 (+4)$   
 $14 \times 4 = 56 (+4)$   
 $15 \times 4 = 60 (+4)$   
 $16 \times 4 = 64 (+4)$

**Figure 108: Using additive thinking in conjunction with the distributive property of multiplication (2 March)**

Despite the instructional focus on using mental strategies, Odette often used standard written algorithms when solving contextual problems. In week one she wrote:

I thought that the algorism way was the best because it was nice quick and easy.  
(2 March)

Odette’s trust in procedural routines was also evident in her description of how she multiplied numbers by ten. She wrote, “I know this is bad but this is how I did it. I added the zero from the 10 to the end of each answer” (15 March). Her initial attempts at division involved repeated multiplicative build-up.

By the end of week four Odette had broadened her range of multiplicative strategies to include use of the distributive and associative properties for multiplication and she applied these properties to division with variable reliability. She adopted arrays as her preferred representation of multiplication and applied this representation to division.

$$\begin{array}{l} 87 \div 3 = 29 \\ 90 \div 3 = 30 \text{ then } -1 = 29 \end{array}$$

Figure 109: Odette solving  $87 \div 3 = \square$  by compensating from  $90 \div 3 = \square$  (18 March)

$$\begin{array}{l} B. 48 \\ \text{i. 6 rows of } 8 \quad 6 \times 8 = 48 \\ \text{ii. 4 rows of } 12 \quad 4 \times 12 = 48 \\ \text{iii. 1 row of } 48 \quad 1 \times 48 = 48 \end{array}$$

Figure 110: Odette deriving how many eights, sixes and 48s equal 48 (23 March)

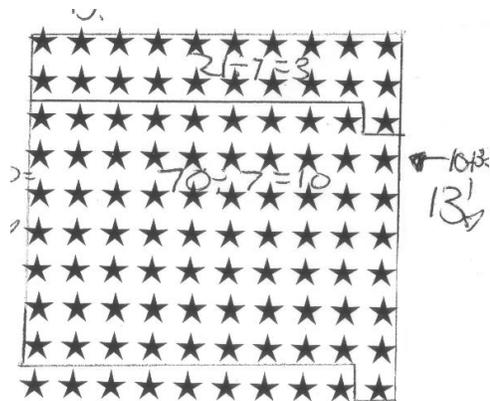


Figure 111: Odette using an array to show  $91 \div 7 = \square$  (18 March)

Despite her willingness to engage in the strategies there were indications that Odette did not trust the conservation of product or quotient when these properties were applied or saw merit in using them in preference to routine procedures. At the end of week four Odette was asked to explain why a hypothetical student had used a particular strategy. She wrote,

She switched the answers to make it easier to work out. You can also check if  $4 \times 7$  actually equals 28 (23 March).

This response signalled Odette's desire for confirmation of answer even when she recognised the simplicity or elegance of the method.

During Term Two Odette was able to use multiplication and division to model unfamiliar situations. For example, she correctly solved  $201 \div 3 = 67$  in a quotative division context. In a whole class lesson students were asked to find an efficient way to count the number of edges and vertices of platonic solids. Odette reasoned that knowing the number and shape of faces gave a total number of sides and vertices available. The number of edges and vertices was found by division (5 June).

Considerable improvements in Odette's understanding of, use of and confidence in the properties of multiplication and division were evident from the start of Term

Three. In week one she was able to use the commutative and associative properties of multiplication relationally without having to check by finding the product, e.g. 15 six-packs equalled 18 five-packs,  $15 \times 6 = 18 \times 5$  (17 July). She showed her ability to calculate multiplication and division answers. She still based her acceptance of multiplicative invariance on procedures for finding the answer rather than structural understanding at times. To solve  $16 \times 12 = \square \times 8$  Odette gave a correct answer of 24. However, she also wrote the total number of eggs, 192, which showed that she performed  $24 \times 8 = 192$  or  $192 \div 8 = 24$  to confirm she was correct (17 July).

She applied this understanding in connecting multiplication and division equations, e.g.  $72 \div 9 = 8$  so  $24 \times 3 = 72$  (18 July), and in solving linear equations, e.g.  $8n + 12 = 260$  (19 July), and in finding rules for given relations from sets of ordered pairs, e.g.  $n^3$  and  $n \div 4 + 2$  (7 August). She developed a stronger understanding of the effect of multiplying and dividing numbers by ten to accompany her add and subtract zeros algorithm. Confidence in calculation seemed to contribute significantly to Odette's belief that properties of multiplication and division worked and could be trusted.

Odette also made learning gains in several other aspects of whole number properties during Term Three. Her use of basic multiplication facts in missing factor, division and fractional representations gained fluency. She was able to apply both recursive and direct methods to find rules for relations presented as sets of ordered pairs, and write these rules using invented algebra. She also applied multiplicative thinking to develop theoretical models for simple situations involving chance, e.g. product of two dice.

### 8.3 Development of Rational Number Sub-constructs

#### 8.3.1 PART-WHOLE AND MEASUREMENT

Instruction in Term Two focused on the concept of equivalent fractions as numbers that describe the same quantity. Odette responded well to the use of a length model. Through reference to diagrams and physical materials, she was able to write equivalence relations within the first week (see Figure 112). She anticipated the result of equal partitioning of a unit fraction, and named the resulting parts using fractions, including in situations where there was no direct access to the unit whole or one (see Figure 113).

$\frac{1}{2} = \frac{4}{8}$ ✓	$\frac{1}{2} = \frac{5}{10}$ ✓
$\frac{1}{2} = \frac{2}{4}$ ✓	$\frac{1}{2} = \frac{3}{6}$ ✓
$\frac{2}{3} = \frac{4}{6}$ ✓	$\frac{3}{4} = \frac{6}{8}$ ✓
$\frac{4}{5} = \frac{8}{10}$ ✓	$\frac{5}{8} = \frac{3}{6}$ ✓

Figure 112: Olivia named equivalent fractions (9 May)

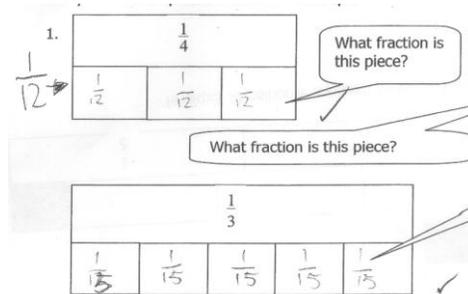


Figure 113: Olivia equally partitioned unit fractions (10 May)

By the end of Term Two Odette co-ordinated the splitting of unit fractions and named the resulting parts, e.g. quarters of thirds as twelfths. However, it was not clear whether she recognised equivalent fractions as numbers for the same quantity. She was unable to co-ordinate equal partitioning and equivalence to find the difference between two fractions, unit or non-unit, e.g. “Which is larger  $\frac{2}{3}$  or  $\frac{3}{4}$ ? By how much is it larger?” Odette relied on a visual image of materials rather than attend to the number properties involved as represented by equations involving fractions, e.g.  $\frac{3}{4} - \frac{2}{3} = \square$ .

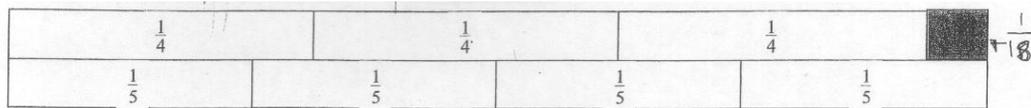


Figure 114: Odette incorrectly finding the difference between three-quarters and four-fifths (10 May)

The pencil and paper test administered at the beginning of Term Three showed the fraction understanding Odette had retained. She correctly solved  $\frac{7}{8} + \frac{3}{4} = \frac{13}{8}$  though initially she wrote  $\frac{10}{12}$  showing that the add numerators and denominators algorithm was still alive in her cueing preferences.

In the interview at the end of term 3 Odette tried to solve three-quarters plus two-thirds in a chocolate bar context. She explained her answer of  $1\frac{3}{9}$  as follows:

O: I wanted to put one-quarter onto two-thirds which is not quite a full one. I thought it might maybe be a ninth if you put it in there (meaning to make up one).

I: So you took that ninth off a half (the remaining half from three-quarters less one-ninth).

O: Yes, something like that.

Her explanation indicated some useful size relation knowledge with thirds and quarters, recognition of the need to identify the missing piece by partitioning,

knowledge that two-quarters was one-half, and one-half less one-ninth was about three-ninths. While Odette had rejected the add numerators and denominator algorithm she had not replaced it with any strategy that was based on finding equivalent fractions with the same denominator.

I revised the concept of equivalent fractions in two ways during Term Three. The focus on decimals involved connection between common fractions and their expression as equivalent fractions with denominators of ten, one hundred and one thousand. One lesson (7 August) involved revisiting equivalence in fixed wholes before extending the idea to variable whole situations.

Odette's response to a problem from her end of Term Four interview (19 November) showed that she accepted fractions as numbers and could rename them as equivalent fractions. The problem was, "You have three-quarters of a tank of petrol left. Each trip to Hamilton uses three-twentieths ( $\frac{3}{20}$ ) of a tank. How many trips can you make?" Odette had no taught or rehearsed procedure for solving this problem and had to rely on structural understanding of the fractions as quantities. The interviewer's suggestion to write information down helped Odette to solve the problem.

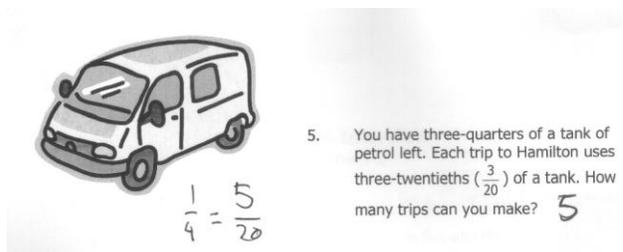


Figure 115: Odette renamed one-quarter as five-twentieths (19 November)

O: Four cut up into twenty is five.

I: So one-quarter is five-twentieths (Odette records this as an equation).

O: So three-twentieths is not a fourth. Would that be like a third (referring to  $\frac{3}{20}$ )?  
No...(recognising that  $\frac{3}{20}$  is not equivalent to one-third).

I: How much petrol have you got? How many twentieths have you got?

O: How many twentieths in a quarter? Um..three-quarters, I have fifteen-twentieths (lots of subvocalising)

I: (Pointing to  $\frac{3}{20}$ ) Each trip takes you three-twentieths and you've got fifteen-twentieths of a tank...

O: (Finger counting) I can do four trips.

I: Okay, why do you say four?

O: Three goes into fifteen...wait...you can do five (laughs).

In her response Odette showed a grasp of equivalent fractions and understood the need to express three-quarters and three-twentieths using a common denominator so the fractions could be compared. Her finger tracking to find out how many three-twentieths were in fifteen-twentieths suggested that she had her working memory fully employed. Recording  $\frac{1}{4} = \frac{5}{20}$  seemed to play a key role in helping her establish the equivalence of fifteen twentieths and three-quarters.

### 8.3.2 QUOTIENT

At the beginning of the year Odette solved sharing problems by drawing partitions on pictures of the pizzas and comparing the size of the shares visually. She received no instruction to develop her thinking in the quotients sub-construct during the first term. The final question in her end of Term One interview (26 March) also involved applying her understanding of fractions in a sharing pizza situation. Odette identified the respective shares as “2 pieces of 3” and “3 slices from 5”. This reflected a proportion view of fractions as quantity representations, i.e.  $x$  out of  $y$ . Her answers appeared to be dependent on her ability to draw on a graphic representation. Odette’s explanation of why the boys got one more slice showed that she applied her preference for rate in the sense that she allocated two-thirds to one girl, four-thirds to two girls, etc. She recognised that at this sharing rate the girls would need ten-thirds of a pizza that was one piece more than the three pizzas available.

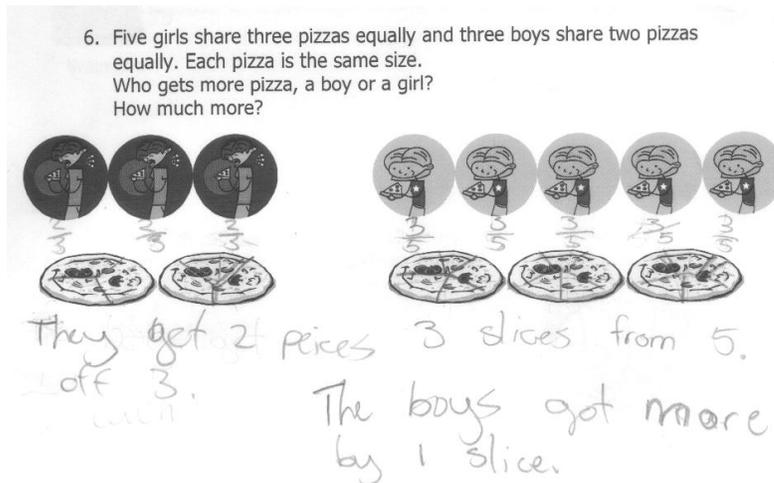


Figure 116: Odette used rate thinking to compare shares in quotient situation (26 March)

The following transcript explains Odette’s reasoning about the comparative size of the shares.

O: And the boys get more by one slice.

I: What do you mean, “They get one slice more”?

O: If you did thirds for the girls it wouldn’t work because...if there were five boys (exchanging boys for girls and getting stuck).

I: Are you talking about chopping these (pointing to three pizzas) into thirds?

O: Yes.

I: And that wouldn’t work. The girls wouldn’t get as many as the boys.

O: And there was just one missing (meaning a piece).

I: So if you did chop those (pizzas) into thirds how many pieces would that be?

O: Um...nine.

I: Oh, and you need ten to give each girl two-thirds.

In a pencil and paper test at the beginning of Term Three (16 July) Odette correctly solved  $\frac{7}{2} = 3.5$ . It is more likely she viewed  $\frac{7}{2}$  as “seven halves” rather than “seven divided by two” in this problem. Solving quotient problems was the outcome for one group lesson in Term Three. The lesson involved sharing contexts with paper circles and I made a link to symbolically recording the calculations. The teaching diary noted;

Students occasionally got lost in counting with materials, e.g. Odette lost one-tenth. Links to symbols seemed more strongly understood after connecting to materials. This idea is still not firmly established – accuracy is inconsistent. This is partly due to division, both knowledge-wise, and conceptual. (30 July)

I did not assess solving quotient problems formally again until the interview at the end of Term Four (19 November). In sharing three pizzas among five girls Odette imaged the pizzas cut into fifths and shared them out by finger pointing among the five girls in the graphic. In this way she obtained the correct answer of three-fifths. Her view of fractions as quantities had shifted from an “out of” view to one of iterations of a unit fraction. Her strategy also showed that she had probably not retained an anticipated process for sharing, i.e.  $a \div b = \frac{a}{b}$ . Alternatively, Odette may have adopted a more secure strategy given the importance she attached to being correct in the interview.

### 8.3.3 OPERATORS

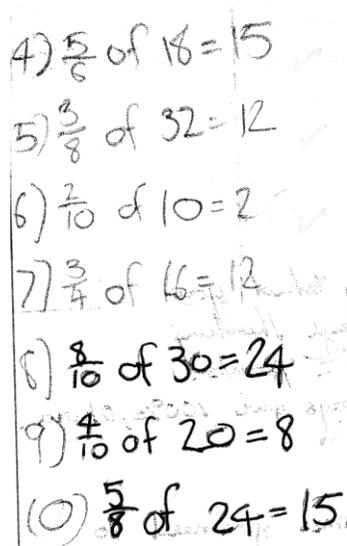
In initial assessments, prior to instruction, Odette was unable to solve problems where a non-unit fraction was used as an operator, e.g. three-fifths of 30. The focus of group instruction was on developing her multiplicative thinking with whole numbers in

Term One and her understanding of fractions as numbers in Term Two. At the beginning of Term Three Odette completed a pencil and paper test that contained an operator item (16 July).

Given a choice of which treasure was more, three-quarters of 660 coins or four-fifths of 550 coins, she correctly chose the first option. Odette provided no working so her choice could have resulted from a correct solution strategy, centration on the number of coins only or the fractions only, with three-quarters assumed to be greater than four-fifths. Her interview in week five of Term Three (16 August) confirmed the retention of her learning of solving operator problems. Odette was able to compare one-third of 36 and five-eighths of 24.

O: That one. If you cut that up into thirds you would have \$12. And if you cut that up into eighths you get three and three times five is \$15.

Further evidence from her bookwork and the anecdotal notes in Term Four (25 October) showed that Odette had retained the use of non-unit fractions as operators, e.g.  $\frac{5}{8}$  of 24 = 15.



A vertical list of seven handwritten mathematical problems, each with a checkmark to its right. The problems are:

- 4)  $\frac{5}{6}$  of 18 = 15 ✓
- 5)  $\frac{3}{8}$  of 32 = 12 ✓
- 6)  $\frac{2}{10}$  of 10 = 2 ✓
- 7)  $\frac{3}{4}$  of 16 = 12 ✓
- 8)  $\frac{8}{10}$  of 30 = 24 ✓
- 9)  $\frac{4}{10}$  of 20 = 8 ✓
- (10)  $\frac{5}{8}$  of 24 = 15 ✓

Figure 117: Odette solved fraction as operator problems (7 November)

Her final interview (19 November) revealed that Odette had improved her understanding of fractions as operators during the year but she still lacked trust in her solutions.

Odette correctly answered  $\frac{5}{8}$  of 40 = 25 in a marbles context.

O: Isn't one-eighth five so it's five times five. Is it twenty-five?

Responding to a problem with a question may have been a way to deal with her lack of confidence in the solution. The final question in her Term Four interview (19

November) involved a comparison of two-thirds of 210 and four-twelfths of 96 in a family car trip scenario.

O: They've done 140 km (by  $(\frac{1}{3} \times 210) \times 2$ ). They've done thirty-two.

I: How did you get that?

O: Divided 96 by twelve...

Odette then found the remaining kilometres left on each family's journey by subtracting rather than using the fractional complement of one, that is  $210 - 140 = 70$  rather than  $\frac{1}{3} \times 210 = 140$  which she has already calculated. In this operator context she did not notice the equivalence of four-twelfths and one-third. However, she retained strong control of her procedure for finding a fraction of a whole number in a problem that involved many calculation steps.

#### 8.3.4 RATES AND RATIOS

In initial assessments Odette showed the ability to solve problems by replicating rates. In her interview at the end of Term One (26 March) she correctly solved the rate problem "18 cakes per 12 hours equals  $x$  cakes in eight hours?" using a strategy based on the unit rate.

O: I went one and a half cakes means 18 cakes in 12 hours. Then I took away four hours which is four one and a halves so that's twelve cakes.

In doing so she found the unit rate, seeing one and a half as the multiplicative operator between eight and twelve, and treated this rate as a co-ordinated unit, i.e.  $1:1\frac{1}{2}$  so 4:6. This matched her performance on rate problems in previous assessments, and this showed that full understanding of the properties of multiplication and division, and strong fraction concepts are not necessary to solve rate problems involving whole number scalars.

In the same interview (26 March) Odette identified fractions for the part-whole relationships in the ratio 4:3, i.e.  $\frac{4}{7}$  and  $\frac{3}{7}$  (see Figure 118).

1. Jay and Rhandal share a pamphlet run. For every four pamphlets Jay delivers Rhandal delivers three. They get paid together. What fraction of the money should each boy get?



Jay would have:  $\frac{4}{7}$   
 Rhandal would have:  $\frac{3}{7}$

I added 4 and 3 together which = 7 then they would get  $\frac{4}{7}$  and  $\frac{3}{7}$  ✓

**Figure 118: Odette identified the part-whole relationships in equivalent ratios (26 March)**

In her interview at the end of Term Three Odette had to compare two ratios in a fruit punch context where orange juice mixed with apple juice. The problem was to find which ratio had the strongest orange taste, 3:5 or 2:3. Her answer referred to the additive difference when orange and apple parts matched in one-to-one correspondence.

O: The second one (2:3) 'cause if you put them both flavours together you get one apple left and the other recipe you get two apple flavours left over.

Previous assessments had shown that Odette could correctly represent the part-whole relationships in a ratio using fractions, but she was unable to connect this idea to the attribute of flavour strength or see how equivalent fractions, or replication of the ratios, serve to find a common comparison.

In Term Four Odette trusted percentages as a way to compare frequencies. For example, she correctly established that 12 out of 16 shots was a better proportion than 16 out of 20 shots.

How do you work out the answer in "best shot" problems like these?

I divided 100 by the amount and timesed the answer to how many any person got. Then you do the same thing to the other person and who ever got the biggest percentage is better. ✓

**Figure 119: Odette calculated percentages to compare frequencies (30 October)**

Despite an apparent rejection of equal differences in frequency situations, Odette's use of additive difference to compare ratios was very resistant to change. She did not readily connect part to whole comparisons in a ratio situation, such as mixing fruit cocktails, to her previous experience with frequency, such as comparing the shooting

success of two netball players. Despite correctly calculating the part-whole relationships in ratios as percentages, Odette did not question her use of differences.

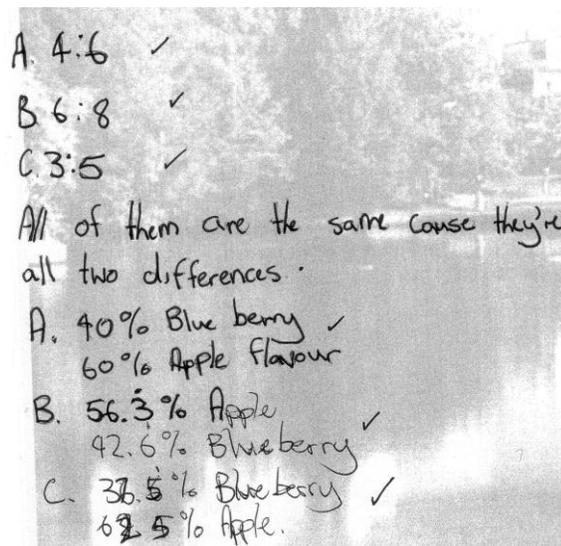


Figure 120: Odette compared ratios by difference of measures (26 October)

Eventually she accepted that percentages were a useful comparison of ratios but did not appear to understand that equal differences between the two measures did not conserve ratio, or the associated attribute of flavour or shooting success. Evidence from her bookwork and the anecdotal notes showed that Odette was able to use common factors to simplify proportions and ratios, e.g. 12:18 as 2:3 (15 November). There was also evidence that when she renamed ratios in terms of the part-to-whole fractions she identified equal ratios, e.g. 1:3 was the same as 3:9 since  $\frac{1}{3} = \frac{3}{9}$  (6 November).

In response to a question of her final interview (19 November) Odette compared two ratios of blueberry and mango juice, 1:2 and 2:3, to determine which had the stronger taste of blueberry. She converted both ratios to percentages to make the comparison. Selection of this strategy may have been an order effect from the previous question that involved calculating a percentage for a given frequency. Firstly Odette recorded the part-whole relationships for the ratios as  $\frac{1}{3}$  and  $\frac{2}{5}$ .

O: Isn't that 27 percent? (referring to  $\frac{1}{3}$ ).

I: If you're right 27 should fit into 100 three times.

O: This is forty (referring to  $\frac{2}{5}$ ). So it's got to be this one (referring to 2:3).

I: Why?

O: Because I know one-third is way less than 40 percent.

I: So what percentage is one third?

O: Thirty-five...no thirty-four...oh thirty-three point three! (recalling that  $\frac{1}{3} = 33.\overline{3}\%$  ).

The ratio problem afforded replication of 1:2 to 2:4 that was of less blueberry taste than 2:3. Although Odette now trusted fractions and percentages as equivalent proportions it begged the question about her understanding of how ratio, and the corresponding attribute of taste, was conserved. She did not connect ratio problems with her preference for rate and was considerably more confident with frequency situations than ratio situations. Odette seemed to be constantly in search of trusted procedures. This acted against her attendance to the structure of ratios and the equivalence of the part-whole relationships.

### **8.3.5 IMPACT OF WHOLE NUMBER STRATEGIES ON FRACTION UNDERSTANDING**

Odette's interview at the end of the Term One indicated that her understanding of fractions had improved to include understanding of fractions as iterations of unit fractions, of part-to-whole relationships in ratios, and the beginnings of an anticipatory scheme for partitive or sharing division where the shares were fractions. Given that the group instruction Odette received aimed at improving her understanding of whole number operations, particularly multiplication and division, this improvement in fraction concepts was surprising.

Possible explanations were that improved understanding of multiplication and division contributed to the development of stronger fraction concepts and /or that Odette benefited from the whole class lessons and the collective sharing of problems solutions by each group at the end of the lessons. The focus of instruction for students in the higher achieving group was entirely on fraction concepts and members of this group shared their solutions with the whole class. It was a popular occurrence for students from other groups to offer solutions. Through listening to students in the higher level group explain their strategies there was considerable potential for Odette to learn.

Evidence of the impact of whole class discussions on Odette's thinking was found in her solution to this problem at the end of Term One, "Remi has 45 calves to feed. Of the calves, 27 are Friesians. What percentage of the calves is Friesians?" Her answer was 53%. For a week I taught percentage, as proportion, to the whole class in the first segment of lessons. Odette received no other instruction in percentages in Term One.

O: It's weird.

I: What's weird about it? What made it hard?

O: I divided it by nine (identified a common factor).

I: And what did you get when you divided by nine?

O: Fifty three. Forty-five is five and 27 is three.

I: So that is where you get 53% (5 and 3).

Odette went on to explain that she knew the equivalence of half of 45 and  $\frac{50}{100}$  or 50%. Twenty-seven was more than half of forty-five so her answer of fifty-three seemed sensible. While the use of common factors was not a feature of instruction in her group it was strongly emphasised with the higher achieving group. Odette had recognised that this problem was one in which common factors were useful but she could not identify  $\frac{27}{45}$  and  $\frac{3}{5}$  as equivalent proportions.

Many other instances occurred during the year that supported the importance of multiplicative thinking, particularly its application to division, as a necessary but not sufficient foundation for the development of fraction concepts. For example, in Term Two Odette was able to establish the equivalence of fractions by splitting using multiplication knowledge. In the end of the Term Three interview she used repeated halving to connect the decimals for  $\frac{1}{4}$ ,  $\frac{1}{8}$  and  $\frac{1}{16}$ . In Term Four bookwork (6 November and 9 November) she used common factors to solve proportion as percentage problems and used division fluently to solve fraction as operator problems (8 November). However, strong multiplicative thinking with whole numbers was not sufficient for Odette to solve ratio comparison problems or express remainders from division as decimals. Conceptual barriers inhibited her solutions to these problems even when the calculation procedures were available.

## 8.4 Decimals

In her initial tests Odette was able to locate a one place decimal, 6.9, on a number line and select the correct answer to  $47.03 + 1.97 = \square$ . These items proved to be inadequate indicators of her decimal knowledge. She was unable to order 5.6, 5.22, 5.315 and 5.08 correctly or identify the position of 5.16 on a number line in the same test, indicating that she based her decimal knowledge on whole number thinking.

In her interview at the end of Term One Odette solved the measurement problem, “Lisa has nine litres of juice for her party. A big glass holds 0.45 litres. How many people get a glass of juice?” She applied additive build-up using the unit rate (see Figure 121). Odette’s strategy reflected her tendency to solve division problems using a combination of multiplicative and additive build-up and her familiarity with the relationship between millilitres and litres. Her recording of 2.9 as 2.900 and  $2 \times 0.9$  as  $2 \times 900$  suggested that she viewed decimals either side of the point as whole numbers. This explained her previous difficulties with ordering decimals.

4. Lisa has 9 litres of juice for her party.  
A big glass holds 0.45 litres.  
How many people get a glass of juice?

20 people ↓

cause 0.45 goes 2 x  
in 1 L with 100 remainder.

so 2.900 for 100 9x then 2 9 times = 18 + 0.90 = 20



Figure 121: Odette used a build-up strategy to solve a decimal division problem (26 March)

No deliberate instruction about decimals occurred until Term Three. At the beginning of Term Three she gave the following answers in a pencil and paper test (16 July).

If the decimal for one-half ( $\frac{1}{2}$ ) is 0.5, what are the decimals for these fractions:

a.  $\frac{2}{1} = 1.1$  (Why?) Because 2 goes into 1 once and so there would be a whole and one extra

b.  $\frac{1}{4} = 0.2$  (Why?) Because 2 is close to half of 5.

c.  $\frac{7}{2} = 3.5$  (Why?) Because 7 goes in to 2 once so it would be three wholes because it goes into 2 too many times over.

Figure 122: Odette's converted fractions to decimals (16 July)

Her responses reaffirmed Odette's limited understanding of conversions between fractions and decimals. Odette appeared to rely on her knowledge of halving and of known facts, i.e.  $\frac{1}{2} = 0.5$ .

The instructional approach to decimals was to present them as a specialised set of equivalent fractions that arose through partitioning of units (ones). Odette was able to use her strength in multiplication and division to achieve considerable success on these decimal related activities. In a context about racing up the steps of the Skytower she was able to rename common fractions as equivalent numbers of thousandths, e.g. five-eighths as 625 thousandths (2 August). Odette was also able to create decimal paper models for fractions and decimals such as  $\frac{5}{8} = 0.625$  (1 August).

Odette learned to order decimals of varied numbers of places through using physical models, e.g.  $0.9 > 0.782$ , and developed a strong anticipated process that was not dependent on access to the physical model. She was able to add and subtract decimals and explain her answers. In describing why  $1.7 + 2.46 = 4.16$  she showed understanding of the canon of decimal place value (24 July).

O: Sue is right because you add the two (lots of) tenths together which is a one and one tenth left then add the six hundredths 'cause you can't add the hundredths which is the six with a tenth 'cause it doesn't add to ten...

Despite her competence in expressing decimal relationship as equalities such as  $\frac{9}{4} = 2.25$  and  $\frac{7}{10} = \frac{70}{100}$ , it was not clear that Odette appreciated the equivalence of fractions and decimals as representing the same quantity. Asked to explain the connection between finding fractions of the way up a Skytower of 1000 steps and the decimals for the same fractions she wrote, "That they're basically both the same answer," referring to the sameness of digits as opposed to the fractions being equivalent.

There were other sources of difficulty in her understanding of fractions and decimals. In her interview at the end of Term Three (16 August) a question read, "If the calculator shows 0.25 as the decimal for one-quarter, what will it show as the decimals for one-eighth and one-sixteenth?" Students did not have access to calculators for this question. Odette speculated on a numeric pattern rather than considering the place value and equivalent fraction structures involved.

O: I kind of just guessed that ( $0.025 = \frac{1}{8}$ ).

I: So you didn't know what to do.

O: Is it 0.250?

I: No, nothing like that. What do you know about a quarter and an eighth?

O: A half.

I: Right, so an eighth is a half of a quarter.

O: So that will be zero point one two five.

I: And a sixteenth will be...

O: Point six... (wrote 0.625).

I: Hasn't this (0.625) got to be smaller than this (0.125)?

O: There must be a zero in here (places zero to record 0.0625).

Calling on Odette's knowledge of ordering decimals caused her to correct her answer for  $\frac{1}{8} = 0.0625$ . She accepted the continuation of the digital pattern produced by repeated halving, appreciated the potential infinity of the system and the significance of zero as a place holder.

In her final AsTTle test Odette correctly ordered 12.13, 11.23, and 12.31 and found  $\frac{2}{5}$  of  $\$2.50 = \$1.00$  and  $\frac{2}{5}$  of  $\$2.80 = \$1.12$ . She incorrectly gave the decimal for  $\frac{16}{20}$  as 4.5, possibly seeing a common factor of four, and gave an answer of  $\$5.10$  to  $6n + \$2.30 = \$14.90$  given in a word story. Similar performance with decimals occurred in her final PAT. By the end of the year Odette's understanding of decimal place value had developed considerably but was still patchy and partially connected.

## 8.5 Percentages

Initial testing showed Odette's inability to solve percentage problems. After limited whole class instruction about percentages she was able to take 25% off \$40 but was unable to name 12 out of 20 as a percentage or calculate the GST inclusive price of \$64 item (9 March). Odette estimated  $\frac{27}{45} = \square\%$  using  $\frac{1}{2} = 50\%$  as a benchmark but she did not calculate the correct answer (26 March). Odette recognized a common factor of nine in both 27 and 45 but was unable to use this knowledge to simplify the fraction.

Term Two was focused on equivalent fraction concepts so percentages were not taught. At the beginning of Term Three her answer to the price of a \$480 chair less 35% was 310 (actual answer 312) (16 July). Again Odette showed no working but the close proximity to the actual answer indicated that she followed a sensible process that did not involve the use of a calculator.

At the end of Term Three (16 August) Odette compared 35%, 387ml and  $\frac{3}{8}$  of a one litre bottle of drink. She did not see any connection between these numbers as representations of equivalent proportions, convertible from one to the other. Her first response was to circle 35 %.

O: No I think it might be that one (pointing to  $\frac{3}{8}$ ).

I: So how much would be in there if it was three-eighths full?

O: Nearly a half.

I: Could you tell me exactly how much three-eighths is in millilitres?

O: It would be about 450.

I: So what would this one be (pointing to 35% full).

O: It would be 300 or so.

In the end of Term Three interview Odette worked out a discount of 25% on a pair of jeans costing \$96 (16 August). She retained her knowledge that 25% was the same as one-quarter.

O: I halved it and halved it again.

In Term Four, I targeted instruction for Odette's group at understanding proportions and how fractions, decimals and percentages are used to represent them. At first, in frequency and ratio contexts she did not see the percentages she calculated being in conflict with her belief in equal differences between measures.

The following notes come from the Teaching diary (25 October):

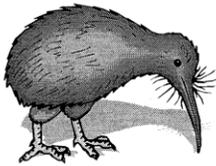
Odette: Given these "Blango" mixtures; A (1:2), B (2:3), C (3:5)

Odette thinks A and B are the same strength of blueberry since both have one different blueberry from mango. C has an extra mango added (two different), therefore a weaker taste of blueberry.

Yet, Odette calculates the percentages correctly, 33%, 40%, 37.5%, but sees no apparent intellectual conflict. What does she think a percentage tells you?

I tried several teaching approaches to create cognitive conflict that would provoke Odette to rethink her additive difference approach to comparing ratios. These approaches included using a computer spreadsheet to model how a pie chart changed as the numbers in a ratio changed. These approaches were partially successful in helping Odette to connect her knowledge of the part-to-whole relationships in ratios with percentages. Gradually she came to see a percentage as representing a proportion, particularly in frequency situations where her "out of" view of fractions was helpful.

In percentage as operator situations Odette had considerable success. This was especially true where the numbers involved were friendly and the problems shared structural relationships.



7. Carla, the kiwi, weighs 2400 grams. Her egg weighs 800 grams. What percentage of Carla's weight is her egg?  $33.3\%$
8. The egg is one-third of the weight of its mother. If the egg weighs 800 grams, how heavy is the mother Kiwi.  $2400$
9. Carla, the kiwi has laid an egg that is 33.3% of her weight. If the egg weighs 800 grams, how heavy is Carla?  $2400$

Figure 123: Odette solved connected percentage problems (6 November)

Odette recognised that both the percentage and the quantity it operates on affect the answer. Asked, "Which is larger, 20% of \$60 or 40% of \$40?" she incorrectly answered, "20% of \$60 because it's larger in dollars but smaller in percentage" (5 November). There were signs that Odette regarded percentages as numbers. Asked, "Is 25% of \$60 the same as 60% of \$25?" she wrote, "It's the same because it's just turned around and yeah" (5 November).

Double number line representations helped Odette to learn fraction to percentage conversions and understand how to derive unknown percentages.

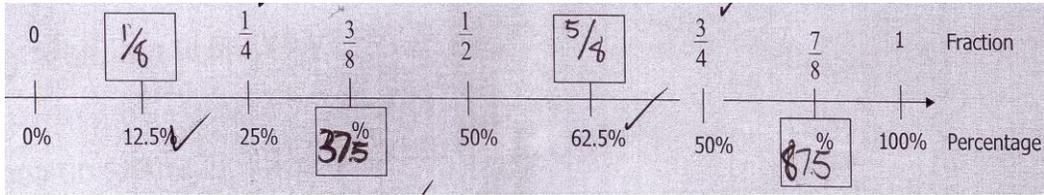


Figure 124 Odette converted fractions to percentages using a double number line (6 November)

Odette had considerable difficulty with sets of percentage problems in which the numbers were unfriendly or finding the unknown involved inverse relationships. She also had difficulty knowing what to do with decimal answers from her calculator (8 November).

The race is 2750 metres long. The horse has run 2008 metres. What percentage of the race has it run?

Odette wrote 73.31 (actual answer is 73.018).

Jamie got about 88% for his AsTTle test. There were 33 questions. How many did he get right?

Odette wrote 28 (correct answer 29).

Her attempts to solve percentage problems with strip diagrams met with variable success. The diagrams provided useful insight into her reasoning. While demonstrating strength in converting between equivalent proportions Odette continued to have difficulty in relating fractions to percentages in operator contexts. Most of the difficulties appeared to be due to knowledge co-ordination, e.g. calculating 70% as seven-eighths rather than seven-tenths.

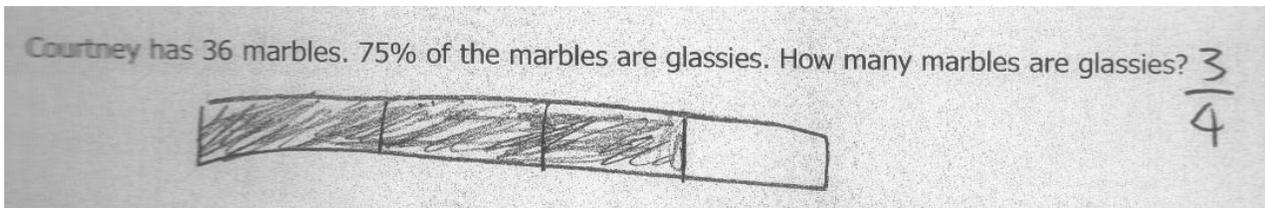


Figure 125: Odette correctly recognised  $75\% = \frac{3}{4}$  but did not calculate 75% of 36 (9 November)

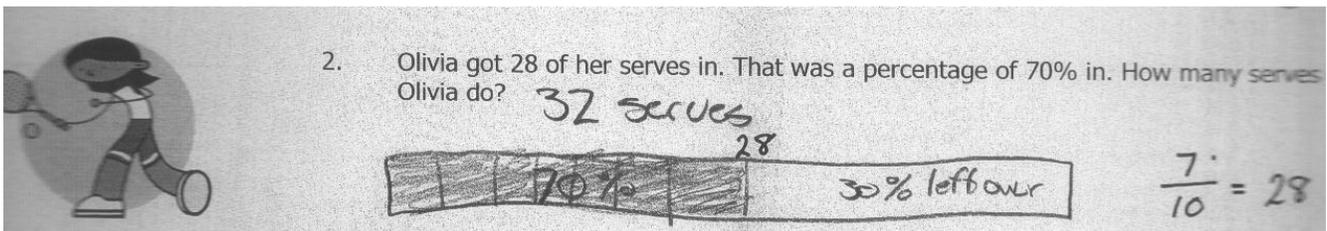


Figure 126: Odette Identified 70% as 7 tenths but incorrectly calculated 70% as seven-eighths of 28 (9 November)

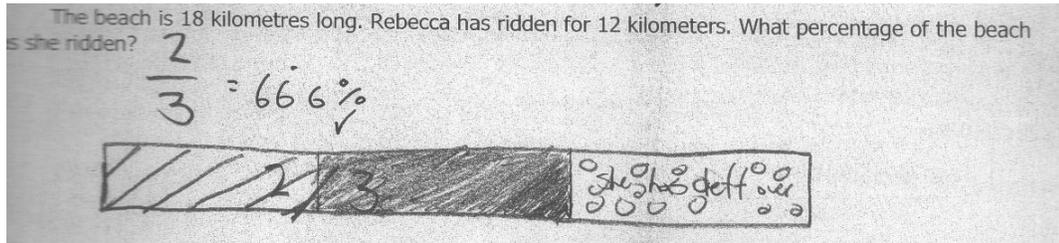


Figure 127: Odette correctly calculated  $\frac{12}{18} = \frac{2}{3} = 66.6\%$  (9 November)

Odette could solve the following problem in her end of Term Four interview (19 November); “You sit a spelling test and get 18 words right. The teacher gives you 60% correct. How many words are in the test?” In her response, Odette established the number of questions that equated to ten percent by successive approximation rather than by division by six.

O: Forty.

I: Okay forty words in the test. How did you work that out?

O: No, that’s not right. That would mean four is ten percent. Maybe twenty. This is so confusing.

I: Tell me what you are thinking.

O: Isn’t one-tenth three words?

I: So ten percent is three words. Let me write that down for you.

O: Yes, six times three is eighteen words so I need...

I: How many percent are in the whole test?

O: One hundred so ten times three is thirty.

Odette’s reluctance to record information in either symbolic or graphic form meant that she was holding all the problem variables in her working memory simultaneously. This appeared to be a significant obstacle to her finding solutions and suggested that she viewed strip diagrams and double number lines as redundant in the testing situation. This disposition for calculating everything mentally disadvantaged her considerably when she solved complex percentage problems.

The final standardised test results from both asTTle and PAT also provided evidence of her growing competence with percentages. Odette correctly solved “Which is more, \$7.00 + \$3.00 or \$7.00 + 30%?”

O: A because he would get more than 30% of his regular pay for three dollars is nearly half of \$7.00 and 30% is way less than half.

In her final PAT Odette was able to “Convert 70 out of 200 to a percentage in a word problem” but was unable to “Find 15% of 300 in a word problem” (chose 30).

## 8.6 Summary

### 8.6.1 Processes as Objects

All of the data collected on Odette pointed to considerable progress in her understanding of multiplicative thinking and proportional reasoning through the course of the year. Her asTTle raw score improved from 523 to 730 during the year, equating to a shift of overall level from 3P to 4P. This was normally two years’ progress. Her PAT stanine improved from five to six. This section explores whether these gains are explainable by process to object theory.

Odette began the year with many anticipated processes. These included use of both iteration and simple multiplicative thinking to anticipate the partitioning of continuous quantities. Her strong register of basic facts were predictions of processes, such as  $6 \times 4 = 24$  anticipating the act of counting six sets of four objects. Odette put faith in written algorithms to anticipate these processes with larger numbers.

There were also examples of processes that Odette enacted on objects rather than anticipating them. For example, in sharing situations she partitioned pictures of pizzas. During the year Odette developed many anticipated processes. These anticipations usually began with actions on physical objects. Trust in reliability of process was significant for Odette in terms of validating any anticipation of that process. For example, cutting up decimats gave her considerable trust in anticipating the relative size of decimals. In quotient situations her inaccuracies in physically partitioning circular regions contributed to her failing to see pattern between the dividend, divisor and quotient. Odette never developed an anticipated process for fractions as quotients.

There were many other examples of reliable actions on physical objects contributing to Odette anticipating a process. For example, she anticipated partitions of length models through finding equivalent fractions by multiplication and division, used multiplication and division to anticipate the replication and partitioning of rates, and anticipated the result of multiplying a quantity by ten.

Instances of Odette using an anticipated action as an object for thought were less common. She connected multiplication and division as inverses through applying the commutative and associative properties, used percentage conversions to compare ratios, and treated fractions as numbers to solve division problems. Yet there was unpredictability about Odette’s use of anticipated processes as objects of thought.

Moments of creative insight were rare. Finding the number of edges and vertices of platonic solids by division was one notable exception. Odette tended to view anticipation of a process as a procedure in itself. She liked the certainty of

procedures. Examples were adding zeros to multiply by ten and dividing 100 by  $a$  then multiplying by  $b$  to convert  $\frac{a}{b}$  to a percentage. Procedures often induced inflexibility and lack of connection. She did not see any conflict in her additive difference concept of ratios with calculating the ratios as percentages. Her reliance and trust in percentages may have prevented her understanding conservation of ratio in the same way that whole number thinking had prevented her understanding of decimal place value.

Several points about anticipated processes arise from Odette's case study. Trust accompanied anticipation before she acted upon that anticipation. Changes in number size greatly affected Odette's trust in percentages as operators. Understanding of structure, *knowing how* and *why* a process works, accompanied Odette's trust before flexible application of the anticipated process as an object of thought. She understood how the process of finding equivalent fractions worked with fixed wholes and was able to use this to combine and separate fractions with different denominators and solve measurement division problems with fractions. Odette did not understand conservation of ratio under replication so she could not compare ratios other than by relying on percentage conversion.

Odette's concepts seemed mostly to be a continual state of condensation rather than reification (Sfard, 1991). Anticipated processes were present in some situations and not in others. Temporally they were often unreliable. Fraction as operator anticipation was present in Term Two, unreliable in Term Three and resurgent in Term Four. Anticipated processes became unanticipated in moments of cognitive load. For example, Odette resorted to finger tracking to solve  $15 \div 3 = 5$  to solve a considerably more complex calculation,  $\frac{3}{4} \div \frac{3}{20} = 5$ .

Even anticipated processes that were reliable did not become objects of thought. Odette knew the part-whole fractions for ratios and learned to find equivalent fractions in fixed whole situations. Yet she never brought the two anticipated processes together to compare ratios. Possession of anticipated process did not lead to Odette having the insight to connect them, to use them as objects of thought.

Odette's case study also provided exemplification of fractions, percentages and decimals as procepts, symbols that embody both process and object. Many problems that Odette had with using symbols as objects of thought were attributable to the multiple processes, and interpretations of those processes, embodied by the symbols. Decimals were a good example.

At the beginning of the year Odette showed some competence with decimals. She correctly added  $47.03 + 1.97$  and located 6.9 on a number line. However, she was unable to order decimals with three places. There was a strong likelihood that Odette had a whole number view of decimals, possibly grounded in operations with money. Her process view of decimals was to transplant the processes she used with whole numbers. This process failed with ordering ragged decimals, i.e. 5.6, 5.22, 5.315, 5.08.

Through instruction Odette connected decimals to fractions. This conflicted with the whole number view initially. She wrote  $\frac{1}{4} = 0.2$  because "two is close to half of five."

In time Odette developed some fraction to decimal conversion facts. She altered her whole number view to include decimal place values so she could order ragged decimals. Co-ordinating the fraction and place value views of decimals was difficult for Odette. With support she connected  $\frac{1}{8} = 0.125$  with  $\frac{1}{16} = 0.0625$ . Integration of these two views of the same symbol was not complete as evidenced by her answer  $\frac{16}{20} = 4.5$  in the end of year AsTTle test.

Odette did not consider other views of decimals that posed problems for higher achieving students. It was common for students to express the remainders for division processes as decimals, e.g.  $18 \div 5 = 3.3$ . Decimals, like fractions and percentages, were complex precepts for Odette partly because they reflected different processes in different situations.

### 8.6.2 Co-ordination Classes

The situational variation, difficulties of knowledge co-ordination and co-existence of conflicting ideas predicted by co-ordination class theory were all obvious in Odette's case study. Examples of situational variation were plentiful. Odette understood three-quarters as iterations of one-quarter and as three out of four at the beginning of the year. Yet she was unable to act on either of these views to find three-quarters of a set of 24 jellybeans.

She developed strong equivalent fraction ideas in fixed whole situations and applied these ideas to complex operations with fractions. Despite knowing the part-whole fraction relationships in ratios, she did not use equivalent fractions to compare ratios. She applied the distributive property in finding percentages of amounts but had difficulty comparing percentages to fractions and decimals as numbers.

Knowledge co-ordination played a significant part in the variation of Odette's strategies from one situation to another. The difference between her treatment of frequency and ratio situations was a good example of the variability created by multiple inferences. Odette had a strong "out of" view of fractions at the beginning of the year. Frequency situations, such as "Odette shoots 12 goals out of 20 shots" required one less inference than establishing the "out of" relationships in a ratio of 12:8. Odette's understanding of comparing frequencies developed considerably more easily than her comparison of ratios. These two situations were different to Odette because they appeared to her as different.

The underpinning views of sub-constructs and symbols that Odette brought to situations also contributed to variation in performance. Her whole number or money view of decimals allowed her to solve some problems and not others. Odette's preference for rate allowed her to compare fractional shares in one pizza situation but her dealing strategy prohibited a solution in another. She found comparison of fractions in a fixed one situation much easier than comparison with variable wholes.

Conflicting ideas co-existed without resolution at times. Odette was content to write  $\frac{2}{1} = 1.1$  and give 27% as one-third until asked if it fitted into 100% three times. She compared ratios by additive difference between measures despite correctly

calculating percentages that showed the comparison to be incorrect, e.g. recipe A 1:2 (33%) is stronger than recipe B 3:5 (37.5%). There was suggestion of a p-prim about ratios in her thinking. Odette possibly believed that measures of one flavour cancelled measures of the other flavour in the same way that adding one measure of each flavour fulfilled the conditions of bigger mix that still tasted of both flavours.

Cueing preferences remained throughout the year. Odette used her strength with rates to compare quotients, to find percentages of quantities and measure one fraction with another. She valued algorithmic procedures for calculation because she viewed them as easy and trustworthy. Additive and multiplicative build-up strategies were common. While she used symbols to record her results there was little evidence that Odette used symbols to derive new results. In her end of year interview Odette relied heavily on her capacity for mental calculation.

### 8.6.3 Span and Alignment

Over the year Odette increased the span of situations to which she applied multiplicative thinking and proportional reasoning. The outward movement of the circles on her Learning Trajectory Map illustrated her improvement. She began the year with strategies for multiplication and division based on written algorithms. Although she knew most basic multiplication facts, her mental strategies were limited. Odette had initial strength in the rate and quotient sub-constructs. She solved simple rate problems by replicating the rate and used physical partitioning to compare shares. Her knowledge of the measure sub-construct was limited to partitioning by imaging iterations of the unit though there were early signs of applications of multiplicative thinking. Odette found unit fractions of quantities but did not know how to find non-units fractions.

#### Multiplication and Division

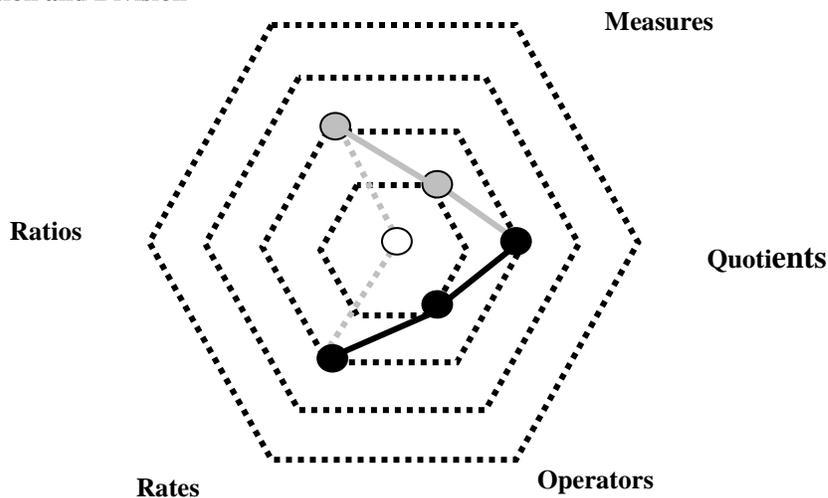


Figure 128: Learning Trajectory Map: Odette beginning of Term One

During Term One there was considerable progress in Odette's application of the properties of multiplication and division to mental calculation strategies (see Figure

129). She applied unit rate strategies to rate problems and identified the part-whole fractions in ratios. No other growth was evident in the other fraction sub-constructs. This matched the teaching focus of the term on strengthening calculation strategies with whole numbers.

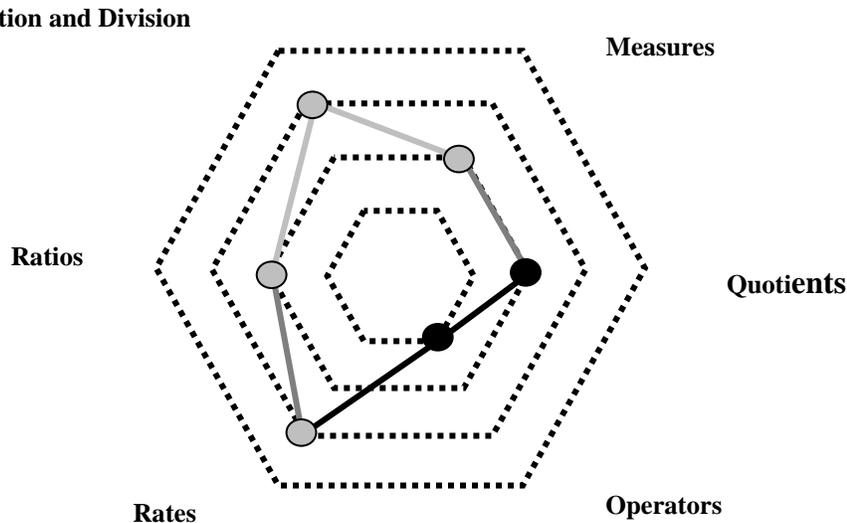
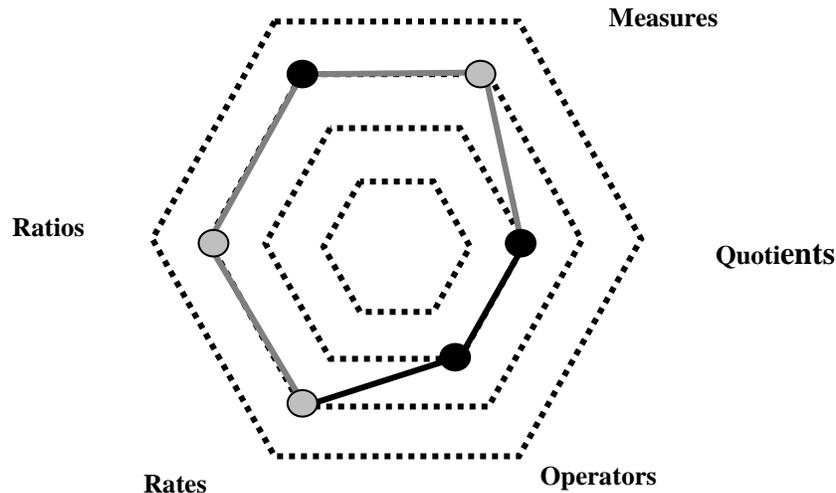


Figure 129: Learning Trajectory Map: Odette End of Term One

By the end of Term Three Odette’s calculation strategies to solve multiplication and division problems were very reliable (see Figure 130). She applied multiplicative thinking to number friendly rate and ratio problems where integral scalars worked and knew that non-unit fractions were iterations of unit fractions. Odette could add and subtract fractions with simple related denominators but did not understand equivalent fractions as the same measure. She made no progress in the quotient and operator sub-constructs during terms two and three.

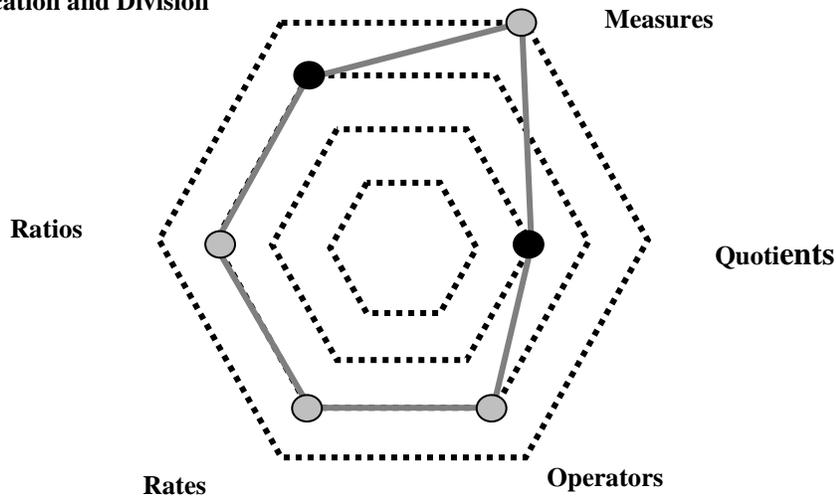
By the end of Term Four Odette understood that equivalent fractions were alternative ways to represent the same measure (number). She applied equivalence to problems of measuring one fraction with another. Her consolidation of non-unit fractions as iterations of unit fractions, combined with strong mental calculation strategies, facilitated her use of non-unit fractions as operators. There was no evidence of Odette’s application of equivalence to comparing shares in quotient situations.

**Multiplication and Division**



**Figure 130: Learning Trajectory Map: Odette End of Term Three**

**Multiplication and Division**



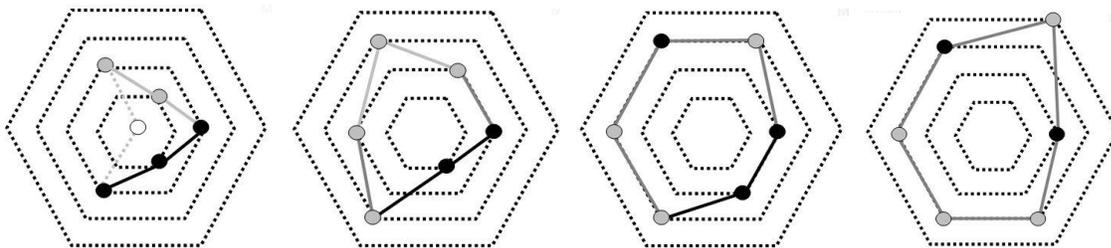
**Figure 131: Learning Trajectory Map: Odette End of Term Four**

A simultaneous view of Odette's learning trajectory maps (Figure 132) showed pronounced growth in the span of her use of multiplicative thinking with whole number and proportional reasoning. The shape of her profile changed over time as her knowledge of some sub-constructs developed while knowledge about other sub-constructs stayed static. Apart from the quotient sub-construct, her profile became more balanced as the year progressed.

While span improved, alignment was inconsistent. Reliable performance in a sub-construct took a long time to develop and was associated with no outwards

progression in the concepts associated with that sub-construct. For example, Odette remained reliable in her ability to compare quotients by physical partitioning but did not progress to using the quotient theorem and equivalent measures to do so. She consolidated her mental strategies for multiplication and division with whole numbers to the point where they were reliable and trusted.

With most sub-constructs, progression in development was associated with less reliable alignment. Odette's case study did not provide any illumination on the ideal balance between development and consolidation of sub-constructs for fractions.



**Figure 132: Simultaneous view of Odette's Learning Trajectory Maps**

## CHAPTER NINE: CASE STUDY OF JASON

### 9.1 Beginnings

#### 9.1.1 PERSONAL CHARACTERISTICS

Jason was 12 years and eight months old at the beginning of the 2007 school year. He was a male student of European ethnicity. His interests included graphic design, hockey and his passion, motocross riding, in which he ranked nationally.

Jason was physically small for his age and was quite shy. He often preferred to work alone though he had one close friend, Len, who was similar in academic ability. In group and whole class teaching sessions, Jason was often inattentive and he was frequently interrupted from daydreams. Pens and other materials were never safe from his experimentation. He liked to manipulate physical materials as he listened. Jason had a determination to succeed and strong personal motivation towards things that interested him.

The class teacher had specifically requested that Jason join her digital class because of his interest in technology. Jason impressed her with his ability to solve problems in creative ways.

#### 9.1.2 INITIAL TESTS

Standardised testing in late February and early March showed that in mathematics Jason was performing at a level slightly above average for his age. His PAT result corresponded to a stanine of six. This was consistent with the asTTle test result that described Jason's overall mathematics achievement at Level 4B of the national curriculum. According to the asTTle result he demonstrated greater competence on number operations tasks than on number knowledge tasks. Jason's ability to derive answers to difficult problems without fluent recall of required items of knowledge was a feature of his performance throughout the year.

In his PAT test Jason solved problems such as  $27 \times 13 = \square$ ,  $3:10$  so  $12: \square$  and  $\frac{1}{3}$  of  $90 = \square$ . He recognised a common factor of seven in connecting  $49 + 14 = \square$  as the answer to  $9 \times 7 = \square$ , solved addition and subtraction problems with decimals to two places and was able to give the number of hundreds in 4826. Jason's performance showed understanding of many properties of multiplication and some place value knowledge of whole numbers and decimals. His weaknesses in number were in solving subtraction and division problems with whole numbers, continuing a sequential number pattern ,i.e. 5, 8, 11 ..., and adding decimals, i.e.  $7.6 + 11.8 = \square$  in a word problem. This indicated potential difficulties with inverse operations and relations, and inconsistency in his understanding of decimals.

### 9.1.3 INITIAL GLOSS INTERVIEW

Jason's initial GloSS interview (20 February) reinforced data from the standardised tests, particularly in respect to his capacity to derive answers in the absence of supporting knowledge. It also highlighted a lack of mental strategies for subtraction, weak knowledge of fractions, decimals and few known basic facts for addition, subtraction, and multiplication.

In the addition and subtraction domain Jason derived the answer to  $8 + 7 = 15$  from  $8 + 8 = 16$  but was unable to solve  $83 - 28 = 55$  mentally. He also derived  $4 \times 8 = 32$  using  $8 + 8 = 16$  then  $16 + 16 = 32$ . Jason used a known doubles fact,  $2 \times 8 = 16$ , to find  $\frac{1}{8}$  of  $16 = 2$  but could not solve  $\frac{3}{5}$  of  $30 = 18$  despite recognising the usefulness of a known fact,  $5 \times 5 = 25$  to do so.

Jason was able to order unit fractions but did not know the size of  $\frac{8}{6}$  as a number. He correctly identified the place of 6.8 on a number line divided into tenths. However he did not know that the eight in 6.8 referred to a number of tenths. To the question "What number is three-tenths less than two?" he replied "1.3". While his answers to some items in the standardised tests suggested knowledge of decimal place value, the GloSS interview showed that Jason did not have a structural understanding of tenths and ones as related units. His knowledge of basic multiplication facts was minimal. Jason successfully derived the answers to  $9 + 9 = 18$ ,  $6 + 9 = 15$ ,  $5 \times 7 = 35$ , and  $17 - 9 = 8$ . His achievement level on standardised tests disguised both his strong capacity for processing calculations mentally and his lack of easily accessible facts.

### 9.1.4 INITIAL NUMBER INTERVIEW

In the number-based interview, prior to instruction Jason also showed strong conceptual understanding without the factual knowledge required to support it. When asked to partition graphics of worms into quarters and fifths respectively he estimated the fractional unit length and iterated it the required number of times. By a process of trial and improvement, he accurately partitioned the worms. When asked which part, one-quarter or one-fifth, was bigger he replied:

J: Which of these? Quarter.

I: So a quarter is bigger than a sixth. A quarter is bigger than a fifth – why?

J: 'Cause there are less things (parts) so it's bigger.

Jason recognised the inverse relationship between number of parts and size of parts given an identical whole, e.g. fifths are larger than sixths. Jason was able to correctly identify quarter of a cake in a diagram that had the cake partitioned into one-half and two-quarters so he was not distracted into considering only the number of parts. The following dialogue described how Jason found three-quarters of 24 in a birthday cake context.

J: (Shares out the counters and counts one quarter) Are there six on each quarter? (checking to see if they are uniformly spread).

I: Yes.

J: (Counts on in sixes using finger tracking but trying to hide it) Seventeen.

I: How did you get that?

J: Six in each quarter. Three lots of that.

T: And that's 17?

J: Yes.

I: If you didn't eat this quarter that would be six pebbles. And there are 24 altogether. So can you figure out what three-quarters is?

J: It's 18.

This interview sequence showed that Jason overcame his lack of multiplication fact knowledge by disciplined perseverance with a cumbersome counting process and that he had considerable conceptual knowledge about fractions, including three-quarters as three iterations of one-quarter, a process for using three-quarters as an operator on 24, and that one less one-quarter is three-quarters.

In a quotient problem that involved comparing the share of pizza for boys and girls Jason systematically partitioned pictures of the pizzas and allocated the correct share to each gender. He used his knowledge of the comparative size of two-thirds and three-quarters to decide on the bigger share. In doing so, he showed understanding that the size of a fraction depended on both the numerator and denominator.

### **9.1.5 SUMMARY OF ENTRY LEVEL**

Jason's performance on assessments at the beginning of the study indicated that he had considerable conceptual knowledge about multiplication and division and fractions. This included concepts such as non-unit fractions as iterations of unit fractions, in part-whole and operator settings, multiplicative grouping by common factors, and anticipating the result of repeated doubling, e.g.  $2 \times 2 \times \square = 4 \times \square$ . He had learned these concepts without the support of knowing number facts and relied heavily on methods of deriving these unknown facts. His ability to solve operational problems diminished considerably where subtraction and division were required as inverse operations.

Some of his solution strategies indicated that Jason also relied on physical or diagrammatic models to solve problems. His performance on GloSS items where he

was asked to solve number problems mentally was considerably weaker than his performance on standardised tests where recording was available. Jason sought meaning and pattern in mathematics but his disposition for sense-making was not matched by an understanding of the need for knowledge of number facts.

## 9.2 Progression in Whole Number Operations

### 9.2.1 DEVELOPMENT OF OPERATIONAL STRATEGIES

Jason's inclusion in the high ability group was short-lived. Due to lack of understanding of fluency with mental calculations with multiplication and division of whole numbers, Jason was unable to cope with the demands of equivalent fractions. From week two he elected to work in the middle achieving group. The instructional focus for this group was operations on whole numbers, particularly multiplication and division. From the first week Jason appreciated the need to learn his basic multiplication facts. He created his own set of flash cards and set about learning the facts with determination.

In week two Jason revealed strength in estimating the answers to addition and subtraction problems. Given a few seconds he was able to estimate the answers to  $172 - 48 = \square$  (answered 127) and  $203 - 97 = \square$  (answered 100) and exactly calculate  $100 - 64 = 36$  and  $83 - 38 = 45$ . He recognised that additive compensation did not conserve product, e.g.  $10 \times 10 > 9 \times 11$ , though he did so by calculating both products. The teaching diary for 7 March documented that Jason's incomplete knowledge of multiplication basic facts still hindered his ability to calculate mentally on occasions.

By the end of week four Jason had generalised many ideas about place value partitioning with multiplication and division and knew the effect of multiplying numbers by powers of ten. Partitioning array models and manipulatives, e.g. unifix™ cubes, seemed important to Jason in validating results. In Figure 133 Jason partitioned an array of 60 to solve  $54 \div 3 = 18$ .

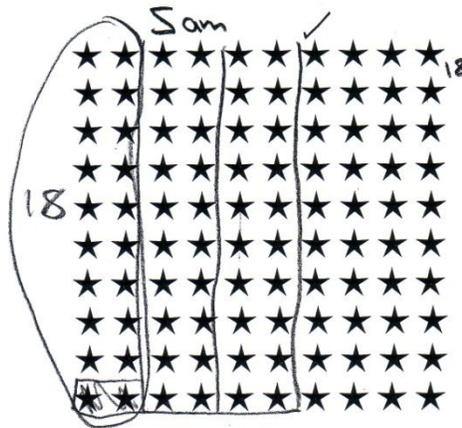


Figure 133: Jason solved 54 divided by three using an array (21 March)

By the end of week four, the confluence of Jason's developing knowledge of multiplication basic facts and the teaching focus on properties of multiplication and division contributed to strong growth in his strategies for solving number problems. From a few examples, he seemed able to understand the arithmetic properties involved and transfer the properties to other examples. In Figure 134 Jason recorded his use of standard place value partitioning to solve  $78 \div 6 = 13$  demonstrating a shift in representation from array drawings to equations.

The diagram shows the number 78 at the top, circled. Two lines branch down from it to 60 and 18. Below 60 is the equation  $60 \div 6 = 10$ . Below 18 is the equation  $18 \div 6 = 3$ . A plus sign is written between these two equations. Lines from both equations point down to a final equation:  $13 \times 6 = \square 78$ . The square is filled with the number 3, and there is a checkmark to the right of the final equation.

Figure 134: Jason used standard place value partitioning for division (20 March)

He was also able to connect division equations with a quotient of 64 by dividing or multiplying both the dividend and divisor by the same factor. Jason was able to solve a series of problems of this type fluently and without any sign of recalculating the quotient each time. This suggested that he was able to transfer partitioning of factors in multiplication problems to the equivalent form in division.

The handwritten text shows a list of equations:  $512 \div 8 = 64$ ,  $128 \div 2 = 64$ , and  $384 \div 6 = 64$ . To the left of these equations, the number 64 is written with a small 'c' next to it.

Figure 135: Jason adjusting the dividend and divisor by a common factor (22 March)

Jason's success in mini-assessment on triangular, square and prime numbers provided further evidence of his development in multiplicative thinking (22 March).

In Term Two the instructional focus for Jason's teaching group shifted to fractions, decimals and percentages. Jason persevered in learning his basic multiplication facts. Whole class work at the start of each lesson promoted mental calculation. An item in the pencil and paper test given at the beginning of Term Three showed that Jason's focus tended to be operational, that is on answer finding, rather than on relational structure. Asked how many eight-packs could be made from 16 dozen eggs he calculated  $16 \times 12 = 192$  then  $192 \div 8 = 24$  using written algorithms (July 16).

Jason's bookwork early in Term Three showed that his use of the properties of multiplication and division was dependable though his calculations were prone to

minor arithmetic slips. Figure 136 below showed Jason's attempts at applying the associative property of multiplication in a packaging context. By this time Jason appeared to be finding his answers by deriving from one equation to the next rather than freshly calculating the answer each time.

3. You have sixteen 8-packs of eggs ( $16 \times 8 = 144$ ).

- a. How many 4-packs can you make? 36
- b. How many 16-packs can you make? 9
- c. How many 2-packs can you make? 72
- d. How many 24-packs can you make? 6
- e. How many 12-packs can you make? 12 ✓

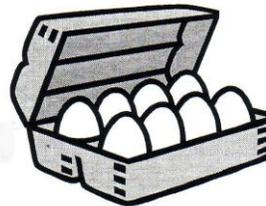


Figure 136: Jason's attempted multiplicative re-unitising with whole numbers (17 July)

In whole class together time at the start of Term Three, all students worked on solving simple linear equations using a strip model. Jason's solutions showed that he understood and could apply the inverse relationship between multiplication and division (see Figure 137). He quickly learned to use algebraic equations in letters as specific unknown contexts. Practice sheets for multiplication basic facts presented the unknowns in various ways, including as missing dividends, divisors or quotients and as missing numerators and denominators of fractions. In two consecutive weeks in Term Three, Jason showed improved recall of three times and nine times basic facts. Of the task types, the fraction representation, e.g.  $\frac{1}{9} = 4$ , presented him with the most difficulty.

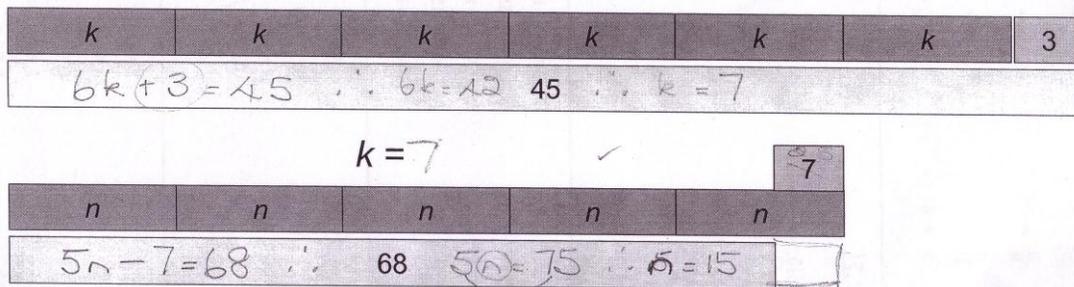


Figure 137: Jason solved linear equations using inverse operations (24 July)

In his end of Term Three interview (16 August) Jason solved the problem  $72 \div 4 = \square$  in a sharing of marbles context. He wrote 14 for his answer.

I: How did you work this out (14)?

J: Just what came to mind.

I: And what did your mind say?

J: That's not right. It's more like twenty something.

I: Write something down so we can follow it.

Jason recorded  $20 = 60$ , 4 then 24, to represent  $20 \times 3 = 60$ ,  $4 \times 3 = 12$  so  $24 \times 3 = 72$ . He correctly solved  $72 \div 3 = 24$  by place value partitioning but seemed oblivious that this was not the original problem. His mental calculations continued to be error-prone but he showed sound understanding of the properties of multiplication and division.

### 9.3 Development of Rational Number Sub-constructs

#### 9.3.1 PART-WHOLE AND MEASUREMENT

Jason began the year with knowledge that non-unit fractions were iterations of a unit fraction. He used this knowledge and repeated halving to partition continuous quantities. However, he had no concept of eight-sixths as a number. Optimism from the teacher and me resulted in Jason's placement in the high-achieving group for instruction at the beginning of Term One. Jason's lack of multiplication fact knowledge interfered with his ability to find equivalent fractions. This optimism was misplaced.

Notes in Jason's learning diary for the end of week one (28 February) stated:

Insecure and lacking in confidence. Tried working in top group on equivalence but the multiplicative nature of fractions troubled him. Recalled the size relation between  $\frac{3}{4}$  and  $\frac{2}{3}$  from lesson but has not generalised this to the value of denominators. Learning basic multiplication facts. The problem may be more knowledge based than conceptual.

The cognitive load associated with deriving multiplication facts in combination with mapping the split of fraction unit back to one appeared to be too great. For example, he recognised that the difference between two-thirds and three-quarters was one-twelfth using fraction strip manipulatives. However, the relationship between three and four, the denominators, as factors of twelve was not established and he consequently did not see any relationship between two-thirds and eight-twelfths or three-quarters and nine-twelfths. The simultaneous double mappings and size comparison were too difficult.

It was also not clear that Jason saw  $\frac{2}{3}$  as a symbol for a number. In contrast, he appeared to view it as an action of combining two units of one-third. Consequently, Jason lost confidence and fraction instruction was delayed until his multiplication and division of whole numbers improved.

In his end of Term One interview I asked Jason to find a fraction between one-quarter and one-third. He added the numerators and denominators to get two-sevenths.

I: Where did you learn to do that?

J: My head. I made it up.

Given Jason's conceptual difficulties with the equivalence of fractions the correctness of his answer was most likely due to good fortune. He seemed to have a sense that his answer was reasonable.

During Term Two Jason continued to connect multiplicative reasoning with partitioning of continuous quantities. During a geometry unit, I asked the students to draw a regular six-sided figure with sides of six centimetres. Jason had drawn Figure 138 when the following teaching interview took place. In the process, he correctly established that the interior angles were 120 degrees that he understood was one-third of a full turn.

I: If you repeated this, what shape would you get?

J: An octagon or hexagon.

I: Which one?

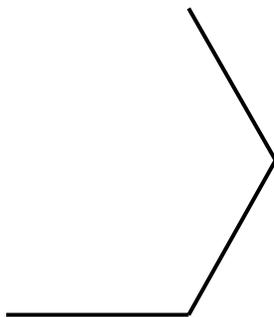
J: Not sure.

I: Look at what you have already. How many sides will the shape have?

J: Six – I don't know what it's called...hexagon?

I: Yes. These angles (pointing), what fraction of a full turn are they?

J: One third.



**Figure 138: Jason drew half of a regular hexagon (6 June)**

Jason's improved competence with multiplication and division appeared to make conceptual understanding of equivalent fractions accessible. He understood that equivalent fractions were the result of equal partitioning of measurement units and was able to map the partitioned units back to one.

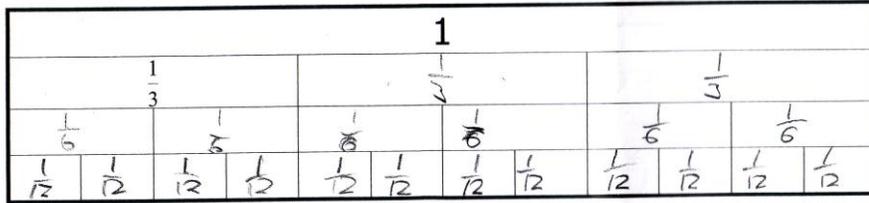


Figure 139: Jason partitioned a length model to establish equivalent units (7 May)

Jason also understood how converting to equivalent fractions allowed fractions to be added. In his pencil and paper test at the beginning of Term Three (16 July) he calculated  $\frac{7}{8} + \frac{3}{4} = 1\frac{3}{8}$  by converting three-quarters into six-eighths.

During Term Three the measure construct was used to develop the concept of decimals as equivalent fractions. Students found tenths, hundredths and thousandths of given continuous amounts by partitioning. Jason successfully partitioned a circle with a circumference divided into 100 parts and a vertical length divided into 1000 steps into simple fractional parts like halves, quarters, and fifths. In dividing the circle into eighths, he estimated each part as twelve-hundredths that resulted in one part being larger than the others were. In these situations Jason was able to adequately apply division with whole numbers and his understanding of unit fractions as operators, e.g.  $\frac{1}{5}$  of 1000.

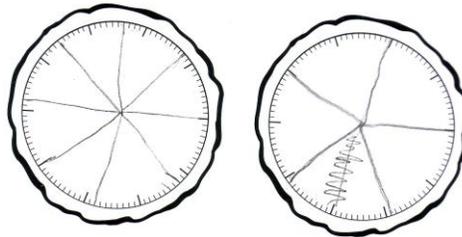


Figure 140: Jason partitioned a percentage pie into fifths and eighths (30 July)

Near the end of Term Three Jason was able to co-ordinate more complex partitions of one based on halving but was erratic in problems involving other numbers of equal parts. In a fraction wall context, he named one-third of two-eighths as one-twenty-sixth then one-twenty-fourth.

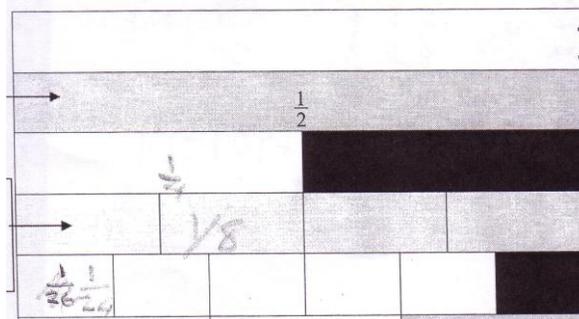


Figure 141: Jason incorrectly named partitions of a length model (8 August)

By the end of Term Three Jason had some control over the addition and subtraction of fractions though he often preferred to convert fractions to decimals. For example when asked  $\frac{1}{3} + \frac{1}{4} = \square$  in context he wrote  $\frac{58}{100}$ , the sum of  $\frac{33}{100}$  and  $\frac{25}{100}$ . His interview response to  $\frac{3}{4} + \frac{2}{3} = \square$  confirmed his improved understanding of decimals and his use of equivalent fractions in fixed whole (measure) situations (16 August).

I: Where did this come from, one point four one?

J: Hmm...three fours. I found them as decimals.

I: I see, you added the decimals together. Is there a way to add them together as fractions?

J: Change them both to twelfths.

I: What are you going to do then?

J: (Writes  $\frac{17}{12}$ ).

I: Is that bigger or smaller than one?

J: Bigger.

I: Does that make sense that it's bigger than one?

J: Yeah...(wrote + 1).

Handwritten work showing the calculation of  $\frac{3}{4} + \frac{2}{3}$ . The student converts  $\frac{3}{4}$  to  $\frac{9}{12}$  and  $\frac{2}{3}$  to  $\frac{8}{12}$ . The sum is  $\frac{9}{12} + \frac{8}{12} = \frac{17}{12}$ . The result  $\frac{17}{12}$  is circled with a plus sign, indicating it is greater than one. There is also a crossed-out  $1.48$  and the number  $12$  written below.

**Figure 142: Jason calculated three-quarters plus two-thirds**

To answer an item from his interview at the end of Term Four (19 November) Jason measured three-quarters with units of three-twentieths in a fuel tank situation. His conversion of three-quarters to fifteen-twentieths and use of division showed he was then able to simultaneously co-ordinate two different referent wholes, one and three-quarters, in the same problem (see Figure 143).

Jason's development of measures showed a progression from treating fractions as iterations of unit fractions, a process, to fractions as numbers or objects. By the end of the year he could add, subtract, multiply and divide fractions using equivalence. Given Jason's early difficulty it seemed that increased fluency with multiplication and division of whole numbers was associated with his progress.

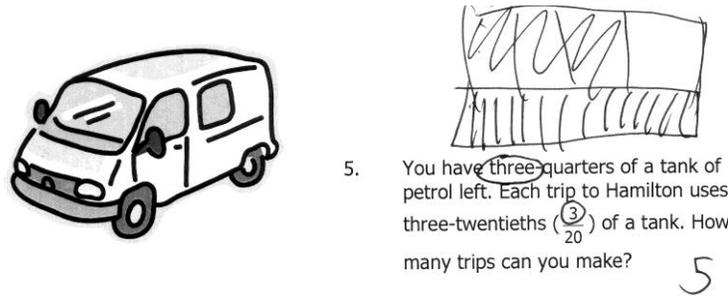


Figure 143: Jason measures three-quarters with three-twentieths

### 9.3.2 QUOTIENT

In his first interview in Term One (21 February) Jason solved a pizza sharing problem. He did so by drawing on the diagram of each pizza to partition it equally into a number of pieces corresponding to the number of people sharing. During Term One he received no instruction focused on solving quotient problems. He experienced instruction aimed at developing ideas about equivalence in part-whole situations.

In the end of Term One interview (26 March) Jason also answered a pizza sharing problem, "Five girls share three pizzas equally and three boys share two pizzas. Each pizza is the same size. Who gets more pizza, a boy or a girl? How much more?" This problem required more difficult partitioning than the problem in his initial interview.

Jason established the shares as two-sixths (of two pizzas) and three-fifteenths (of three pizzas) respectively. He was unaware of the significance that these fractions referred to different wholes. Jason showed strong spatial partitioning strategies and attempted to apply equivalence in relating the fractions back to a common referent, one pizza.

He found fifteenths of one pizza by finding fifths of thirds indicating a strong connection to splitting whole number factors multiplicatively. His diagram showed use of rate thinking to find equivalent fractions, i.e.  $\frac{2}{6} = \frac{4}{12} = \frac{6}{18}$  ( $2:6 = 4:12 = 6:18$ ), and to find a common source of comparison, i.e. numerators in  $\frac{6}{18} > \frac{6}{30}$ . Jason based his answer that boys got one-tenth more of a pizza on a visual estimate from his diagram (see Figure 144).

Jason created a mathematical model and acted on the model on the assumption that transformations on it mapped to the original situation. While the model was flawed it produced a correct size comparison of shares in this instance.

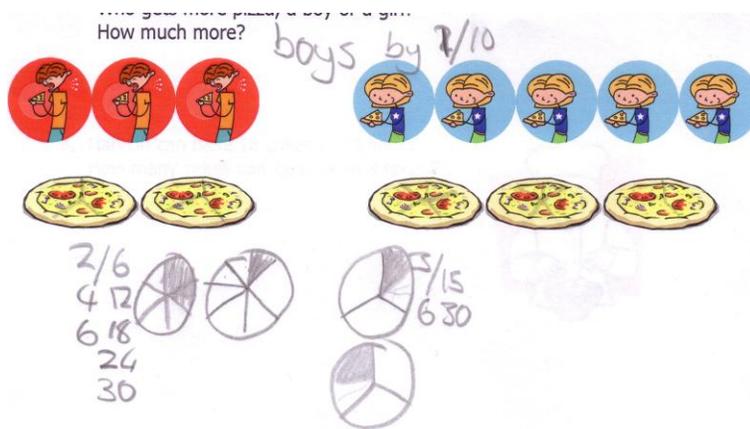


Figure 144: Jason solved a quotient problem using inverse rate thinking (26 March)

I explicitly taught solving quotient problems during Term Three. Teaching diary notes (27 July) reported Jason's preference for solving quotient problems using decimals. His answers suggested acceptance of the quotient theorem,  $a \div b = \frac{a}{b}$ , and some competence in expressing fractions as decimals. Jason's fraction to decimal conversions were not always correct and he tended to satisfice with one decimal place by rounding down, e.g.  $8 \div 5 = 1.6$ ,  $9 \div 7 = 1.2$  (1.29). This suggested that he used a calculator to obtain his answers. By Jason's final interview at the end of Term Four quotient problems were routine. Asked to share three pizzas among five girls Jason wrote  $\frac{3}{5}$  with no sign of partitioning of the pizza graphics to do so.

### 9.3.3 OPERATORS

In his first interview (21 February), Jason's lack of basic multiplication fact knowledge restricted his ability to solve a fraction as operator problem, three-quarters of 24 in a pebbles (sweets) on a cake context. While he understood the roles of denominator as divisor and numerator as multiplier, Jason used tracked counting to obtain an answer of 17.

During Term Two the instructional focus was on strategies for multiplying and dividing whole numbers and on fractional numbers as measures rather than operators. In his pencil and paper assessment at the beginning of Term Three (16 July) Jason attempted a fraction as operator problem. In a pirate crew context, he compared three-quarters of 600 gold coins with four-fifths of 550 gold coins. Jason circled the incorrect answer with no working. He may have attended to the larger fraction in doing so.

Instruction in Term Three focused Jason on other aspects of rational number such as equivalence, fractions to decimal conversions, and ratios. In his end of Term Three interview (16 August) Jason showed that the fraction as operator sub-construct was not as secure as he had demonstrated previously. In a piggy bank context, he compared one-third of \$36.00 with five-eighths of \$24.00. His initial answer was to

circle one-third of \$36.00 (incorrect). As previously, Jason did not consider the significance of both the fraction operator and the quantity operated on.

I: Do you know how to find five-eighths of something?

J: Find four-eighths and put the other bit on.

I: So how do you find one-eighth?

J: You halve it and halve it and halve it.

I: So to get eighths you divide by eight so what do you do to get thirds?

J: Cut them into threes, I presume. (Writes \$12 for one-third of \$36) Um...six...three.

I: So an eighth would be three and you're getting five-eighths.

J: (Writes 3 12 15) Yes (indicating  $\frac{5}{8} \times 24$  is greater).

Comparison of two fraction operators on two different quantities presented obstacles for Jason in terms of cognitive load. The interview showed that he had sufficient knowledge resources to solve these problems given scaffolding support and encouragement from me. Jason showed understanding that the distributive property of multiplication applied to fractions as operators. In his end of Term Three interview (16 August) Jason correctly found three-eighths of 1000ml as 375 ml. In this case, familiarity with common fraction to decimal conversions alleviated his need to perform any calculation.

Jason's responses to two questions in his interview at the end of Term Four (19 November) confirmed his level of understanding about fractions as operators. In a marbles context he calculated five-eighths of 40 as "5 = 25". Though Jason solved the problem easily, he showed no sign of seeing it as equivalent to  $\frac{5}{8} \times 40 = 25$  symbolically.

In the second problem, Jason compared the remaining travel distance for two families. The Smith family had travelled two-thirds of 210 kilometres and the Hohepa family had travelled four-twelfths of 96 kilometres. Jason realised that the Smith family had one-third of 210 kilometres left to travel and attempted to divide 200 by three in chunks, i.e.  $100 \div 3 = 33$ , so  $200 \div 3 = 66$ ,  $10 \div 3 = 3$  so  $210 \div 3 = 69$ .

I: Can I just stop you there? I bet you know how many threes are in 21.

J: Yes, seven.

I: So how many threes are in 210?

J: 69 (Staying with his original answer).

I: That can't make sense if 21 divided by three is seven.

J: (Reconsiders – builds up in counts of thirty finger tracking). 70.

Jason's inability to solve division problems with whole numbers compromised the process of finding a fraction of a number. He did not connect  $21 \div 3 = 7$  with  $210 \div 3 = 70$  using place value. The calculation restraint also applied when Jason found four-twelfths of 96. His first action was to attempt division of 96 by 12.

I: Do you notice something about  $\frac{4}{12}$  that might help?

J: It's a third.

I: Does that help?

J: Yes (writes  $9/6$  (place value split of 96) and records 32).

Jason's attempts suggested that solving operator problems depended on understanding of the denominator as divisor and the numerator as multiplier. This concept Jason had. Success also depended on calculation, both division and multiplication, with whole numbers. With some problems, this was a significant constraint for Jason. In the end of year standardised tests, AsTTle and PAT, Jason successfully solved four out of five problems involving fractions as operators. The problems he solved successfully were calculating one-half and one-third of 24, identifying 30 as one-third of 90, comparing  $\$7.00 + \$3.00$  and  $\$7.00 + 30\%$ , and finding two-fifths of  $\$2.50$ .

Yet he miscalculated two-fifths of  $\$2.80$ , which further supported the dual knowledge requirement of fractions as operators, numerator and denominator as multiplier and divider, and multiplication and division of whole numbers. By the end of the year there was little evidence that Jason treated fractions as numbers in operator situations, as he did in other situations. The only sign of transfer between properties of multiplication and division with whole numbers to fractions was use of the distributive property, albeit in a pseudo additive way.

#### 9.3.4 RATES AND RATIOS

There was little evidence gained at the start of the year with regard to Jason's understanding of ratios. His interview at the end of Term One (26 March) showed that he understood that the part-whole relationships in a 4:3 ratio are represented by  $\frac{4}{7}$  and  $\frac{3}{7}$  respectively. He also showed understanding of co-ordinating rate pairs and

finding the unit rate. For the problem in Figure 145, he wrote 12 as his answer then crossed it out, possibly to disguise the fact he had used his calculator.

3. Hamish can bake 18 cakes in 12 hours.  
How many cakes can he bake in 8 hours?

~~12~~



Figure 145: Jason used a unit rate (26 March)

The interview transcript captured his strategy.

J: I used a calculator.

I: That's fine. What did you do?

J: I went 18 divided by 12 (appeared unsure of the answer).

I: (Writing 1.5). The calculator would have given you this.

J: Yes, that's right.

I: And what did that tell you?

J: One and a half cakes in one hour.

Jason's end of Term Three interview (16 August) and bookwork at the start of Term Four confirmed his knowledge of the fractional part-whole relationships in ratios (see Figure 146). He successfully converted the fractions to percentages but did not connect the percentage as a representation of the proportion within the ratio or the associated attribute, i.e. flavour strength in fruit cocktails. In the example below, Jason identified recipe A as having the strongest taste of blueberry, successfully converted the part-whole fractions to percentages but saw no contradiction in his answers. His comparison of ratios focused on the additive difference between parts or on one measure in the ratio.

1 a 1:2 b 2:3 c 3:5 ✓

2 ~~Recipe~~ A ×

3 A. blueberry = 33.33% ✓  
 mango = 66.66% ✓  
 B blueberry = 40% ✓  
 mango = 60% ✓  
 C blueberry = 37.5% ✓  
 mango = 62.5% ✓

Figure 146: Jason converted ratios to percentages (25 October)

After a week of encountering ratio problems in various contexts Jason appreciated the need for a common whole for comparison and heavily relied on percentages to provide that common referent whole.

In his final interview in Term Four Jason's answer to a ratio comparison question, 1:2 with 2:3, showed his strength in diagrammatic modelling, knowledge of the part-whole relationships in ratios and an understanding that ratio comparison required a common referent whole.

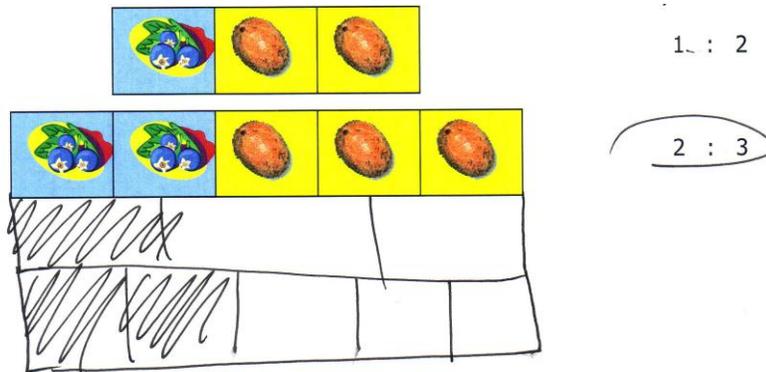


Figure 147: Jason compared two ratios for strength of blueberry flavour (19 November)

From this response, it was unclear how Jason approached comparison of ratio problems in which his diagrammatic strategy was inadequate. In this situation, his diagram satisfied.

### 9.3.5 IMPACT OF WHOLE NUMBER STRATEGIES ON FRACTION UNDERSTANDING

There was much evidence during the early part of the year that Jason's lack of mental calculation strategies with whole numbers greatly inhibited his ability to solve

problems with fractions. Yet at the beginning of the year he could solve complex problems such as  $27 \times 13 = \square$  and  $\$30.00 - \$26.70 = \square$  where he had access to written recording. His mental strategies at the start of the year were limited to simple addition and repeated doubling. He only had rapid recall of simple addition facts.

The teaching journal and notes from his bookwork described how this lack of mental calculation strategies adversely affected Jason's ability to see patterns in equivalent fractions. His mental strategies for solving multiplication and division problems with whole numbers progressed considerably in terms one and two. By the end of Term Three he applied the commutative, distributive and associative properties of multiplication with increased fluency, and understood the application of these properties to division. Increased competence with multiplication and division of whole numbers was associated with improved understanding of fractions. I made this comment in Jason's personal record towards the end of Term Two (18 May).

Jason has understood that equivalent fractions, as numbers, are derived from splitting measurement units. This was his second attempt at this and his greater strength in multiplication and division appears to have helped with his conceptual understanding. He is able to integrate equivalence with a referent whole and iterations of a unit to solve problems.

Further evidence in the terms three and four confirmed that calculation capability either enabled or prohibited the solving of problems with fractions even when the conceptual understanding of a valid process was present. For example, at the end of Term Four Jason could solve five-eighths of 40 but could not solve two-thirds of 210 without support.

Access to calculation without conceptual knowledge of process also resulted in lack of success. Despite understanding of equivalent fractions in fixed whole situations and being able to express the ratios 1:2, 2:3, and 3:5 as percentages, Jason still ordered the ratios by difference between the numbers in each pair, e.g. 3:5 had a difference of two. It seemed that understanding of process and mental calculation enjoyed a symbiotic relationship in Jason's validation of ideas.

## **9.4 Fractions, Decimals and Percentages**

### **9.4.1 INITIAL DIFFICULTIES**

In his early assessments Jason correctly calculated  $0.6 + 0.4 = \square$  but could not calculate  $7.6 + 11.8$ . He was unable to order a set of decimals to two decimal places or identify the decimal halfway between 4.0 and 4.2. In his first GloSS interview, Jason ordered unit fractions but did not know the size of eight-sixths. He correctly identified 6.8 as the number residing in a given position on a number line divided into tenths. However, he answered 1.3 to the problem "Two take away three tenths." This suggested that Jason's knowledge of fractions was restricted to unit fractions and that his knowledge of decimals, even to one decimal place, was not well established.

### 9.4.2 DECIMALS

In his end of Term One interview (26 March) Jason solved the problem, “Lisa has 9 litres of orange juice for her party. A big glass holds 0.45 litres. How many people get a glass of juice?” His strategy was to treat the problem as a rate, i.e. 1:0.45, 2:0.90, 4:1.80, etc., using additive build-up. He worked with the decimals to two places inspired by the divisor of 0.45 and seemed to treat decimals like whole numbers as indicated by his recording of redundant zeros. Jason understood that sufficient build-up to the right of the point resulted in the creation of units of one litre. It was not clear that he thought of the decimal places as tenths and hundredths. Rather his contextual knowledge about capacity measure compensated for a lack of decimal understanding.

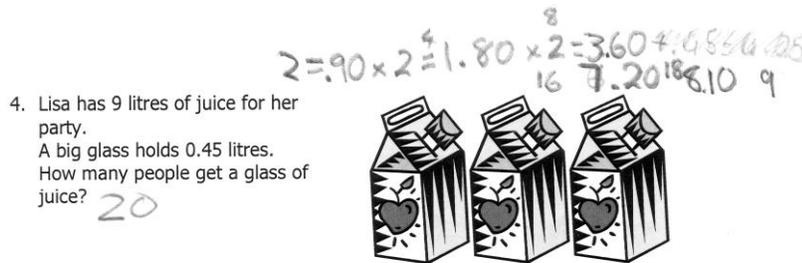


Figure 148: Jason treated decimal division as a rate problem (26 March)

A pencil and paper test (16 July) confirmed Jason’s preference for treating decimals like whole numbers in writing  $\frac{4}{5} = 0.800$  and  $\frac{1}{4} = 0.250$ .

In Term Three Jason’s teaching group received instruction about decimal place value using length and area models. This teaching included ordering, adding and subtracting decimals and developing common fraction to decimal conversions. With partitive division problems, Jason expressed his answers in decimals, e.g.  $2 \div 3 = 0.69$  (incorrect) and  $7 \div 4 = 1.75$ . His preference for expressing remainders as decimals distracted him from noticing the quotient theorem. His calculations in difference problems with decimal measures were unreliable at first but became consistent after two days of instruction. Jason was able to combine and decompose decimal place values, e.g. difference between 53.006 and 55.124 is 2.118 and  $4.567 - 2.8 = 1.767$ . He quickly learned to order ragged decimals with varied numbers of decimal places, e.g.  $3.45 > 2.999$ .

The visual approach to teaching fraction to decimal conversions included use of continuous and discrete models. Jason’s improvement in computational fluency for multiplication and division supported him in learning common conversions for halves, quarters, eighths, fifths and thirds. In the following bookwork sample, He used a decimat model to convert common fractions to decimals (1 August).

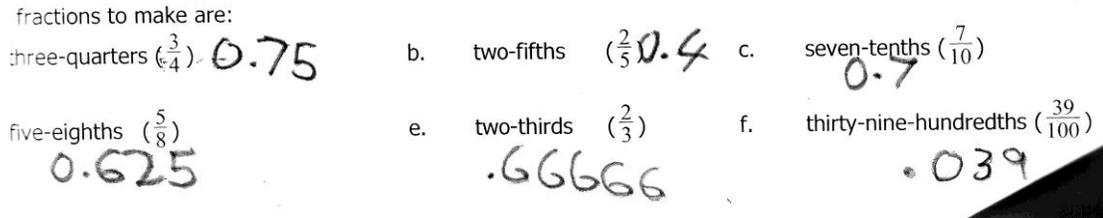


Figure 149: Jason converted fractions to decimals (1 August)

Jason preferred to add fractions as decimals at times, e.g.  $\frac{1}{3} + \frac{1}{4}$  as  $0.33 + 0.25 = 0.58$ . He continued to struggle with finding equivalent fractions where the required multiplication and divisions calculations were difficult. In his end of Term Three interview (16 August), Jason showed improved understanding of fraction to decimal conversions. He was asked, “If  $\frac{1}{4} = 0.25$  then  $\frac{1}{8} = \square$  and  $\frac{1}{16} = \square$ ?” Jason correctly gave  $\frac{1}{8}$  as 0.125, possibly from his memory of common conversions.

I: So how did you decide what one sixteenth was?

J: I halved this (pointing to 0.125).

I: So you ended up with .0075. Why the two zeros? Can you explain?

J: Half of .025 would be .0025 and half of .1 would be .05 so I added them up.

His response showed an understanding of how zeros act as place-holders and a correct halving of 0.1 to get 0.05. It was not clear if his incorrect halving of .025 to get .0025 was a slip or a sign of conceptual confusion. In the same test Jason successfully added three-quarters and two-thirds in his preferred way as decimals ( $0.75 + 0.66 = 1.41$ ) and also as fractions.

During Term Four Jason began to accept decimals as numbers that could be operated on in the same way as whole numbers. In whole class starter sessions he opted for the hardest examples. One exercise required finding three factors that multiplied to a given product. Jason experimented with decimal factors with mixed success.

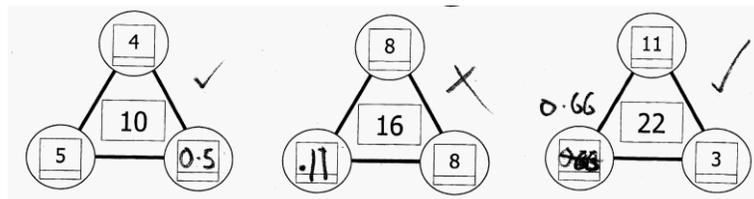


Figure 150: Jason experimented with decimal factors (6 November)

Jason became a member of the high achieving group and used decimals when solving trigonometry problems. His PAT responses at the end of the year showed improved understanding of decimals from the beginning of the year. Jason correctly solved a

decimal ordering task and identified the number halfway between 4.0 and 4.2. However, he maintained his previous choice of 18.4 to the calculation  $7.6 + 11.8 = \square$ . This item had a low level of difficulty that indicated something about it provoked his non-attendance to renaming ten tenths as one. Jason's asTTle test responses were not consistent with his PAT responses. He was able to correctly order a set of decimals, choose equivalent fractions shown using visual models, and write sixteen-twentieths as a decimal. He incorrectly answered  $\$14.90 = 6n + \$2.30$  in a story context as  $n = \$1.15$ . When asked to explain a strategy for checking an incorrect answer to  $2.93 + 49.5$  in algorithmic form Jason worked out the correct answer using place value partitioning. Jason wrote 5126.9 for the question, "Using the digits 1,2,5,6, and 9 write a number that has a 1 in the hundredths column, a 2 in the tens column, a 5 in the thousandths column, a 6 in the ones column and a 9 in the tenths column."

It was likely that Jason's error with this item was due to misreading thousands for thousandths rather than it being due to a conceptual problem.

By the end of the year, Jason showed strong knowledge of decimal place value to thousandths in ordering tasks but was prone to not "carrying" ten-tenths into the ones column in addition tasks. He appeared to understand how decimal place values could be decomposed but was inconsistent in his application of this principle to places to the right of the hundredths column.

### 9.4.3 PERCENTAGES

Jason's initial assessment indicated that he realised that a percentage was a fraction out of one hundred. After four short whole-class mini-lessons on percentages in the second teaching week, he answered all three of the following calculations correctly (9 March):

1. Something normally costs \$40.00. You buy it at a 25% off sale. How much do you pay?
2. If you get 12 out of 20 on a test, what percentage do you get?
3. Something costs \$48 without GST. What does it cost with 12.5% GST added on?

Jason appeared confident in using a percentage as an operator where the percentage to fraction conversion was known, e.g.  $25\% = \frac{1}{4}$ , or the scalar was an easy integer,

e.g.  $\frac{12}{20} = \frac{60}{100}$ .

In his end of Term One interview Jason was unable to convert 27 out of 45 to a percentage in a feeding calves scenario. The inaccessibility of this problem may have been due to Jason's calculation inhibited difficulties with equivalent fractions. Forty-five as the denominator did not lend itself to easy scaling to 100 so common factor thinking was required, e.g.  $\frac{27}{45} = \frac{3}{5}$  using nine as a common factor or  $\square\%$  of  $45 = 27$  so  $\square\% = \frac{27}{45}$ .

Jason had greater success with percentage problems when he treated the percentage as an honorary whole number and applied the distributive property. At the beginning of Term One he found the cost of a \$480 item at 35% discount using a rate approach by partitioning 35% as 10% + 10% + 10% + 5%. Jason used a similar approach in his end of Term Three interview to find the cost of a \$96 pair of jeans at 25% off. This indicated that he viewed percentages in the same way as whole number multipliers whilst still retaining some sense of the magnitude of the answers.

Jason solved the problem in Figure 151 at the beginning of Term Three test (16 July). He treated the percentage like a whole number operator to which the distributive property applied. Jason used ten percent as a foundation for build-up, i.e.  $10\% \times 480 = 48$ ,  $3 \times 48 = 144$ ,  $144 + 24 = 168$ ,  $480 - 168 = 312$ . He successfully co-ordinated a complex sequence of calculation steps and showed understanding that  $100\% - 35\% = 65\%$ . His strategy also showed a connection to rates, i.e.  $10\%:48$ ,  $30\%:144$ ,  $5\%:24$ ,  $35\%:168$ .

4. Olivia wants to buy this armchair for her Mother.  
It is on sale at 35% off.  
How much will Olivia have to pay?

312



\$480  
168 ✓  
 $144 = 30\%$   
 $\frac{24}{168}$

Figure 151: Jason solved a percentage as operator problem using rate (16 August)

In the same interview (16 August), Jason calculated the net cost of a \$96.00 pair of jeans with 25% discount. Again, he used a rate-based strategy.

J: I got 10%, doubled it, then halved it to get 5%.

(Written as 9.60 (10%), 19.20 (20%), 4.8 (5%), 18 and 23 as sub-totals. Gets \$24 and subtracts it to get \$72).

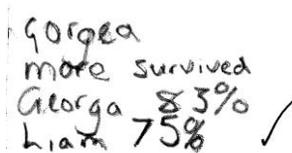
At this time percentages, expressed as whole number operators, evoked a different response from Jason than operators expressed as fractions. Jason did not see fractions and percentages in the same way.

In Term Four, his group instruction focused Jason on percentages as proportions in ratio contexts. These notes from his bookwork (24 October) described Jason's conceptual difficulties in connecting percentages and ratios.

Correctly labels part-part relationships as ratios but unable to order fruit cocktail recipes by taste. Successfully converts the ratios to percentages but does not see the conflict with his previous answer. What does he think percentages are? Similar occurrence on page 2. Issues: Jason does not see percentage as a proportion that describes the attribute of “strength of taste”.

I addressed the preference of students in Jason’s teaching group for treating percentages as honorary whole numbers rather than proportions during Term Four, particularly in ratio contexts. Using percentages as operators was discussed as a special case of fractions as operators, e.g. 40% of  $n$  as  $\frac{40}{100}$  of  $n$ . A teaching diary entry (5 November) and Jason’s bookwork samples showed that he was able to fluently solve fraction of whole number problems. However, in comparison situations he tended to rely on calculation rather than on relational thinking. When asked if 25% of \$60 was the same or different to 60% of \$25 Jason suspended judgment until both answers were calculated. Jason had not recognised that properties of multiplication and division applied with fractions, including percentages, in the same way as they did with whole numbers.

Several graphic and manipulative models were used to develop students’ ideas about percentages as proportions. These included use of a double number line in the form of percentage strips and a Microsoft Excel™ spreadsheet that automatically adjusted pie graphs as ratios were changed. Problems involved several contexts including fruit cocktail mixes, changing bar graphs to pie graphs, and plant survival rates. Jason learned to view percentages as proportions through these approaches and change his previous singular focus on one value or the additive different between the two values in ratios. He had faith in percentages as common referents.



A handwritten note on lined paper comparing survival rates. It reads: 'Gorgea more survived', 'Georgia 83%', and 'Liam 75%' with a checkmark to the right.

Figure 152; Jason used percentages to compare proportions (31 October)

At times seeing percentages as equivalent fractions Jason’s seemed to interfere with his previous focus on successful calculation through seeing percentages as operators. He was cautious about whether  $x$  percent of  $y$  was the same as  $y$  percent of  $x$ . In whole unknown problems Jason did not understand how the unit rate was found. He enhanced his view of percentages in one respect, as proportions, which caused new reluctance at times to view them in the same way as whole numbers operationally.

His final interview in Term Four (19 November) revealed the connection Jason forged between equivalent fractions in fixed one and variable one (ratio) situations. Jason answered, “You sit a spelling test and get 18 words right. The teacher gives you 60% correct. How many words are in the test?” The item required him to establish the whole given a part (18) and its equivalent proportion of the whole (60%). Jason applied rate thinking.

$$\begin{array}{l}
 18 = 60\% \\
 9 = 30\% \\
 \rightarrow \\
 30
 \end{array}$$

Figure 153: Jason recorded his answer to a whole unknown percentage problem (19 November)

I: So what percentage is 27?

J: 90.

I: Is that a problem?

J: 29 or 30.

I: Which one?

J: I think it's 30.

I: Why 30?

J: Because that is what it's most likely to be in a test.

I: Does it check out? Is 18 out of 30 60%?

J: Yes.

Jason's response indicated some convergence of operator and proportional views of percentages. His initial preference was for a rate strategy, i.e. 60%:18, 30%:9, 90%:27 so 100%:29 or 30. Jason adjusted his initial estimate of 29 using contextual knowledge that tests tended to involve friendly numbers. It was not clear whether Jason based his certainty about 30 test items being the whole on seeing  $\frac{18}{30}$  and 60% as equivalent fractions or an estimate of reasonableness, e.g. 18 out of 30 is 10% more than 50%.

In end of year testing Jason was able to write 20% as a decimal and compare \$7.00 + \$3.00 to \$7.00 + 30%. Neither item measured the bounds of his understanding. His end of year performance suggested that Jason was learning to view percentages as both operators and proportions depending on the demands of the given situation.

## 9.5 Summary

### 9.5.1 PROCESSES TO OBJECTS

Norm referenced tests at the end of the year gave conflicting data about Jason's progress through the year. His overall asTTle result, Level 4B, showed no change from the beginning of the year. Jason's PAT result showed a significant shift from stanine six to stanine nine. This result placed him in the top four percent of students for his age. It was difficult to rationalise these two results.

The detailed data presented in this case study showed that Jason progressed significantly in his thinking about fractional numbers during the year. The reason for his progress was an alignment between Jason's knowledge of anticipated processes and the associative knowledge required to enact the anticipation and apply it to ideas that were more complex.

Jason's disposition was to anticipate actions on physical or diagrammatic embodiments. This was ironic given his enjoyment of physical manipulation and strong capacity for imagery. He began the year with many anticipated actions. For example, he knew that distributing one factor additively in multiplication preserved the product, compensation of one addend from another conserved the sum in addition and that a non-unit fraction was composed of iterations of a unit fraction.

Jason acted on these anticipated actions as objects of thought. For example, he applied the distributive property in finding five-eighths of 24, as half of 24 and one-eighth of 24. He applied additive compensation to add three-quarters and seven-eighths. Not all of Jason's invented anticipated processes were valid. For example, he compared quotients using different referent wholes. He added numerators and denominators to find a fraction between two fractions. The former was flawed, the latter correct. Both theorems stood up to scrutiny of size relationship in the situations.

Jason's ability to act with an anticipated process was heavily dependent on the knowledge needed to enact it. Most but not all of his missing knowledge was factual. Lack of basic fact knowledge for multiplication adversely affected his ability to find equivalent fractions by partitioning. When this associated knowledge was well-established by the end of Term Two Jason constructed strong equivalent fraction understanding in measure situations. Some absent associated knowledge involved schemes or *know-hows*, particularly in respect to place value. For example, Jason struggled to find  $210 \div 3 = 70$  but knew  $21 \div 3 = 7$  and found  $9000 \div 450 = 20$  by repeated subtraction when  $450 \times 10 = 4500$  was accessible.

Recording, both symbolic and diagrammatic, played a key role for Jason in supporting the anticipation of processes. His development of division of whole numbers progressed from reliance on physical or visual models, such as place value materials, to equations. Early in the year, he used partitioning of circular regions to shade three-fifteenths of a pizza and at the end of the year compared the ratios 1:2 and 2:3 using strips of equal length. Chains of calculations allowed Jason to access answers that he could not calculate mentally.

A reasonable conjecture was that the development of Jason's number knowledge and mental arithmetic had a two-fold effect. Firstly, it enabled him to act with the anticipated processes he already possessed and secondly, it allowed him to see symbolic and diagrammatic relationships that became the material for further anticipated processes. The idea of a procept (Gray & Tall, 2001) seemed an appropriate model for Jason's interpretation of symbols.

Rather than viewing fractions, decimals, percentages and ratios as single processes embodied by a symbol Jason came to accept the co-existence of multiple processes within the same symbol. At the beginning of the year Jason had some understanding of the fraction  $\frac{a}{b}$  in number form, not algebraic form. He knew  $\frac{a}{b}$  meant "a out of b" and a iterations of  $\frac{1}{b}$ . These different views were applicable at different times.

Iteration of a unit fraction was critical for the construction of fractions as operators and measures, "out of" was important for ratio and frequency. However, Jason struggled to co-ordinate these two processes in determining the size of  $\frac{8}{6}$ . In time he came to understand  $\frac{a}{b}$  as a number that embodied both processes but also was available as an object to think with, just as any whole number was.

He accepted that  $\frac{a}{b} = \frac{na}{nb}$  by a process of equal partitioning and in turn used that to divide  $\frac{a}{b}$  by other fractions. His co-ordination of multiple processes was not consistent by the end of the year. While accepting that the distributive property applied to fractions, and percentages, as multiplicative operators he did not readily accept the commutative property. Jason remained more circumspect about equivalence of fractions in variable whole (frequency and ratio) situations than fixed whole (measure) situations.

From the data, it was hard to see Jason as having a reified, encapsulated idea of  $\frac{a}{b}$  as a multiple procept. Rather the idea that he did possess was adequate for some purposes and inadequate for others. The multiple procept seemed to be continually evolving through connection to new situations.

### 9.5.2 CO-ORDINATION CLASSES

Situational variation, as anticipated by co-ordination class theory, was in evidence throughout the year. Much of the situational disturbance appeared to be due to absence of knowledge elements, an inability to choose an appropriate anticipatory process or a synthesis of the two. Jason's construction of a co-ordination class for percentages best exemplified situational variation.

Jason began the year with knowledge that percentage meant "out of 100" and that a fraction operated by dividing the quantity into the denominator number of parts and multiplying by the numerator. From an ontological perspective, the co-ordination of these two knowledge elements was co-ordination of measure, ratio and operator sub-constructs with percentages, in the same way as fractions, e.g. 40% of \$120 as  $\frac{40}{100}$  of \$120. In contrast, Jason's early successes with finding percentages of amounts

involved avoiding percentages as fractions. He treated them like honorary whole numbers as he applied the distributive property of multiplication and rate thinking. To find 35% of \$480 he split 35% additively and built up the calculation as a rate. Jason's choices may have been strategic in terms of his accessibility to the multiplication and division knowledge needed to enact them.

At that time problems involving percentages as equivalent fractions were inaccessible to Jason, especially where common factors were required, e.g. 27 out of 45 as a percentage. Jason's co-ordination of two understandings coupled with stronger mental calculation allowed him access to other ways to consider percentages while not diminishing his initial rate based preferences. From work with measures, Jason understood equivalent fractions as the same number. From work with frequencies and ratios, he realised that proportions with different wholes were only compared with the same referent. In the midst of co-ordinating these views Jason was content to accept attention to one measure, the amount of mango juice, as a way to order the "blueberriness" of three ratios when the correctly calculated percentages said different. Contradictory answers resided together without resolution.

This development took a prolonged period, nine months, in which Jason encountered many different situations involving percentages. By the end of the year he was able to simultaneously consider a percentage as a whole number operator and an equivalent fraction to solve 18 out of  $\square$  equals 60%. His cueing preference remained with rate-based strategies and the co-ordination of operator, rate and measure sub-constructs was by no means secure. In the language of co-ordination class theory Jason improved the span of his class for percentages, the range of situations to which he could apply it. His alignment also improved. He solved percentage problem more reliably at the end of the year compared with the beginning.

Jason's case study also provided useful insights into the role of cueing preferences as both enablers and inhibitors of conceptual development. His preference for and strength in creating visual models helped Jason to solve problems on many occasions. There was no evidence to suggest that these models inhibited his development of more elaborate concepts through over-reliance. However, Jason's preference for deriving rather than knowing basic facts was another matter. The cognitive load associated with working out simple multiplication answers interfered with his development of equivalent fraction concepts.

In Term Three Jason developed strong decimal understanding through manipulation of materials. From that point on, the calculator became Jason's close companion. He converted fractions to decimals to add them and expressed the answers to division as decimals. His preference for decimals did not prohibit the development of an algorithm for adding and subtracting fractions and the quotient theorem. Cueing preferences both supported and in some cases delayed conceptual growth.

There was some suggestion of Jason's possession of p-prims and *met-befores*. In ratios two were identified:

- For a ratio  $a:b$ , addition of  $n$  measures of both  $a$  and  $b$  gives a ratio of  $a + n:b + n$  that still tastes of  $a$  and  $b$ .
- For two ratios  $a:b$  and  $c:d$ , if  $b$  is less than  $d$  then  $a:b$  will taste more strongly of the first measure as it will have more effect.

Jason also used the p-prim, more parts of the same whole means smaller parts in comparing  $\frac{6}{30}$  and  $\frac{6}{18}$ .

Practical experience outside school contributed to his strong growth in the ratio sub-construct in Term Four. The following discussion occurred during his end of Term Four interview (19 November).

J: My Dad and I bought an 85 motorbike and I mix oil and petrol, I mean petrol and oil.

I: Yes, so you mix them to get your two-stroke mixture.

J: It's a forty to one ratio.

I: So you're a gun on ratios.

J: And I have to be able to afford it.

I: That can make you think carefully about stuff.

In the everyday context of mixing fuel Jason saw a practical necessity for learning about ratio. This motivated him and provided a specific situation from which Jason appeared to generalise his ratio construct.

## 9.6 SPAN AND ALIGNMENT

The following three-dimensional graphs summarised Jason's progress in multiplication and division and Kieren's sub-constructs (Kieren, 1980, 1988, 1993) during the course of the year. The graphs derived from Jason's learning trajectory map. At the start of the year Jason's lack of basic fact knowledge and his disposition for using cumbersome, additive methods to solve problems restricted his proficiency in multiplication and division of whole numbers (see figure 154). In comparison, Jason had relatively strong development in the fraction constructs, particularly rate and quotient. No data about his understanding of ratio was available at the start of the year.

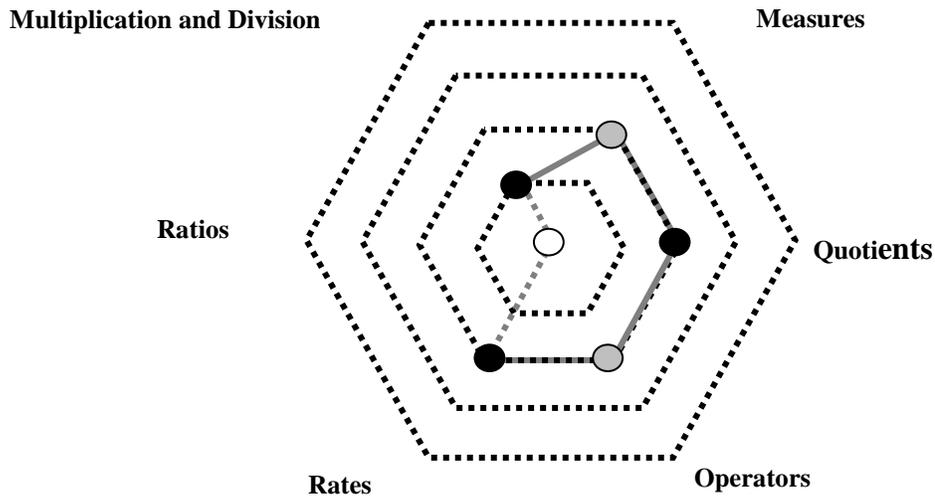


Figure 154: Learning Trajectory Map: Jason beginning of Term One

During Term One Jason made considerable progress in span for multiplication and division of whole numbers. He used the properties of multiplication to derive answers using his incomplete range of basic facts and place value knowledge. Absence of some factual knowledge contributed to a lack of reliability in his strategies (alignment). There was also progress in the measure and rate constructs as Jason understood equivalent fractions as the same measure and applied unit rate strategies. Improvements in alignment did not accompany the improvements in span. The newness of ideas coupled with the demands of mental calculation contributed to unreliability of his responses in all but the quotients construct.

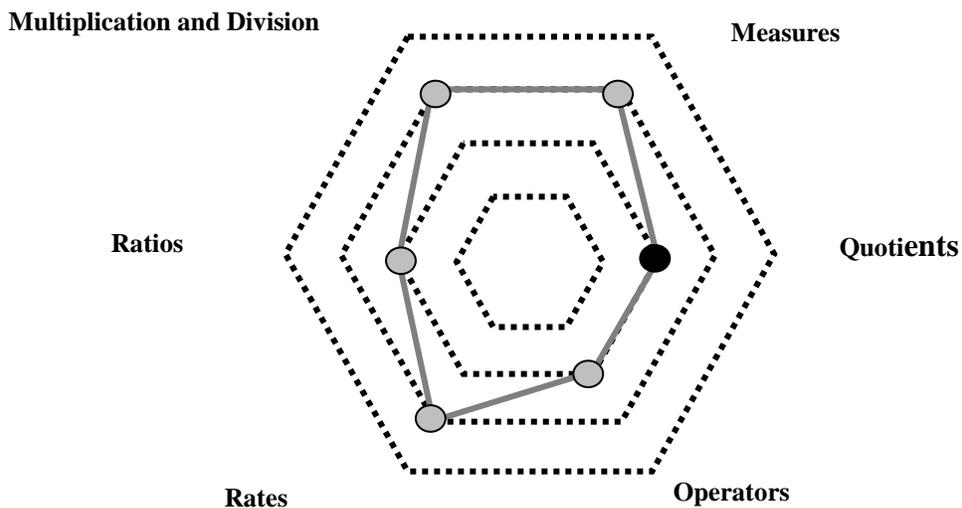


Figure 155: Learning Trajectory Map: Jason end of Term One



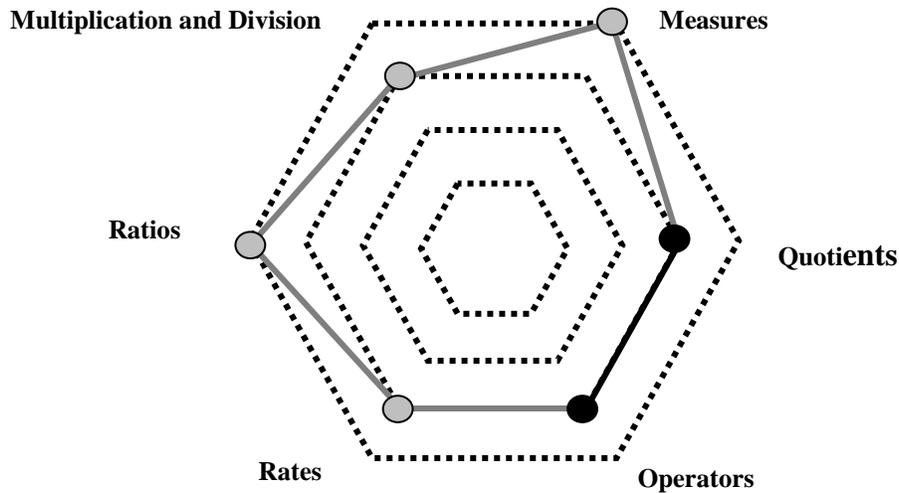


Figure 157: Learning Trajectory Map: Jason end of Term Four

Viewing Jason’s graphs simultaneously confirmed that he made considerable progress in all sub-constructs during the year. Progress was characterised by rapid increases in span in particular constructs, without demonstrable growth in others. The shape of his map changed markedly between time points.

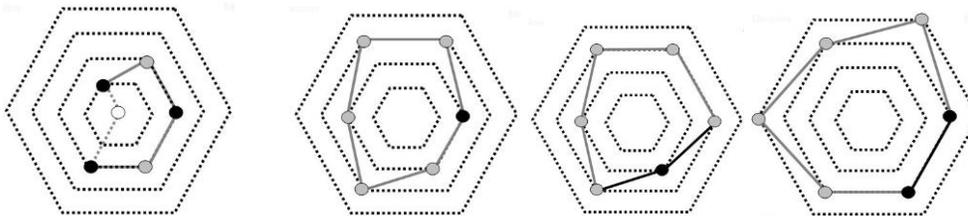


Figure 158: Simultaneous view of Jason’s Learning Trajectory Maps

Growth in some sub-constructs appeared to be dependent on growth in others and transfer between sub-constructs. There was pronounced teaching effect in this result since Jason tended to improve in the sub-constructs that were the focus of instruction. His attempts to learn equivalent fractions as measures were a notable exception. It was conjectured that weak multiplication strategies and knowledge restricted Jason’s learning of fraction concepts early in the year. Mental calculation strategies with whole numbers based on knowledge of the properties of multiplication and division, and on basic fact and place value knowledge, seemed necessary to Jason’s understanding of equivalent fractions. Whole number strategies facilitated Jason’s learning of fractions as operators.

Once he understood equivalent fractions as measures, Jason transferred the concept to ratios. This required acceptance of the need for a common referent whole. Knowledge of equivalent fractions as measures was also requisite in comparing fractions through addition, subtraction and division. At the end of the year, Jason provided no evidence that he transferred equivalent fraction as measures to comparing quotients and rates.

While the graphs showed that the span of Jason's application of fraction sub-constructs broadened they also showed that alignment was seldom secure. Sometimes this was due to aspects of a given sub-construct not being taught or learned. With the exception of the ratio sub-construct that Jason learned from his father, there was no evidence of learning without deliberate acts of teaching. Sometimes lack of alignment appeared due to task variables in conjunction with the solution strategy Jason adopted. Absence of knowledge to enact a chosen strategy was often a constraint.

## **CHAPTER TEN: CASE STUDIES OF LINDA AND SIMON**

### **10.1 Abbreviated Case Studies**

I selected nine students for the case study group. The previous chapters present four of these case studies. For reasons of space this chapter presents two abbreviated case studies. The two students, Linda and Simon, represent opposite ends of the achievement range in the research class. Their stories present interesting contrasts about transfer and the potential for object theory and co-ordination class theory to model and explain it.

### **10.2 Linda**

#### **10.2.1 PERSONAL CHARACTERISTICS**

Linda was 13 years and two months old at the beginning of the school year. She classified herself as a pakeha of European ethnicity. Her interests included singing, drama, dance and hockey. She was a formidable goalkeeper in hockey and represented her province in that position.

Linda was the consummate organiser. Her mother ran a catering business and Linda learned to be self-sufficient and to organise herself and others. Other students did not always appreciate her assertiveness, especially the low achieving boys from her instructional group. Linda was outgoing and talkative. She tended to form friendships with one or two classmates rather than participate in a wider group. Linda struggled academically across most areas of the curriculum. Her mother offered strong support from home and Linda was highly motivated to succeed.

#### **10.2.2 INITIAL TESTS**

Standardised norm-referenced tests at the beginning of the school year confirmed that Linda was achieving at below average levels for her age and class. Her AsTTle test result (1 March) reported her achievement on number knowledge at Level 2P (two proficient) and on number operations at Level 3B (three basic). The average expectation for Linda's class level was 4P and each level was expected to take about two years to progress through.

The PAT (21 February) provided more detail on Linda's knowledge of number. She scored at stanine three for the test. This result placed her in the lower quartile for her class. Linda had significant difficulty with items that involved fractions, decimals and simple proportions, and with interpreting different strategies for whole number calculation. Linda succeeded on applications of simple multiplication facts, giving change in a money context and calculating the price of two items.

### 10.2.3 INITIAL NUMBER INTERVIEWS

Linda's performance on her initial GloSS interview (20 February) was consistent with the below average achievement reported in the results to standardised tests. Without access to her preference for written algorithms, her mental strategies were limited. She performed consistently at an Early Additive stage across all three domains, deriving  $6 + 9 = \square$  from  $5 + 10 = 15$  and  $17 - 9 = \square$  from  $17 - 10 + 1 = 8$  by compensation strategies. She did not know  $4 \times 8 = 32$  to solve an array based problem but derived the fact by leveraging off  $4 \times 10 = 40$ , calculating  $40 - 8 = 32$ . Linda found one-eighth of 16 using her two times tables for multiplication. This indicated Linda had some knowledge of the distributive property of multiplication and an inclination to derive unknown facts. Her knowledge of basic facts was limited to doubles, teen numbers like  $10 + 7 = 17$ , and the five times multiplication facts.

Linda's initial number interview occurred on 21 February, 2007. It involved questions based on the measures, operators and quotients sub-constructs for fractions. In two partitioning tasks there was no evidence that Linda knew that iteration of the unit was a useful way to determine the size of a fractional part. Her attempt to partition a second worm into fifths produced unequal parts. She noted the parts were "not very even" which suggested her recognition that equal parts were required.

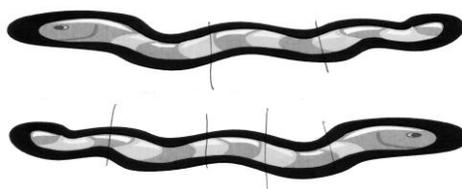


Figure 159: Linda partitioned worms into thirds and fifths (21 February)

The interview revealed that Linda relied on a circular region image of fractions.

I: How do you go about the job of cutting this into equal pieces?

L: It's difficult because I always think of circles and this is a different shape.

Asked which fraction was bigger, Linda correctly identified that one-quarter was larger than one-fifth. Her explanation suggested she possessed a p-prim that the size of parts decreased as the number of parts increased. Given her inability to explain the size relation of one-quarter and one-fifth in symbolic form (GloSS test) Linda did not easily connect symbols for fractions with words, images or quantities.

L: One quarter because with a quarter you are giving only four pieces and with a fifth you are giving ... five pieces.

In question four Linda looked at the strip diagram below. She could not find "How many sixths are the same as one-half?" without support from the interviewer. She

knew that sixths meant six equal parts and recognised the symbol for one-sixth, but was unable to write an equation to describe the relationship, i.e.  $\frac{1}{2} = \frac{3}{6}$ .

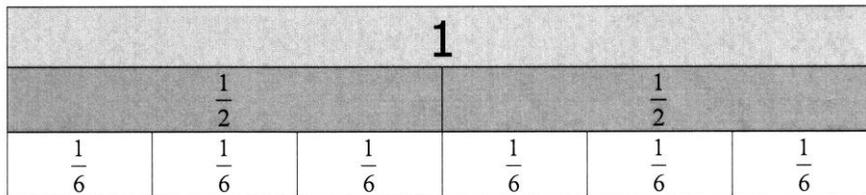


Figure 160: Linda referred to a fraction strip diagram (21 February)

I: Why are three-sixths the same as one half?

L: You kind of look at the picture and it gives it away.

There was no evidence that Linda mapped either one-half or one-sixth back to the referent one. Both fractions appeared to exist as unrelated units in her thinking at this point.

Next Linda was asked what fractions a cake was cut into. She replied that the parts were thirds because there were three of them. She appeared confused by the conflict that two pieces appeared to be quarters and yet there were three not four pieces in the whole cake. She sought clarification from the interviewer.

L: Is three-quarters the same as one third?

Asked to anticipate the result of finding three-quarters of a set of 24 pebbles (sweets) she partitioned one-half of the cake into quarters. She concluded that each person would get “three each.” Linda appeared to confuse finding three-quarters of 24 with one-quarter of 12 through changing the referent whole to one-half. She used the visual arrangement of the pebbles in the diagram to find an answer.

5. What fractions is the cake cut into?

There are 24 pebbles on the cake.  
If you eat three quarters of the cake,  
how many pebbles do you get?

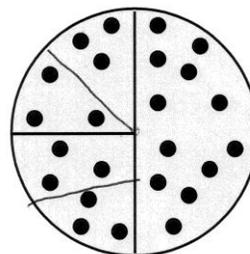
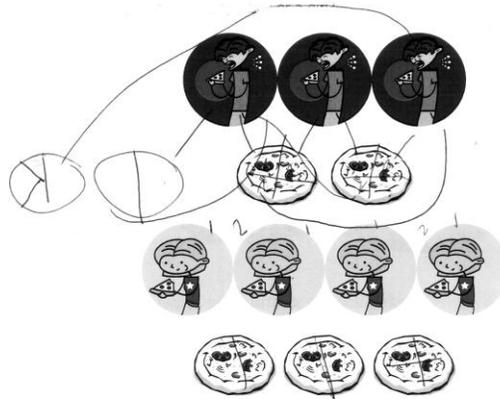


Figure 161: Linda divided a cake into eighths to find a fraction of a set (21 February)

In a quotient task Linda used one-half as a unit for sharing the pizzas (see Figure 162). For the girls she gave each person one-half and cut the remaining pizza into

quarters. For the boys she cut both pizzas in half and further subdivided one of the halves into thirds (sixths of one pizza).



**Figure 162: Linda shared pizzas equally by partitioning into halves (21 February)**

Linda decided that both boys and girls were getting the same amount of pizzas as “both are getting the same- two pieces”. In this situation, she recognised the importance of equal partitioning in establishing fair shares but did not understand that the significance of the piece sizes in comparing shares.

In summary, Linda’s understanding of the fraction sub-constructs was poor. She was unable to equally partition continuous quantities apart from halving, did not recognise the importance of equality and iteration in mapping fractions onto the referent one, and focused on the number of pieces without attendance to size in sharing situations. Her connection of fraction symbols to physical representations was limited and she tended to rely on visual images of circular regions.

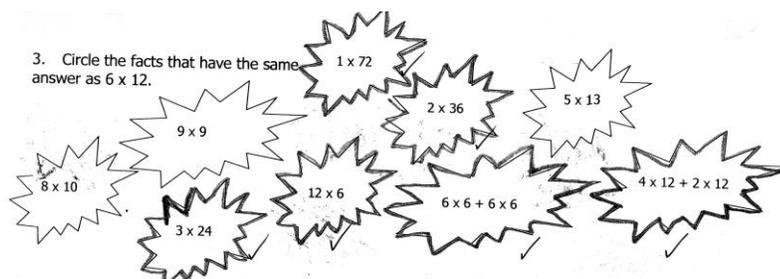
#### **10.2.4 PROCESSES TO OBJECTS**

Linda made progress during the year, mainly with her strategies for operating on whole numbers rather than her understanding of fraction sub-constructs. This reflected the focus of group instruction. Results from norm-referenced tests at the end of year were inconsistent. She showed no gain in achievement as measured by PAT but increased her performance on number knowledge and operations tasks by one curriculum level as measured by asTTle. The asTTle result equated to two years progress. These two assessment tools reflected very different interpretations of the curriculum in their sampling and item types. Linda’s progress is now analysed in relation to process to object theory.

Fundamental to object theory is condensation of actions on objects, real or imaged, into anticipation of a process. Linda’s GloSS and number interviews in February showed some evidence of anticipated processes. For example, she derived basic addition and subtraction facts and used knowledge of five times seven to anticipate counting objects in an array. She based anticipation on secure knowledge of facts, e.g. doubles and teens, and invariance properties, e.g. compensation between addends

and the distributive property for multiplication. She trusted in vertical written algorithms to add and subtract whole numbers and applied these trusted symbolic processes in context.

The early data also showed that in the absence of secure knowledge Linda was strongly disposed towards acting on objects with little expectation of result. For example, her approach to partitioning a worm into thirds and fifths showed no expectation that the unit should iterate to fill the whole. As the year progressed, Linda acquired a considerable repertoire of anticipated processes. She knew that multiplication facts allowed her to anticipate the result of sharing and that rates could be replicated using a common factor.



**Figure 163: Linda used properties of multiplication to identify facts with the same product (12 March)**

Linda’s ability to use the anticipated process was very dependent on the availability of secure knowledge. In Term Four she learned that the result of finding a non-unit fraction of an amount was anticipated by dividing by the denominator then multiplying by the numerator, e.g.  $\frac{2}{3}$  of 36 as  $(36 \div 3) \times 2$ . In her final interview Linda devoted her working memory to finding one-third of 12 and in doing so forgot the chain of operations she imagined herself doing.

At times Linda seemed to regard symbols as entities in themselves removed from the quantities they represented and existing independently. Physical materials were integral to instruction and used as a vehicle for predictions of the results of actions and validation of those predictions. Yet Linda acted on symbols with no evidence of mapping back to physical actions. For example, she was content to take lower from higher digits in the algorithm below (Figure 164).

$$\begin{array}{r}
 534 \\
 - 286 \\
 \hline
 \$352
 \end{array}$$

**Figure 164: Linda subtracted lowest digit from highest digit in vertical algorithm (16 July)**

Two days later she compared an incorrect vertical algorithm with a correct adding-on method shown on an empty number line. The calculations were  $152 - 88 = \square$  and

$88 + \square = 152$  respectively. Linda chose the adding-on method “because you cannot take 8 away from 2”. Her attention was on the impossibility of taking higher from lower rather than on the reasonableness of the answer or the structural validity of the method. On a pencil and paper test (16 July) Linda recorded  $\frac{6}{8}$  as her answer to  $\frac{3}{4} + \frac{3}{4} = \square$  and placed one in its prototypical position on a number line despite the presence of a supportive strip model.

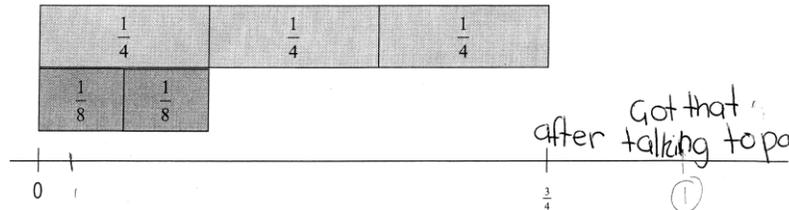


Figure 165: Linda incorrectly placed one on a number line (16 July)

Linda’s lack of symbolic conscience mixed with occasions in which symbols were helpful to her anticipation of physical processes. These successful occasions were simple derivations using her preferred and trusted theorems. For example, in Term Two Linda came to trust doubling and halving with the distributive property of multiplication. She used the distributive property to find  $12 \times 24 = 288$  and remembered a rule for multiplying multiples of ten.

- a.  $12 \times 24 = 288$  Why?  $12 \times 25 = 300 - 12 = 288$  so that make the equation  $12 \times 24$
- b.  $6 \times 50 = 300$  Why?  $6 \times 5 = 30$  put 0 - back = 300

Figure 166: Linda used the distributive property and a rule to solve multiplication problems (16 July)

She extended her growing knowledge of multiplication to using distributed place value to solve problems. With the support of written recording, Linda performed reliable calculations for multiplication involving one single digit factor.

$4 \times 18 =$   
 place value because  
 this is how i would  
 work it out

$4 \times 10 = 40$   
 $4 \times 8 = 32$   
 $32 + 40 = 72$

Figure 167: Linda used the distributive property to solve multiplication problems (1 August)

Linda's use of symbols had benefits and drawbacks. Symbols allowed her to organise information in ways that took load off her working memory. It also gave her a language of communication. Balancing the advantages were the added connections required between the symbols and the quantities symbolised, and the potential for Linda to attend to incorrect relationships. Her work with equivalent fractions and fractions as operators exemplifies this tension.

Students used region models to illustrate the concept of equivalent fractions and systematic recording to support the connection between equivalent fractions in symbolic form. Linda was mostly successful at expressing fractions in their simplest form, given a routine procedure to follow. The examples below indicated she was comfortable with using common factors to simplify fractions but examples where no such common factor existed easily confused her.

$$c \frac{6}{8} = \frac{3}{4} \quad d \quad 2$$

$$a \frac{15}{30} = \frac{1}{2} \quad b \frac{29}{30} = \frac{4}{5}$$

Figure 168: Linda simplified fractions to their simplest form (25 October)

In Term Four the class teacher introduced a systematic method of finding the result of fractional operator problems. The method used a table to record the steps in a calculation as rate pairs. Linda enjoyed considerable success with the method.

D) Tama

42 cubes  $42 \div 7 = 6$   
 $6 \times 3 = 18$

$\frac{3}{7}$  are white

fraction	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{7}$	$\frac{3}{7}$
number	42	21	6	18

Figure 169: Linda used a table to record an operator problem as rate pairs (11 October)

Her working on operator problems quickly reduced to recording the essential steps. The method was procedural and Linda was again highly consistent in getting correct answers (see Figure 170).

$$\frac{5}{7} \text{ of } 35 = 25$$

$$35 \div 7 = 5$$

$$5 \times 5 = 25$$
  

$$\frac{2}{3} \text{ of } 36 = 24$$

$$36 \div 3 = 12$$

$$12 \times 2 = 24$$

**Figure 170: Linda recorded her procedures for solving fraction as operator problems (16 October)**

In her interview at the end of Term Four, Linda solved a contextual problem that required finding two-thirds of 36 marbles. Firstly, Linda drew a circular region and attempted to divide it into thirds by area not angle or arc length.

L: Two-thirds is something of a fraction but it's only two times (referring to the numerator two)...The number 36 is coming back as something to do with my times tables.

I: What times tables?

L: Six and three... six times six is 36.

I: Is it in your three times table?

L: Yes, three times...(long pause).

I: It's six times six so it must be three times.

L: 10 (knows not enough)..no...12.

I: So does that help you solve the problem?

L: Yes, oh...not sure.

The episode showed that Linda anticipated being able to solve the problem, and applied the associative property as an object of thought in trying to do so. She also appeared to know that she needed to find two measures of one-third, so applied a theorem that non-unit fractions are iterations of unit fractions. Her cognitive effort went into finding one-third of 36 by dividing by three. After an extended yet successful process of doing so Linda was unable to enact the operational sequence she first imagined. Computational demands diminished the working memory space she had left and with it her confidence in the solution path. In teaching situations, recording symbols helped her enact the process. The recording systems also appeared to play a part in Linda establishing a trusted anticipation of a physical process.

However, she never owned the recording method in the sense that it became a cued preference.

The episode also highlights that in anticipating processes Linda often chose methods that were inefficient at first and not helpful for generalisation. This reflected her state of transition between additive thinking and multiplicative thinking. Her early attempts at division were either applications of her few known multiplication facts or repeated subtractions. In Figure 171 below, she used a model of tens and ones to work out the answer to the problem, “Nat has baked 96 biscuits for the gala day. He puts four biscuits in each packet. How many packets can he make?” Linda used equal chunking (quotitive) strategies irrespective of whether the context involved sharing (partitive) or measurement (quotitive) division.

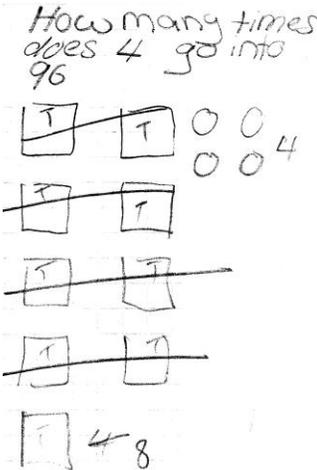


Figure 171: Linda used a place value representation to solve a division problem (22 March)

By the end of the year Linda attempted diverse strategies for division, was considerably more efficient in her use of the distributive property and was prepared to connect division with multiplication. In a marble sharing scenario she misinterpreted the problem as 72 divided by three instead of four but got a correct answer of 24. A recreation of her erased paper working is below. Her recording showed Linda leveraged off known multiplication facts and used a combination of multiplication and addition to solve a division problem.

$$\begin{array}{l}
 + 6 \times 3 = 18 \\
 + 3 \times 3 = 9 \\
 \hline
 18 + 9 = 27 \\
 \hline
 27 + 45 = 72
 \end{array}$$

Figure 172: Linda recorded her partial working for finding  $72 \div 3 = 24$  (16 August)

Linda interpreted questioning to mean that her answer was incorrect. Her explanation showed that she regarded using the distributive property was easy, the “cheat” way, and was not as valued as other methods such as splitting factors multiplicatively, e.g.  $72 \div 9 = 8$  so  $72 \div 3 = 24$ . Her inaccurate recall of  $8 \times 8 = 72$  restricted her potential use of alternative strategies.

I: How did you get \$24?

L: Oh, have I got that one wrong?

I: No. Just tell me what you did.

L: I know that eight times eight equals 72 (looking for an application of the associative property). But it needs to be into three. So what I did was... I did it the cheat way and I broke it all the way down and gave them twenty each. And each one of those I wrote down the side I add three. Then I have three left (after  $3 \times 3 = 9$ ) so I add one and get 24 between four... Oh, I didn't answer it properly (recognises the calculation should have been  $72 \div 4 = \square$ ).

This scenario suggests that simply asking Linda to anticipate a physical process, while worthy in itself, was not sufficient. A prolonged period of engagement with the process to make anticipation more efficient was required to create theorems that were available for further application.

Her case study data suggested that Linda became more disposed towards anticipating processes and using that anticipation as a basis of finding new results as the year progressed. Even by the end of the year, her willingness to anticipate was dependent on secure and trusted knowledge. Linda did not remember information easily so learning was a doubly fraught process. Lack of access to plentiful knowledge elements restricted her ability to anticipate actions even when she imagined the required operations. In turn, the lack of application of anticipated actions to unfamiliar situations restricted her independent generation of more knowledge elements.

### 10.2.5 CO-ORDINATION CLASSES

Linda's progress on multiplicative thinking and proportional reasoning was an excellent fit with co-ordination class theory. While Linda knew more and could solve a broader range of problems by the end of the year, her progress was relatively slow in comparison to other students. This was foreseeable given her weak initial knowledge and the complex demands of knowledge co-ordination in these domains.

Linda was highly susceptible to variations in context and task variables. She based her preferred model of fractions on images of circular regions. Linda used the model to identify equivalence relations and wrote  $\frac{12}{12} = 1$  and  $\frac{6}{12} = \frac{1}{2}$  (7 May). Asked if  $\frac{12}{4}$  was another name for three she relied on an image of circular regions.

3. He thinks that  $\frac{12}{4}$  is another name for 3. Is he right? Explain why. *yes because 3 guitars 4 pizzas that have 4 quarters make 12.*

Figure 173: Linda explained why twelve-quarters was the same as three wholes (9 May)

Yet she could not replicate the same thinking with a length model when asked to place improper fractions on a number line.

In another instance at the end of Term Four Linda attempted two problems involving rates. The first of these problems was “Josh can run 4 laps in 3 minutes. If he doesn’t get tired how many laps can he run in 15 minutes?” Linda began a table of rate pairs to solve the problem. There were signs of both repeated addition and multiplication in her recording. While she co-ordinated the rate pairs Linda did not realise that eight laps in six minutes and 12 laps in nine minutes could be combined to give 20 laps in 15 minutes.

8 laps 6 min  
12 laps 9 min

4 8 12

$4 \times 3 = 12$

Figure 174: Linda created a rate table (19 November)

The second rate problem involved a context about the price of apples. Linda showed strong connection to food and money contexts previously. In this situation she did not know how to compare the price of four apples for three dollars with five apples for four dollars. Given a calculator, Linda keyed in  $4 \div 5 = 0.8$ .

I: What does that mean?

L: 80 cents (without certainty)?

While connecting 0.8 with 80 cents Linda did not recognise one apple to \$0.80 as the unit rate. In contrast to co-ordinating rate pairs in the first problem Linda attended separately to both measures, number of apples or amount of money, in the second.

Conflicting ideas co-existed without any apparent recognition by Linda of the inconsistency. For example, she named pieces of a cake as thirds because there were three pieces then immediately treated the pieces as quarters to share out jelly beans.

She could visualize quarters as pieces of circles but wrote  $\frac{1}{4} + \frac{1}{4} = \frac{2}{8}$  (7 December).

She recognized the need for equal parts when partitioning a length but attended to the number of parts rather than their equality in naming circular parts, i.e. three parts meant thirds. In her final PAT Linda did not apply two-thirds as an operator to

correctly locate the fraction on a vertical wall divided into nine divisions after being successful previously with operator problems.

The co-ordination of multiple inferences and pieces of knowledge presented considerable difficulty to Linda. Regularly the demands of co-ordination on complex tasks were too high for her working memory. For example, in May she attempted to locate  $\frac{28}{6}$  in relation to whole numbers. She knew that non-unit fractions were iterations of unit fractions so Linda carried out repeated subtraction  $28 - 6 - 6 - 6 \dots$ . In doing so she was unable to track the number of one units created as she carried out the process. The teacher noted that Linda's lack of fluency with whole number multiplication prohibited her from seeing patterns in equivalent fractions and in calculating the size of fractions (7 May).

Co-ordination of words, fraction symbols and the quantities to which they referred required lengthy learning time. At first, she associated the words and symbols for some unit fractions with partitions of ones. By the end of the year she understood the meaning of numerator as a count and denominator as a result of equal partitioning. Linda transferred her understanding about non-unit fractions to using fractions as operators and to describing the part-whole relationships in frequencies.

Preferences dominated Linda's thinking and were often difficult to modify. For example, her application of a p-prim for more pieces therefore more quantity was very resilient even in the face of contradictory physical experience. She applied this idea consistently in quotient and frequency situations. More pieces of pizza or more goals meant more pizza and better shooting respectively.

The first data about Linda's knowledge of fractions in frequency contexts was on July 16 when she stated who was the better shot in basketball, Stephen with seven out of 10 shots in or Jacob with nine shots out of 15. She concentrated on the number of goals scored by each player without consideration of their total number of shots. This aligned to her concentration on the number of parts (numerator) in comparing fractions without considering the size of the parts (denominators). Her recording showed she knew how to write frequencies as fractions.

9/15 Jacob because he go 9 and Stephen got 7  
Stephen 7/16

**Figure 175: Linda represented proportions as fractions (16 July)**

Linda solved a ratio problem in her interview at the end of Term Three (16 August). The problem compared ratios of orange and apple juice. Instead of comparing 3:3 and 1:2, Linda compared the total amount of apple juice with the total amount of orange juice. She wrote, "Apple because there is 5 apples (apples)."

The more pieces therefore more pizza p-prim manifested in her approach to quotient problems. At the end of Term One Linda's interview (26 March) showed that improved strategies for whole number calculation had not impacted significantly on

her approach to fractions as quotient problems. She compared the shares of three boys with two pizzas with four girls with three pizzas.

L: I cut the pizzas into fourths. So they get one piece from that pizza, one piece from that pizza, and one piece from that pizza. So girls get more.

I: But what do boys get?

L: I haven't worked out the boys but I'm certain they will get less than the girls because there are only two pizzas.

Other cueing preferences were strong. Additive thinking dominated her approaches to multiplication and division problems, her partitioning strategies with continuous quantities, and her attempts to extend patterns in algebra instruction. She spotted elaborate recursion patterns in tables and equation sets. In the equation set in Figure 176 she was also able to see relationships across the equations in writing an equation in the set that was a long way down.

All of the expectations of co-ordination class theory were in evidence with Linda. Situational variation, co-existing conflicting knowledge, cueing preferences and difficulties with co-ordinating knowledge elements were commonplace. By the end of the year Linda had emerging co-ordination classes for multiplication and division, and the sub-constructs of fractions. The process of development promised to be lengthy.

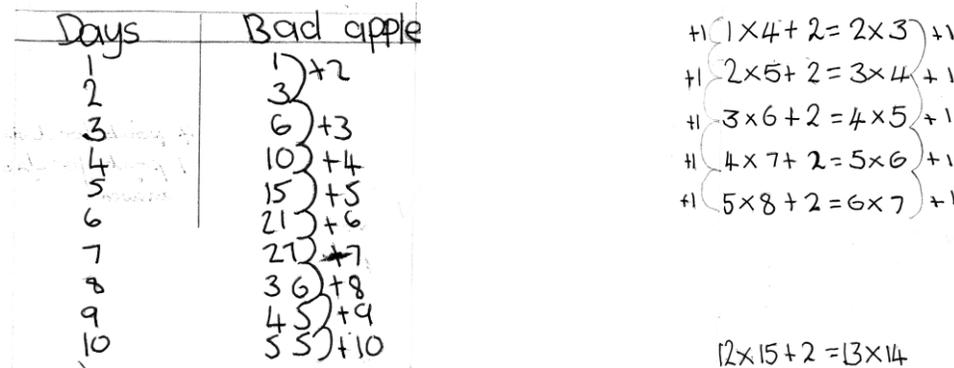


Figure 176: Linda used recursive methods based on addition (2 August)

### 10.2.6 SPAN AND ALIGNMENT

At the beginning of the year Linda reliably used additive thinking to derive unknown multiplication facts. Where appropriate fact knowledge existed, she also used the distributive property, and doubling and halving. Her sub-constructs for fractions were poorly developed. Linda knew how to use halving to find equal measures and that more parts resulted in smaller equal shares. She transferred halving based strategies to quotient problems. No data was available about her knowledge of rates and ratios.

Her data supported a conjecture that Linda’s additive thinking and halving strategies related closely at the time (see Figure 177).

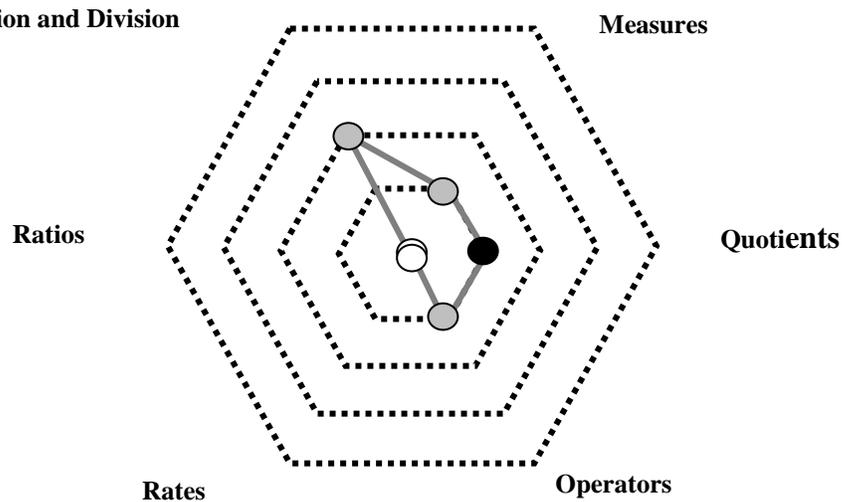


Figure 177: Learning Trajectory Map: Linda beginning of Term One

The instructional focus on whole number operations in Term One contributed to broadening of Linda’s strategies, including improved knowledge of basic multiplication facts and division by multiplicative and additive build-up. This was not evident from the graph as Linda’s strategies were not reliable. Partial alignment shows as grey circles.

There was limited transfer of improved whole number strategies to the fraction sub-constructs. Linda improved her understanding of fractions as measures to include the meaning of numerator and denominator and determining the size of improper fractions. She also learned how to find a non-unit fraction of a quantity (operator) though her strategies were constrained by limited reliability in her whole number calculations.

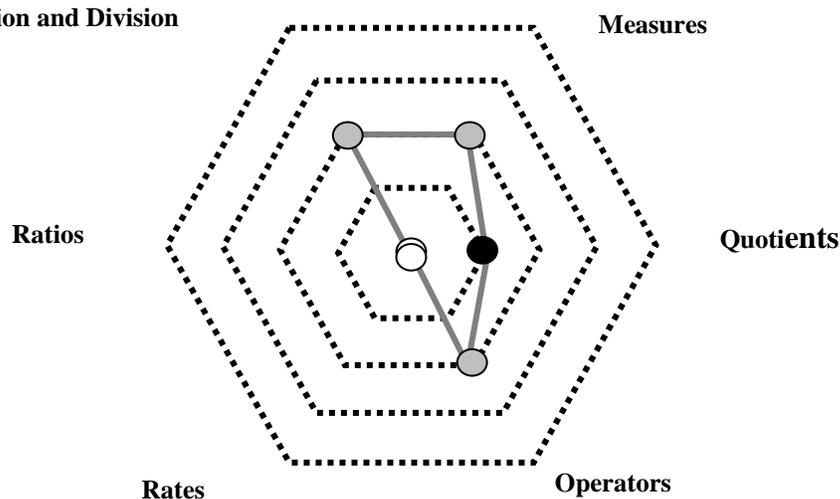


Figure 178: Learning Trajectory Map: Linda end of Term One

Linda's maps for the end of Term One and end of Term Three (Figures 177 and 178) were almost identical. There was no evidence of substantive shift in span or alignment in that time. Attempts to teach her about equivalent fractions as equal measures did not work. Linda had forgotten her previous knowledge that a non-unit fraction was composed of iterations of a unit fraction. There was no assessment of her retention of understanding about the size of improper fractions. Retention was unlikely.

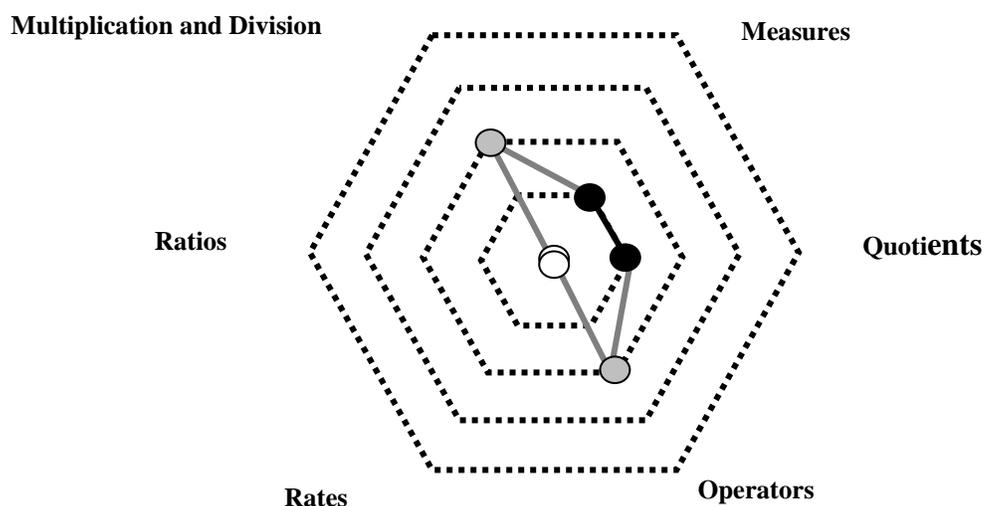


Figure 179: Learning Trajectory Map: Linda end of Term Three

By the end of Term Four, targeted instruction about fractions was associated with developments of the measure, operator and rate sub-constructs (see Figure 180). Linda relearned the iterative unit interpretation of non-unit fractions and applied this concept to using non-unit fractions as multiplicative operators. She understood replication of rates by addition. There was no growth in her knowledge of the quotient construct but some evidence that Linda understood the fractional part-whole relationships in frequency ratios.

The simultaneous view of Linda's maps showed that she made limited progress in both multiplicative thinking with whole numbers and proportional reasoning (see Figure 181). There was little impact on fraction sub-constructs until Linda understood and gained greater mental computation fluency with multiplication and division of whole numbers. Her understanding of measures regressed between the end of Term One and the end of Term Three then resurfaced in Term Four. A feature of Linda's graphs was her "spiky" profile. The sub-constructs did not appear to develop in harmony indicating that Linda did not readily connect across sub-constructs.

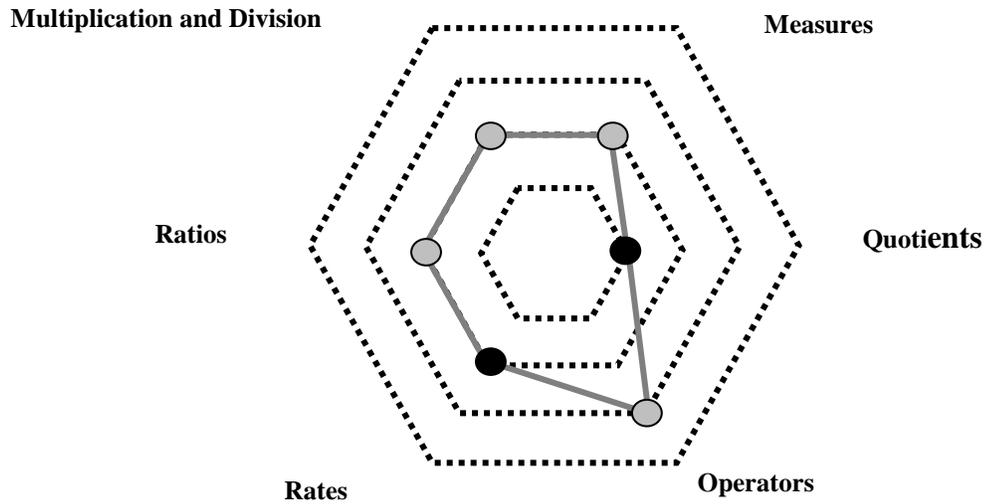


Figure 180: Learning Trajectory Map: Linda end of Term Four

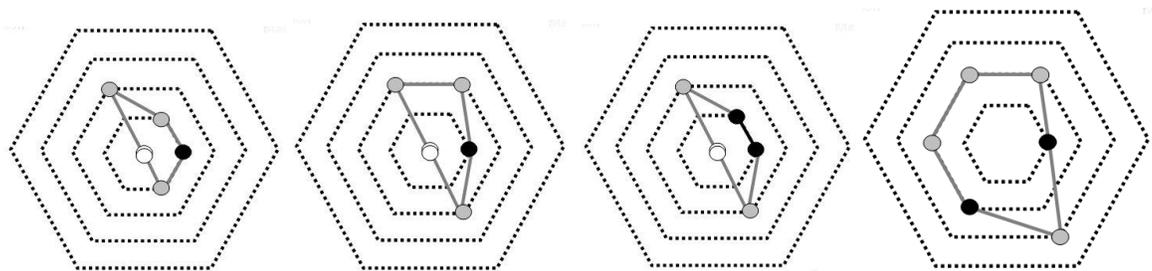


Figure 181: Simultaneous view of Linda's learning trajectory maps

## 10.3 Simon

### 10.3.1 PERSONAL CHARACTERISTICS

Simon was 12 years nine months old at the beginning of the school year. He classified himself as Maori. Simon's twin sister Odette was also in the class.

Simon listed his favourite interests as sports and reading. The sports included rugby, soccer, table tennis and basketball. He was the school tennis champion. Simon also enjoyed dance and won the opportunity to train with a company during the school year.

Simon was highly motivated, curious and able. He enjoyed intellectual challenge and any potential competition that went with it. Simon was modest and very popular with his peers. He related easily to both boys and girls and preferred friendships with other high-achieving students.

### 10.3.2 INITIAL TESTS

Norm-referenced tests of mathematics used at the beginning of the year suggested that Simon was working at a level marginally above average for his class level. In an asTTle test (1 March) he was rated at 4P for number, meaning that he was proficient in number knowledge and operations at level four of the National Curriculum. Level four was considered the average target level for year eight, Simon's class at the beginning of the year. A PAT (21 February) found a similar level of achievement. Simon rated at stanine six, placing his achievement in the top 40% of students for his class.

He showed strength in most whole number calculations, sequential patterning and problem solving with simple decimals, percentages and integers. His incorrect answers were to problems that involved more complex decimal, percentage and fraction relationships, a difficult rate, and extended whole number place value.

### 10.3.3 INITIAL NUMBER INTERVIEWS

The initial GloSS and number interviews revealed that Simon's understandings of proportional reasoning were out of phase. He applied equivalence fluently in quotient situations but did not transfer his knowledge to other situations such as finding a fraction between two fractions (measures). In the example below, Simon compared the shares for five boys with three pizzas and three girls with two pizzas. His answer was complete. He applied the quotient theorem to establish the shares, e.g.  $3 \div 5 = \frac{3}{5}$ , and converted both fractions to equivalent form to establish which was larger.

The image shows handwritten mathematical work. At the top left is the fraction  $\frac{3}{5}$  with an arrow pointing to the calculation  $\frac{2 \times 5}{3 \times 5} = \frac{10}{15}$ . At the top right is the fraction  $\frac{2}{3}$  with an arrow pointing to the calculation  $\frac{3 \times 3}{5 \times 3} = \frac{9}{15}$ . A large arrow points from these two calculations to the text: " $\frac{10}{15}$  is bigger girl gets more."

Figure 182: Simon solved a quotient problem using equivalent fractions (21 February)

Simon did not see decimals and percentages as related to equivalent fractions and had little knowledge of the part-whole relationships in ratios. Simon preferred to find unit rates and avoided decimal place value by using whole numbers.

For example, Simon used a unit rate strategy to solve the problem, "Hamish can bake 48 cakes in 16 hours. How many cakes can he bake in nine hours?"

Simon used a division algorithm to find  $48 \div 16 = 3$  cakes per hour, then multiplied by nine to find the correct answer (see Figure 183).

$$16 \overline{)48}^3 \quad 3 \times 9 \text{ hours} = 27$$

$$27 \text{ cakes}$$

Figure 183: Simon used a unit rate strategy (21 February)

Simon's belief was that pattern existed in the symbols and that number problems had a simple elegant symbolic solution. He relied heavily on imaging written algorithms for mental calculations and many of these calculations were incorrect. For example, Simon compared the engine capacities of two cars. His answer was  $4.2 - 1.29 = 3.09$  due to place value confusion with the hundredths. Simon seemed content to apply rules with little understanding of how or why these rules worked. In a calf context, he found nine out of 30 as a percentage. His strategy suggested that he equated percentage with difference (see Figure 184).

$$\begin{array}{r} 2 \\ 30.00 \\ - 09.00 \\ \hline 21.00 \\ 21\% \end{array}$$

Figure 184: Simon calculated a percentage as a difference (21 February)

Questioning showed that Simon knew that percentages had something to do with one hundred and that he relied on an algorithmic procedure.

I: So what do you usually do work out problems that have percentage in them? What do you do?

S: With a dot. So I move it back.

Simon's profile at the beginning of the year was of a student who knew many isolated knowledge elements but had not learned to apply his plentiful knowledge resources to many complex types of problem. While he knew all of his basic facts for addition, subtraction, multiplication and division, Simon was unable to apply these facts reliably to mental calculation with whole numbers and decimals.

#### 10.3.4 PROCESSES TO OBJECTS

Simon's gain in achievement during the school year was significant by any of the measures used in this study. By the end of the year, his performance on standardised norm-referenced tests placed him in the top five percent of students for his class. His story provided a detailed case study of the construction of knowledge in a complex

domain, proportional reasoning. The anticipation of processes and construction of theorems was a significant feature of Simon's mathematical behaviour.

He appeared to have unyielding faith in the existence of pattern and considerable confidence in his ability to find it. Simon paid minor attention to actions on physical or imaged objects unless provoked; only enough until he felt that anticipation of that action was obvious. For example, after two instances of partitioning decimats to anticipate fraction to decimal conversions Simon abandoned physical action. From then he relied on symbolic pattern to find his answers (see Figure 185).

Symbols were Simon's language of choice. He saw relationships in equations and expressions that allowed him to derive new results without any obvious mapping to physical or imaged actions. For example, the Teaching Diary of 1 August recorded that Simon recognised the reciprocal symbolic pattern in division of one by a fraction after one example supported with fraction manipulatives,  $1 \div \frac{3}{5} = \frac{5}{3}$ . He subsequently applied the division by a fraction algorithm, invert and multiply, to a variety of contexts correctly (see Figure 186).

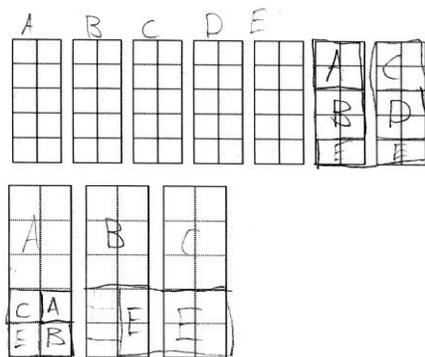


Figure 185: Simon drew decimat models for  $7 \div 5 = 1.4$  and  $3 \div 4 = 0.75$  (15 March)



5. Alana has two-thirds of a tank of petrol left. It takes four-ninths of a tank to drive to and from Tauranga. How many return trips can she make?

$$\frac{2}{3} \div \frac{4}{9} = \frac{18}{12} = \frac{6}{4} = 1.5 \quad \checkmark$$

Figure 186: Simon solved a problem by dividing a fraction by a fraction (3 August)

In another episode, practice of multiplication of fractions became trivial as he saw the pattern of multiplying numerators and multiplying denominators after a few examples with an array model. Simon built theorem upon theorem on the basis of trust. Sometimes his trust was unwarranted. Sometimes it was misplaced. During Term One he retained a preference for finding the fraction between two fractions by adding numerators and denominators, e.g. between  $\frac{2}{3}$  and  $\frac{3}{4}$  lay  $\frac{5}{7}$ . Simon knew the size

relations looked reasonable. A proof of the correctness of the theorem was beyond Simon at this stage but a few successful examples were sufficient for him to establish trust in it. Absence of verification did not prohibit Simon from trusting a conjecture until the availability of evidence to the contrary. He also displayed confidence in measures. Simon recognised that measures of centrality, median and mean, allowed for comparison of distributions of different samples and that percentages were an easy measure for comparing ratios.

Alongside a disposition that mathematics was about pattern and that new results could be built on old, Simon had strong mechanisms for validation. He knew that there was consistency between physical reality and the symbols that represented it and expected consistency between the symbols themselves, particularly around reasonableness of answer size. On one occasion, Simon thought that adding the same measure to each side enlarged a triangle while preserving its similarity. After drawing three triangles based on additive difference, he accepted that multiplicative thinking applied.

At times validation through physical action enhanced Simon's understanding of theorems he already held to be true. This occurred in his early work with equivalent fractions. Simon had a trusted procedure for finding equivalent fractions but did not appreciate them as representing the same number.

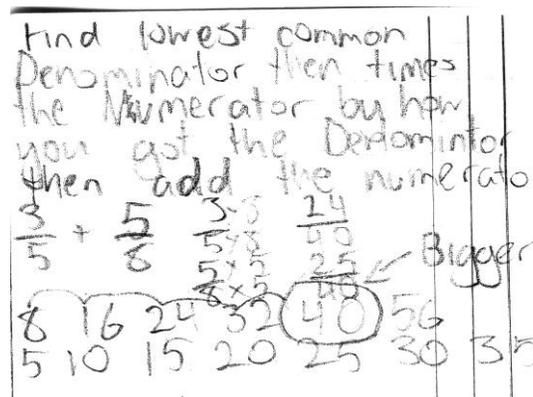
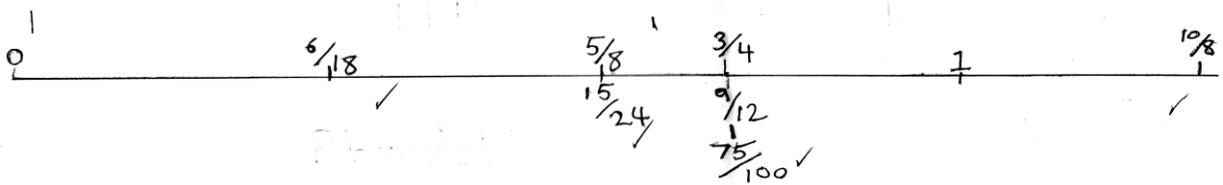


Figure 187: Simon described his method for finding equivalent fractions (5 March)

Work with physically splitting fixed ones enhanced his understanding of equivalent fractions as the same number. Initially Simon found co-ordination of the splitting difficult but over time made sense of the symbolic consistency that multiplying numerator and denominator by the same factor was like multiplying the fraction by one. He recognised that equivalent fractions represented the same number.



**Figure 188: Simon created a fraction number line to scale (7 March)**

Simon's case study provided many examples of anticipated processes becoming objects for concept creation. At the start of the year, he applied equivalent fractions to quotients. During the year, he also applied equivalence to the addition and subtraction of fractions, to percentages as proportions, to division by a fraction and to finding unknown operators between two whole numbers. By the year end, he was yet to fully connect equivalent fractions to the properties of operations with fractions and decimals. He did not see that  $\frac{1}{3}$  of 96 kilometres was less than  $\frac{8}{12}$  of 210 kilometres despite noting the equivalence of  $\frac{1}{3}$  and  $\frac{4}{12}$  saying "they (Hohepas) have  $\frac{2}{3}$  and they (Smiths) have  $\frac{1}{3}$  left to go". He calculated  $\frac{1}{3}$  of 210 km and  $\frac{2}{3}$  of 96 km correctly. Simon did not notice that the distances are comparable using doubling and halving of factors, i.e.  $\frac{1}{3} \times 210 = \frac{2}{3} \times 105 > \frac{2}{3} \times 96$ . Treating fractions as numbers to which the properties of operations applied, as they did to whole numbers, was not apparent to Simon at this point.

This case study supported the idea that anticipation of process and the creation of theorems or objects for further thought are critical to the development of mathematical thinking. However, it also suggested that Simon needed only to trust his theorems to think with them. Alongside the power of his belief in mathematics as a game played in his mind, stripped away from but isomorphic with actions on physical objects, Simon believed in consistency of the evidence. He readily adapted his ideas from experience.

### 10.3.5 CO-ORDINATION CLASSES

Simon's growth in understanding throughout the year represented an improvement in both the span and alignment of his co-ordination classes for proportional reasoning. He enhanced both the range of situations to which he applied proportional reasoning and the consistency with which he obtained correct solutions.

Many examples of variation were evident as Simon constructed mathematical models for situations he identified as similar. Situational variation was not just a product of different conditions and contexts. It often resulted from Simon's choice of action, the availability of knowledge he needed, and the co-ordination of sub-tasks along the way. Ratio and rate situations provide an excellent example. Kieren (1980, 1988, 1993) classified ratios and rates under the same sub-construct to reflect their structural similarity. There was little evidence that Simon saw ratios and rates as structurally similar, at least until the end of the year.

In Term One Simon grappled with the appropriateness of additive or multiplicative relationships as the model for ratios and rates. Initial assessment showed he preferred to convert the given rate pair to a unit rate and use scaling strategies. In early work on frequencies he applied an additive model incorrectly when comparing two shooters with frequencies of 12 out of 18 ( $\frac{12}{18}$ ) and 15 out of 24 ( $\frac{15}{24}$ ).

Jee hyun better by  
three shots cause  
add six to eighteen  
is 24 so add six  
to 12 which equals  
 $18, 15/18 = 3$

Figure 189: Simon compared two frequencies additively (28 February)

Connection with equivalent fractions sometimes allowed Simon to recognise multiplicative relationships as appropriate in situations, sometimes it did not. On 8 March, he compared two basketball shooters in the following scenario, “Jessie gets 32 out of 40 shots in, Rachel gets 39 out of 50 in.” He changed both frequencies into equivalent fractions with denominators of 100 and renamed the difference as a fraction out of 50.

$4/5$  Jee       $39/50$  Rebecca  
 $80/100$  Jee     $78/100$  Rebecca  
Jee by  $1/50$  ✓

Figure 190: Simon compared two shooters using equivalent fractions

Simon continued to find difficulty in subtle qualitative comparisons, for example, “Is 18 out of 24 better or worse than 18 out of 25?” (12 March). He also struggled with the homogeneity of equivalent ratios and rates, and sometimes applied additive rather than multiplicative thinking. For example, in a horizontal bungee context designed to develop percentage concepts he did not accept that the cord stretched uniformly (13 March). He equated 80:100 with 55:75 and 30:50 as the cord reduced in length.

Simon’s choice of model for ratios and rates was variable throughout terms two and three. He treated rates additively but ratios generally multiplicatively. There were moments of revelation. Initially Simon believed that a triangle enlarged by adding the same measure to each side. After drawing a few examples he realised that additive models resulted in dissimilar triangles and converted to multiplicative thinking with conviction (Teaching Diary 15 May).

There were also indications that Simon recognised the constant of proportionality in ratios. At first he derived the constant using common factors in a pseudo-additive way. In describing a cement mixture, 24 tonnes gravel: 9 tonnes cement: 15 tonnes water, he noted that the gravel was eight-thirds heavier than the cement (18 July).

During Term Four, Simon successfully applied the ratios within and between similar triangles to trigonometry. On 21 November he found the height of a tree in Figure 191.

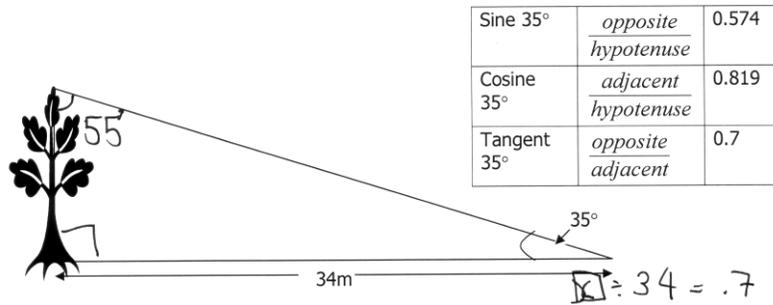


Figure 191: Simon solved a trigonometry problem about the height of a tree

S: So it's opposite and adjacent, that's tangent. (Writes  $x \div 34 = 0.7$ ) I need to work backwards. (Keys in  $34 \div 0.7 = 48.571$ ) No.

I: So how will you undo the divided by 34 on the left-hand side?

S: 34 times 0.7 equals 23.8. That's right.

Successful transfer with ratios and rates mixed with unsuccessful transfer. In some inverse rate situations, Simon recognised conservation of product. This was particularly true in contexts where the rates involved actions such as painting murals and running laps. During group lessons Simon recognised that balance situations had similar structure but did not retain this understanding into the end of term interview (18 August).

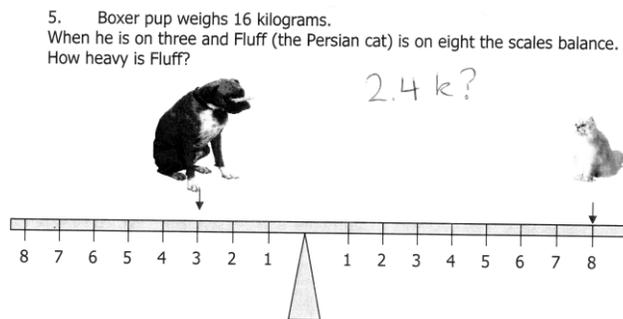


Figure 192: Simon speculated about the answer to a balance problem

S: 2.4kg.

I: How did you get that?

S:  $8 \times 3 = 24$  so 2.4.

Graphing of rates was introduced in Term Three. Simon saw that a point represented a rate pair but did not recognise that slope represented rate of change, the constant of proportionality. In the end of Term Three interview he compared the speed of three runners displayed as points on a number plane. Asked who was the fastest runner Simon wrote, “Odette ran the fastest ’cause she did more laps but Ben ran MORE ’cause he did more minutes.” His confusion over which measure to attend to, number of laps, time or speed, was further revealed in his interview.

I: Have you completely discounted Jee? Why can’t she be the fastest?

S: Because she hasn’t run for as many minutes or for as many laps.

I: But the question is not for how long she has run, it is how fast she’s run. How do you decide how fast she has run?

S: Measure how many laps she’s done.

I: How do you measure how fast a car goes?

S: K’s per hour...oh so minutes per lap.

He successfully solved the problem by converting each rate pair to a unit rate almost in avoidance of dealing with both measures simultaneously. By the end of Term Four, Simon recognised that comparison of rates involved considering both measures but still saw no connection between slope and unit rate. Again, he converted to unit rates to compare rate pairs with hours and money earned shown as points on a graph.

I: Is there any way you could look at the graph and tell without using your calculator?

S: Yes. Ben has worked a lot more hours and only got a little bit more money (uses fingers to measure the gap).

Simon’s selection of solution path for rate and ratio problems was susceptible to situational variables. The availability of a calculator in his end of year interview (21 November) appeared to promote Simon’s zeal to use it, often in speculative ways. He attempted the problem, “Fifteen pineapples cost 10 dollars. At the same rate how much would you pay for six pineapples?”

S: (Keys in  $15 \div 10 = 1.5$ )

I: I’d like you to use your head to work this out.

S: You could divide six by 15 and divide 10 (incorrect operation) by that.

I: Yes, you could.

S: (Gets lost in the calculation)

I: Look at the relationship between pineapples and dollars. Does a pineapple cost more or less than a dollar?

S: More, no less.

I: So how many pineapples do you get for a dollar?

S: One point five, oh (Counts up in 1.5s co-ordinating fingers as dollars). That's four dollars.

I: Now you know the answer can you think of another way to work it out?

S: (Records  $\frac{15}{10} \frac{6}{4} \frac{3}{2} 1 \frac{1}{3}$ )

I: So what does this mean in terms of our dollars and pineapples?

S: Three pineapples cost two dollars.

I: And that makes one and a third?

S: (Self correcting) oh, one and a half (recognising the connection to the unit rate).

There was complexity and flexibility in the way Simon thought about rates in this situation although his initial strategy was procedural. While he had access to unit rate-based strategies, he also treated other rates as composites with which he could work, for example 3:2. Simon used his knowledge of equivalent fractions and common factors.

Simon's development in his understanding of ratios and rates provides rich insight about situational variation. He did not treat ratios and rates as structurally similar because he did not see them as such. Additive and multiplicative relations competed for preference over time and their selection was unpredictable. Simon sought to develop theorems but these theorems were not explicit and were situation specific. For example, he eventually saw enlargement as an application of multiplicative relations but did not connect unit rate with slope of a graph.

New contexts, new models and new representations were understood through connection to existing ideas that were trusted. Simon trusted readily. Knowing the expression of speed of a car in kilometres per hour helped him see the corresponding rate in laps per minute. This connection perhaps helped him to consider both

measures in comparing rates of pay. Some contexts seemed supportive of Simon seeing the underlying mathematical structure. Some situations, like balance scales, provided no phenomenological clue.

Partial acts of transfer as well as successful solutions punctuated Simon's attempts at problem solving. In-depth analysis of both types proved illuminating in understanding his development of concepts.

### 10.3.6 STRATEGIC CHOICE AND CO-ORDINATION OF KNOWLEDGE

Simon's attempts to co-ordinate his knowledge also provides interesting insights. Given the considerable capacity of his working memory, Simon's case was far from typical. Yet on many occasions, the demands of inference were too great. Strategic choice initially influenced these demands. On some occasions, Simon readily saw a solution path and enacted it. On most occasions, there was little evidence that the process of solving complex problems was pre-ordained. Rather it appeared that Simon explored a solution path to successful completion or opted for another path if the first proved fruitless.

A good example of this was his attempt to share \$250 in a ratio of 48:32 (16 July). Using common factors Simon simplified the ratio to 6:4. However, his concentration on the part-part relationship of 6:4 meant he missed the part-to-whole relationships of six-tenths and four-tenths necessary to solve the problem. Strategic choice and available knowledge acted symbiotically in the process of solution. Simon's attention to conditions in given situations influenced his strategic choice. For example, availability of a calculator shifted his attention from the structure of the operation required to the reasonableness of the answer.

Cognitive load occasionally resulted in items of knowledge readily available independently were not easily retrievable in solving complex problems. Sometimes Simon did not see that knowledge elements were useful. For example, given  $\frac{3}{8} = 0.375$  and asked to convert  $\frac{3}{24}$  to a decimal he could not see the relationship in denominators, i.e.  $\frac{3}{24}$  is  $\frac{1}{3}$  of  $\frac{3}{8}$  (16 July).

Simon showed strong cueing preferences. He commonly applied unit rate strategies and use of common factors to find equivalent fractions. He converted decimals to whole numbers to operate on them and changed decimals to percentages by "moving the dot". By Term Four, Simon embraced decimals as his numbers of choice particularly if a calculator was available. His preferences were amenable to change but were complemented rather than subsumed. For example, in Term Four he operated with decimals in division without changing them to whole numbers yet thought of \$14.00 as 1400 in a rate context.

Connecting between classes was a strong feature of Simon's thinking. That is not to say that Simon could explicitly explain borrowing from one class to inform another even though he had mathematical register to do so. These borrowings were natural acts of connection as Simon scanned his available knowledge resources for

something useful to satisfy situations. His development of a co-ordination class for percentages serves as a good example.

At the beginning of the year, Simon had little understanding of percentages as either rates or proportions. Yet Simon already possessed equivalence fraction knowledge commensurate with the measures sub-construct. After a sequence of four mini-lessons with a Slavonic abacus he readily carried out percentage conversions early in week four of Term One. For example, he wrote  $\frac{4}{5} = 80\%$  and  $\frac{13}{20} = 65\%$  (28 February).

His early attempts to solve percentage problems reflected rate build-up strategies. In his end of Term One interview (25 March) Simon expressed  $\frac{27}{45}$  as a percentage in a context about calves. He found the problem difficult because he could not scale 45 up to 100 or find a common factor. Simon attempted to build 45 up to 100 in successive increments. The cognitive load associated with double tracking the amounts contributed to arithmetic slips in his calculations.

S: It's annoying.

I: Why is it annoying?

S: Times 27 isn't very tidy.

I: Neither is 45.

S: So I just doubled both of them...90...that's 54...and nine more is 99 so I halved the nine to get 4.5 and then I need to get two more percent ...So 1% would be .04...

Simon made a strategic choice to operate on 27 out of 45 as a rate in an effort to scale it to  $x$  out of 100. In doing so, he resorted to additive thinking and did not conserve the original rate in his addition of increments.

During terms two and three Simon developed and used the connection between percentages and part-whole ratios to add to his preference of rate-based strategies.

If you are given a mark out of 40, what do you multiply it by to get the percentage?

(times it by 2.5)

Rachel bought a skirt that usually cost \$50. She got \$15 off the price. What percentage discount was that? (30% off)

Zach has 36 ewes on his farm. This year his lambing percentage was 150%. How many lambs were born on his farm? (54 lambs are born so altogether there are 90 ewes sheep) (9 May)

He also developed flexibility in treating percentages as rates. At the beginning of Term Three, Simon used strategies based on the distributive property to compare the price of jeans from two stores thereby treating percentages as operators (16 July). To calculate the price of a \$119.95 pair of jeans, with 35% off, he found 50% and 20% of \$120. He combined 50% and 20% to get 70% of \$120 and halved this to calculate the 35% discount. He estimated the price of a \$79.95 pair of jeans with 7% discount by finding 10% of the price.

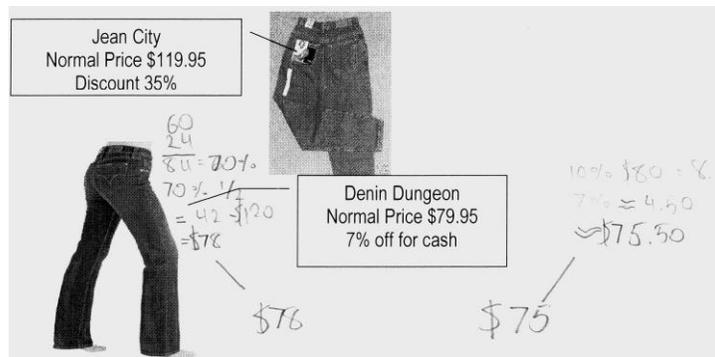


Figure 193: Simon compared the price of jeans after percentage discounts

Simon also connected quotients, fractions as measures, and ratios in some percentage situations. For example, to calculate a calving percentage he compared 485 calves (whole) to 321 cows (whole). He divided to get a decimal, which he converted to a percentage.

$$485 \div 321 = 1.5$$

$$\text{then times } 321 \text{ by } 1.5 = 481.5$$

$$= 150\%$$

Figure 194: Simon calculated a calving percentage (8 August)

In his end of Term Four interview (21 November) Simon found the whole in a frequency problem given a part and its equivalent percentage. The problem was in the form  $\frac{18}{n} \approx 67\%$  in a test question scenario. Simon revealed flexibility of strategies for solving percentage problems that involved his knowledge about percentages as numbers (measures) i.e.  $\frac{18}{n} \approx \frac{67}{100}$ , as operators, i.e. 67% of  $n = 18$ , and as ratios, i.e. 18 out of  $n$  is about equivalent to 67 out of 100. He also showed his tendency at that time to try several calculations with the key numbers in a problem in search of a reasonable answer.

S: (Enters  $18 \div 67 = 0.26$  on a calculator). That's not much. Ummm...18 right, 67% so doesn't that mean that one equals 2%. (Enters  $67 \div 18 = 3.7$ ) So 3.7 (finding unit rate).

I: Okay, so each question is worth 3.7%. So how are you going to find out how many questions there are?

S: (Enters  $100 \div 3.7 = 27.02$ )... So 27 questions (some uncertainty).

I: Well, does that check out? Can you find another way of checking that out?

S: (Enters  $27 \times 3.7 = 99.9$ )

I: Pretty close. So if you wrote 18 out of 27 as a fraction, can you just write that now? S: (Writes  $\frac{18}{27}$ ) Do you see anything about 18 out of 27?

S: (Writes  $\frac{2}{3}$ )

I: So does that check out?

S: Yes.

I: So is two-thirds the same as 67%? Why does the teacher put 67%?

S: It's 33.3%...no 66.6%. It's rounded up.

Of the case studies in this thesis, Simon's contains the fullest description of construction for a co-ordination class about percentages by connection of sub-constructs. His expertise rested on flexibility in how he treated percentages. Simon had the capacity to view percentages as operators to which the distributive property applied. In these situations, he applied both between and within strategies to percentages as rates. In other situations, Simon treated percentages as equivalent ratio fractions in both part-whole (frequency) and whole-whole (comparison) situations subject to multiplication and division by common factors. He was also able to connect quotients to determine the appropriate operation to solve whole-unknown percentage problems.

Not all co-ordination between classes was well-developed. Simon did not readily connect fractions as measures with operators, or rates as slopes of linear graphs by the end of the year. In line with co-ordination class theory it seemed that some classes were sufficient for some problems and sufficient for co-ordination with other classes. Some classes were not sufficient. The construction of classes was never contained and complete.

### 10.3.7 SPAN AND ALIGNMENT

Simon began the year with strong knowledge for most fraction sub-constructs that resulted in broad span of situations to which he could apply proportional reasoning (see Figure 195). He already knew how to find equivalent fractions by multiplication and division and applied equivalence in conjunction with the quotient theorem to compare shares in sharing situations. Development of his mental strategies for multiplication and division of whole numbers was incomplete as he relied on imaging written algorithms. Evidence of his understanding of ratios was limited as he was unable to identify and use the part-whole fractions. He reliably found equivalent rates and non-unit fractions of quantities by multiplication and division.

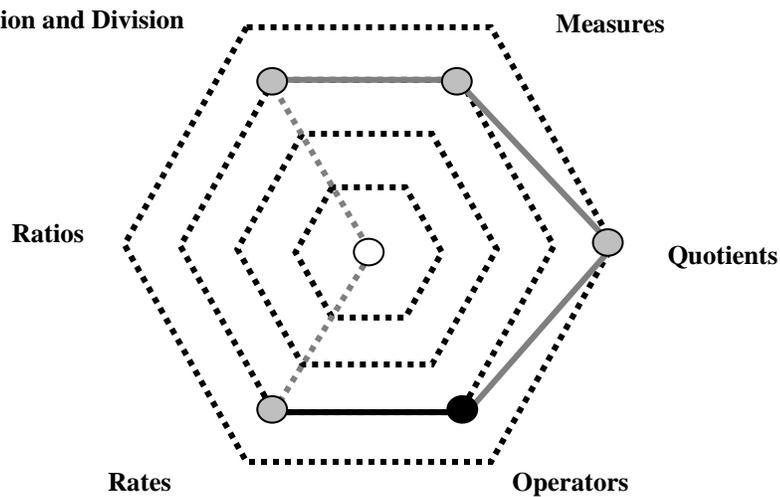


Figure 195: Learning Trajectory Map: Simon Term One

During Term One improvement in Simon's span for proportional reasoning occurred in the ratio and measure sub-constructs (see Figure 196). He applied his understanding of equivalent fractions as the same measure to ordering, comparing and measuring with fractions. He identified the part-whole fractions in ratios. There was little evidence in growth of the other sub-constructs though his mental strategies for multiplication and division of whole numbers developed. His reliability of response (alignment) remained stable in fraction as operator situations.

During terms two and three Simon made further gains in span (see Figure 197). His solutions to problems involving fractions as measures and comparison of quotients became reliable. He understood the need for a common referent in ratio situations and used multiplication and division to scale rates and ratios. Simon also used his strong knowledge of common factors to identify the unknown operator between two whole numbers.

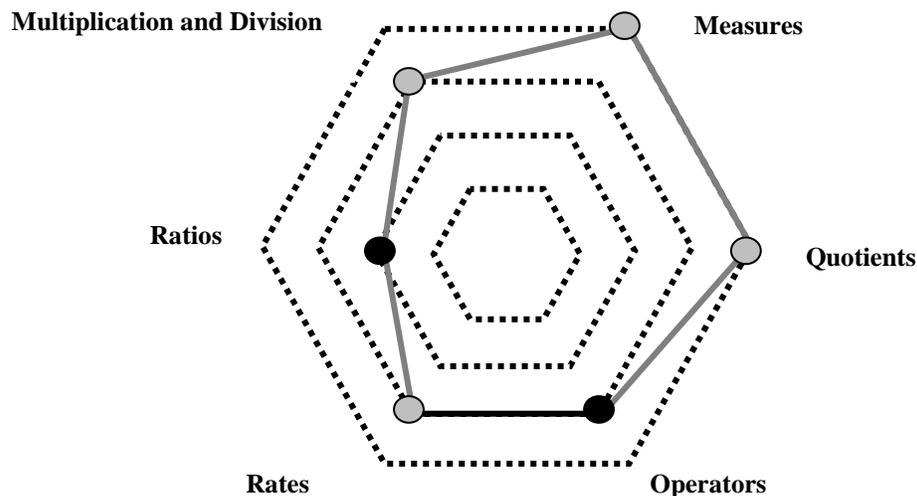


Figure 196: Learning Trajectory Map: Simon end of Term One

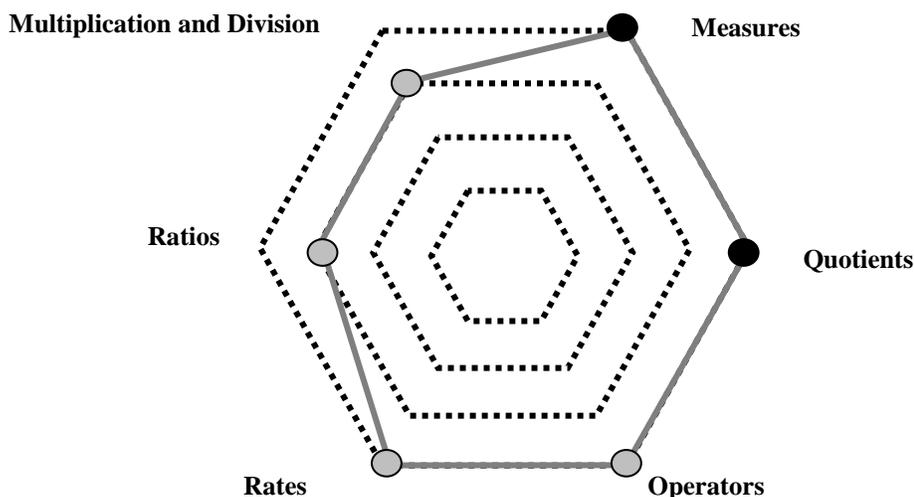


Figure 197: Learning Trajectory Map: Simon end of Term Three

There was little change to Simon’s learning trajectory map during Term Four. He consolidated his understanding of ratios to include whole-to-whole comparisons. Other sub-constructs appeared to be in a state of consolidation while measures and quotients seemed limited by the scope of the trajectory (see Figure 198).

A simultaneous view of the maps reveals three features of Simon’s learning. Firstly, his increase in span for multiplication and division, and proportional reasoning occurred rapidly in terms one to three. Secondly, Simon also achieved strong alignment for most sub-constructs quickly as well. That is not to say that all sub-constructs were reliable by the end of the year. His understanding of ratios and rates was not fully developed. Thirdly, Simon’s profile was well-balanced by the end of Term Three. This was possibly a result of the ceiling effect of the trajectory map but

it more likely reflected Simon's ability to co-ordinate between constructs to solve problems.

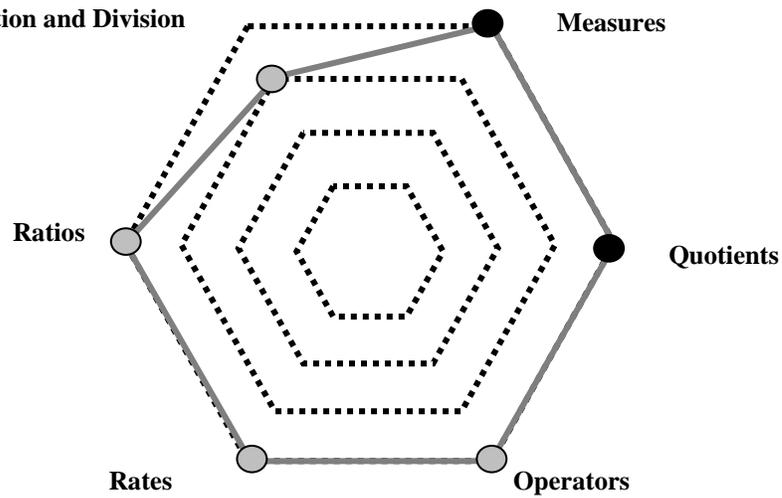


Figure 198: Learning Trajectory Map: Simon end of Term Four

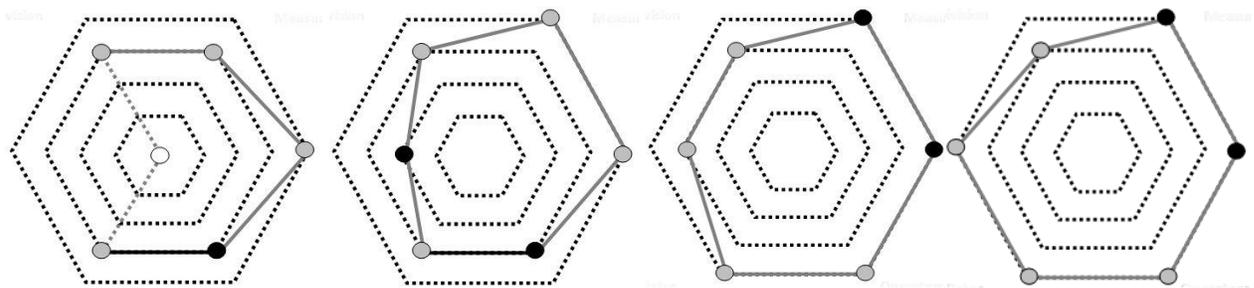


Figure 199: Simultaneous views of Simon's learning trajectory maps

The maps suggest that co-ordinated development of Kieren's (1980, 1988, 1993) sub-constructs were a useful frame to view Simon's development of proportional reasoning. The sub-constructs were good examples of co-ordination classes subject to situational variation associated with Simon's attempts to co-ordinate knowledge elements. The emerging regularity of the maps also suggests that co-ordination across the sub-constructs played a vital role in Simon's successful development of proportional reasoning.

## CHAPTER ELEVEN: DISCUSSION

### 11.1 Introduction

This study aimed to answer the following research question with two parts reflecting different but complimentary issues:

**How do students develop their multiplicative thinking and proportional reasoning?**

- a. **Does a Hypothetical Learning Trajectory (HLT) reflect the actual learning trajectory of students?**
- b. **What model of conceptual learning, object theory or co-ordination class theory, best represents the growth of multiplicative thinking and proportional reasoning, and the transfer of knowledge within and between situations?**

Section Two of this chapter addresses the first issue through contrasting similarities and differences between the Hypothetical Learning Trajectory (HLT) and the actual learning trajectories of the case study students. The purpose is to determine the extent to which the development of multiplicative thinking and proportional reasoning has a generalised growth path that holds for all learners or is variable and unpredictable for individual learners.

Section Three addresses the second issue. It summarises the findings of the study in respect to conceptual learning and discusses which model, object theory or co-ordination class theory, best describes the changes observed in students' cognition. The final section poses some implications for research and classroom practice that arise from the findings.

### 11.2 Generic Progression in Multiplicative Thinking and Proportional Reasoning

#### 11.2.1 HYPOTHETICAL LEARNING TRAJECTORY

The purpose of constructing the HLT was to create a framework for instruction and a structure for assessment of student learning. In both respects, the trajectory proved useful. It integrated many significant ideas from research about the conceptual development of multiplicative thinking and proportional reasoning. This was new ground since the research in these areas was specialised and mostly unconnected.

The breadth of the trajectory seemed adequate in providing a broad range of problem types and conceptual ideas. Students progressed broadly through the stages of the trajectory for particular sub-constructs though achievement for individual students at

points in time was often out of phase between sub-constructs. Instruction based on the trajectory was associated with gains in student achievement.

A complicating feature of the trajectory was the inclusion of different knowledge types and strategies alongside one another (Mason & Spence, 1999). This duality may be prone to misinterpretation. For example, some knowledge involves factual association, such as multiplication and division facts. Other knowledge involves processes such as finding equivalent fractions and ratios by multiplication and division. Both types of knowledge involve performance and are relatively easy to target from a pedagogical perspective. Other performance indicators in the HLT represent conceptual understanding assessed by observing the types of strategies students use during problem solving. These strategies are often described as schemes in the literature (Vergnaud, 1998; Steffe, 1994, 2003). Examples include recognising equivalent fractions as representing the same number and equivalent ratios as representing the same proportional relationship. In the quotients sub-construct the only description of phases is by strategies, such as half-based sharing or equal partitions that are indicators of conceptual understanding. A possible misinterpretation of knowledge and strategies is inappropriate teaching and assessment of different knowledge types in the same way. It is arguable that teaching factual association and processes in an ordered way is appropriate. Ordered instruction is not necessarily appropriate for conceptual understanding. For example, it may be pedagogically sound to provide problems that develop conceptual understanding but unsound to teach students to solve the problems using particular strategies. This is especially true if strategies are unsophisticated and if practising those limits more advanced conceptual development.

Finer details about knowledge within the trajectory emerged through the research. There was strong evidence that proficiency with variance and invariance properties of multiplication and division with whole numbers was foundational to proficiency with equivalent fractions. This is consistent with Lamon's (2005) central multiplicative structures and the views of Olive (1999) and Steffe (2003) that fraction schemes require children to reorganise, not displace, their whole number schemes. The trajectory did not reflect the order of this relationship. This was also true of other ordinal relationships such as unknown operators being foundational to finding constants of proportionality in rates and ratios. Understanding of fractions as measures or numbers seemed to precede applying properties of multiplication and division to fractions as ratios, operators and quotients. Students brought intuitive understanding of replicating rates from everyday life but struggled with applications that are more complex until they grasped unknown fractional operators.

The HLT represents the sub-constructs as discrete. Findings from the case studies suggest that the development of strong multiplicative thinking and proportional reasoning involves connection between sub-constructs. A possible misinterpretation of the trajectory in this respect is that situations can only ever involve one sub-construct and that students will activate the expected sub-construct in response to carefully orchestrated problems. Consideration of actual learning trajectories shows that this is certainly not the case. Rather students, both low and high achievers,

frequently used different sub-constructs from those intended by the problem structure.

### **11.2.2 ACTUAL LEARNING TRAJECTORIES**

Views of the simultaneous Learning Trajectory Maps offer insight about consistency and variability of students' learning during the year (see Figure 200). In comparing maps, it is important to recognise that factors such as teaching effect and the limits of the HLT, especially at the highest stage, account for consistency to some extent.

This may account for the similarity of shape of the maps for the high achieving students, Ben, Rachel and Simon, by the end of the year. All three students showed strong connection between sub-constructs in their solutions to problems. A more likely rationale for the regularity is that proportional reasoning involves the integration of the sub-constructs into a super co-ordination class that allows students to draw on multiple views of rational numbers in solving problems in context.

In contrast to the maps of high achieving students, those for the middle achievers, Odette and Jason, and the lower achiever, Linda, are markedly different. All three maps display irregularity in that some sub-constructs develop without corresponding advance in others. The evolution of the maps is different for all three students, which supports the idea that development of the sub-constructs is variable and unconnected in the early stages. Linda's map is the most irregular and shows an instance of regression as previously learned knowledge of a sub-construct is absent later.

Irregularity might be associated with the developments at the earlier stages of the HLT being out of phase temporally with students' actual growth paths. There is no clear pattern in the irregularities for the maps of Odette, Jason and Linda to suggest this. Growth in understanding and use of the properties of multiplication and division with whole numbers uniformly precedes significant growth in the sub-constructs for rational number. Teaching was not the cause, as the relationship was independent of the order of instruction. For Jason, absence of multiplication basic fact knowledge restricted his ability to make sense of instruction about rational numbers as measures and ratios during Term One. Similarly, for Linda, her unreliable calculation and inability to see multiplicative relationships with whole numbers restricted her capability to understand and use fractions as equivalent measures.

The final maps of Rachel, Simon and Ben all display the same shape. The upper stages of the HLT place bounds on the representation of conceptual development though the grey dots reveal a lack of consistent alignment. This suggests the emergence of generalised proportional reasoning is based on transfer between the sub-constructs. There is clear intersection between applying invariance and variance properties with multiplication of whole numbers and applying the same properties with fractions, decimals and percentages (Post, et al., 1993; Thompson & Saldanha, 2003). Only Simon made that connection. The HLT needs modification to reflect this temporal development from whole number to fractional operators.

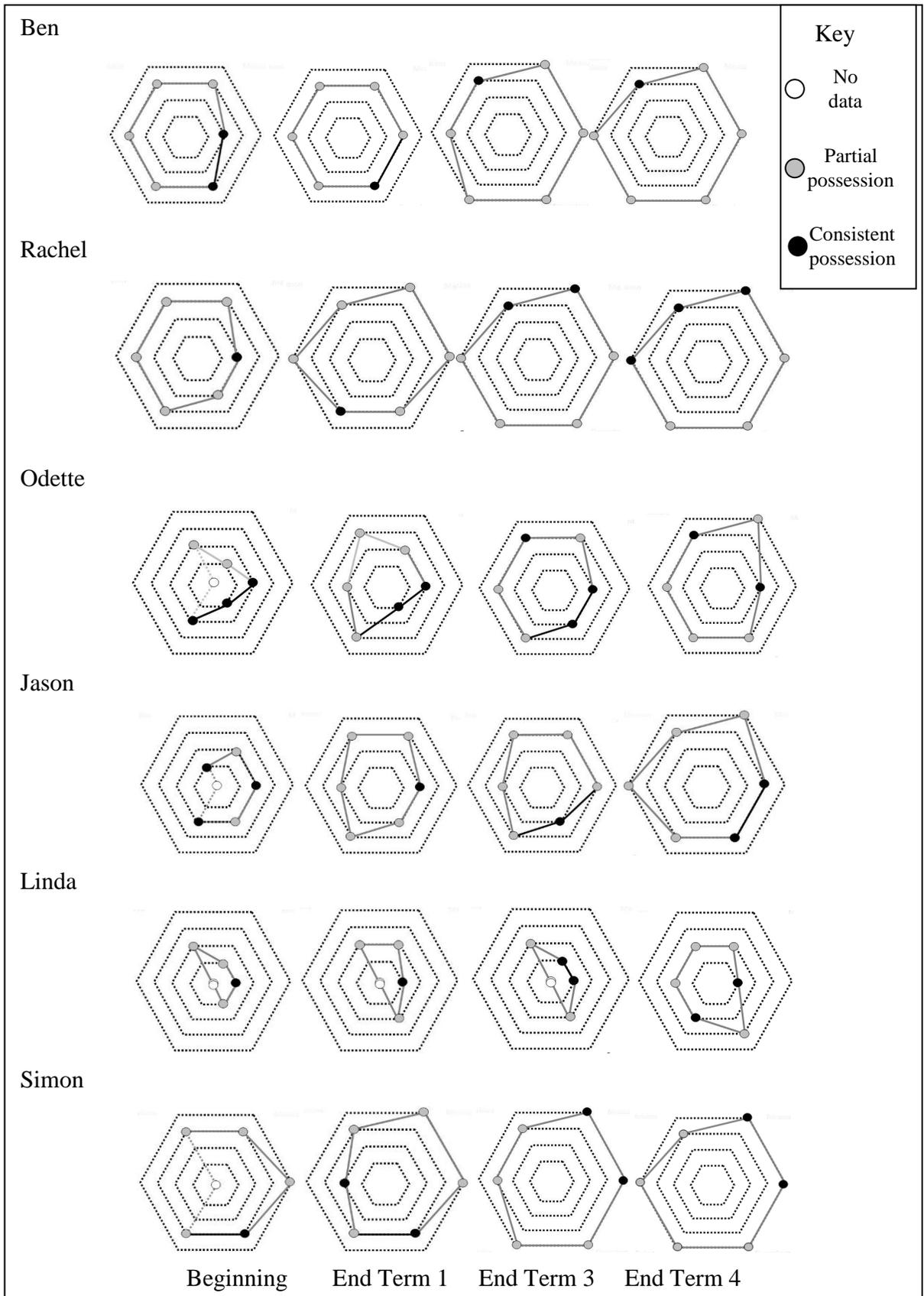


Figure 200: All learning trajectory maps viewed simultaneously

The generalised growth across all six learners suggests that proficiency with rational numbers as measures, particularly in respect of equivalent fractions as representing the same quantity, is foundational to progress in the other constructs. There is also a pattern that transfer of equivalence to quotients and operators tends to occur before corresponding developments with rates and ratios. This pattern partially reflected teaching emphasis in that students encountered measures situations at first. Teaching emphasis does not explain the relative difficulty of rates and ratios at the upper phases of the HLT as these sub-constructs received attention before quotients and operators. A more likely explanation is that quotients and operators facilitate the recognition of constants of proportionality in rates and ratios. There is evidence from the case studies of the high achieving students that they recognised unknown operators in number pairs using common factors.

### **11.2.3 ADEQUACY OF HYPOTHETICAL LEARNING TRAJECTORY**

The HLT combines multiplicative thinking with whole numbers and Kieren's sub-constructs for rational number as key dimensions. This raises issues of the adequacy of the trajectory, particularly in proportional reasoning. Lobato and Siebert's (2002) concept of situation, model and representation as nodes for transfer is useful in considering the issue of adequacy.

Alongside multiplication and division of whole numbers, Kieren's (1980; 1988; 1993) sub-constructs comprise the models in the HLT. The case studies report considerable evidence that students transferred between sub-constructs and that fluent transfer was the sign of well-developed proportional reasoning. For example, the use of rate based strategies was common with quotients problems. Yet some transfer of sub-constructs was potentially detrimental, for example, rate thinking often satisfied the demands of quotients problems and reduced learner attendance to the quotient theorem which was critical for more complex applications.

The adequacy of Kieren's sub-constructs as models, in the sense of Lobato and Siebert's types of transfer, is a matter of perspective. In terms of conceptual field theory proportional reasoning problems appear framed within individual sub-constructs or combinations of them. From a situated learning perspective, the sub-constructs are inadequate. Varying situations involving the same sub-construct seemed to require different distances of transfer for students at both an individual and collective level (DiSessa & Wagner, 2005).

For example, all the students in this study treated rates and ratios respectively in different ways. They engaged differently with part-whole and whole-whole ratios. There were clear differences in learner behaviour towards ratios as frequencies and ratios in which the part-whole relationships required inference. Probability situations involve ratios as frequencies and transfer between measures, and operators. Of the high achieving students only Simon accepted the duality of theoretical prediction and experimental variation, albeit hesitantly. Linda was unable to deal with sample proportions in a meaningful way.

From a conceptual field perspective, these differences in student response are examples of transfer between models and situations. Situational variation is anticipated by co-ordination class theory and explainable by learners' personal engagement with the complexity of readout and inference, and the knowledge co-ordination required (DiSessa, 2008; Wagner, 2006). Yet, there is merit in the parsimony of the sub-constructs in terms of elegance and simplicity.

Parsimony can come at the expense of informative detail. Some situational variation was consistent across the six case studies. To complement the learning trajectory a framework of situation types is needed similar to Greer's (1994) semantic types for multiplication and division. Adjige & Pluvineage (2007) and Alatorre & Figueras (2004, 2005) made a useful initial contribution to this work by differentiating the characteristics of proportional reasoning problems. It is also clear that the connections learners make between sub-constructs is of considerable significance in understanding the development of proportional reasoning. The Trajectory may promote an impression that the sub-constructs are discrete.

Similar issues of adequacy surround transfer between representations and between representations and models and situations. Representations include symbols, diagrams, physical materials and words. The register for proportional reasoning is broad and high-achieving students learned and used specific vocabulary such as "a b-ths" (Andrew) and "two-thirds per boy" (Rachel). This vocabulary seemed to provide them with tools to think with as well as ways to describe occurrences in a way consistent with Thomas, Davis, Gray, & Simpson's (2000) view of concept as conceived idea.

In this study consistent use was made of physical equipment and diagrams as teaching tools. Yet there was inconsistent evidence of students imaging equipment or diagrams in solving problems unless explicitly asked to do so. Linda and Odette used circular images early in the year that constrained their strategies. In contrast, Jason employed diagrams as powerful thinking tools, including a moment of insight when he compared ratios using a strip diagram late in the year.

Graphs appeared to pose interpretive challenges as representations of rate. Most case study students accepted linear graphs as indicative of constant difference but none associated slope with rate of change. Problems involving graphs of rates resulted in high achievers resorting to primitive strategies (for them), such as calculating unit rates or equalising of one measure. Progression in multiplicative thinking and proportional reasoning for the case study students was facilitated by transfer between representations, particularly symbols, models, and situations.

There was considerable evidence of students abandoning actions on equipment or images in favour of using symbols. The presence of symbolic conscience that mapped operations on symbols back to the quantities they embodied was a common disposition of Simon, Rachel, and Ben that was often absent with Linda and Odette. The use of physical equipment and diagrams was significant. High achieving students seemed to appreciate the purpose of equipment and diagrams was to validate anticipatory actions on symbols in terms of the corresponding actions on quantities. Linda frequently treated manipulation of materials as an end in itself. Most recording

by students in problem solving situations was symbolic and included algorithms, tables, equations and words. Some recording systems modelled in instruction, such as double number lines and strip diagrams, were not commonly adopted by students.

The common use of symbols across diverse situations implies transfer between representations and models. For high achieving students like Ben, Simon and Rachel symbols seemed to aid in their construction of similarity across situations, as a way to park information to ease memory load, and as markers for thinking with. Their use of fractions, decimals and percentages suggested the presence of multiple procepts (Tall, et al., 2000), embodiment of multiple processes within a single symbolic form, as explanatory. The table below summarises significant processes that seemed embodied in students' interpretation of fraction symbols. Construction of similar tables for percentages and decimals is possible.

**Table 15: Fractions as multiple procepts**

Model	Process
Measures	$\frac{a}{b}$ is $a$ iterations of $\frac{1}{b}$ and $\frac{a}{b} = \frac{na}{nb}$ or $\frac{n}{n} \times \frac{a}{b}$ by equal splitting or combining
Operator	$\frac{a}{b} \times c$ is found by $c \div b \times a$ or $c \times a \div b$ since $\div b$ establishes $\frac{1}{b}$ of $c$
Quotient	$\frac{a}{b}$ is the share (or measure) resulting from $a \div b$
Ratio	$\frac{a}{b}$ is the part-part relation between $b$ and $a$ in the ratio $a:b$ and $\frac{a}{a+b}$ is the part-whole relation of $a$ to the whole
Rate	$\frac{a}{b} M_1 / M_2$ means $a M_1$ units for every $b M_2$ units ( $M_1$ and $M_2$ are measurement spaces)

The case studies show many examples of students noticing and extending patterns they see in symbols. Yet all students, including the high achievers, had considerable difficulty connecting different symbolic representations of numbers, especially when additional inferences were required. For example, no student could use knowledge of  $\frac{3}{8} = 0.375$  to derive the decimal for  $\frac{3}{24}$  at the beginning of Term Three or use the associative property to compare two-thirds of 210 km with four-twelfths of 96 km at the end of the year. Transfer between symbolic forms was difficult for these students.

#### 11.2.4 MODIFICATION AND FUTURE RESEARCH

In summary, the HLT goes some way in providing a framework for the development of multiplicative thinking and proportional thinking. The trajectory neglects the range of situational types as commonly interpreted by students and the transfer demands of

connecting representations, particularly symbols, with models, situations and other representations. The trajectory is a simplistic growth path accompanied by an expectation that progress through it by different students will be variable.

While the global phases of the HLT appear to hold within each sub-construct, several alterations arise from the findings. The descriptors need to be schemes, anticipations of actions or “*know-tos*”. This modification removes the confusion of different knowledge types and their treatment pedagogically. A braided network approach to multiplicative strategies with whole numbers and the rational number sub-constructs has merit. This allows for key temporal and conceptual connections that are important in treating the sub-constructs as integrated rather than independent. Schemes in different sub-constructs connect temporally as knowledge elements used in more advanced schemes for other sub-constructs. Figure 201 illustrates how schemes for multiplication and division of whole numbers and iterating unit fractions facilitate schemes for both equivalent fractions as measures and fractions as operators.

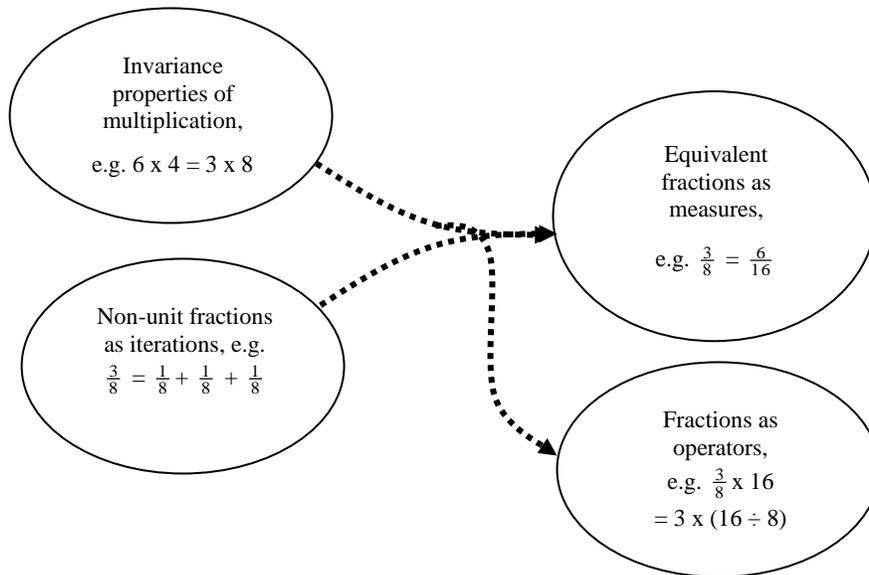


Figure 201: Temporal relationships in schemes

Connection of nodes at the same phase might also illustrate how schemes from one sub-construct facilitate schemes for other sub-constructs without that facilitation being significant developmentally. For example, at the unit co-ordination phase students used rates to solve quotient and operator problems.

A braided approach to the sub-constructs begs the question of the multiple situation types and representations that make proportional reasoning so complex. It also does not address the importance of other knowledge types, “*knowing-that*” and “*knowing-why*”, to the schemes. “*Knowing to*” is assumed in the situation types. Indexing key knowledge to schemes at each scheme (node) of the trajectory using computer database technology is potentially one approach. This would retain the relative simplicity of the HLT while documenting contributory knowledge. However, every

act of categorisation is an act of segregation with consequence. For example, treating decimals and percentages as simply representations of rational number indexed to schemes may inadequately represent the development of thinking about decimals and percentages. There are particular issues in play with these developments, such as how a learner modifies their place value knowledge of whole numbers to deal with the demands of decimals. Allowing a flexible approach to the HLT that permits selecting development progressions in various ways offers potential, for example, by selecting either representation or situation. Facility to take different “slices” of the trajectory, e.g. by decimals, by situation type, increases its flexibility of use.

To further enhance the HLT, there is need of two main areas of future research. Firstly, there is need for a large-scale longitudinal study to establish the degree to which the growth path of individuals conforms or deviates from that anticipated by the HLT. The issue is whether individual variation is so significant that no generic map is possible or desirable. Secondly, instructional use of the HLT warrants investigation. The purpose of the HLT is to assist teachers in providing targeted instruction that improves educational outcomes. In this sense, it is tool, which is only as good as its functionality, not its form.

## **11.3 Theories of Learning**

### **11.3.1 OBJECT THEORY**

Object theory describes the anticipation of actions on physical or imaged embodiments. These anticipations become objects of thought for the generation of new results (Sfard, 1991, 1998; Tall, et al., 2000). Table 16 summarises important characteristics of the case study students in terms of object theory. All six students displayed strategic variation in response to different situations and vulnerability resulting from their choices where those choices involved absent knowledge and excessive working memory demands. Linda was particularly vulnerable to overload in her working memory. Distinguishing high-achievers from moderate and low achievers were also their ability to anticipate actions after few examples, trust in those anticipations until proven otherwise, make powerful connections between symbols and the embodied quantities, and act creatively and flexibly with anticipated actions as objects of thought.

This research reveals many occasions in which students appeared to use symbols, words and other representations to anticipate actions and use the anticipations as objects for thought without necessarily folding back to those potential actions on physical or imaged entities (Pirie & Kieren, 1994). There was clear evidence of detachment of anticipations from actions in this study. The detachment came with natural consequences. Students frequently created anticipated actions that were inaccurate, usually based on properties of representations, particularly symbolic patterns. For example, Ben trusted that  $\frac{15}{24} = \frac{16}{25}$  and  $\frac{18}{27} > \frac{18}{26}$  based on symbolic pattern

alone. Students were happy to think with these false anticipations. Trust was the basis of applying anticipations not validity.

**Table 16: Case study students' characteristics for creating mathematical objects**

	Ben	Rachel	Odette	Jason	Linda	Simon
Anticipation of physical action	frequent	frequent	sometimes	frequent	sometimes	frequent
Application in object-like ways	frequent	frequent	sometimes	frequent	rare	frequent
Trust disposition	little validation	little validation	prolonged validation	little validation	prolonged validation	little validation
Situational variation	frequent	frequent	frequent	frequent	frequent	frequent
Use of symbols to anticipate and theorise	frequent	frequent	sometimes	sometimes	rare	frequent
Vulnerability to strategic choice and cognitive load	sometimes	sometimes	often	sometimes	often	Sometimes
Flexibility and creativity of solutions	high	high	moderate	high	moderate	high

Trust was often, but not always, based on understanding of isomorphic actions on quantities. Connecting materials with symbols and words was a key feature of the teaching. At the beginning of the year many students trusted algorithms for calculation without understanding of the conservation or transformation of quantity represented by the symbols. For example, Simon and Rachel multiplied fractions by  $\frac{n}{n}$  to create equivalent fractions without appreciating that the resulting numbers represented the same quantity, or recognising  $\frac{n}{n}$  as a name for one, the identity element for multiplication. Trust was fundamental in students seeing themselves in the position of solving a problem (Mason & Spence, 1999). At the beginning of the year Rachel mistrusted her knowledge of percentages so could not act on the problem about percentages.

Reliability of trust in anticipated actions seemed to be a combination of a conscience for validity, and noticing consistency of results. Students who had a disposition for seeking consistency between their beliefs and evidence, particularly between symbols and the quantities represented, tended to achieve more highly than those for whom the disposition was less established. Incompatible co-existing knowledge was evident with all six students. For example, Odette did not rationalise her incorrect comparison of two ratios by additive difference with her correct calculation of the part-whole relationships as percentages. High achieving students accepted contradiction of their ideas through evidence and sought to rationalise any contradictions. They trusted an anticipation of action until proven otherwise. In contrast, lower achieving students seemed to view contradiction as a blow to trust. Calculation inaccuracy and forgetting undermined their trust that, in turn, adversely affected their search for consistency. Many students in the moderate and lower achieving groups of the study class described mathematics as “wierd” and untrustworthy which was consistent with Ell’s (2005) finding.

The case study evidence does not support the idea that an object of thought is complete, stable and separated from context. Students forgot concepts that they had previously applied reliably. Rachel used equivalent fractions to compare shares late in Term One but was unable to do so at the end of Term Three. New situations and representations frequently resulted in students being uncertain about concepts that previously appeared well established. For example, graphical representation caused Ben, Rachel and Simon to apply more primitive thinking than they normally did in solving rate problems. Simon and Rachel used multiplicative thinking in most rate contexts but abandoned it in favour of additive thinking in their first dealings with enlargement. Students modified concepts through experience with situations, upon connection that the situations were similar and that the same knowledge elements were useful.

These observations were consistent with the idea of folding back to physical actions, imaging and anticipation of action when new affordances for a concept are first encountered by the learner. The elegant simplicity of object theory does not match the variability in learning that occurred in this study.

### **11.3.2 CO-ORDINATION CLASS THEORY**

Co-ordination class theory anticipates and explains much of the case study evidence presented and throws light on the difficulty of transfer within and between situations. Throughout this work, reference is made to multiplication and division of whole numbers, and Kieren’s sub-constructs for rational numbers as candidates for co-ordination classes. Decimals, percentages, graphs and probability have also been included. Two points are relevant here. Firstly, the above classes were in my mind and not necessarily in the minds of students. Secondly, co-ordination classes are so named because development of these types of concept requires the co-ordination of multiple knowledge elements (DiSessa & Wagner, 2005). The case studies show ample evidence that a fine-grained analysis of the knowledge resources used and

needed by students in construction of the concepts was useful (DiSessa, 2002). For example, higher achievers used knowledge of common factors to identify unknown operators between rate pairs.

Highly consistent with actor-orientated perspectives there was evidence that for these students the mathematical structure afforded by a situation was only relevant if the student perceived the opportunity (Greeno, et al., 1993; Royer, et al., 2005; Wagner, 2010). For example, Simon did not see a balance situation as affording use of constant product or a graph situation as affording slope as rate of change. Yet he saw opportunities to use mathematical structure in many other situations in which rates were involved.

Co-ordination class theory presents two functions, readout and the inferential net. Readout is the perception by the learner of affordances in the situation that lead to strategic choice and the inferential net is the co-ordination of sub-goals, involving the activation of knowledge resources (DiSessa & Wagner, 2005). It was very rare in this study for the students not to identify useful conditions in problems. Readout was usually sound. Inference was the most common source of difficulty.

B: I understood it. I just couldn't get 80% of 35 shots.

R: I drew up a number line because I didn't get it, then I got it.

The inferential net involved the interaction of strategic choice and the availability of knowledge resources in a symbiotic way. A section of an interview with Jessie, another student in the class, illustrates the relationship. Jessie compared the equal shares of three girls with two pizzas and five boys with three pizzas.

J: The girls.

I: How much do they get?

J: (Writes  $2\overline{)3}$  ) One and a half.

I: Does that make sense?

J: No. I don't know how much they're going to get but I know the girls are going to get more.

I: Why's that?

J: Because there's five boys and three pizzas, and if there's no boy that's four boys and three pizzas and they get the same as three girls and two pizzas.

Firstly, Jessie correctly decides that division is a productive strategy to begin with. This is her first strategic choice. Her lack of knowledge for interpreting the division of three by two in terms of a rate (one and a half girls per pizza) derails her strategic choice. Secondly, she adopts an equalising strategy, also potentially productive. She knows that five boys with three pizzas get less than four boys with three pizzas. However, she also knows that three girls with two pizzas get the same share as four boys with three pizzas (both have a difference of one). Unfortunately, what she knows is incorrect.

Knowledge does not just enable Jessie's strategic choices in the sense of providing functions to exercise that choice. Knowledge also influences what Jessie sees as potential affordances in the situation. The high-achieving students in this study, and Jessie was one of those, all knew more to begin with than the moderate and low achievers. Though the high achievers experienced the same blocks associated with lack of knowledge and/or seeing its applicability to situations, they exercised greater flexibility in attempting alternative strategic choices if initially unsuccessful. In the course of doing so they learned more. This is consistent with the finding of Simpson (2009) that strategic choice and knowledge possession have a mutual bootstrapping effect.

There is evidence in all six case studies for cueing preferences of knowledge resources by students (Pratt & Noss, 2002). Linda's preference for visualising circular regions, Ben's view of decimals as money, Rachel's fondness for written algorithms, and Simon's use of unit rate were examples. Students displaced these preferences to lower priority as they solved problems for which their preferences would not satisfy. However, the preferences did not go away and re-emerged from time to time. Knowledge preferences were both useful and counter-productive. For example, trust in percentage as a measure of proportion allowed Odette to compare ratios but it also stopped her attendance to conservation of ratio.

This study identified at least four types of knowledge used by students in the construction of co-ordination classes that were consistent with Mason and Spence's (1999) model of *knowing-to* act in the moment.

1. *Knowing-that* – associational knowledge such as  $\frac{3}{4} = 0.75 = 75\%$  or  $a:b$  represents  $a$  ratio of  $a$  measures to  $b$  measures.
2. *Knowing-how* – process knowledge such as simplifying a fraction using common factors or multiplying two numbers and interpreting the answer or seeing pattern in a set of equations
3. *Knowing-why* – explanatory knowledge such as why doubling and halving factors conserves product or why decimals of some fractions recur
4. *Knowing-to* – situational knowledge such as knowing to use percentages to compare shooting statistics of netballers or knowing to use quotients to compare population densities

In many instances students were content to trust *that* and *how* knowledge and apply it to situations in a *knowing-to* way. Early in the year, Simon applied equivalent

fractions to quotients without *knowing why* equivalent fractions represented the same quantity. His coming to *know-why* was associated with application of equivalence across the sub-constructs but causality is a brave assertion. It was clear from this study that *knowing-that* and *knowing-how* was sufficient basis for *knowing-to* apply the knowledge in different situations for some students. It is seductive to believe that *knowing-why* is important for transfer (Skemp, 1971). However, students knew to use knowledge in situations without fully understanding why that knowledge worked.

The relationship between strategic choice and knowledge possession explains the variability of concept application within situations and to different situations (Schwartz, Bransford, & Sears, 2005). All of the case studies revealed multiple examples of partial transfer within situations and variable transfer between situations. Hypothesised sources of difficulty in transfer for learners were:

1. Inability to perceive a useful strategy due to the absence of knowledge resources;
2. Inability to connect the features of a new situation with the features of previously encountered situations;
3. Choice of a strategy for which inadequate knowledge existed to enact it;
4. Choice of a strategy that involves complex co-ordination of knowledge that leads to excessive load on working memory;
5. Choice of a strategy for which knowledge exists but the knowledge is incorrect.

Combinations of the above sources contribute to difficulties in transfer as Jessie's interview segment exemplifies. The sources of difficulty explain why transfer is difficult. Students in this study learn to transfer in and between situations. Their span and alignment for multiplicative thinking and proportional reasoning improves but transfer is hard-won. The next section considers the implications of object theory and co-ordination class theory as models of transfer.

### **11.3.3 IMPLICATIONS OF OBJECT THEORY AND CO-ORDINATION CLASS THEORY FOR TRANSFER**

In one scenario, I presented the high achieving group of students with a sequence of word problems potentially modelled by division with whole numbers. The problems reflected both partitive and quotative division, exact divisibility and remainder calculations, and a range of semantic types (Greer, 1992, 1994). Students attempted the problems in small groups of two or three then shared the solution strategies with the whole group. The strategies used by students were diverse, fluent, successful, and strongly rooted in variance and invariance properties of multiplication and division. I focused the students on common structure across the problems.

T: You guys solved these problems pretty easily. What was it about the problems that told you that they were about division?

J: Experience.

There is considerable wisdom in Jay's cynical reply. For these students the problems were alike and they were like many problems they had encountered before. Viewing the episode as a snapshot might lead the observer to say that the students had constructed an abstract object that they mapped onto the situations, stripped away from specific contexts.

This scenario is in marked contrast to the majority of problem solving attempts by students reported in these case studies. Their strategies were characterised by variability in the face of different situations and task conditions even when the same concept was involved. Structure of a situation is in the eye of the beholder. Situations afford opportunities for concept projection (Wagner, 2006, 2010) that are or are not attended to by the student. The knowledge resources available affect the recognition of affordance.

So, is it helpful to view abstraction as the creation of mathematical objects? Yes and no. The thinking of high-achieving students in this study shows their disposition that mathematics is a game played in the mind. Rachel, Ben and Simon did not dwell on actions with physical or imaged material though they responded to these actions for validation. These students anticipated the actions, mostly through using tools, and used that anticipation to create *theorems in action* (Vergnaud, 1983, 1988, 1994, 1998) which were connections between anticipated actions. In contrast, Linda the lower achiever was more inclined to be satisfied with the result of real or imaged physical actions rather than anticipation of them. Odette seemed to be more inclined to theorem creation as the year progressed. The metaphor of mathematical objects is useful pedagogically since it guides disposition from actions to ideas.

There are three major weaknesses in object theory. Firstly, it gives an illusion that objects are complete and contextually detached in order to be available for further thought. Evidence from this research contradicts that view. Concepts appeared to be continually evolving. Secondly, use of an anticipated action as an object for thought involves adequacy rather than completeness or detachment. In fact, the attachment to original situation played a part in the development of concepts. For example, Jason related his conversations with his father about mixing fuel to explain his strategy for comparing ratios of fruit juice. Simon connected rate of pay with speed of a car in kilometres per hour. Thirdly, object theory neglects to describe how the learner becomes experienced to the point that a variety of situations involving the same concept appears familiar. Understanding the process of conceptual change is of educational significance.

This study supports two hypotheses derived recently from co-ordination class theory about the recognition of structure by students (Wagner, 2010). Structure in this sense refers to elements of knowledge and the relationships between those elements as observed by a learner's application of a mathematical concept or concepts to situations. Recognition of structure facilitates transfer in and between situations. Firstly, as discussed previously, the same student attends to different features of different problems that involve the same concept. Situational variation is a common

feature of all six case studies. Secondly, different learners attend to different features in the same problem. To illustrate this, consider the responses of students to this problem:

Screecher shoes are 80 percent of the price of Petrol shoes;

Nykie shoes are 75% of the price of Screechers;

What percentage of the price of a pair of Petrol shoes is a pair of Nykies?

Andrew calculated three-quarters of four-fifths and converted the result, three-fifths, to sixty percent. Ben saw no way to solve the problem. Jessie misunderstood the question and calculated  $75\% + 80\% = 155\%$ . Simon set the price of Petrol shoes at \$10 and calculated  $\frac{4}{5}$  of \$10 = \$8 then  $\frac{3}{4}$  of \$8 = \$6 and converted six out of ten to 60%. Rachel used a similar strategy in setting the price at \$100 which made it easy for her to find 80% of \$100 = \$80 then 75% of \$80 = \$60.

Attendance to different features in turn activates different affordances for action. Success or non-success of action, as perceived by the learner, depends on availability and co-ordination of appropriate knowledge resources that are associated with the perceived features. Success promotes the cueing of the useful knowledge resources associated with that particular situation. History of success in several situations regarded as similar explains why students change their preferences, as Odette does in using percentages to measure ratios in comparison tasks.

Exposure of the same learner to different situations involving the same concept may or may not result in the learner seeing a common structure in the situations. Consider different situations ( $S_1, S_2, S_3$ ) that potentially involve a common concept. The situations have features ( $F_1, F_2, F_3, \dots$ ) that may or may not be attended to (readout) by the learner as significant (see Figure 202). Attendance to features is learner dependent. Associated with the features are cued knowledge resources ( $K_1, K_2, K_3, \dots$ ) which are the result of individual learners' experiences. For example, noting the feature of equal sharing with pizza cued the p-prim, more pieces more pizza, for Linda and the associational knowledge that three equal parts are called thirds for Odette. In ideal circumstances, transfer occurs when the learner connects selected features in each situation to the concept. Many connections unrelated to the concept are also possible.

The process of recognition of similarity is potentially problematic from a learner perspective for several reasons. Firstly, features that are extraneous to the concept may form the locus of attendance. For example, Linda connected pizza contexts to catering where the number of pieces was more significant than the size of pieces. Secondly, the learner may achieve no successful resolution of the situations so there is no provocation of cueing priority for useful knowledge resources ( $K_1, K_4$ ). Success enables trust. Failure disables trust. Thirdly, the recognition of similarity requires classification by common features, transfer between situations, which in the face of multiple distracting features, may not occur. Inability to see similarity of structure in rates was common to all six students.

The risks of non-recognition of common structure between situations and the inferential demands of knowledge co-ordination make transfer a fraught process and explain why it is so difficult. Co-ordination class theory seems a better fit to the data in this study than object theory in terms of explaining the variability observed in students' responses to situations and the process by which the students' became sufficiently experienced to demonstrate improved span and alignment for multiplicative thinking and proportional reasoning.

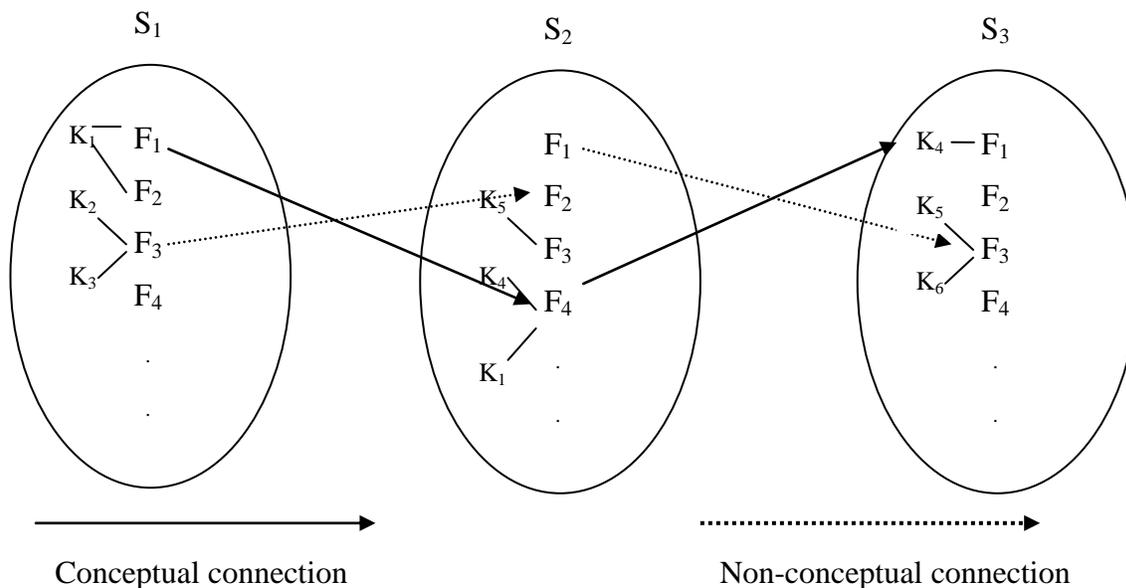


Figure 202: Model for transfer between situations

## 11.4 Implications

### 11.4.1 LIMITATIONS OF GENERALISATION

These case studies reflect a specific setting that limits any claims to generalisation to the wider population. This study represents the learning of six students that occurred within a classroom situation so that exact replication is impossible. Findings in respect to changes in student thinking reflect the social-cultural environment in which the study occurred. The actors and situation are unique. Nevertheless, even given the small sample of students, the results revealed findings that are of significance to all classrooms in which multiplicative thinking and proportional reasoning are taught and learned.

Caution regarding generalisation is tempered with acknowledgement of the strong research base on which this study is based. The HLT drew on a broad range of research that added weight to the significance and relative difficulty for students of key ideas in multiplicative thinking and proportional reasoning. This discussion describes actual learning trajectories in contrast to those hypothesised. Similarly, the

discussion on learning theory and transfer rests on the ideas of many researchers who have gone before. It considers the theories in light of the evidence to evaluate how well the models describe what occurred. The topic is of considerable significance in mathematics education in terms of the essential nature of multiplicative thinking and proportional reasoning and the difficulty it poses for learners and teachers.

#### **11.4.2 MULTIPLICATIVE THINKING AND PROPORTIONAL REASONING**

These case studies revealed limitations in the current research about the advanced phases of development for multiplicative thinking with whole numbers. The research base is consistent about the early progressions but descriptions of further development are either unclear or cover only specific aspects. This research suggests at least one phase of multiplicative relations in advance of the derivation of basic multiplication facts for which there is limited large-scale evidence. Among the multiplicative thinking knowledge required for proportional thinking are:

- Conceptual understanding of both partitive and quotative division;
- Transfer of the properties of multiplication to division, particularly the associative property that is needed for re-unitising;
- Recognition of divisibility, in particular common factors; and
- Attendance to both additive and multiplicative relationships in number pairs, including the features of situations to which these relations apply.

There is also a need to integrate Hypothetical Learning Trajectories for multiplicative thinking and proportional reasoning as attempted in this research and validate the trajectories on a larger scale. Considering the variation in students' actual learning trajectories reported here the research should consider whether the variation is so significant that no generic growth path exists or whether Behr, Harel, Post, and Lesh (1994) are correct in their claim that the development of proportional reasoning is initially localised in context but later generalised in nature. Evidence from this study suggests that the HLT is a useful generic growth path and that there may be some temporal sequence to students' development of sub-constructs in proportional reasoning.

#### **11.4.3 WHERE TO FOR OBJECT THEORY AND CO-ORDINATION CLASS THEORY?**

This research found support for both object and co-ordination class theory in the learning of concepts. However, the weight of evidence suggests that the establishment of encapsulated, complete, contextually detached objects is an unlikely metaphor. Use of anticipated processes as objects for thought only requires the objects to be sufficient for purpose. Learners modify concepts in response to reflective experience and connect the concepts to situations in which they are developed. The contextual features of situations may provide knowledge resources

that are of use by learners in noticing similarity between situations. All of the learning behaviours anticipated by co-ordination class theory were present in the case studies, including high degrees of situational variation and the co-existence of incompatible knowledge. The in pieces metaphor was more prevalent than the metaphor of coherent, encapsulated objects of thought. The inferential demands of knowledge co-ordination accounted for the partial transfer exhibited by the students on numerous occasions.

The two theories are compatible in some respects. Sfard (1991) discusses the duality of process and object as one of strange co-existence. Object theorists have different names for the transition stage between provocation of conceptual change and the state of integration of the new concept. All agree the process is lengthy but the descriptions of what occurs are sketchy. The development of co-ordination classes is explanatory of the conceptual change process as the learner becomes more knowingly experienced in projecting a concept onto different situations.

Object theory and co-ordination class theory part ways in two respects that are hard to reconcile, the completeness and detachment of concepts as objects of thought and the focal distance from which educators view conceptual change. Concepts only need to be adequate for purpose rather than complete. Learners amend concepts as new situations provoke new connections and features of situations are integrated. This research is in agreement with diSessa (2008) that fine grained analysis of the knowledge elements required, and the inferential processes for co-ordinating that knowledge, are critical to understanding how particular concepts are learned and that viewing change from a macro perspective is less helpful.

#### **11.4.4 LEARNING AND TEACHING**

This research has significant implications for teaching approaches that assist students to transfer in and between situations. The observation that the same students see different affordances in different situations and that different students see different affordances in the same situation is rich ground. Processing students' ideas collectively makes use of different affordances. The socio-mathematical norms, particularly around the nature of what constitutes a mathematical argument, appear fundamental in maximising the potential of the classroom in which learning is socially constructed (Cobb, et al., 1995).

Integration of the research about classroom culture and co-ordination class theory offers considerable potential for finding teaching approaches that facilitate transfer of concepts through the strategic use of situations. The data from this study strongly suggest that this work needs to be concept specific and use design experiment methods that acknowledge the instructional significance of student ideas (Cobb, et al., 2003). Attention to all four types of knowledge elements identified in this study and common issues for students in co-ordinating this knowledge is imperative, as is the use of situational variables that may advantageously alter students' cueing preferences. The HLT identifies a large number of fine-grained knowledge elements that are necessary for students' understanding of rational number and proportional reasoning. Teachers need to be aware of this knowledge and how it connects.

The prevalent incidence of partial transfer in situations shown in the case studies supports the perspective of Schwartz, Bransford, & Sears (2005) about preparation for future learning. Assessment occupied solely with endpoint performance neglects the balance between efficiency and innovation. Teachers potentially gain insight from the partially correct and incorrect attempts at transfer made by students both in and between situations. Competence displayed by correct answers is often illusory as shown by Ben's money based thinking with decimals. While students in the study made solid progress as measured by standardised tests, all of them had concepts that were partially developed. Teaching for conceptual understanding involves taking a long-run approach to learning (Lamon, 2007). Another finding of this study is that students' tendency to satisfice in meeting the demands of an assessment task often means they do not reveal the most advanced thinking of which they are capable.

Object theory also has pedagogical implications. Students who achieve highly pass rapidly beyond actions on physical embodiments to anticipation of action and thinking with anticipations. Use of embodiments as a means to an end, ideas in the mind, seems an essential feature of classroom practice supported by cultural tools like words, symbols and diagrams. Embodiments also offer mechanisms for validation of ideas through folding back which is essential given students' tendency to create both illegitimate and legitimate theorems in action (Vergnaud, 1998).

The single most compelling feature of the data in this study was the variability in student learning and engagement with tasks. While this variability may sit easily with constructivist learning theory and actor-oriented views about transfer it presents significant challenges. Understanding and acting on observations of many learners all engaging with tasks in their own way, in a field as complex as proportional reasoning, presents significant challenge to teachers.

## CHAPTER TWELVE: SUMMARY OF FINDINGS

### 12.1 Research questions

This study aims to answer the following research question with two parts:

**How do students develop their multiplicative thinking and proportional reasoning?**

- a. **Does a Hypothetical Learning Trajectory (HLT) reflect the actual learning trajectory of students?**
- b. **What model of conceptual learning, object theory or co-ordination class theory, best represents the growth of multiplicative thinking and proportional reasoning, and the transfer of knowledge within and between situations?**

### 12.2 Method

The study is a design experiment conducted with a class of 11-13 year old students at a middle school located in a large rural town in New Zealand. The students show a considerable range in achievement at the start and end of the year. I (the researcher) co-teach with the class teacher for a total of 14 weeks during 2007 with the aim of getting shifts in the students' understanding of multiplicative thinking and proportional reasoning.

The case study group includes nine students reflecting a range of gender, ethnicity and initial level of achievement. Six of the case studies are included in the thesis. Data sources are standardised test results, interview transcripts, work samples, teaching diary notes, modelling books and weekly plans.

### 12.3 Development of multiplicative thinking and proportional reasoning

#### 12.3.1 HYPOTHETICAL AND ACTUAL LEARNING TRAJECTORIES

From the literature a Hypothetical Learning Trajectory (HLT) is created. The HLT consists of four broad phases of progression in multiplicative thinking and proportional reasoning from grouped counting/additive strategies to advanced multiplicative strategies involving constants of proportionality. Kieren's sub-constructs (1980, 1988, 1993) form the dimensions of the progression for proportional reasoning with a large body of other work providing detail for the phases.

Students' actual learning trajectory are summarised at four points during the year using a tabular form of the HLT. The summaries are further synthesised onto a Learning Trajectory Maps to provide a visual tool for analysing the synchronicity of individual students' progress through the phases by sub-construct.

### **12.3.2 RESULTS**

The HLT is a useful tool for planning and assessment purposes. It provides a broad scope of situations and problem types. Progression against the HTL is consistent with the anticipated phases within sub-constructs but there is considerable variability between sub-constructs for individual students. Higher achieving students display greater regularity in their learning trajectory maps than lower achieving students. This indicates that co-ordination of sub-constructs is significant for the development of multiplicative thinking and proportional reasoning. Use of sub-constructs to inform others, such as solving quotient problems using rates, sometimes supports the growth of proportional reasoning and sometimes appears to restrict development. Advances in span, the range of situations to which sub-constructs are applied, is usually followed by periods of consolidation as the student achieve better alignment, reliability of concept application.

The HLT has several limitations. The sub-constructs reflect a conceptual field approach that neglects the affordance patterns of the students. Situation types affect the responses of students even when the same sub-construct is involved. There is inadequate description of the transfer between and within models (sub-constructs), representations and situations. The HTL contains a mix of knowledge and strategy types that demand different pedagogical attention. Knowledge demands deliberate teaching. Strategies are indicators of conceptual growth and are sometimes unsophisticated. Teaching of unsophisticated strategies that may restrict later progress seems questionable.

The data also shows some ordinal patterns of development such as understanding of the properties of multiplication and division with whole numbers, supported by basic fact knowledge, being an essential requisite to understanding equivalent fractions as the same measure. These ordinal patterns are sometimes within the same phases of the HLT.

## **12.4 Theories of Learning and Transfer**

### **12.4.1 OBJECT THEORY AND CO-ORDINATION CLASS THEORY**

Object theory involves the idea of anticipated actions on physical or imaged entities being used by learners as objects of thought to obtain new ideas separate from physical action. There is considerable commonality between theoretical descriptions of this hypothesised development and agreement that the process is complex and lengthy. The descriptions all have a final stage of abstraction in which the concept is

accessible for reliable application across diverse situations. Debate centres on the co-existent relationship of processes and objects and the extent to which objects of thought are encapsulated and detached from the contexts on which they were founded.

Co-ordination class theory describes the development of concepts in which plentiful fine-grained knowledge elements must be co-ordinated. The recognition by the learner of common features in different situations (concept projections) is the cause of improved span and alignment. Successful experience creates cueing preferences of knowledge elements aligned to the structural features which act symbiotically in the learner seeing affordances in situations (readout) and co-ordinating the sub-goals to resolution (inferential net). Co-ordination class theory anticipates considerable variation within and between individual learners, the co-existence of incompatible knowledge and the strong influence of prior knowledge and experience in how learners perceive situations.

The study contrasts the two views of concept development in terms of their explanatory power as models of transfer, that is, the application of the same concept in different situations.

#### **12.4.2 RESULTS**

The use of anticipated actions as objects of thought is more common in the high achieving students than it is with the lower achieving students. High achievers more readily trust and apply their theorems, often from few examples of the physical phenomena. Low achievers tend to be more distrustful of patterns and are satisfied with enacting physical actions. All learners trust ideas that are incorrect at times, fail to connect symbols to physical quantities, and use algorithms without fully understanding why the strategies work. High achievers change their ideas in the face of contradictory physical or symbolic evidence without losing trust and flexibility to create new theorems. Cognitive conflict adversely affects lower achievers' trust. Symbols play a key role as tools in enabling students to anticipate actions and use the anticipations as objects of thought.

The data do not support the idea that concepts are encapsulated, complete and situation-free. Students are vulnerable to situational variation indicating that concepts are continually evolving. This supports models of conceptual development that include folding back to physical or imaged actions.

Co-ordination class theory seems explanatory of the plentiful examples of partial transfer within and between situations in this study. Available knowledge influences students' strategic choice and possession of knowledge enables or disables the successful use of a strategy. The complexity of co-ordinating multiple fine-grained knowledge elements also partially accounts for the difficulty of transfer. Cueing preferences are common such as money views of decimals and circular embodiments of fractions. The study identifies four types of knowledge, *knowing that*, *knowing how*, *knowing why* and *knowing to*. Students are often able to *know to* apply a concept to a situation without *knowing why* the concept works.

### 12.4.3 SUMMARY AND IMPLICATIONS

With acknowledgement of individual variation the HLT shows promise as a generalised growth path for multiplicative thinking and proportional reasoning. There is a need for more research to validate the later phase/s of multiplicative thinking with whole numbers, and connect these phases with proportional reasoning. Revision of the HLT should include ordinal effects, model-representation-situation transfer and the rationalisation of knowledge and strategy types.

The pedagogical implication of object theory that the creation of theorems is the aim of mathematics is useful. Shared understanding with students that anticipation of action is powerful when used to derive new results seems critical to success in learning mathematics.

Co-ordination class theory seems a better model of the process by which a learner becomes experienced at applying a given concept. Student in the study develop concepts by reflective experience with situations. This involves reading out key features of the situations and co-ordinating a diversity of knowledge elements. There is a dynamic relationship between knowing and seeing affordance in situations. The fine-grained knowledge elements used by students are concept specific and include helpful and unhelpful ideas from prior experience. Use of design experiment research connecting students' development of co-ordination classes and socio-mathematical norms in classrooms appears to be potentially fruitful.

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# APPENDICES

## Appendix 1: Example of test

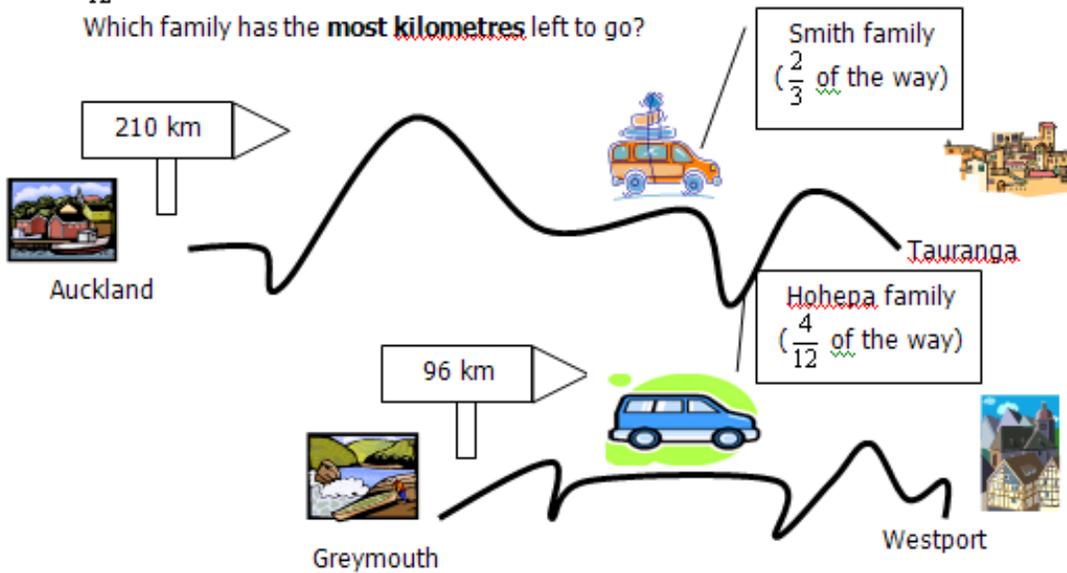
### Term 4 Interview Group 2

Name:

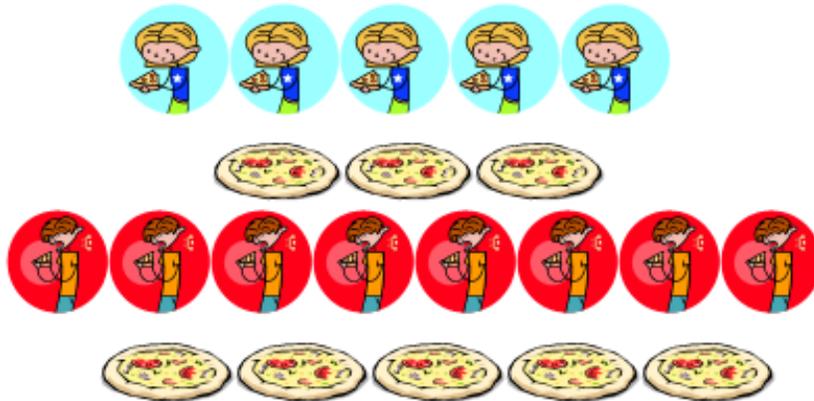
Date:

1. The Smith and the Hohepa families are driving home from their holidays. The Smith family has driven  $\frac{2}{3}$  of the way home and the Hohepa family has driven  $\frac{4}{12}$  of the way home.

Which family has the **most kilometres** left to go?



2. Five girls share three pizzas equally and eight boys share five pizzas equally. All the pizzas are the same size. Who gets more pizza, a girl or a boy? How much more?



## Appendix 2: Example of test

### Group 2 Assessment Term 3

Name:

Date:

1. Which piggy bank would you rather get for your pocket money?



One-third of this piggy bank



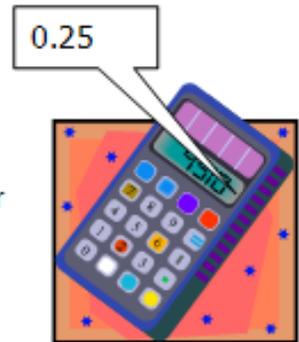
Five-eighths of this piggy bank

Explain your answer.



2. Jamie has a bag of 72 marbles to share among himself and three friends. All of the four people must get equal shares. How many marbles should each person get?

3. If the calculator shows 0.25 as the decimal for one-quarter, what will it show as the decimals for one-eighth and one-sixteenth?



4. Olivia buys a pair of jeans that usually cost \$96.00. She gets 25% discount. How much does Olivia pay for the jeans?

## Appendix 3: Example of a weekly plan

**Mathematics Plan**                      **Term: One**                      **Week: Six**                      **Dates: 12/3 -15/3**

Starter: Estimation game at 2 different levels (calculators), Flash cards, Loopje (4 versions) Conversations: Odd/Even numbers, Square numbers, triangular numbers, rectangular numbers, nonrectangular (primes)					
Group	Monday	Tuesday	Wednesday	Thursday	Friday
<b>Group One</b>	T: Properties of multiplication (comm./distrib./assoc.)	T: Deriving using the distrib. Property. x10 to x5 to x9 x5 to x6 to x7 to x 10 FIO 7-8 N.1 Fives and tens (p.4-5)	T. Deriving nine times tables from ten times tables	T: Doubling to get multiplication facts	
	I: Circle the same facts Box Games	I: Rainforest (Number)-Grade 4- Mult/Div-Doubles, extensions, split strategy	I: Worksheet on Jumping from ten Rainforest Group games	I: N 7/8 Link Table Tricks (2) N 7/8 Link factor Fun (5) Rainforest	
<b>Group Two</b>	I: Fair deals Area and perimeter Town populations 360 2400 18 000 150 000	T: Arrays with imaging Construct the factors Calculate Multiplicative place value, e.g. 40 x 50 = 2000	T: Using Maddy the Multiplier  Efficient ways to partition the areas	T: Multiplying and dividing by ten as moving digits. Place of decimal point. Division strategies	
	I: Six Shooters Maddy the multiplier Wishball	I: Solve five problems on Maddy the Multiplier (Easy) and five problems on Hard Write down your working Group Box games	I: The power of ten Worksheet Group Box games	I: N 3.3 Arcade Adventure (18) Number on Math Support Scale Matters Wishball	
<b>Group Three</b>	I: FIO PR 3-4.2 Flavoursome (p. 6-7) Common Factor Climb Factor	I: FIO N3.1 Stretch and Grow (p.9) Bean brains (p.9) Factor (maths support)	I: Probability- Proportional reasoning associated with variability Worksheet	I: Continue with probability worksheet on coins and spinners	
	T: Ratios blue to yellow Direction of change 1:3 vs 2:8 3:2 vs 5:3 4:3 vs 3:2	T: Compare by cloning Ratios Convert to fractions and percentages Qualitative judgment 2:3, 3:5, 4:6, 5:7	T: Fractions/decimals as the result of division - decimats - paper circles	T:	

### Internet Games/Activities

#### Groups One and Two

Addition/subtraction Facts

[http://www.wmnet.org.uk/wmnet/custom/files\\_uploaded/uploaded\\_resources/851/Hitthebutton7.swf](http://www.wmnet.org.uk/wmnet/custom/files_uploaded/uploaded_resources/851/Hitthebutton7.swf)

Bonds within twenty -Choose from:

+ bonds

- bonds

u + u + u

Place Value

<http://www.rainforestmaths.com/>

Click on Grade 4

Click on either abacus, parrot or slider

#### Group Three

[http://www.wmnet.org.uk/wmnet/custom/files\\_uploaded/uploaded\\_resources/851/Hitthebutton7.swf](http://www.wmnet.org.uk/wmnet/custom/files_uploaded/uploaded_resources/851/Hitthebutton7.swf)

Times table- Choose from:

Hit the answer or hit the question

<http://www.freewebs.com/weddell/Equiv%20Fractions%20Contents.html>

## Appendix 4: Learning trajectory table

Whole Number Operations	Counting on and simple additive partitioning	Complex additive and emerging multiplicative	Complex multiplicative
Multiplication	Skip counting and repeated additive composition Hybrids based on known facts and counting/adding	Deriving multiplication facts using distributive property and doubling/halving Known multiplication facts	Deriving multiplication answers using all properties of multiplication Relational thinking about multiplication (operating on the operator)
Division	Equal sharing using composite counts or additive build-up	Division by multiplicative build-up	Known division facts Common factors Connection of <u>partitive</u> and <u>quotitive</u> division Division by connecting multiplication properties and inverse
Construct/Phase	Unit splitting and replication	Unit co-ordination	Fractional equivalence
Part-Whole	Halving based splits More equal parts so smaller parts	Repeated splitting preserving equivalence Numerator as iterations – Denominator as iterated	Equivalence as multiplicative relation Equivalent fractions as same number
Measure	Physical comparison of equivalent measure	Size of improper fractions in relation to whole numbers	Combining and separating fractions Measurement of one with non-unit fractions
Operator	Unit fraction related to amount	Non-unit fraction of amount as iterations of unit fraction	Non-unit fraction multiplicatively (variable unknown) Operator and operated affect size of result
Quotient	Practical equal sharing by halving (restricted shares)	Practical equal sharing (open shares)	Quotient theorem, $a \div b = \frac{a}{b}$ Equal and unequal shares
Rate/Ratio	Practical duplication of rates	Equivalent rates by replication	Conversion to unit rate Equivalent rates by multiplication and division
	Practical duplication of ratios	Part-whole relationship in ratios Equivalent ratios by replication	Equivalent ratios by multiplication and division Part: part Whole: whole
Ben: End of term one			Comparison of shares using equivalence Remainders as fractions Comparison of rates (measures unchanged) Graphs of linear rates Comparison of rates (measures changed) Inverse rates Comparison of ratios Comparison of frequencies

Learning Trajectory Table