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Maxwell's Equations on a
10-Dimensional Manifold with Local
Symmetry $so(2,3)$

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Abstract

The Hawthorn model [1] is built upon the idea that the Lie algebra $so(2, 3)$ is a more natural description of the local structure of spacetime than the Poincaré Lie algebra. This model uses a 10-dimensional spacetime referred to as an *ADS manifold*. We find the model (as it stands in [1]) to be inconsistent with Maxwell's equations. We investigate why this is so and proceed to revise the model so as to restore consistency with electromagnetic theory. Consequently we find that the Faraday-Gauss equations (a subset of Maxwell's equations) arise naturally from the geometry of an ADS manifold.

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Chapter 1

Introduction

This thesis looks at the **Hawthorn model** [1] which is built upon the idea that the Lie algebra $so(2, 3)$ is a more natural description of the local structure of spacetime than the Poincaré Lie algebra. This model uses a 10-dimensional spacetime referred to as an **ADS manifold**. The Dirac equation works very nicely on an ADS manifold. From the Dirac equation we can deduce an electromagnetic potential which satisfies equations similar to Maxwell's. However we find that these equations satisfy unphysical constraints. This thesis investigates the problem of making (what should be) Maxwell's equations work on an ADS manifold. In the process adjustments to the Hawthorn model are made. We manage to revise the model so as to make it consistent with a working form of Maxwell's equations. Furthermore we find that in this revised model the Faraday-Gauss equations are simply geometric identities, i.e. they arise naturally and necessarily from the structure of an ADS manifold.

1.1 Thesis overview

Chapter 2 introduces the classical forces; electromagnetism and gravity. This is followed by a presentation of Kaluza-Klein theory which is an attempt to unify these two classical interactions. After pointing out some weaknesses of the theory (as we have presented it) the chapter concludes with some instructive principles which are relevant to the rest of the thesis. The main references

used are [6], [8], [10], [12], [18], [20], [21], [23] and [24].

Chapter 3 introduces and explores the Lie group $SO(2, 3)$ and its corresponding Lie algebra $so(2, 3)$. The relationship between $SO(2, 3)$ and the Poincaré group via Lie group contraction is outlined. It is argued that we are at liberty to use $so(2, 3)$ rather than the Poincaré Lie algebra to describe the local symmetries of spacetime, and have reason to do so. The main references used are [1], [5], [7], [8], [9], [13], [14] and [25].

Chapter 4 develops what we call the **Hawthorn model**, which attempts to define the action of $so(2, 3)$ on a curved manifold in a natural way. The archetype manifold is the Lie group $SO(2, 3)$ which is 10-dimensional. This leads us to the use of an **ADS manifold**, a 10-dimensional manifold with local structure $so(2, 3)$. Each point on the 10-dimensional manifold is interpreted as an inertial frame.¹ The chapter concludes with a summary of the main assumptions and a justification for each. This chapter follows [1] very closely for the following reasons. As [1] is the only source for this material, it would create unnecessary confusion to alter things significantly, in particular the notation. Furthermore the Hawthorn model is central to this thesis and the only reference for it is currently unpublished. It is therefore prudent to check its correctness carefully, especially since we shall seek to make adjustments to the model as it stands in [1]. The other references used are [3] and an updated draft of Hawthorn's work [2].

Chapter 5 defines some terminology for the low dimensional representations of $so(2, 3)$. Useful mathematical results are derived, in particular with regard to the curvature. As in chapter 4, the results found here are presented very much as they are in [1] and [2].

¹In physics an inertial frame is specified by 10 parameters: one temporal, three spatial, three Lorentz boost and three rotational.

Chapter 6 shows how the Dirac equation works very well on an ADS manifold. Benefits of using a Dirac equation defined on an ADS manifold are considered, including how the issue of Zitterbewegung can be resolved. We perform a decomposition of the connection term found in the covariant formulation of the Dirac equation on an ADS manifold. One of the irreducible components of the connection is identified as a 10-dimensional electromagnetic potential requisite for building Maxwell's equations. Again, this chapter draws heavily from the work of [1]. Other important references are [15], [16], [17], [22] and [26].

Chapter 7 uses the electromagnetic 10-potential from chapter 6 to construct an appropriate 10-dimensional generalisation of the electromagnetic field tensor F_{ij} . Similar 10-dimensional analogues of Maxwell's equations are given and referred to as the **extended Maxwell equations**. It is found that the extended Maxwell equations do not reduce to the usual Maxwell equations in the limit that $so(2, 3)$ becomes the Poincaré Lie algebra.

After investigating what might have produced this failure we find that the problem arises because of assumption 4.9. The process of lifting this assumption then ensues with the subtle expense of permitting the existence of quantities on the ADS manifold which are like scalars in every respect except that they parallel transport non-trivially. These unusual quantities are referred to as **bullet scalars**, see [2].

Chapter 8 considers the effects on the Hawthorn model from the inclusion of these bullet scalars. Subsequently the extended Maxwell equations are re-considered and it is shown that they do in fact reduce to the usual Maxwell equations in the appropriate way.

Furthermore in our new approach there is a direct link between the curvature and the electromagnetic field tensor. This prompts a more natural definition of the electromagnetic field tensor in terms of the curvature. This

new definition is subtly different from our previous one, yet it does not alter the fact that the extended Maxwell equations reduce in the proper manner. The identification does however mean the Faraday-Gauss equations 7.4 are a direct consequence of one of the Bianchi identities 8.19. Hence not only does the (revised) Hawthorn model permit Maxwell's equations, but one could say in some sense that "half" of Maxwell's equations arise purely from the geometry of spacetime and do not need to be postulated independently. The relevant references for this chapter are [2], [8], [10] and [11].

Chapter 2

The Classical Forces

The fundamental forces (or interactions) of electromagnetism and gravity are known as the *classical forces*. In this chapter we briefly consider these two forces. We then introduce Kaluza-Klein theory, which has the goal of unifying these fundamental interactions. This requires one to show that they are both in fact special cases of, or follow from some more general, overarching physical interaction. This chapter draws from references [6], [8], [10], [12] and [23].

2.1 Electromagnetism

The development of electromagnetic theory climaxed in 1865 with Maxwell adjusting the existing set of laws to make them self-consistent. His alteration to the former set of experimental laws implied the existence of hitherto unknown physical processes. The addition of this new phenomenon was verified by subsequent measurements, see p. 177 of [10]. These laws are expressed in the following section and unite the electric and magnetic forces into one theory.

2.2 Maxwell's equations

Let $\nabla = (\partial_x, \partial_y, \partial_z)$. The microscopic Maxwell equations in vacuo are

$$\text{Gauss's Law} \quad \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad (2.1)$$

$$\text{Absence of magnetic charges} \quad \nabla \cdot \mathbf{B} = 0 \quad (2.2)$$

$$\text{Faraday's Law} \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (2.3)$$

$$\text{Ampère's Law} \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t} \quad (2.4)$$

These are the basic laws of classical electrodynamics (given in SI units). The quantities \mathbf{E} and \mathbf{B} are the *electric and magnetic fields* respectively, ρ is the *electric charge density* and $\mathbf{J} = \rho \mathbf{v}$ is the *current density* with $\mathbf{v} = (v_x, v_y, v_z)$, the velocity of the flow of the charge. The continuity equation

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \quad (2.5)$$

is a consequence of equations 2.1 and 2.4. It expresses the fact that electric charge is a locally conserved quantity and is true in any inertial frame. We can write equation 2.5 in the more loquacious form

$$\frac{1}{c} \frac{\partial}{\partial t}(c\rho) + \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = 0 \quad (2.6)$$

Since $\partial_i = (c^{-1}\partial_t, \nabla)$ transforms as a 4-vector in Minkowski spacetime, it follows that $J^i \equiv (c\rho, \mathbf{J})$ must also be a 4-vector in order to ensure equation 2.5 is Lorentz covariant. We may now concisely write equation 2.6

$$\partial_i J^i = 0 \quad (2.7)$$

Consider an *electric scalar potential* ϕ and a *magnetic vector potential* \mathbf{A} satisfying

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \quad (2.8)$$

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (2.9)$$

These potentials do not uniquely determine the (physical) fields \mathbf{E} and \mathbf{B} , viz. a transformation of the form

$$\mathbf{A} \rightarrow \mathbf{A} + \nabla\chi \quad \phi \rightarrow \phi - \frac{\partial \chi}{\partial t}$$

for an arbitrary function χ , leaves \mathbf{E} and \mathbf{B} unchanged. Such a transformation is called a *gauge transformation*. \mathbf{E} and \mathbf{B} (and hence Maxwell's equations) are said to be *gauge invariant* (with respect to the afore stated gauge transformation). This means we are free to choose our potentials so that they satisfy the *Lorenz gauge condition*

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0 \quad (2.10)$$

Define the 4-vector $A^i \equiv (\phi/c, \mathbf{A})$ which we shall refer to as the *4-potential*. From the 4-potential we build the electromagnetic field tensor F_{ij}

$$F_{ij} \equiv \partial_i A_j - \partial_j A_i$$

Using the notation $B_{j,i} \equiv \partial_i B_j$ and the contravariant form of the Minkowski metric g^{ij} with signature $(-+++)$, we are now in a position to see that

$$g^{jk} F_{ij,k} = \mu_0 J_i$$

describes equations 2.1 and 2.4. While

$$F_{ij,k} + F_{jk,i} + F_{ki,j} = 0$$

encapsulates equations 2.2 and 2.3. Using this notation a gauge transformation looks like $A^i \rightarrow A^i + \partial^i \chi$, and the gauge condition 2.10 is $\partial_i A^i = 0$. These equations are built from Lorentz covariant quantities therefore they too are Lorentz covariant. To make them generally covariant we simply replace the partial derivatives with covariant ones (denoted with a semicolon) and no longer restrict g_{ij} to be Minkowskian.

$$F_{ij} = A_{j;i} - A_{i;j} \quad \text{Definition of the field tensor.} \quad (2.11)$$

$$g^{jl} F_{jk;l} = \mu_0 J_k \quad \text{Source equation.} \quad (2.12)$$

$$F_{ij;k} + F_{ki;j} + F_{jk;i} = 0 \quad \text{Faraday-Gauss equation.} \quad (2.13)$$

$$J_{;i}^i = 0 \quad \text{Continuity equation.} \quad (2.14)$$

The covariant derivative $A_{i;j}$ of a vector A_i is defined by $A_{i;j} = A_{i,j} - \Gamma_{ij}^k A_k$ where $\Gamma_{ij}^k = \frac{1}{2} g^{kl} (g_{li,j} + g_{jl,i} - g_{ij,l})$. Note again that equation 2.14 is a consequence of equation 2.12.

2.3 Gravitation

In 1915 Einstein published his general theory of relativity, see pp. 431-434 of [11]. *Einstein's equations* govern this theory of gravitation and determine the geodesics of both massive and massless particles. In their full generality they are

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R - \Lambda g_{\alpha\beta} = \kappa T_{\alpha\beta} \quad (2.15)$$

where $R_{\alpha\beta}$ is the Ricci curvature tensor, R is the curvature scalar, $\kappa = -8\pi G/c^2$ is the Einstein constant of gravitation (in SI units), $g_{\alpha\beta}$ is the metric tensor, and Λ is the cosmological constant. Cosmological models based on the Friedmann metric require the current value of Λ to be very small, see [24], indeed for physical situations dealing with smaller than galactic distance scales it is common to set $\Lambda = 0$, see p. 411 of [11]. $T_{\alpha\beta}$ is the stress-energy tensor describing the energy-density of spacetime, which we may write as the sum of stress-energy tensors for matter fields and electromagnetic fields

$$T_{\alpha\beta} = M_{\alpha\beta} + E_{\alpha\beta}$$

where the stress-energy of the electromagnetic field is

$$E_{\alpha\beta} = -\frac{1}{\mu_0} \left(F_{\alpha\lambda} F_{\beta}{}^{\lambda} - \frac{1}{4} g_{\alpha\beta} F_{\delta\sigma} F^{\delta\sigma} \right) \quad (2.16)$$

Hence via equation 2.15 the electromagnetic fields will determine, though not necessarily completely, (if e.g. matter is present) a test particle's trajectory. However the converse is not the case, viz. we cannot determine how the electromagnetic fields will evolve using only equation 2.15. We must therefore postulate equations 2.12 and 2.13 independently. The set of equations 2.12, 2.13, 2.15 and 2.16 form the *Einstein-Maxwell equations*.

The trace of equation 2.15

$$R - \frac{1}{2}4R + 4\Lambda = \kappa T \quad \Rightarrow \quad R = 4\Lambda - \kappa T$$

leading to an alternative form for equation 2.15

$$R_{\alpha\beta} = -\frac{\kappa}{2}Tg_{\alpha\beta} + 3\Lambda g_{\alpha\beta} + \kappa T_{\alpha\beta} \quad (2.17)$$

In the following work we shall neglect the cosmological constant ($\Lambda = 0$), thus Einstein's equation reduces to

$$R_{\alpha\beta} = \kappa \left(-\frac{1}{2}Tg_{\alpha\beta} + T_{\alpha\beta} \right) \quad (2.18)$$

2.4 The Einstein-Hilbert action

At the same time that Einstein presented his general theory of relativity, Hilbert showed the Einstein equations¹ could also be derived using a variational principle, see pp. 132-136 of [12]. The independent variables in the action integral are the components of the metric tensor. His approach was to find the extremum of the Einstein-Hilbert action

$$I = -\frac{1}{2\kappa} \int_{V_4} (\sqrt{-g}R + \mathcal{L}) d^4x$$

where $g = \det(g_{\alpha\beta})$, R is the curvature scalar, $\kappa = 8\pi G/c^4$ and \mathcal{L} is the Lagrangian for any fields containing energy. V_4 is a region of spacetime on whose boundary the variations $\delta g_{\alpha\beta} = 0$.

This variational approach is what Kaluza made use of in 1921 when he proposed what is now known as Kaluza-Klein theory. His theory attempted (though not for the first time, see [19]) to unite the only two well understood interactions of the day, gravity and electromagnetism. The aim is to deduce both Maxwell's and Einstein's (4-dimensional) equations from the 5-dimensional Einstein-Hilbert action, given a specific metric. The main sources for this section are [8] and [12].

¹Actually Hilbert presented a subclass of Einstein's equations where the energy-momentum tensor was that for the electromagnetic field only, and not general distributions of matter.

2.5 Kaluza-Klein theory

In this section upper-case Latin letters A, B can take on values $0, 1, 2, 3, 4$ and refer to coordinate indices on a 5-dimensional Riemannian manifold \mathcal{R}_5 while lower-case Greek letters α, β can take the values $0, 1, 2, 3$ and refer to coordinate indices in \mathcal{R}_4 (4-dimensional Riemannian space). Thus the first four components of any vector $V^A \in \mathcal{R}_5$ correspond to a vector $V^\alpha \in \mathcal{R}_4$. Strictly speaking \mathcal{R}_4 and \mathcal{R}_5 are actually pseudo-Riemannian manifolds since we wish to consider metrics which are not positive definite.

Kaluza-Klein theory is built on \mathcal{R}_5 with a metric k_{AB} of signature $(+, -, -, -; -)$. It is essentially 5-dimensional general relativity determined by the Einstein-Hilbert action

$$I = -\frac{1}{2\hat{\kappa}} \int_{V_5} \sqrt{-k} \hat{R} d^5x \quad (2.19)$$

where $k = \det(k_{AB})$, \hat{R} is the curvature scalar and $\hat{\kappa}$ is essentially the gravitational constant of \mathcal{R}_5 . The equations of motion for this action are

$$\hat{R}_{AB} = 0$$

Equation 2.19 is invariant under general coordinate transformations

$$\bar{k}_{AB}(\bar{x}^M) = \frac{\partial x^C}{\partial \bar{x}^A} \frac{\partial x^D}{\partial \bar{x}^B} k_{CD}(x^M)$$

However, to ensure the fifth dimension is unobservable it must be assumed that the components of the metric k_{AB} are all independent of x^4 , which is to say

$$\frac{\partial}{\partial x^4}(k_{AB}) = 0$$

This is known as the *cylinder condition*. Such a condition is not generally covariant, however it remains true under the following class of transformations

$$x^\alpha \rightarrow \bar{x}^\alpha = \bar{x}^\alpha(x^\mu) \quad (2.20)$$

$$x^4 \rightarrow \bar{x}^4 = \rho x^4 + \xi(x^\mu) \quad (2.21)$$

where ρ is a constant. A symmetric k_{AB} has 15 independent components. They can be grouped into 10 which describe gravity $k_{\alpha\beta}$, 4 which describe electromagnetism $k_{\alpha 4}$, and 1 scalar field $k_{44} = \phi$ which appears to be a redundant degree of freedom. To justify these relations consider how these quantities transform under 2.20

$$\bar{k}_{\alpha\beta} = \frac{\partial x^\mu}{\partial \bar{x}^\alpha} \frac{\partial x^\nu}{\partial \bar{x}^\beta} k_{\mu\nu} \quad \bar{k}_{\alpha 4} = \frac{\partial x^\mu}{\partial \bar{x}^\alpha} k_{\mu 4}$$

We see that they transform as usual covariant tensors of ranks 2 and 1 respectively. And under transformation 2.21

$$\begin{aligned} \bar{k}_{\alpha\beta} &= k_{\alpha\beta} - \partial_\alpha \xi k_{4\beta} - \partial_\beta \xi k_{\alpha 4} + \phi \partial_\alpha \xi \partial_\beta \xi \\ \bar{k}_{\alpha 4} &= k_{\alpha 4} - \phi \partial_\alpha \xi \end{aligned}$$

In order to assert that $k_{\alpha 4}$ transforms like the electromagnetic 4-potential we make the further assumption that ϕ is a constant function. We are thus free to write $k_{\alpha 4}$ as any scalar multiple of the electromagnetic 4-potential A_α . We choose $k_{\alpha 4} = \phi A_\alpha$. The ordinary 4-dimensional metric of physical spacetime ought to be invariant under translations along x^4 , which is not the case for $k_{\alpha\beta}$. We pick

$$g_{\alpha\beta} = k_{\alpha\beta} - \phi^{-1} k_{\alpha 4} k_{4\beta}$$

as the metric of \mathcal{R}_4 since it satisfies this requirement. We are now in a position to write the 5-dimensional Kaluza-Klein metric in terms of physical familiar quantities (with the exception of ϕ).

$$k_{AB} = \begin{pmatrix} g_{\alpha\beta} + \phi A_\alpha A_\beta & \phi A_\alpha \\ \phi A_\beta & \phi \end{pmatrix}$$

and its inverse

$$k^{AB} = \begin{pmatrix} g^{\alpha\beta} & -A^\alpha \\ -A^\beta & \phi^{-1} + A_\lambda A^\lambda \end{pmatrix}$$

where $g^{\alpha\beta}$ is the inverse of the metric $g_{\alpha\beta}$, used to raise 4-dimensional indices.

To calculate the determinant $k = \det(k_{AB})$, write the metric as

$$k_{AB} = \begin{pmatrix} I_4 & \phi A_\alpha \\ \mathbf{0}^T & \phi \end{pmatrix} \begin{pmatrix} g_{\alpha\beta} & \mathbf{0} \\ A_\beta & 1 \end{pmatrix}$$

where I_4 is the 4×4 identity matrix and $\mathbf{0}$ is the column zero vector of \mathbb{R}^4 .

This form makes it easy to find the determinant.

$$\begin{aligned} k &= \det \begin{pmatrix} I_4 & \phi A_\alpha \\ \mathbf{0}^T & \phi \end{pmatrix} \det \begin{pmatrix} g_{\alpha\beta} & \mathbf{0} \\ A_\beta & 1 \end{pmatrix} \\ &= \phi \det(I_4) \det(g_{\alpha\beta}) \\ &= \phi g \end{aligned}$$

where $g = \det(g_{\alpha\beta})$.

If one performs the laborious task of writing out the Christoffel symbols, see pp. 165-166 of [12], then the various components of the Ricci tensor can be calculated.

$$\begin{aligned} \hat{R}_{\alpha\beta} &= R_{\alpha\beta} - \frac{1}{2}\phi F_\alpha^\rho F_{\rho\beta} + \frac{1}{2}\phi A_\alpha F_{\beta;\rho}^\rho + \frac{1}{2}\phi A_\beta F_{\alpha;\rho}^\rho + \frac{1}{4}\phi^2 A_\alpha A_\beta F^{\sigma\rho} F_{\sigma\rho} \\ \hat{R}_{4\beta} &= \frac{1}{2}\phi F_{\beta;\rho}^\rho + \frac{1}{4}\phi^2 A_\beta F^{\sigma\rho} F_{\sigma\rho} \\ \hat{R}_{44} &= \frac{1}{4}\phi^2 F^{\sigma\rho} F_{\sigma\rho} \end{aligned}$$

Contraction with k^{AB} yields

$$\hat{R} = R + \frac{1}{4}\phi F^{\mu\nu} F_{\mu\nu}$$

where \hat{R} is the curvature scalar of \mathcal{R}_5 and R is the curvature scalar of \mathcal{R}_4 . If we pick $\phi = -2\kappa/\mu_0$, the 5-dimensional Einstein vacuum equation $\hat{R}_{AB} = 0$ will yield the Einstein-Maxwell equations in the absence of matter and charge/current. Unfortunately the additional restriction $F^{\sigma\rho} F_{\sigma\rho} = 0$ is also imposed, thus not even the source-free Maxwell equations are produced in their full generality.

With a sleight of hand we can remove this unwanted restriction. Since $k_{AB} \neq k_{AB}(x^4)$ the integrand 2.19 will not depend on x^4 either. In order to make the action 2.19 finite, the fifth coordinate must have finite measure. We can achieve this by postulating the extra spatial dimension to be compact, with the geometry of a circle. Thus $x^4 \in [0, L]$, where $L \in \mathbb{R}$ is the circumference

of the circle.

$$\begin{aligned}
I &= -\frac{1}{2\hat{\kappa}} \int_{x^4=0}^{x^4=L} \left(\int_{V_4} \sqrt{-k} \hat{R} d^4x \right) dx^4 \\
&= -\frac{L}{2\hat{\kappa}} \int_{V_4} \sqrt{-k} \hat{R} d^4x \\
&= -\frac{1}{2\kappa} \int_{V_4} \sqrt{-g} \phi \left(R + \frac{1}{4} \phi F^{\mu\nu} F_{\mu\nu} \right) d^4x \quad \kappa \equiv \hat{\kappa}/L \quad (2.22)
\end{aligned}$$

Earlier on we could have chosen to scale the 5-dimensional metric by what is known as the *Weyl factor*: $k_{AB} \rightarrow \phi^{-1/3} k_{AB}$. Since this would have been a messy substitution to keep track of we shall simply make use of the result here. Making this substitution will remove of the factor of ϕ from under the square root sign in equation 2.22 while leaving it identical in all other respects, see [20] and [21]. We write down this modified version of equation 2.22.

$$I = -\frac{1}{2\kappa} \int_{V_4} \sqrt{-g} \left(R + \frac{1}{4} \phi F^{\mu\nu} F_{\mu\nu} \right) d^4x \quad (2.23)$$

If we pick $\phi = -1/\mu_0$, this leads to the Einstein equation (in the absence of matter) by the principle of least action, and to the Maxwell equations (in the absence of charge/current) via the Euler-Lagrange equations for the dynamics of the field A_μ . Note the negative sign in the value chosen for ϕ in order to give the correct Maxwellian Lagrangian. This is why the extra dimension is spatial.

2.6 Disadvantages of Kaluza-Klein theory

The weaknesses of Kaluza-Klein theory are as follows:

- The formulation of Kaluza-Klein theory is not covariant with respect to 5-dimensional coordinate transformations. This is due to the additional symmetry $k_{AB,4} = 0$.
- The original action 2.19 has equations of motion $\hat{R}_{AB} = 0$. The new action 2.23 no longer satisfies these, yet it was derived from 2.19.

- Kaluza-Klein theory does not produce Maxwell's equations in their full generality, i.e. the source equation 2.12 is given only in the limited case $J_k = 0$.
- This unification of gravity and electromagnetism does not include gravitational fields induced by the presence of mass. Since all known massless particles are of neutral charge, see [18], this makes sense of why Kaluza-Klein theory imposes the stringent condition $J_k = 0$. It is because the existence of a charge distribution requires the presence of massive particles - of which we have none.
- The significance of the fifth spatial dimension is unclear. Here it has simply been employed as a mathematical device to achieve a given purpose. Should we attribute to it any physical significance? Indeed we have presented no natural explanation for the employment of this method other than 'it works'.

Ultimately, unity of all the fundamental interactions is sought after, not just gravity and electromagnetism. The goal of this thesis is not to salvage Kaluza-Klein theory. Rather the point of considering it has been to illuminate the, or more correctly, *a* process of unification. In light of this we can observe some guiding principles which we see fit for the pursuit of any physical model.

- The addition of dimensions should be clearly motivated and, if possible, be accompanied by a physical interpretation.
- Once a framework has been developed, results ought to follow naturally, rather than by ad hoc maneuvers. Being forced into an ad hoc position may indicate the necessity to revise the theory.

Chapter 3

The Lie algebra $so(2, 3)$

In physics the local symmetries of spacetime are described by the Poincaré group. We can approximate the Poincaré group with the Galilean group, in the limit that the speed of light is infinite. In a similar manner the Poincaré group itself approximates a group called $SO(2, 3)$, often referred to as the *anti-de Sitter group*. We are interested in the consequences of choosing to use the group $SO(2, 3)$ to describe local spacetime symmetries. While the group $SO(2, 3)$ is of relevance to the study of anti-de Sitter/conformal field theory (or AdS/CFT) correspondence, AdS/CFT is not something considered here. Much of the notation and explanation has been adapted from [1].

3.1 The Lie algebra $so(2, 3)$

For coordinates λ, t, x, y, z in \mathbb{R}^5 we define $SO(2,3)$ as the Lie group of 5×5 real matrices which conserve the bilinear form

$$F(u, v) = u_\lambda v_\lambda + u_t v_t - u_x v_x - u_y v_y - u_z v_z \quad u, v \in \mathbb{R}^5$$

i.e. if $A \in SO(2, 3)$ then $F(Au, Av) = F(u, v)$. We can of course write the bilinear form $F(u, v)$ as $u^T F v$ where F is now the matrix

$$F = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

Hence for $A \in SO(2, 3)$

$$\begin{aligned} u^T F v &= (Au)^T F (Av) \\ &= u^T A^T F A v \end{aligned}$$

but since this is true for any $u, v \in \mathbb{R}^5$ we can just say

$$A^T F A = F \tag{3.1}$$

In principle we can find what the elements of $SO(2, 3)$ look like using 3.1, however it is better for us to consider the Lie algebra $so(2, 3)$. The Lie group $SO(2, 3)$ is also a matrix Lie group, hence following the approach found on p. 39 of [7], the (matrix) Lie algebra $so(2, 3)$ consists of all matrices X such that $e^{\theta X}$ is in $SO(2, 3)$ for all real numbers θ . Finding what a general matrix $X \in so(2, 3)$ looks like will enable us to find a basis for $so(2, 3)$. Since θ can be any real number, we take it to be small. Thus we shall only need to consider an element of the Lie group up to first order in θ , i.e. $e^{\theta X} = I + \theta X$. Substitute $I + \theta X$ into 3.1

$$\begin{aligned} F &= (I + \theta X)^T F (I + \theta X) \\ &= (I + \theta X^T) F (I + \theta X) \\ &= F + \theta X^T F + \theta F X + \theta^2 X^T F X \\ &= F + \theta X^T F + \theta F X \quad \text{(First order in } \theta \text{.)} \end{aligned}$$

The form of X can therefore be determined by the following relation.

$$X^T F = -F X \tag{3.2}$$

Let us write these matrices as

$$F = \begin{pmatrix} I_2 & 0 \\ 0 & -I_3 \end{pmatrix} \quad X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

each entry in X having equal dimensions to the corresponding entry in F .

Thus 3.2 (upon simplification) is

$$\begin{pmatrix} A^T & -C^T \\ B^T & -D^T \end{pmatrix} = \begin{pmatrix} -A & -B \\ C & D \end{pmatrix}$$

The matrices A , $B = C^T$ and D must therefore take the form

$$A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \quad B = \begin{pmatrix} b & c & d \\ e & f & g \end{pmatrix} \quad D = \begin{pmatrix} 0 & h & i \\ -h & 0 & j \\ -i & -j & 0 \end{pmatrix}$$

Thus X can have up to 10 independent entries.

$$X = \begin{pmatrix} 0 & a & b & c & d \\ -a & 0 & e & f & g \\ b & e & 0 & h & i \\ c & f & -h & 0 & j \\ d & g & -i & -j & 0 \end{pmatrix}$$

i.e. $so(2,3)$ has basis of dimension 10. We choose a particular basis which is given in table 3.1. Table 3.2 gives the commutators for these matrices.

Table 3.1: A basis for the canonical representation of $so(2,3)$.

$$T = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$X = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$I = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad J = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix} \quad K = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

3.2 Anti de Sitter space

Following [8], let M_5 refer to the 5-dimensional flat space with metric signature $(+, +, -, -, -)$. In such a space

$$\begin{aligned} ds^2 &= d\lambda^2 + c^2 dt^2 - dx^2 - dy^2 - dz^2 \\ &= d\lambda^2 + \eta_{ij} dx^i dx^j \end{aligned} \quad (3.3)$$

η_{ij} is the Minkowski metric. Consider the hypersurface S_4 embedded in M_5 given by the equation of a hypersphere of ‘radius’ a

$$\lambda^2 + \eta_{ij} x^i x^j = a^2 \quad (3.4)$$

Table 3.2: Commutation relations for the Lie algebra $so(2, 3)$.

	T	X	Y	Z	A	B	C	I	J	K
T	0	A	B	C	-X	-Y	-Z	0	0	0
X	-A	0	-K	J	-T	0	0	0	Z	-Y
Y	-B	K	0	-I	0	-T	0	-Z	0	X
Z	-C	-J	I	0	0	0	-T	Y	-X	0
A	X	T	0	0	0	-K	J	0	C	-B
B	Y	0	T	0	K	0	-I	-C	0	A
C	Z	0	0	T	-J	I	0	B	-A	0
I	0	0	Z	-Y	0	C	-B	0	K	-J
J	0	-Z	0	X	-C	0	A	-K	0	I
K	0	Y	-X	0	B	-A	0	J	-I	0

It is the maximally symmetric subspace of M_5 and is known as *anti de Sitter space* or the AdS manifold (this is not the same as an ADS manifold defined in chapter 4). When acting on the AdS manifold, $SO(2, 3)$ is known as the *anti de Sitter group* (AdS group). On S_4 we can write λ as a function of the other four coordinates. To express the invariant interval (on S_4) independent of the λ coordinate, differentiate 3.4 and substitute it into 3.3

$$ds^2 = \eta_{ij} dx^i dx^j + \frac{(\eta_{ij} x^i dx^j)^2}{a^2 - \eta_{mn} x^m x^n} \quad (3.5)$$

S_4 inherits natural time and distance scales from \mathbb{R}^5 as follows. The radius of S_4 is a , informally we call it the *radius of the universe*, hence it makes sense to define a metres = 1 natural distance unit. It then follows that a/c seconds = 1 natural time unit, given that c , the speed of light, is the natural unit for velocity. Following Hawthorn [1] we define $a = rc$ so that r is the radius of the universe as measured in seconds. Accordingly

- rc metres = 1 natural distance unit.
- r seconds = 1 natural time unit.

The operators in table 3.1 can be identified by examining the way in which

they act on the AdS manifold.

Consider the neighbourhood of the point $\lambda = a$ on S_4 or equivalently $\lambda = 1$ in natural units, this is precisely the neighbourhood of $x^i = 0$. In this region the metric tensor is closely approximated by

$$g_{ij} = \eta_{ij} + \frac{x^i x^j}{a^2} \quad (3.6)$$

Consider applying the group element $e^{\theta T}$ where θ is small, to a point in the neighbourhood of $\lambda = 1$ (we are using natural units). Such a point is given by the coordinate vector $(1, t, x, y, z)^T$, where t, x, y, z are small.

$$\left(\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} + \theta \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right) \begin{pmatrix} 1 \\ t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 - \theta t \\ t + \theta \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ t + \theta \\ x \\ y \\ z \end{pmatrix}$$

The θt term can be ignored as it is second order in small terms. The group operation $e^{\theta T}$ has translated the time coordinate by θ natural units (of time). Similarly we find that X, Y, Z are related to translation through space, A, B, C to Lorentz boost, and I, J, K to rotation.

Let us now consider the basis in which these operators translate an inertial frame through space, time etc. by 1 ordinary unit (e.g. metres, seconds). This is the basis $\{\frac{1}{r}T, \frac{1}{rc}X, \frac{1}{rc}Y, \frac{1}{rc}Z, \frac{1}{c}A, \frac{1}{c}B, \frac{1}{c}C, I, J, K\}$. If we commute the elements of this new basis we find some of the results have extra factors of $1/c$ or $1/r$ than is shown by table 3.2 if we simply substitute in the new basis elements. In particular the commutation relations between T, X, Y, Z produce an additional factor of $1/r^2$, hence in the limit $r \rightarrow \infty$ these elements will in fact commute. Furthermore, the commutator table for $so(2, 3)$ reduces to the commutation relations for the generators of the Poincaré Lie algebra. This process is called *contraction* [14] and we say that the AdS group contracts to

the Poincaré group as the radius of the universe tends to infinity. In a similar fashion the Poincaré group contracts to the Galilean group as the speed of light tends to infinity. This is why the Galilean and Poincaré groups are functionally equivalent for practical purposes when the velocities involved are much less than the speed of light. Likewise $SO(2, 3)$ and the Poincaré group are functionally equivalent provided the distance scales considered are much less than rc metres.

We note also that the AdS metric 3.6 reduces to the Minkowski metric η_{ij} as r (and hence a) tends to infinity. The curved anti de Sitter space becomes the flat Minkowski spacetime as the radius of the universe extends.

3.3 Why $so(2, 3)$?

The AdS metric permits the existence of closed timelike curves, e.g. the curve $(\lambda, ct, x, y, z) = (a \sin \tau, a \cos \tau, 0, 0, 0)$, where τ is the proper time. These curves contradict the notion of causality hence the metric 3.5 is typically inappropriate for the purposes of a cosmological model. However we are not trying to claim anything about the global structure of spacetime. These closed timelike curves are then no problem to us since we wish to employ $so(2, 3)$ to describe *local* symmetry. Indeed *locally* the causal structure of metric 3.5 is of the same qualitative nature as Minkowski spacetime, see p. 195 of [9].

We shall assume the parameter r is sufficiently large so that the action of $so(2, 3)$ on spacetime is locally indistinguishable from that of the Poincaré Lie algebra. We are then free to use $so(2, 3)$ as the locally symmetry group of spacetime for the purposes of classical physics. Hence we shall explore the following assumption.

Axiom 3.1 *The Lie algebra $so(2, 3)$ describes the local symmetry of spacetime*

At this point one may ask: what benefit is there in adopting assumption 3.1 if we cannot make any practical distinction? To answer this question we need to consider quantum mechanics. The symmetry group of spacetime can act not

just *extrinsically*, on spacetime itself, but also *intrinsically*, on the range space of the wave function of a particle. For example, rotation operators can act both on the spacetime in which a particle sits, giving eigenvalues of angular momentum, and on (the range space of the wave function of) the particle itself, giving discrete eigenvalues of spin (intrinsic angular momentum).

Consider now the set of compatible observables $\{T, I\}$. In the terminology of [13] the operator I (we mean $e^{\theta I}$) is cyclic (for either group), hence it gives discrete (spin) eigenvalues when acting on a wave function.

But the operator T from the Poincaré group is a hyperbolic (not cyclic) operator and so it has a continuous spectrum of eigenvalues when acting on the range space of a wave function. Since we do not know of a quantum number with a continuous spectrum which would correspond to this action, we conclude that this particular element of the group does not act intrinsically. While it comes as a surprise that this second sort of (intrinsic) action should exist at all, it is bizarre that it should be allowed or disallowed in a seemingly unclear fashion.

In the case of $SO(2, 3)$, T is indeed a cyclic operator giving rise to discrete intrinsic eigenvalues. In fact the fundamental representation of $so(2, 3)$ is 4-dimensional with two quantum numbers: intrinsic angular momentum taking values $\pm\frac{1}{2}$, and intrinsic energy also taking values $\pm\frac{1}{2}$ (in natural units).

Solutions to the Dirac equation (fermions) are characterised by the two quantum numbers, spin and charge. If we identify intrinsic energy as charge this fits precisely with the intrinsic spectrum of $so(2, 3)$. Such a link is quite reasonable, indeed positrons can be described as electrons travelling backwards in time, see [27].

3.4 The Lie algebra $sp(4, \mathbb{R})$

In this section we define the symplectic group $Sp(4, \mathbb{R})$ and consider how it relates to $SO(2, 3)$.

Definition 3.1 A **symplectic form** is a bilinear form $\Omega : \mathbb{R}^4 \times \mathbb{R}^4 \rightarrow \mathbb{R}$ satisfying:

Total isotropy, $\Omega(v, v) = 0, \forall v \in \mathbb{R}^4$; and

Nondegeneracy, if $\Omega(u, v) = 0, \forall v \in \mathbb{R}^4$, then $u = 0$.

Proposition 3.2 A symplectic form is antisymmetric.

Proof. The symplectic form Ω is totally isotropic, hence $\Omega(u + v, u + v) = 0$ for all $u, v \in \mathbb{R}^4$. But Ω is a bilinear form

$$\begin{aligned}\Omega(u + v, u + v) &= \Omega(u, u) + \Omega(u, v) + \Omega(v, u) + \Omega(v, v) \\ &= \Omega(u, v) + \Omega(v, u)\end{aligned}$$

Thus $\Omega(u, v) = -\Omega(v, u)$ for all $u, v \in \mathbb{R}^4$. □

The **symplectic group** $Sp(4, \mathbb{R})$ is the Lie group of 4×4 real matrices which preserve a symplectic form. Elements of $Sp(4, \mathbb{R})$ are called **symplectic matrices**. Consider the matrix

$$\Omega = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

it defines a symplectic form (with respect to a particular basis) given by $\Omega(x, y) = x^T \Omega y$. A matrix $A \in Sp(4, \mathbb{R})$ must satisfy $A^T \Omega A = \Omega$. Similarly elements X of the Lie algebra $sp(4, \mathbb{R})$ must satisfy $(I + X)^T \Omega (I + X) = \Omega$, or simply (neglecting second order terms)

$$\Omega X = -X^T \Omega \tag{3.7}$$

Let

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{and} \quad \Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

where $A, B, C, D, I, 0$ are 2×2 real matrices, in particular I is the identity and 0 the zero matrix. From 3.7 we can deduce the dimension of the Lie algebra.

$$\begin{aligned} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} &= - \begin{pmatrix} A & B \\ C & D \end{pmatrix}^T \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \\ \Rightarrow \begin{pmatrix} C & D \\ -A & -B \end{pmatrix} &= \begin{pmatrix} C^T & -A^T \\ D^T & -B^T \end{pmatrix} \end{aligned}$$

So $A = -D^T$, $B = B^T$ and $C = C^T$. Thus B and C each have 3 independent entries and the 4 independent entries of A completely determine D (and vice versa). In total any matrix $X \in sp(4, \mathbb{R})$ has up to 10 independent entries, hence the Lie algebra - and therefore the Lie group - have bases of dimension 10. Table 3.3 gives one such basis, the elements of the basis are given the familiar names T, X, Y, \dots etc. since they commute with each other in the exact manner prescribed by table 3.2. Hence the Lie algebra $sp(4, \mathbb{R})$ is isomorphic to $so(2, 3)$.

Table 3.3: A basis for the Lie algebra $sp(4, \mathbb{R})$

$$\begin{aligned} T &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} & X &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\ Y &= \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} & Z &= \frac{1}{2} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \\ A &= \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} & B &= \frac{1}{2} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \end{aligned}$$

$$C = \frac{1}{2} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad I = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$J = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad K = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

Let us call a matrix P satisfying $P^T\Omega = \Omega P$, an Ω -**symmetric** matrix. If P is Ω -symmetric and $M \in sp(4, \mathbb{R})$ then, using the usual notation $[,]$ for the commutator

$$\begin{aligned} \Omega[M, P] &= \Omega MP - \Omega PM = -M^T\Omega P - P^T\Omega M \\ &= -M^T P^T\Omega + P^T M^T\Omega \\ &= (-M^T P^T + P^T M^T)\Omega \\ &= (-PM + MP)^T\Omega \\ &= [M, P]^T\Omega \end{aligned}$$

Thus $[M, P]$ is Ω -symmetric as well which means the 4×4 Ω -symmetric matrices form a Lie algebra representation of $sp(4, \mathbb{R})$. This representation is 6-dimensional, a basis is given in table 3.4. It is reducible and is the direct sum of the irreducible representations of dimensions 1 and 5. We have shown that the 16 dimensional space of 4×4 matrices can be decomposed into irreducible representations of dimension 1, 5 and 10 under the action of $sp(4, \mathbb{R})$.

Table 3.4: A basis for the 6-dimensional representation of the Lie algebra $sp(4, \mathbb{R})$

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad P_\lambda = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$P_T = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad P_X = \frac{1}{2} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P_Y = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad P_Z = \frac{1}{2} \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

3.5 The adjoint representation

The **adjoint** map

$$\mathbf{ad} : \mathfrak{g} \rightarrow \mathfrak{g}$$

defined by

$$\mathbf{ad}_X(Y) = [X, Y]$$

is a Lie algebra endomorphism and therefore a representation of \mathfrak{g} . The adjoint map is also linear

$$\mathbf{ad}_X(fY) = [X, fY] = f[X, Y] = f\mathbf{ad}_X(Y)$$

$$\mathbf{ad}_X(Y + Z) = [X, Y + Z] = [X, Y] + [X, Z] = \mathbf{ad}_X(Y) + \mathbf{ad}_X(Z)$$

where f is a scalar and X, Y, Z are in \mathfrak{g} . We can build a basis for the adjoint representation of $so(2, 3)$ from our knowledge of how commutation relations

on table 3.2. Let us consider the operator

$$\text{ad}_T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which acts on 10-vectors $(t, x, y, z, a, b, c, i, j, k)^T$. We know $[T, X] = A$, hence ad_T should map $(0, 1, 0, 0, 0, 0, 0, 0, 0, 0)^T$ to the vector $(0, 0, 0, 0, 1, 0, 0, 0, 0, 0)^T$. This is the function of the 1 entered in row 5, column 2 in ad_T , the other five non-zero commutators involving T explain the five remaining non-zero entries. A basis for the adjoint representation is given in appendix A.

3.6 The representation theory of $so(2, 3)$

In this section we briefly summarise some important results from the representation theory of $so(2, 3)$. These are outlined in [1] which makes use of [7] and [25].

The Lie algebra $so(2, 3)$ is simple, we may therefore construct weight diagrams for every irreducible representation (although this is only practical for ones of low dimension). In particular we will be able to deduce the fundamental representation of $so(2, 3)$. From the fundamental representation one can ‘build’ all other irreducible representations by finding the invariant subspaces of tensor products of the fundamental representation and its dual.

In the adjoint representation the operators T and I are diagonalisable over \mathbb{C} . It can be shown therefore, that the images of these operators in any finite

dimensional representation will be diagonalisable over an algebraically closed field. In particular the matrices of T and I are diagonalisable over any finite dimensional complex representation.

The operators T and I form a maximal Cartan subalgebra, viz. the largest possible set of mutually commuting elements from the basis of $so(2, 3)$, so they are simultaneously diagonalisable and indeed simultaneously diagonalisable in any finite dimensional complex representation, which means $so(2, 3)$ has a basis of simultaneous eigenvectors for T and I .

Definition 3.2 *If π is a representation of $so(2, 3)$ on V , then an ordered pair $\mu = (q, s)$ is called a **weight** for π if there exists $v \neq 0$ in V such that*

$$\begin{aligned}\pi(T)v &= iqv \\ \pi(I)v &= isv\end{aligned}\tag{3.8}$$

*(the factors of i are inserted to make the weights real and thereby maintain consistency with the way physicists talk about spin). The vector v is called a **weight vector** corresponding to the weight μ . The set of all weight vectors for a particular weight together with the zero vector is a vector subspace of V called the **weight space**.*

From our previous statements we conclude that the weight space gives a basis for V . We establish an ordering on weights by saying $(q_1, s_1) > (q_2, s_2)$ if $q_1 > q_2$ or if $q_1 = q_2$ and $s_1 > s_2$. A finite dimensional representation of $so(2, 3)$ is characterised by its highest weight.

Definition 3.3 *The **degree** of a weight in a representation (π, V) is defined as the dimension of corresponding subspace (with respect to V) of weight vectors.*

In a finite dimensional representation the degree of a weight is calculated by Konstant's formula (see Theorem 7.42 in [7]). The sum of the degrees of all weights gives the dimension of the representation. Weights together with their degrees can be depicted in *weight diagrams*, see pp. 13-15 of [1].

Chapter 4

The Hawthorn model

In this chapter we develop what shall be referred to as the *Hawthorn model*. This development will closely follow that provided by chapter 2 in [1] since this is our only source.

4.1 The spacetime manifold

We have examined a few of the representations of $so(2, 3)$ and seen that observation does not rule it out as a candidate local symmetry group. On the contrary discrete eigenvalues for charge and spin arise quite naturally out of the fundamental representation. It is now time for us to make mathematically precise what it means for us to use $so(2, 3)$ to describe local symmetry.

We could tack the Lie algebra $so(2, 3)$ onto an arbitrary manifold, but such an approach would be unsatisfactory. We wish to attach the Lie algebra to a manifold in a natural fashion so that it arises from the structure of the manifold itself. In particular the local symmetry at every point on the manifold ought to be described by the Lie algebra. The manifold must give rise to the tangent space $so(2, 3)$ at every point, so in order to make things work properly and naturally, it is best for the manifold to be of the same dimension as the Lie algebra.

It may seem that our choice of a 10-dimensional manifold (as opposed to the usual 4-dimensional spacetime) will create more problems than it is worth

when it comes to explaining away the extra six dimensions. But this is not what we are trying to do here. Rather than bothering ourselves with how to abandon these extra dimensions once they have served a given purpose, we wish to embrace the extra dimensions as physical dimensions - those of rotation and Lorentz boost. We shall call them *Lorentz dimensions*. Our manifold is thus the 10-dimensional manifold of inertial frames. To properly locate an event on this manifold one must specify the orientation and instantaneous reference velocity of the inertial frame (in addition to the position and time coordinates).

We are immediately confronted with a conundrum when we consider curvature. Curvature describes the failure of parallel transport to commute - we are not guaranteed things will look the same if we take an alternate route to the same point on the manifold i.e. translations do not commute. But the Lie algebra itself also describes the nature of translation's failure to commute. It seems we have two different mathematical structures competing to describe the same thing. To resolve this clash we must first develop some clear notation.

4.2 The covariant derivative

To do physics we need a means by which we may compare tensor quantities at two different points of the manifold. On a curved manifold partial derivatives will not (in general) suffice for this task as they do not transform like tensors. We define a more general operator satisfying some familiar conditions.

Definition 4.1 *A tensor derivation D on a manifold \mathcal{M} is a linear map $D : \text{tensors} \rightarrow \text{tensors}$ that obeys the Leibniz condition on tensor products and commutes with contraction (of the tensors it operates on).*

Definition 4.2 *An ordinary derivation is a tensor derivation which maps scalar functions to scalar functions.*

There is no guarantee that the composition of tensor derivations will obey the Leibniz condition, and hence be another tensor derivation. However we can

establish the following proposition.

Proposition 4.1 *If D and E are tensor derivations, then $[D, E]$ is also a tensor derivation where $[D, E](X) = D(E(X)) - E(D(X))$.*

Proof. For tensor derivations D and E , tensors X and Y and scalar f we need only to show that $[D, E]$ is linear, Leibniz and commutes with contraction.

Linearity

$$\begin{aligned} [D, E](fX) &= D(E(fX)) - E(D(fX)) \\ &= D(fE(X)) - E(fD(X)) \\ &= fD(E(X)) - fE(D(X)) \\ &= f[D, E](X) \end{aligned}$$

$$\begin{aligned} [D, E](X + Y) &= D(E(X + Y)) - E(D(X + Y)) \\ &= D(E(X) + E(Y)) - E(D(X) + D(Y)) \\ &= D(E(X)) + D(E(Y)) - E(D(X)) - E(D(Y)) \\ &= [D, E](X) + [D, E](Y) \end{aligned}$$

Leibniz

$$\begin{aligned} [D, E](XY) &= D(E(XY)) - E(D(XY)) \\ &= D(E(X)Y + XE(Y)) - E(D(X)Y + XD(Y)) \\ &= D(E(X)Y) + D(XE(Y)) - E(D(X)Y) - E(XD(Y)) \\ &= D(E(X))Y + E(X)D(Y) + D(X)E(Y) + XD(E(Y)) \\ &\quad - E(D(X))Y - D(X)E(Y) - E(X)D(Y) - XE(D(Y)) \\ &= [D, E](X)Y + X[D, E](Y) \end{aligned}$$

Contraction

The tensor X may contract in some fashion (either with itself or with another tensor Y). The operator $[D, E]$ can be reduced down to a combination of a series of operations, all of which conserve contraction. Hence $[D, E]$ itself will also conserve contraction. Thus the map $[D, E]$ is seen to satisfy the three properties of a tensor derivation. \square

Definition 4.3 Let D be a tensor derivation and f a tensor of rank zero. Then $D(f)$ is a tensor of rank $\binom{i}{j}$, say. We define $\binom{i}{j}$ to be the **rank** of D . We can therefore write D using index notation: $D_{\beta_1\beta_2\dots\beta_j}^{\alpha_1\alpha_2\dots\alpha_i}$.

Proposition 4.2 Every tensor derivation of rank $\binom{i}{j}$ maps tensors of rank $\binom{k}{l}$ to tensors of rank $\binom{i+k}{j+l}$.

Proof. Let the tensor derivation D be of rank $\binom{i}{j}$ and the tensors f and T be of ranks zero and $\binom{k}{l}$ respectively. Using the Leibniz condition

$$D(fT) = D(f)T + fD(T)$$

The first term (on the RHS) clearly has rank $\binom{i+k}{j+l}$, hence so does the second term. Given that f is of rank zero it follows that $D(T)$ is of rank $\binom{i+k}{j+l}$. \square

Proposition 4.3 If D is a tensor derivation and S is any tensor, then $S \otimes D$ is a tensor derivation, where $(S \otimes D)(T) = S \otimes D(T)$.

Proof. Similar to that of proposition 4.1. \square

The forthcoming propositions contribute to determining what a general tensor derivation might look like.

Proposition 4.4 $D - a^i \frac{\partial}{\partial x^i}$ is a tensor derivation of rank $\binom{0}{0}$ which maps all functions to the zero function, where D is a tensor derivation of rank $\binom{0}{0}$ and a^i are (real) vector components.

Proof. Every ordinary derivation $a^i \frac{\partial}{\partial x^i}$ can be extended to a tensor derivation of rank $\binom{0}{0}$ by allowing it to act on the components of a tensor. Conversely, every tensor derivation D of rank $\binom{0}{0}$ acts on functions as an ordinary derivation. Hence in any coordinate system we can find a vector field a^i such that for any function f , $D(f) = a^i \frac{\partial}{\partial x^i}(f)$. It follows that $D - a^i \frac{\partial}{\partial x^i}$ is a tensor derivation of rank $\binom{0}{0}$ and $(D - a^i \frac{\partial}{\partial x^i})(f) = 0$. \square

Proposition 4.5 *Let E be a tensor derivation of rank $\binom{0}{0}$ with $E(f) = 0$ for all functions f on \mathcal{M} . There exists a tensor Γ_j^i of rank $\binom{1}{1}$ such that*

$$E(X_{\beta_1\beta_2\dots\beta_n}^{\alpha_1\alpha_2\dots\alpha_m}) = \sum_s \Gamma_{\hat{\alpha}_s}^{\alpha_s} X_{\beta_1\beta_2\dots\beta_n}^{\alpha_1\dots\hat{\alpha}_s\dots\alpha_m} - \sum_t \Gamma_{\hat{\beta}_t}^{\beta_t} X_{\beta_1\dots\hat{\beta}_t\dots\beta_n}^{\alpha_1\alpha_2\dots\alpha_m}$$

Proof. If v is a vector field and f is a scalar field on \mathcal{M} then $E(fv) = E(f)v + fE(v)$. But $E(f) = 0$ hence $E(fv) = fE(v)$ which means E acts linearly on the tangent vector fields of \mathcal{M} . Such an action is given by contraction of a vector field with a tensor of rank $\binom{1}{1}$ (local matrix multiplication).

Consider the vector fields $\{\mathbf{e}_i\}$ which form a basis of the tangent spaces at each point of \mathcal{M} . We can thus write the vector field in terms of this basis $v = v^i \mathbf{e}_i$ in order to explicitly find this tensor. Thus

$$E(v) = E(v^i \mathbf{e}_i) = v^i E(\mathbf{e}_i) = v^i \Gamma_i^j \mathbf{e}_j$$

for some Γ_i^j . If we describe the tensor simply in terms of coordinates

$$E(v^j) = v^i \Gamma_i^j$$

Since

$$0 = E(u_i v^i) = E(u_i) v^i + u_i E(v^i) = E(u_i) v^i + u_i \Gamma_i^t v^t$$

for all v , it follows that

$$E(u_j) = -\Gamma_j^i u_i$$

It is an exercise in mathematical induction to show this for a tensor of arbitrary rank. □

Conversely, if Γ is a tensor of rank $\binom{1}{1}$, then $\Gamma \binom{*}{*}$ defined by

$$\Gamma \binom{*}{*} (X_{\beta_1\beta_2\dots\beta_n}^{\alpha_1\alpha_2\dots\alpha_m}) = \sum_s \Gamma_{\hat{\alpha}_s}^{\alpha_s} X_{\beta_1\beta_2\dots\beta_n}^{\alpha_1\dots\hat{\alpha}_s\dots\alpha_m} - \sum_t \Gamma_{\hat{\beta}_t}^{\beta_t} X_{\beta_1\dots\hat{\beta}_t\dots\beta_n}^{\alpha_1\alpha_2\dots\alpha_m}$$

is a tensor derivation. Thus we have shown that $(D - a^i \frac{\partial}{\partial x^i})(f) = -\Gamma \binom{*}{*} (f)$, therefore all rank $\binom{0}{0}$ tensor derivations are of the form

$$a^i \frac{\partial}{\partial x^i} + \Gamma \binom{*}{*}$$

Proposition 4.6 *Every tensor derivation of rank $\binom{m}{n}$ is of the form*

$$D_{\mu_1 \dots \mu_n}^{\lambda_1 \dots \lambda_m} = a_{\mu_1 \dots \mu_n}^{\lambda_1 \dots \lambda_m i} \frac{\partial}{\partial x^i} + \Gamma_{\mu_1 \dots \mu_n}^{\lambda_1 \dots \lambda_m} (*)$$

where

$$\Gamma_{\mu_1 \dots \mu_n}^{\lambda_1 \dots \lambda_m} (*) (X_{\beta_1 \beta_2 \dots \beta_n}^{\alpha_1 \alpha_2 \dots \alpha_m}) = \sum_s \Gamma_{\mu_1 \dots \mu_n \hat{\alpha}_s}^{\lambda_1 \dots \lambda_m \alpha_s} X_{\beta_1 \beta_2 \dots \beta_n}^{\alpha_1 \dots \hat{\alpha}_s \dots \alpha_m} - \sum_t \Gamma_{\mu_1 \dots \mu_n \hat{\beta}_t}^{\lambda_1 \dots \lambda_m \beta_t} X_{\beta_1 \dots \hat{\beta}_t \dots \beta_n}^{\alpha_1 \alpha_2 \dots \alpha_m}$$

Proof. Each component of $D_{\mu_1 \dots \mu_n}^{\lambda_1 \dots \lambda_m}$ is individually a general tensor derivation of rank $\binom{0}{0}$. \square

Of importance to us are the tensor derivations of rank $\binom{0}{1}$, viz.

$$D_i = a_i^j \frac{\partial}{\partial x^j} + \Gamma_i (*)$$

In particular we shall work with a distinguished covariant derivative ∇_i which is the tensor derivation given by $a_i^j = 1_i^j$. Because of the way it appears in the covariant derivative, we identify Γ_{ij}^k as the affine connection.

4.3 The Bianchi identities

Let \mathcal{M} be a manifold with distinguished covariant derivative ∇_i . The commutator of two tensor derivations is (as we have shown) a tensor derivation, in particular $[\nabla_i, \nabla_j]$ is a tensor derivation of rank $\binom{0}{2}$. We may thus write

$$[\nabla_i, \nabla_j] = T_{ij}^k \frac{\partial}{\partial x^k} + K_{ij} (*) \quad (4.1)$$

If we let both sides operate on a function f

$$-\Gamma_{ij}^k \frac{\partial f}{\partial x^k} + \Gamma_{ji}^k \frac{\partial f}{\partial x^k} = T_{ij}^k \frac{\partial f}{\partial x^k}$$

and therefore

$$T_{ij}^k = -(\Gamma_{ij}^k - \Gamma_{ji}^k)$$

which is the negative of the torsion tensor as it is usually defined. Similarly by applying $[\nabla_i, \nabla_j]$ to a vector field v^k and comparing the terms which do not involve partial derivatives of v^k

$$K_{ij}^k = \frac{\partial \Gamma_{jx}^k}{\partial x^i} - \frac{\partial \Gamma_{ix}^k}{\partial x^j} + \Gamma_{it}^k \Gamma_{jx}^t - \Gamma_{jt}^k \Gamma_{ix}^t + T_{ij}^t \Gamma_{tx}^k$$

We see that K_{ijx}^k is the usual Riemann curvature tensor plus an extra torsion term - in the torsion-free case it is the Riemann tensor.

Smooth operators will obey the Jacobi identity

$$[[\nabla_i, \nabla_j], \nabla_k] + [[\nabla_j, \nabla_k], \nabla_i] + [[\nabla_k, \nabla_i], \nabla_j] = 0$$

We can use this to deduce the well known Bianchi identities which place crucial constraints on curvature. However, evaluating the Jacobi identity by expressing $[\nabla_i, \nabla_j]$ in terms partial derivatives is not a nice way to proceed with the calculation so we seek to express $[\nabla_i, \nabla_j]$ in terms of ∇_k . We therefore define

$$[\nabla_i, \nabla_j] = T_{ij}^k \nabla_k + R_{ij}^{\quad (*)} \quad (4.2)$$

where the coefficients T_{ij}^k are the same as those in equation 4.1 because covariant and partial derivatives act identically on functions. From this definition

$$\begin{aligned} T_{ij}^k \frac{\partial}{\partial x_k} + K_{ij}^{\quad (*)} &= T_{ij}^k \nabla_k + R_{ij}^{\quad (*)} \\ &= T_{ij}^k \frac{\partial}{\partial x_k} + T_{ij}^k \Gamma_k^{\quad (*)} + R_{ij}^{\quad (*)} \end{aligned}$$

$$\Rightarrow R_{ij}^{\quad (*)} = K_{ij}^{\quad (*)} - T_{ij}^l \Gamma_l^{\quad (*)}$$

It follows from the linearity of the operators $K_{ij}^{\quad (*)}$ and $T_{ij}^l \Gamma_l^{\quad (*)}$ that $R_{ij}^{\quad (*)}$ is also linear. Letting both sides operate on a vector v^k we find that R_{ijx}^k is the usual Riemannian tensor even when the manifold is not torsion-free. Having attained an expression of the commutator of nablas in terms of nabla we shall

now derive the Bianchi identities. Consider the first term of the Jacobi identity

$$\begin{aligned}
& [[\nabla_i, \nabla_j], \nabla_k](v^x) \\
&= [T_{ij}^l \nabla_l + R_{ij}^{\quad (*)}, \nabla_k](v^x) \\
&= (T_{ij}^l \nabla_l \nabla_k + R_{ij}^{\quad (*)} \nabla_k - \nabla_k T_{ij}^l \nabla_l - \nabla_k R_{ij}^{\quad (*)})(v^x) \\
&= T_{ij}^l \nabla_l (\nabla_k(v^x)) + R_{ij}^{\quad (*)} (\nabla_k(v^x)) - \nabla_k (T_{ij}^l \nabla_l(v^x)) - \nabla_k (R_{ij}^{\quad (*)} (v^x)) \\
&= T_{ij}^l \nabla_l (\nabla_k(v^x)) - R_{ijk}^t \nabla_t(v^x) + R_{ijs}^x \nabla_k(v^s) - \nabla_k (T_{ij}^l \nabla_l(v^x)) - T_{ij}^l \nabla_k (\nabla_l(v^x)) \\
&\quad - \nabla_k (R_{ijs}^x) v^s - R_{ijs}^x \nabla_k(v^s) \\
&= T_{ij}^l [\nabla_l, \nabla_k](v^x) - R_{ijk}^t \nabla_t(v^x) - \nabla_k (T_{ij}^l \nabla_l(v^x)) - \nabla_k (R_{ijs}^x) v^s \\
&= T_{ij}^l (T_{lk}^u \nabla_u + R_{lk}^{\quad (*)})(v^x) - R_{ijk}^t \nabla_t(v^x) - \nabla_k (T_{ij}^l \nabla_l(v^x)) - \nabla_k (R_{ijs}^x) v^s \\
&= (T_{ij}^l T_{lk}^t - \nabla_k (T_{ij}^t) - R_{ijk}^t) \nabla_t(v^x) + (T_{ij}^l R_{lks}^x - \nabla_k (R_{ijs}^x))(v^s)
\end{aligned}$$

If we cyclically permute the indices i, j, k in this expression we get the other two terms from the Jacobi identity. By the Jacobi identity, the sum of these three expressions is zero. We can now use the coefficient of $\nabla_t(v^x)$ to get the first Bianchi identity

$$\boxed{T_{ij}^l T_{lk}^t - \nabla_k (T_{ij}^t) - R_{ijk}^t \stackrel{(ijk)}{=} 0} \quad (4.3)$$

Here we have used the notation $Q_{ijk} \stackrel{(ijk)}{=} 0$ to abbreviate $Q_{ijk} + Q_{kij} + Q_{jki} = 0$. Note the relation $\stackrel{(ijk)}{=}$ sums together permutations of the LHS only. Similarly the coefficient of v^s gives the second Bianchi identity

$$\boxed{T_{ij}^l R_{lks}^x - \nabla_k (R_{ijs}^x) \stackrel{(ijk)}{=} 0} \quad (4.4)$$

4.4 Local Lie manifolds

Consider the manifold \mathcal{M} which is also a Lie group. A covariant derivative on a manifold can be defined in a natural manner from the action of the Lie algebra on the Lie group. In such a case the components of the torsion tensor T_{ij}^k will be precisely the structure coefficients c_{ij}^k . Hence the torsion obeys the Jacobi identity $T_{ij}^x T_{kx}^y \stackrel{(ijk)}{=} 0$. Given that the Lie algebra structure is the same

anywhere on the manifold (the Lie algebra is the tangent space at any point) it follows that the torsion tensor is invariant, $\nabla_l(T_{ij}^k) = 0$ everywhere.

Our interest is in manifolds possessing the aforementioned structural properties but without stipulating that they bear an entire Lie group structure. More precisely

Definition 4.4 *A Local Lie manifold is a manifold \mathcal{M} together with a covariant derivative ∇_k where the torsion T_{ij}^k satisfies*

$$\nabla_l(T_{ij}^k) = 0 \quad (4.5)$$

$$T_{ij}^x T_{kx}^y \stackrel{(ijk)}{=} 0 \quad (4.6)$$

On a local Lie manifold the torsion gives a Lie algebra structure on each tangent space. This Lie algebra is the same across the whole manifold. The trace of operators from the adjoint representation defines the killing form

$$k_{ij} = T_{ib}^a T_{ja}^b$$

which is a bilinear form. It is invariant

$$\nabla_x(k_{ij}) = \nabla_x(T_{ib}^a T_{ja}^b) = \nabla_x(T_{ib}^a) T_{ja}^b + T_{ib}^a \nabla_x(T_{ja}^b) = 0$$

If the Lie algebra is semisimple, the bilinear form will be non-degenerate and define a pseudometric on the manifold.

On a local Lie manifold the first Bianchi identity reduces to

$$\boxed{R_{ijk}^l \stackrel{(ijk)}{=} 0}$$

4.5 The Hawthorn universe

Axiom 4.7 *The universe of the Hawthorn model is a local Lie manifold for the Lie algebra $so(2,3)$. We call such a manifold an **ADS manifold**.*

Note that an ADS manifold is different from the AdS manifold talked about in chapter 3. We interpret the ADS manifold as the manifold of local inertial

frames, which is ten dimensional. Our motivations for choosing $so(2, 3)$ as the local symmetry group (for the local Lie manifold which we have postulated describes our universe) are summarised below.

- $so(2, 3)$ contracts to the Poincaré Lie algebra, hence they are locally indistinguishable.
- As $so(2, 3)$ is semisimple it defines an intrinsic distance scale (unlike the Poincaré Lie algebra). Thus a non-degenerate metric naturally arises. Using the basis in table A.1 the metric k_{ij} is diagonal with values (in natural units)

$$k_{TT} = k_{II} = k_{JJ} = k_{KK} = -6$$

$$k_{XX} = k_{YY} = k_{ZZ} = k_{AA} = k_{BB} = k_{CC} = 6$$

This metric agrees with the Minkowski metric (up to a factor of 6) on the spacetime dimensions.

- Four component spinors arise naturally via the action of $sp(4, \mathbb{R}) = so(2, 3)$.

On an ADS manifold parallel transport, described by the connection Γ_{ij}^k is neither completely symmetric nor antisymmetric (with respect to the covariant indices). The non-commutativity of the symmetric part of Γ_{ij}^k is encapsulated by the Riemann curvature tensor (which describes curvature) whilst the non-commutativity of the antisymmetric part constitutes the torsion tensor which we have identified as the Lie algebra structure, which in turn describes the failure of translations to commute. So the apparent clash in the preliminary section of this chapter is resolved once we realise we are dealing with two separate objects, namely curvature and torsion.

On an ADS manifold curvature is not merely between the spacetime dimensions, but the Lorentz dimensions also. For this reason the forces being described by such curvature are expected to be more than purely gravitational in nature. To progress with this model we need to consider how matter should

be described in this universe, in particular fermions, which arise from the Dirac equation. This will require a theory of spinors on ADS manifolds. The following definitions will be employed.

4.6 V -tensors

Let V be any vector space with basis $\{e_\alpha\}$ and \mathcal{M} a manifold. A V -**vector field** on \mathcal{M} is defined to be a map $v : \mathcal{M} \rightarrow V$. A typical V -vector field can be denoted by its components v^α with respect to the basis of V .

The **dual** of a V -vector field is a V^* -vector field so that if $\{e_\alpha\}$ is a basis of V , then $\{e^\alpha\}$ is a basis of V^* where $e^\alpha e_\beta = \delta^\alpha_\beta$. Similarly a V^* -vector field is denoted by its components u_α with respect to this dual basis and maps v^α to $u_\alpha v^\alpha$ via the summation convention.

The **tensor product** (the most general bilinear operation) of a U -vector field and a V -vector field is a $U \otimes V$ -vector field. If $\{e_\alpha\}$ is a basis of U and $\{f_\beta\}$ is a basis of V then $\{e_\alpha \otimes f_\beta\}$ is a basis of $U \otimes V$ and $U \otimes V$ -vector fields can be denoted by their components $w^{\alpha\beta}$ (with respect to this basis).

A pointwise linear map from U -vector fields to V -vector fields is a $V \otimes U^*$ -vector field or a $\mathbf{Hom}(U, V)$ -vector field.

A V -**tensor** of rank $\binom{m}{n}$ is a $V \otimes \dots \otimes V \otimes V^* \otimes \dots \otimes V^*$ -vector field.

4.7 \mathfrak{X} -tensors

If \mathfrak{X} is a set of vector spaces then an \mathfrak{X} -**tensor** is a $X_1 \otimes \dots \otimes X_k$ -vector field where either $X_i \in \mathfrak{X}$ or $X_i^* \in \mathfrak{X}$ for each $i \in [1, k] \subset \mathbb{N}$. One can talk about the V -rank of an \mathfrak{X} -tensors by considering each $V \in \mathfrak{X}$, however this may be non-unique if the vector spaces in \mathfrak{X} are related in some way. So V -tensors are \mathfrak{X} -tensors with $\mathfrak{X} = \{V\}$ and ordinary tensors are V -tensors where V is the tangent space of the manifold. The tensor product of \mathfrak{X} -tensors is again an \mathfrak{X} -tensor, similarly the set of all \mathfrak{X} -tensors is closed under Hom and dual.

If we require that \mathfrak{X} includes the tangent space of the manifold then all

ordinary tensors are \mathfrak{X} -tensors, thus \mathfrak{X} -tensors are indeed extended tensors (abbreviated *extensors*).

Consider now the case where \mathfrak{X} consists of the tangent space \mathcal{T}_M and one additional space V . The more general situation where \mathfrak{X} contains the tangent space and a collection of additional spaces is mathematically straightforward, but notationally cumbersome. It is simply stated that the following results can be extended without (mathematical) difficulty to this more general case.

If an \mathfrak{X} -tensor has ordinary rank $\binom{m}{n}$ and V -rank $\binom{p}{q}$ then we say it has \mathfrak{X} -rank $\binom{m,p}{n,q}$. Such a tensor can be denoted by

$$X_{j_1 \dots j_n \mu_1 \dots \mu_q}^{i_1 \dots i_m \lambda_1 \dots \lambda_p} \in \underbrace{\mathcal{T}_M \otimes \dots \otimes \mathcal{T}_M}_m \otimes \underbrace{\mathcal{T}_M^* \otimes \dots \otimes \mathcal{T}_M^*}_n \otimes \underbrace{V \otimes \dots \otimes V}_p \otimes \underbrace{V^* \otimes \dots \otimes V^*}_q$$

in terms of a basis of V and a coordinate system on \mathcal{M} . Of course if V is related in some way to \mathcal{T}_M then X may have more than one \mathfrak{X} -rank, an obvious example is when $V = \mathcal{T}_M$. We adopt the convention that Greek indices refer to the fundamental space V and Latin indices give components with respect to a coordinate system on the manifold.

4.8 The covariant derivative of \mathfrak{X} -tensors

We now extend previous definitions and propositions in order to deduce the manner in which the covariant derivative of our local Lie manifold will act on \mathfrak{X} -tensors.

Definition 4.5 *An \mathfrak{X} -tensor derivation on a manifold \mathcal{M} is a map from \mathfrak{X} -tensors to \mathfrak{X} -tensors which satisfies*

1. *Linearity.*
2. *The Leibniz condition on tensor products.*
3. *Commutates with contraction.*

The trace of $\text{Hom}(V, V)$ or $V^* \otimes V$ maps X_β^α to $\sum_\alpha X_\alpha^\alpha$, and is basis independent. This can be extended (via tensor product) to an operation on \mathfrak{X} -tensors

which we shall call *contraction* of Greek indices. Most of the propositions from section 4.2 can be extended quite easily and so we shall simply state them here.

Proposition 4.8 *If D and E are \mathfrak{X} -tensor derivations, then $[D, E]$ is also an \mathfrak{X} -tensor derivation.*

Definition 4.6 *Let D be an \mathfrak{X} -tensor derivation and f an \mathfrak{X} -tensor of rank $\binom{0,0}{0,0}$. Then $D(f)$ is a tensor of rank $\binom{i,m}{j,n}$, say. We define $\binom{i,m}{j,n}$ to be the **rank** of D .*

Proposition 4.9 *Every \mathfrak{X} -tensor derivation of rank $\binom{i,m}{j,n}$ maps \mathfrak{X} -tensors of rank $\binom{k,p}{l,q}$ to \mathfrak{X} -tensors of rank $\binom{i+k,m+p}{j+l,n+q}$.*

Of course for the same reason that an \mathfrak{X} -tensor may have a non-unique rank (if V is linked in some way to the tangent space) an \mathfrak{X} -tensor derivation may also have more than one rank.

Proposition 4.10 *If D is an \mathfrak{X} -tensor derivation and S is any \mathfrak{X} -tensor, then $S \otimes D$ is an \mathfrak{X} -tensor derivation, where $(S \otimes D)(T) = S \otimes D(T)$.*

Employing a similar version of previous arguments for an \mathfrak{X} -tensor derivation D , the difference $D - a^i \frac{\partial}{\partial x^i}$ is an \mathfrak{X} -tensor derivation of rank $\binom{0,0}{0,0}$ which maps all functions to the zero function.

Proposition 4.11 *Let E be an \mathfrak{X} -tensor derivation of rank $\binom{0,0}{0,0}$ with $E(f) = 0$ for all functions f on \mathcal{M} . There exists an \mathfrak{X} -tensor Γ_j^i of rank $\binom{1,0}{1,0}$ and an \mathfrak{X} -tensor Γ_β^α of rank $\binom{0,1}{0,1}$ such that $E = \Gamma \binom{*}{*}$, where $\Gamma \binom{*}{*}$ operates on Greek indices in the obvious way.*

We now have an \mathfrak{X} -tensor of rank $\binom{0,0}{1,0}$

$$\frac{\partial}{\partial x^k} + \Gamma_k \binom{*}{*}$$

which functions as a covariant derivative of \mathfrak{X} -tensors, defining for us parallel transport, not only of tangent vectors, but also vectors in V .

4.9 Attaching spinors to an ADS manifold

We are interested in \mathfrak{X} -tensors which are of physical significance. In particular we expect there will exist a representation of $so(2,3)$ on the space V . Now ‘combining’ representations of a Lie algebra via \oplus , \otimes , Hom or dual will yield another representation of the Lie algebra. Hence if V is a representation of a Lie algebra \mathfrak{g} then \mathfrak{g} will act quite naturally on \mathfrak{X} -tensors for $\mathfrak{X} = \{\mathcal{T}_M, V\}$.

The Lie algebra $so(2,3)$ is isomorphic to $sp(4, \mathbb{R})$ and has a faithful representation on \mathbb{R}^4 . The extension of this representation to \mathbb{C}^4 is fundamental as all other finite dimensional representations of $so(2,3)$ can be generated as subspaces of tensors products of this representation and its dual. If we choose the representation on the space V to be that of $sp(4, \mathbb{R})$ on \mathbb{R}^4 , then the set of all \mathfrak{X} -tensors will include maps from the manifold into arbitrary finite dimensional representations of $so(2,3)$. Furthermore we shall be able to identify the tangent space with its representation on \mathbb{R}^4 . We do this by asserting the existence of an \mathfrak{X} -tensor $T_{i\alpha}^\beta$ of rank $\binom{0,1}{1,1}$ such that

$$\boxed{T_{i\lambda}^\beta T_{j\alpha}^\lambda - T_{j\lambda}^\beta T_{i\alpha}^\lambda = T_{ij}^k T_{k\alpha}^\beta} \quad (4.7)$$

where T_{ij}^k are the Lie structure constants of $so(2,3)$. The name T is chosen so that we may write the action of the Lie algebra on any \mathfrak{X} -tensor as $T_i \binom{*}{*}$ where either T_{ib}^a or $T_{i\beta}^\alpha$ is used depending on whether Latin or Greek indices are acted on.

The matrices $T_{i\alpha}^\beta$ are elements of the Lie algebra $sp(4, \mathbb{R})$ and preserve a symplectic form. We must therefore also require an antisymmetric, non-degenerate \mathfrak{X} -tensor of rank $\binom{0,0}{0,2}$ denoted $s_{\alpha\beta}$ satisfying

$$T_i \binom{*}{*} (s_{\alpha\beta}) = T_{i\alpha}^\lambda s_{\lambda\beta} + T_{i\alpha}^\lambda s_{\alpha\lambda} = 0 \quad (4.8)$$

The conservation of this symplectic form is what characterises the Lie algebra $sp(4, \mathbb{R})$.

The trace form associated with this representation is $g_{ij} = T_{i\alpha}^\beta T_{j\beta}^\alpha$. Since $so(2,3)$ is simple, an invariant bilinear form on any representation is unique up

to a scalar multiple, i.e. g_{ij} ought to be some multiple of k_{ij} . Upon examination of the particular representations we do indeed find that $k_{ij} = 6g_{ij}$. We choose to use the trace form g_{ij} rather than the Killing form for our metric on the manifold since it shall free us from needlessly carrying about trivial constants in subsequent processes. We are free to choose either for our metric since the factor of 6 can easily be accommodated by a different choice of units. We define the contravariant form of the metric tensor g^{ij} by the equation $g^{ij}g_{jk} = \delta^i_k$.

We have required that on an ADS manifold the covariant derivative conserve the Lie algebra structure and that this structure is equal to the torsion. To extend this definition to a manifold with \mathfrak{X} -tensors we must consider if there are further conditions we want to place with respect to how $so(2,3)$ acts on other \mathfrak{X} -tensors. We shall define some terminology as we consider the two types of action we have defined.

- The **local action** of $so(2,3)$ on the manifold is that specified by T_i^* as applied to \mathfrak{X} -tensors. An \mathfrak{X} -tensor X is said to be **locally invariant** if $T_i^*(X) = 0$.
- The **global action** of $so(2,3)$ on the manifold is that specified by the covariant derivative ∇_i as applied to \mathfrak{X} -tensors. An \mathfrak{X} -tensor Y is said to be **globally invariant** if $\nabla_i(Y) = 0$.
- An \mathfrak{X} -tensor is said to be **totally** or **fully invariant** if it is both locally and globally invariant.

4.10 Extending the physical assumptions

A local Lie manifold can thus be defined as a manifold together with a covariant derivative where the torsion is totally invariant. Let us investigate upon what physical basis assumptions 4.5 and 4.6 stand in the hope that this would expose how we might properly extend these assumptions now that we have

spinors defined on the manifold. A similar investigation can be found in [2].

In general relativity the metric tensor g_{ij} is assumed to be globally invariant (in the sense that we have defined), see [3]. This ensures the associated bilinear form, in particular the infinitesimal interval

$$ds^2 = g_{ij}dx^i dx^j$$

is invariant with respect to parallel transport. The bilinear form can distinguish between spacelike and timelike coordinates, so the global invariance of g_{ij} means this distinction will be conserved under parallel transport.

Local invariance of the torsion means T_{ij}^k defines a Lie bracket i.e. the Lie algebra structure. The metric we have chosen to use on the ADS manifold is generated from the Lie structure, viz. $g_{ij} = \frac{1}{6}T_{ia}^b T_{jb}^a$. Thus global invariance of g_{ij} is a consequence of our (hence stronger) assumption that T_{ij}^k is globally invariant. The equation $\nabla_m(T_{ij}^k) = 0$ means that T_{ij}^k and hence the Lie bracket which it defines is conserved under parallel transport. This means that if $[X, Y] = Z$ for tangent vectors X, Y, Z at any point, and we parallel transport these to obtain tangent vectors X', Y', Z' at another point then $[X', Y'] = Z'$. The Lie structure provides more information than the metric, it allows us to distinguish between e.g. a displacement and a boost coordinate. We expect that to be able to identify the nature of our coordinates on an ADS manifold. It therefore seems physically reasonable to require that such an identification is conserved under parallel transport.

The local invariance of $T_{i\alpha}^\beta$ expressed in equation 4.7 follows from the fact that $T_{i\alpha}^\beta$ describes the action of the Lie algebra on a given space. Let us refer to the representation on this space as the *spinor representation*. Assuming global invariance of $T_{i\alpha}^\beta$ means the action of the Lie algebra on spinors will be conserved under parallel transport. More specifically if $\phi = X(\psi)$ for tangent vector X and spinors ϕ and ψ defined at some point, then the parallel trans-

port of these quantities X' , ϕ' and ψ' will satisfy $\phi' = X'(\psi')$ at an adjacent point. This assumption can be viewed as allowing the local and global actions on spinors to commute, an idea we choose to adopt.

Axiom 4.12

$$\boxed{\nabla_m(T_{i\alpha}^\beta) = 0}$$

Now the total invariance of the metric $g_{ij} = T_{i\alpha}^\beta T_{j\beta}^\alpha$ follows from the total invariance of $T_{i\alpha}^\beta$. Indeed the total invariance of $T_{ij}^k = g^{kt} T_{t\beta}^\alpha (T_{i\lambda}^\beta T_{j\alpha}^\lambda - T_{j\lambda}^\beta T_{i\alpha}^\lambda)$ is also a consequence of axiom 4.12 together with requirement 4.7.

Up until now we have talked about \mathfrak{X} -tensors with Greek indices without considering how we might raise or lower these indices. The candidate quantity for this job is the bilinear form $s_{\alpha\beta}$ (and its inverse). We must however be conscientious about our ordering of indices so as to avoid unsolicited negative signs (due to the antisymmetry of $s_{\alpha\beta}$). Define $s^{\alpha\beta}$ by the equation $1_\alpha^\beta \equiv s^{\beta\lambda} s_{\alpha\lambda} = s^{\lambda\beta} s_{\lambda\alpha}$ where 1_α^β is the identity map. The convention we adopt for raising and lowering indices shall be to

- lower indices on the left: $v_\alpha = s_{\alpha\beta} v^\beta$.
- raise indices on the right: $v^\alpha = v_\beta s^{\beta\alpha}$

thus raising and lowering are inverse operations. However our current assumptions do not guarantee the process of raising/lowering a Greek index will be conserved under parallel transport, viz. $\nabla_m(v^\alpha)$ will not equal $\nabla_m(v_\beta) s^{\beta\alpha}$ in general. We do not wish for the raising or lowering of a Greek index in a globally invariant equation to destroy its invariance, hence we make the assumption

$$\boxed{\nabla_m(s_{\alpha\beta}) = 0} \tag{4.9}$$

Furthermore, when considering the decomposition of the space of spinor transform in chapter 5 we shall identify $s_{\alpha\beta}$ as an intertwining map. (Intertwining maps are globally invariant, see section 4.12.) We shall later find (in chapter

7) that assumption 4.9 turns out to be too rigid as it does not permit the existence of Maxwell's equations on an ADS manifold.

4.11 Higher order representations

In this section we generalise our previous work to consider all finite dimensional, irreducible representations of the $so(2,3)$. We extend the set \mathfrak{X} of vector spaces (used to define \mathfrak{X} -tensors) so that it includes all other finite dimensional irreducible representations of the Lie algebra. Since all finite dimensional representations of $so(2,3)$ can be constructed from the fundamental representation it follows that any \mathfrak{X} -tensor can be built from tensor products, direct sums and duals of vectors and spinors.

If v^Σ is a vector into one of these representations then the Lie algebra acts on v^Σ via the matrices $T_{i\Lambda}^\Sigma$ (a generalisation of T_{ij}^k and $T_{i\alpha}^\beta$). These matrices must satisfy

$$T_{i\Theta}^\Sigma T_{j\Lambda}^\Theta - T_{j\Theta}^\Sigma T_{i\Lambda}^\Theta = T_{ij}^k T_{k\Lambda}^\Sigma$$

In order to show that $T_{i\Lambda}^\Sigma$ is in fact globally invariant consider the following theorem.

Theorem 4.13 *If the local action T_i^* and global action ∇_j commute and are defined on \mathfrak{X} -tensors U and V , then actions which are defined on $U \oplus V$, $U \otimes V$ and V^* will commute with each other.*

Proof. The proof is routine for the tensor product and direct sum cases.

$$\begin{aligned} T_i^* (\nabla_j (U \oplus V)) &= T_i^* (\nabla_j (U) \oplus \nabla_j (V)) \\ &= T_i^* (\nabla_j (U)) \oplus T_i^* (\nabla_j (V)) \\ &= \nabla_j (T_i^* (U)) \oplus \nabla_j (T_i^* (V)) \\ &= \nabla_j (T_i^* (U \oplus V)) \end{aligned}$$

$$\begin{aligned}
T_i \binom{*}{*} (\nabla_j(U \otimes V)) &= T_i \binom{*}{*} (\nabla_j(U) \otimes V + U \otimes \nabla_j(V)) \\
&= T_i \binom{*}{*} (\nabla_j(U) \otimes V) + T_i \binom{*}{*} (U \otimes \nabla_j(V)) \\
&= T_i \binom{*}{*} (\nabla_j(U)) \otimes V + \nabla_j(U) \otimes T_i \binom{*}{*} (V) \\
&\quad + T_i \binom{*}{*} (U) \otimes \nabla_j(V) + U \otimes T_i \binom{*}{*} (\nabla_j(V)) \\
&= \nabla_j(T_i \binom{*}{*} (U)) \otimes V + T_i \binom{*}{*} (U) \otimes \nabla_j(V) \\
&\quad + \nabla_j(U) \otimes T_i \binom{*}{*} (V) + U \otimes \nabla_j(T_i \binom{*}{*} (V)) \\
&= \nabla_j(T_i \binom{*}{*} (U)) \otimes V + U \otimes T_i \binom{*}{*} (V) \\
&= \nabla_j(T_i \binom{*}{*} (U \otimes V))
\end{aligned}$$

It remains to show that $T_i \binom{*}{*}$ and ∇_j commute on the dual space V^* . For a general \mathfrak{X} -tensor derivation D and \mathfrak{X} -tensors ϕ and v

$$D(\phi v) = D(\phi)v + \phi D(v) \quad (4.10)$$

we are interested in \mathfrak{X} -tensor derivations applied to ϕ so we shall rearrange equation 4.10

$$D(\phi)v = D(\phi v) - \phi D(v) \quad (4.11)$$

Let $\phi = V^*$ and $v = V$, so $\phi : V \rightarrow \mathbb{R}$ hence $\phi v \in \mathbb{R}$. We consider the \mathfrak{X} -tensor derivation $T_i \binom{*}{*}$.

$$\begin{aligned}
T_i \binom{*}{*} (\phi)v &= T_i \binom{*}{*} (\phi v) - \phi T_i \binom{*}{*} (v) \\
&= -\phi T_i \binom{*}{*} (v)
\end{aligned}$$

since the local action on scalars is trivial. And the \mathfrak{X} -tensor derivation ∇_j

$$\nabla_j(\phi)v = \nabla_j(\phi v) - \phi \nabla_j(v)$$

Therefore

$$\begin{aligned}
(\nabla_j T_i^*)(\phi)v &= \nabla_j(T_i^*(\phi))v \\
&= \nabla_j(T_i^*(\phi)v) - T_i^*(\phi)\nabla_j(v) \\
&= \nabla_j(-\phi T_i^*(v)) - T_i^*(\phi)\nabla_j(v) \\
&= -\nabla_j(\phi)T_i^*(v) - \phi\nabla_j T_i^*(v) - T_i^*(\phi)\nabla_j(v) \\
&= -\nabla_j(\phi)T_i^*(v) - \phi T_i^*(\nabla_j(v)) - T_i^*(\phi)\nabla_j(v) \\
&= -\nabla_j(\phi)T_i^*(v) + T_i^*(\phi)\nabla_j(v) - T_i^*(\phi)\nabla_j(v) \\
&= -\nabla_j(\phi)T_i^*(v) \\
&= T_i^*(\nabla_j(\phi))v \\
&= (T_i^*(\nabla_j)(\phi))v
\end{aligned}$$

so the local and global actions commute on V^* , which completes the proof. \square

All \mathfrak{X} -tensors are built from vectors and spinors using these operations and the local and global actions on vectors and spinors commute. Hence by theorem 4.13 the local and global actions on general \mathfrak{X} -tensors must also commute, i.e.

$$\nabla_m(T_{i\Lambda}^\Sigma) = 0$$

4.12 Intertwining maps

Any irreducible representation is a direct sum of tensor products of the fundamental representation. We use \mathfrak{X} -tensors $s_{\Sigma}^{\alpha_1\alpha_2\dots\alpha_n}$ and $s_{\alpha_1\alpha_2\dots\alpha_n}^\Sigma$ to alternate between these descriptions.

$$x^\Sigma = x^{\alpha_1\alpha_2\dots\alpha_n} s_{\alpha_1\alpha_2\dots\alpha_n}^\Sigma \quad (4.12)$$

$$x^{\alpha_1\alpha_2\dots\alpha_n} = x^\Sigma s_{\Sigma}^{\alpha_1\alpha_2\dots\alpha_n} \quad (4.13)$$

These tensors define intertwining maps between Lie algebra representations, the composition of these gives either the identity map

$$1_\Lambda^\Sigma = s_\Lambda^{\alpha_1\alpha_2\dots\alpha_n} s_{\alpha_1\alpha_2\dots\alpha_n}^\Sigma$$

on the irreducible representation, or a projection map

$$\Pi_{\alpha_1\alpha_2\dots\alpha_n}^{\beta_1\beta_2\dots\beta_n} = s_{\Sigma}^{\beta_1\beta_2\dots\beta_n} s_{\alpha_1\alpha_2\dots\alpha_n}^{\Sigma}$$

onto the corresponding component of the tensor space. Intertwining maps describe an equivalence between representations. This is essentially a relabelling and ought to commute with the local and global actions of the Lie algebra

$$T_i^*(*) (s_{\Sigma}^{\alpha_1\alpha_2\dots\alpha_n}) = 0 \quad (4.14)$$

$$T_i^*(*) (s_{\alpha_1\alpha_2\dots\alpha_n}^{\Sigma}) = 0 \quad (4.15)$$

$$\nabla_m (s_{\Sigma}^{\alpha_1\alpha_2\dots\alpha_n}) = 0 \quad (4.16)$$

$$\nabla_m (s_{\alpha_1\alpha_2\dots\alpha_n}^{\Sigma}) = 0 \quad (4.17)$$

4.13 Summary

In this chapter we have assumed that our universe is an ADS manifold, viz. a local Lie manifold for the Lie algebra $so(2, 3)$. This means we have chosen to use $SO(2, 3)$ (over the Poincaré group) as the local symmetry group of spacetime. Following this decision we have built up a sufficiently elaborate mathematical structure upon which we might formulate physical theories. The physical assumptions included can be summarised as follows.

There exists a local action of the fundamental representation of $so(2, 3) = sp(4, \mathbb{R})$ on the space \mathbb{R}^4 which is locally invariant

$$\boxed{T_{i\lambda}^{\beta} T_{j\alpha}^{\lambda} - T_{j\lambda}^{\beta} T_{i\alpha}^{\lambda} = T_{ij}^k T_{k\alpha}^{\beta}}$$

There also exists a global action ∇_i on the space \mathbb{R}^4 which defines the connection $\Gamma_{i\alpha}^{\beta}$ that describes the parallel transport of maps from the manifold into the space \mathbb{R}^4 . The global action satisfies

$$\boxed{[\nabla_i, \nabla_j] = T_{ij}^k \nabla_k + R_{ij}^*(*)}$$

where $R_{ij}(\ast)$ is a linear map defined on V at each point. These local and global actions commute

$$\boxed{\nabla_m(T_{i\alpha}^\beta) = 0}$$

We have a globally invariant bilinear form on \mathbb{R}^4 which enables us to raise and lower spinor indices (the components of vectors from the spinor representation) in a manner that is consistent with parallel transport

$$\boxed{\nabla_m(s_{\alpha\beta}) = 0}$$

Chapter 5

Representations of Low Dimension

Following on from the mathematical framework which was developed in the previous chapter, we now turn our attention in particular to the representations of low dimension. We shall follow chapter 3 in [1] and [2] which give a more extensive investigation of the low dimensional representations of $so(2, 3)$. We begin by clarifying some terminology.

- The word **tensor** shall now be used to refer to a general \mathfrak{X} -tensor, where the set \mathfrak{X} contains all the irreducible representations of $so(2, 3)$ as well as the tangent space of the ADS manifold.
- The word **scalar** refers to a tensor associated with the trivial representation. Scalars are denoted by the index \circ although we typically denote them without indices (unless we find it useful to do so). So $f_{\circ} = f = f^{\circ}$.
- The word **vector** refers to a tensor associated with the (regular) 10-dimensional irreducible representation. Vectors are denoted by lower-case Latin indices; e.g. v^i or v_j . If we need to use the word vector to refer to something other than a 10-vector, the context shall make this clear.

- The word **spinor** refers to a tensor associated with the 4-dimensional irreducible representation $sp(4, \mathbb{R})$. Spinors are denoted by lower-case Greek indices; e.g. v^α or v_β .
- The word **versor** refers to a tensor associated with the canonical 5-dimensional irreducible representation. Versors are denoted by upper-case Latin indices; e.g. v^A or v_B .

5.1 Spinor transformations

We seek to find intertwining maps for a smooth decomposition of the 16-dimensional space of spinor transformations $\{X_\beta^\alpha\}$. This will reflect the decomposition of 4×4 matrices into irreducible representations carried out in chapter 3.

- The set of \mathfrak{X} -tensors $\{T_{k\beta}^\alpha\}$ span a 10-dimensional irreducible subspace of $\{X_\beta^\alpha\}$ under the local action. This representation is isomorphic to the regular one. If we choose for our basis the matrices $T_{k\beta}^\alpha$ we can write down intertwining maps

$$s_{k\beta}^\alpha = T_{k\beta}^\alpha$$

$$s_\delta^{k\sigma} = g^{kl} T_{l\delta}^\sigma$$

along with the projection and injection maps

$$\Pi_{\beta\delta}^{\alpha\sigma} = g^{kl} T_{l\delta}^\sigma T_{k\beta}^\alpha$$

$$1_j^i = g^{il} T_{l\delta}^\sigma T_{j\beta}^\alpha$$

- The fully invariant transformation $T_{\circ\beta}^\alpha = \frac{1}{2} 1_\beta^\alpha$ provides a basis for the 1-dimensional irreducible component of $\{X_\beta^\alpha\}$ and behaves as a trivial representation under the local action on spinor transformations. Elements of this representation will be denoted as scalars. We have intertwining

maps

$$s_\beta^\alpha = \frac{1}{2}1_\beta^\alpha \quad (\text{or } s_{\circ\beta}^\alpha = T_{\circ\beta}^\alpha)$$

$$s_\alpha^\beta = \frac{1}{2}1_\alpha^\beta \quad (\text{or } s_\alpha^{\circ\beta} = g^{\circ\circ}T_{\circ\alpha}^\beta)$$

and projection/injection maps

$$\Pi_{\beta\delta}^{\alpha\sigma} = \frac{1}{4}1_\beta^\alpha 1_\delta^\sigma \quad (\text{or } g^{\circ\circ}T_{\circ\beta}^\alpha T_{\circ\delta}^\sigma)$$

$$1_\circ^\circ = \frac{1}{4}1_\beta^\alpha 1_\alpha^\beta = 1 \quad (\text{or } g^{\circ\circ}T_{\circ\beta}^\alpha T_{\circ\alpha}^\beta)$$

where is the trace form $g_{\circ\circ} \equiv T_{\circ\beta}^\alpha T_{\circ\alpha}^\beta$ and the equation $g^{\circ\circ}g_{\circ\circ} \equiv 1$ defines $g^{\circ\circ}$, the inverse of the trace form, although these definitions are trivial since $g_{\circ\circ} = 1$.

- We choose a basis $\{T_{A\beta}^\alpha\}$ for the remaining 5-dimensional irreducible component of $\{X_\beta^\alpha\}$. This defines for us the 5 matrices $T_{A\beta}^\alpha$. Since they map trivially onto the trivial and vector components we have $T_{A\beta}^\alpha 1_\alpha^\beta = T_{A\alpha}^\alpha = 0$ and $T_{A\beta}^\alpha T_{k\alpha}^\beta = 0$ respectively. We now define

$$g_{AB} \equiv T_{A\beta}^\alpha T_{B\alpha}^\beta \quad (5.1)$$

which we use to construct the intertwining maps

$$s_{A\beta}^\alpha = T_{A\beta}^\alpha \quad (5.2)$$

$$s_\delta^{B\sigma} = g^{AB}T_{A\delta}^\sigma \quad (5.3)$$

and projection/injection maps

$$\Pi_{\beta\delta}^{\alpha\sigma} = g^{AB}T_{A\delta}^\sigma T_{B\beta}^\alpha \quad (5.4)$$

$$1_C^A = g^{AB}T_{B\alpha}^\beta T_{C\beta}^\alpha = g^{AB}g_{BC} \quad (5.5)$$

Equation 5.5 shows that g_{AB} is non-singular with inverse g^{AB} . The total invariance of these intertwining maps necessitates the total invariance of $T_{A\beta}^\alpha$ and hence g_{AB} and its inverse g^{AB} . So we can use g^{AB} and its covariant counterpart to raise and lower versor indices. The total invariance of $T_{A\beta}^\alpha$ defines the local action T_{iA}^B by the equation

$$T_i \begin{matrix} (*) \\ (*) \end{matrix} (T_{A\beta}^\alpha) = 0$$

and the connection Γ_{iA}^B which describes the parallel transport of versors is defined by the equation

$$\nabla_i(T_{A\beta}^\alpha) = 0$$

The sum of these three projection maps is the identity map

$$1_\delta^\alpha 1_\beta^\sigma = g^{kl} T_{l\delta}^\sigma T_{k\beta}^\alpha + \frac{1}{4} 1_\beta^\alpha 1_\delta^\sigma + g^{AB} T_{A\delta}^\sigma T_{B\beta}^\alpha \quad (5.6)$$

5.2 Two component spinors

We could similarly perform the decomposition of the space of tensors with two contravariant spinors indices $\{X^{\alpha\beta}\}$ into irreducibles of dimension 10, 5 and 1. We assume the 1-dimensional irreducible component is a scalar representation with intertwining map $s_{\alpha\beta}^\circ : X^{\alpha\beta} \rightarrow X^\circ$. Given that intertwining maps are totally invariant, this must define a totally invariant bilinear form for spinors. We can therefore identify $s_{\alpha\beta}$ as $s_{\alpha\beta}^\circ$ since they also transform in the same way, lending support to assumption 4.9. Indeed in order for us to relinquish assumption 4.9 we would have to reinterpret this 1-dimensional irreducible component as something other than a scalar representation.

5.3 Casimir identities

The quadratic operator

$$-T^2 + X^2 + Y^2 + Z^2 + A^2 + B^2 + C^2 - I^2 - J^2 - K^2$$

commutes with every element in $\{T, X, Y, Z, A, B, C, I, J, K\}$. It is called the **quadratic Casimir operator**. Such an operator will be scalar in every irreducible representation. This gives an identity for every irreducible representation. We call the collection of these identities the **Casimir identities**. We

give some of the low dimensional ones here.

$$g^{ij}T_{i\lambda}^\beta T_{j\alpha}^\lambda = \frac{5}{2} \cdot 1_\alpha^\beta \quad (5.7)$$

$$g^{ij}T_{iX}^B T_{jA}^X = 4 \cdot 1_A^B \quad (5.8)$$

$$g^{ij}T_{ix}^b T_{ja}^x = 6 \cdot 1_a^b \quad (5.9)$$

5.4 The reduced curvature tensor

The curvature tensor $R_{ij}(\ast)$ can act on vectors or spinors and is described by the tensors R_{ijt}^s or $R_{ij\alpha}^\beta$ respectively. We can again use the Jacobi identity to derive the Bianchi identities by acting on a spinor (instead of a vector). The first Bianchi identity remains the same, but the second gives a new result involving the curvature tensor with spinor indices

$$T_{ij}^l R_{lk\alpha}^\beta - \nabla_k (R_{ij\alpha}^\beta) \stackrel{(ijk)}{=} 0 \quad (5.10)$$

We call $R_{ij\alpha}^\beta$ the **spinor curvature tensor**. Using equation 4.2 we can express $R_{ij}(\ast)$ in terms of covariant derivatives only. It then follows that any globally invariant quantities will be invariant with respect to $R_{ij}(\ast)$ as well. In particular

$$R_{ij}(\ast)(s_{\alpha\beta}) = 0$$

$$R_{ij}(\ast)(T_{i\alpha}^\beta) = 0$$

which leads us to the following theorem.

Theorem 5.1 *There exists a tensor R_{ij}^s such that $R_{ij\alpha}^\beta = R_{ij}^s T_{s\alpha}^\beta$*

Proof. Since $R_{ij}(\ast)(s_{\alpha\beta}) = 0$ this means that $R_{ij\alpha}^\beta$ is a matrix in $sp(4, \mathbb{R})$, for fixed i and j . Therefore we can write $R_{ij\alpha}^\beta$ as a linear combination of basis elements of $sp(4, \mathbb{R})$. Hence $R_{ij\alpha}^\beta = R_{ij}^s T_{s\alpha}^\beta$. \square

The strength of this result is displayed in the following theorem.

Theorem 5.2 *If R_{ij}^s is defined as above then $R_{ijk}^t = R_{ij}^s T_{sk}^t$*

Proof. $R_{ij}^*(*) (T_{k\alpha}^\beta) = 0$ hence it follows that

$$\begin{aligned} R_{ijk}^t T_{t\alpha}^\beta &= R_{ij\lambda}^\beta T_{k\alpha}^\lambda - R_{ij\alpha}^\lambda T_{k\lambda}^\beta \\ &= R_{ij}^s (T_{s\lambda}^\beta T_{k\alpha}^\lambda - T_{s\alpha}^\lambda T_{k\lambda}^\beta) \\ &= R_{ij}^s T_{sk}^t T_{t\alpha}^\beta \end{aligned}$$

Since $T_{t\alpha}^\beta$ is generally non-zero, we conclude that $R_{ijk}^t = R_{ij}^s T_{sk}^t$. \square

These results may be extended to all irreducible representations of $so(2,3)$ so that $R_{ij}^*(*) = R_{ij}^s T_s^*(*)$ in general. We call R_{ij}^s the **reduced curvature tensor**. We can write the Bianchi identities for the reduced curvature tensor.

$$\begin{aligned} R_{ij}^s T_{sk}^l &\stackrel{(ijk)}{=} 0 \\ R_{is}^l T_{jk}^s + \nabla_i (R_{jk}^l) &\stackrel{(ijk)}{=} 0 \end{aligned}$$

We now consider contractions of the curvature and reduced curvature tensors since the properties of these tensors are of paramount importance if one is interested in building a theory of gravitation on an ADS manifold. We name these contractions

The curvature scalar	$R = R_{ij} g^{ij}$
The curvature vector	$R_i = R_{ij}^j$
The Ricci tensor	$R_{ij} = R_{ib}^a T_{ja}^b$

Employing the Bianchi identities we verify the following results.

Proposition 5.3

- 1) *The Ricci tensor is symmetric.*
- 2) $\nabla_k (R_{ij}^k) = 0$
- 3) $\nabla_k (R) = 2\nabla^t (R_{tk})$

See pp. 52-53 of [1] for a proof of these results. The last result here is significant. It shows that the 10-dimensional generalisation of the Einstein tensor

$$R_{ij} - \frac{1}{2} g_{ij} R$$

is divergenceless. It is the zero divergence of the ordinary Einstein tensor which is a necessary condition for its involvement in the Einstein field equations.

Chapter 6

The Dirac equation

After developing and exploring our mathematical framework let us now consider how the equations of physics sit on our manifold. In particular this chapter shall give its consideration to the Dirac equation.

Let us consider the usual (specially covariant) Dirac equation

$$i\gamma^\Sigma\partial_\Sigma\psi = \frac{mc}{\hbar}\psi \quad (6.1)$$

where ψ is a Dirac spinor and capital Greek indices shall refer to the usual four spacetime dimensions of relativity, $\Sigma = 0, 1, 2, 3$. The γ^Σ are (of course) the 4×4 gamma matrices, defined by

$$\gamma^\Sigma\gamma^\Delta + \gamma^\Delta\gamma^\Sigma = -2\eta^{\Sigma\Delta}I_4 \quad (6.2)$$

where $\eta^{\Sigma\Delta}$ is the Minkowski metric tensor of signature $(-+++)$, and I_4 is the identity matrix. If we consider the first 4 basis elements T, X, Y, Z of $sp(4, \mathbb{R})$, it is not hard to demonstrate that $2i$ multiples of them obey precisely the anticommutation relations which define the gamma matrices. For this reason we make the following identifications

$$\gamma_0 = 2iT, \quad \gamma_1 = 2iX, \quad \gamma_2 = 2iY, \quad \gamma_3 = 2iZ \quad (6.3)$$

using this notation equation 6.1 is

$$\left(-\frac{1}{c}T\partial_T + X\partial_X + Y\partial_Y + Z\partial_Z\right)\psi^\alpha = \frac{mc}{2\hbar}\psi^\alpha$$

Consider now the following invariant equation on an ADS manifold (section 4.5)

$$g^{ij}\nabla_i T_j^*(\psi^\alpha) = \lambda\psi^\alpha \quad (6.4)$$

where λ is a constant. In a locally flat basis we can write the operator $g^{ij}\nabla_i T_{j\beta}^\alpha$ in natural units

$$-T\partial_T + X\partial_X + Y\partial_Y + Z\partial_Z + A\partial_A + B\partial_B + C\partial_C - I\partial_I - J\partial_J - K\partial_K$$

where $T, X, Y, Z, A, B, C, I, J, K$ are the matrices $T_{i\beta}^\alpha$. We convert the derivatives into natural units

$$-rT\partial_T + rcX\partial_X + rcY\partial_Y + rcZ\partial_Z + cA\partial_A + cB\partial_B + cC\partial_C - I\partial_I - J\partial_J - K\partial_K$$

and divide through by rc in 6.4, neglecting the terms with factors of $\frac{1}{r}$ (since we have always assumed r to be very large)

$$\left(-\frac{1}{c}T\partial_T + X\partial_X + Y\partial_Y + Z\partial_Z\right)\psi^\alpha = \frac{\lambda}{rc}\psi^\alpha$$

This is indeed equation 6.1 provided

$$\frac{\lambda}{r} = \frac{mc^2}{2\hbar}$$

We could alternatively arrive at the same conclusion by assuming that the Dirac spinor ψ^α is a function of spacetime dimensions only (and not Lorentz boost or rotation).

6.1 Benefits of the new Dirac equation

- Equation 6.4 is built from purely tensorial quantities, hence it is a tensor equation valid in all frames. Normally one would have to justify that equation 6.1 is indeed Lorentz covariant by investigating how the spinor ψ transforms under arbitrary Lorentz transformations. This is the process of finding the so-called S -matrix transformations.

- Every quantity in equation 6.4 has a direct physical interpretation, in particular the matrices $T_{i\alpha}^\beta$. Whereas their counterpart, the gamma matrices of equation 6.1 are chosen purely for their geometric properties. As a result, typical formulations of the Dirac equation use gamma matrices which are complex. However as we have shown this need not be the case in order to satisfy the relationships given in equation 6.2.

6.2 The speed of electrons

In the usual formulation of the Dirac equation the Hamiltonian for a free particle is given by

$$\hat{H} = c(\gamma^0)^{-1}\gamma^k\hat{p}_k + \gamma^0 mc^2$$

where c is the speed of light, $\hat{p}_k = -i\hbar\partial_k$ is the momentum operator, and the index $k = 1, 2, 3$. In the Heisenberg picture of quantum mechanics, state vectors ψ are time independent while operators \hat{Q} are time dependent, and satisfy the equation of motion

$$\frac{d\hat{Q}(t)}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{Q}(t)] + \frac{\partial\hat{Q}}{\partial t}(t) \quad (6.5)$$

Operators \hat{R} from the Schrödinger picture are related to operators \hat{Q} from the Heisenberg picture by $\hat{Q} = \hat{U}\hat{R}\hat{U}^\dagger$ where $\hat{U} = e^{i\hat{H}t/\hbar}$. If \hat{R} is a physical observable then it will be time independent and \hat{Q} will have no explicit time dependence. Thus a physical observable \hat{Q} will satisfy

$$\frac{\partial\hat{Q}}{\partial t} = 0$$

Hence in particular $\partial_t(\hat{x}_k(t)) = 0$ for the time dependent position operator $\hat{x}_k(t) = e^{i\hat{H}t/\hbar}\hat{x}_k e^{-i\hat{H}t/\hbar}$ where \hat{x}_k is the position operator from the Schrödinger

picture. Using equation 6.5 we consider the velocity operator

$$\begin{aligned}
\frac{d\hat{x}_k(t)}{dt} &= \frac{i}{\hbar} [\hat{H}, \hat{x}_k(t)] = \frac{i}{\hbar} [c(\gamma^0)^{-1}\gamma^k\hat{p}_k + \gamma^0 mc^2, \hat{x}_k(t)] \\
&= \frac{i}{\hbar} [c(\gamma^0)^{-1}\gamma^k\hat{p}_k, \hat{x}_k(t)] + \frac{i}{\hbar} [\gamma^0 mc^2, \hat{x}_k(t)] \\
&= \frac{i}{\hbar} \hat{U} [c(\gamma^0)^{-1}\gamma^k\hat{p}_k, \hat{x}_k] \hat{U}^\dagger + \frac{i}{\hbar} \hat{U} [\gamma^0 mc^2, \hat{x}_k] \hat{U}^\dagger \\
&= \frac{i}{\hbar} c \hat{U} (\gamma^0)^{-1} \gamma^k [\hat{p}_k, \hat{x}_k] \hat{U}^\dagger \\
&= \hat{U} c (\gamma^0)^{-1} \gamma^k \hat{U}^\dagger
\end{aligned} \tag{6.6}$$

Define $\alpha_k = (\gamma^0)^{-1}\gamma^k$. The matrix $c\alpha_k$ acts on fermions ψ and has a purely discrete spectrum of eigenvalues: $\pm c$. It is unitarily equivalent to the operator $c\alpha_k(t) = ce^{i\hat{H}t/\hbar}\alpha_k e^{-i\hat{H}t/\hbar}$ hence $c\alpha_k(t)$ has eigenvalues $\pm c$ for all time t , see p. 19 of [26]. This is a paradox. These are supposed to be eigenvalues for the velocity operator acting on a fermion. How can a massive object (e.g. an electron) travel at the speed of light?

6.3 Zitterbewegung

A possible way of dealing with this paradox involves finding and interpreting $\hat{x}_k(t)$. Consider the operator $\alpha_k(t)$. Since it is time-dependent we cannot (at least not easily) integrate equation 6.6. We therefore will find it useful to consider the Heisenberg equation of motion for the operator $\alpha_k(t)$.

$$\frac{d\alpha_k(t)}{dt} = \frac{i}{\hbar} [\hat{H}, \alpha_k(t)] = \frac{2i}{\hbar} (c\hat{p}_k - \alpha_k(t)\hat{H})$$

We may integrate this with respect to time since \hat{p}_k and \hat{H} are time-independent.

$$\begin{aligned}
&\int_0^t \frac{d\alpha_k(t')}{c\hat{p}_k\hat{H}^{-1} - \alpha_k(t')} = \int_0^t \frac{2i\hat{H}}{\hbar} dt' \\
\Rightarrow &\ln(c\hat{p}_k\hat{H}^{-1} - \alpha_k(t)) - \ln(c\hat{p}_k\hat{H}^{-1} - \alpha_k(0)) = \frac{2i\hat{H}}{\hbar}t \\
\Rightarrow &\alpha_k(t) = c\hat{p}_k\hat{H}^{-1} - \left(c\hat{p}_k\hat{H}^{-1} - \alpha_k(0)\right) e^{2i\hat{H}t/\hbar}
\end{aligned} \tag{6.7}$$

Substitute equation 6.7 into equation 6.6.

$$\begin{aligned}
\frac{d\hat{x}_k(t)}{dt} &= c^2\hat{p}_k\hat{H}^{-1} - \left(c^2\hat{p}_k\hat{H}^{-1} - c\alpha_k(0)\right) e^{2i\hat{H}t/\hbar} \\
\Rightarrow \hat{x}_k(t) &= \hat{x}_k(0) + c^2\hat{p}_k\hat{H}^{-1}t - \frac{\hbar c}{2i} \left(c\hat{p}_k\hat{H}^{-1} - \alpha_k(0)\right) \left(e^{2i\hat{H}t/\hbar} - 1\right)
\end{aligned} \tag{6.8}$$

The first two terms describe a linear evolution of the position operator as is expected for a free particle. The final term is oscillatory and may induce what is called *Zitterbewegung*, which means trembling motion in German. To resolve the paradox it is envisaged that the particle's speed alternates between c and $-c$ at a very high frequency so that we observe some averaged velocity that is less than the speed of light.

Consider for example an electron at rest. The *Zitterbewegung* frequency would be $f = 2m_e c^2 / \hbar = 1.5 \times 10^{21} \text{s}^{-1}$. An apparatus able to measure time intervals of $6.5 \times 10^{-22} \text{s}$ would be required in order to detect such an effect. Given that the record for the smallest measured time interval is about $12 \times 10^{-18} \text{s}$ [15], such a verification is, at least for the time being, out of the reach of observation. The effect has however been produced in a quantum simulator for a trapped ion set to behave as a free relativistic quantum particle, see [16].

Apart from the experimental challenges there are also theoretical limitations on measurement. The magnitude of the frequency of the oscillatory term in equation 6.8 is $f = 2\hat{H}/\hbar$. For any particle $H \geq mc^2$, so the frequency of the *Zitterbewegung* $f \geq 2mc^2/\hbar$. According to the relationship $c = f\lambda$, this frequency corresponds to a wavelength of $\lambda \leq \hbar/(2mc)$. This is the reduced Compton wavelength which is often interpreted as the smallest measurable distance for a *single* particle. This follows from an uncertainty in the energy large enough to allow the creation of particles, see [22].

6.4 Hawthorn's interpretation

Let us see how this issue is addressed by the Hawthorn model. In the view of equation 6.4, the gamma matrices T, X, Y, Z represent the intrinsic action of translation by one natural unit along the t, x, y, z directions. This is the interpretation they bear *a priori* to their involvement in the Dirac equation.

Extrinsic energy and extrinsic linear momentum (along the x -axis, say) are given by the eigenvalues of the operators $i\hbar\partial_t$ and $-i\hbar\partial_x$ respectively. We expect therefore that $i\hbar T$ and $-i\hbar X$ are the operators whose eigenvalues give intrinsic energy and intrinsic momentum respectively. These should relate to ordinary energy and momentum in much the same way as spin relates to ordinary angular momentum. We note that intrinsic energy cannot be *rest mass* since rest mass is a combination of the extrinsic properties, energy and momentum, and therefore is itself extrinsic.

Intrinsic energy and momentum are not simultaneously observable on account of the failure of T, X, Y, Z to commute. Disregarding any factors of i , the eigenvalues of each of these operators are $\pm\frac{1}{2}$ in natural units (where $\hbar = 1$). Thus in ordinary units intrinsic energy is $\pm\frac{\hbar}{2r}$ and intrinsic momentum (along any axis) is $\frac{\hbar}{2rc}$.

The most natural way of defining intrinsic velocity in a particular direction (should we be interested in such a thing) would be as the quotient of intrinsic energy and intrinsic momentum in the given direction. Given that

$$i\hbar T(-i\hbar X)^{-1} = -TX^{-1}$$

consider now the eigenvalue equation

$$(-TX^{-1})\phi = v\phi$$

which has eigenvalues $v = \pm 1$ in natural units, or $v = \pm c$ in ordinary units. The operator TX^{-1} is essentially $\alpha_k = (\gamma^0)^{-1}\gamma^k$ (just use identity 6.2).

We have shown that in our formalism it also makes sense to interpret $c\alpha_k$ as a velocity operator. The contrast is that we interpret it as an *intrinsic* velocity operator due to the quantities from which it is constructed. For this reason there is no longer any need to explain away the discrete velocity eigenvalues of $\pm c$ since we are no longer talking about extrinsic velocity. The extrinsic velocity is free to take any physically acceptable value independent of the intrinsic velocity. The problem of Zitterbewegung is thus avoided.

Table 3.2 shows that the rotation operators I, J, K are commutators of X, Y, Z . We do not need to define spin operators as commutators of gamma matrices with spatial indices (see p. 8 of [26]) since the operators I, J, K already bear this interpretation.

In chapter 3 $\{T, I\}$ is (the basis of) a maximal Cartan subalgebra of $so(2, 3)$. Thus intrinsic energy and spin are simultaneously observable. Each of these observables takes on both a positive and a negative eigenvalue. There are four linearly independent Dirac spinors characterised by positive/negative values of charge and spin. Hence it is natural to interpret intrinsic energy as charge. Thus the link between time reversal and charge inversion makes perfect sense.

6.5 Investigating the connection

We have up until now, neglected the idea of curvature. However, given that our Dirac equation is defined in terms of the covariant derivative, it must include a description of how curvature affects the evolution of spinors. Let us write out equation 6.4 more explicitly

$$g^{ij}T_{j\gamma}^{\beta}(\partial_i + \Gamma_{i\beta}^{\alpha})(\psi^{\gamma}) = \lambda\psi^{\alpha}$$

The curvature of the manifold is expressed by the connection $\Gamma_i^{(*)}$. In general relativity the connection Γ_{ij}^k gives rise to the gravitational force. Our model includes quantities such as $\Gamma_{i\alpha}^{\beta}$ which describes the parallel transport of spinors. We expect that more than just gravitational forces are contained in $\Gamma_i^{(*)}$.

The connection $\Gamma_{i\alpha}^{\beta}$ has two spinors indices. According to chapter 5 this may be decomposed into scalar, vector and versor components as follows.

$$\begin{aligned}\Gamma_{i\alpha}^{\beta} &= \Gamma_{i\sigma}^{\delta}1_{\delta}^{\beta}1_{\alpha}^{\sigma} \\ &= \Gamma_{i\sigma}^{\delta}(g^{kl}T_{l\delta}^{\sigma}T_{k\alpha}^{\beta} + \frac{1}{4}1_{\alpha}^{\beta}1_{\delta}^{\sigma} + g^{AB}T_{A\delta}^{\sigma}T_{B\alpha}^{\beta})\end{aligned}$$

We make the expression more compact

$$\Gamma_{i\alpha}^{\beta} = A_i1_{\alpha}^{\beta} + N_i^A T_{A\alpha}^{\beta} + G_i^k T_{k\alpha}^{\beta} \quad (6.9)$$

where we have defined the quantities

$$A_i = \frac{1}{4}\Gamma_{i\gamma}^\gamma, \quad N_i^A = \Gamma_{i\delta}^\gamma T_{B\gamma}^\delta g^{BA}, \quad G_i^k = \Gamma_{i\delta}^\gamma T_{j\gamma}^\delta g^{jk} \quad (6.10)$$

Although $\Gamma_{i\alpha}^\beta$ is not a tensor, this does not render the above decomposition invalid. It does of course mean that we cannot expect these newly defined quantities to transform as tensors. To write down their transformation properties we choose local bases for vectors, spinors and versors and a new set of bases denoted by primed indices. We define change of basis matrices at every point of the manifold, $\delta_{j'}^i$, $\delta_{\beta'}^{\alpha'}$ and $\delta_{B'}^A$, and their respective inverses $\delta_{i'}^j$, $\delta_{\beta'}^\alpha$ and $\delta_{B'}^A$. The spinor connection thus transforms according to the equation

$$\Gamma_{i'\alpha'}^{\beta'} = \delta_{i'}^i \delta_{\alpha'}^\alpha \delta_{\beta'}^\beta \Gamma_{i\alpha}^\beta - \delta_{i'}^i \delta_{\alpha'}^\alpha \frac{\partial}{\partial x^i} (\delta_{\alpha'}^{\beta'})$$

Hence we obtain the following transformations

$$A_{i'} = \delta_{i'}^i A_i - \frac{1}{4} \delta_{i'}^i \delta_{\alpha'}^\alpha \frac{\partial}{\partial x^i} (\delta_{\alpha'}^{\beta'}) \quad (6.11)$$

$$G_{i'}^{k'} = \delta_{i'}^i \delta_{k'}^{k'} G_i^k - \delta_{i'}^i \frac{\partial}{\partial x^i} (\delta_{\alpha'}^{\beta'}) T_{j\beta}^\alpha g^{jk} \delta_{\beta'}^\beta \delta_k^{k'} \quad (6.12)$$

$$N_{i'}^{A'} = \delta_{i'}^i \delta_{A'}^A N_i^A - \delta_{i'}^i \frac{\partial}{\partial x^i} (\delta_{\alpha'}^{\beta'}) T_{B\beta}^\alpha g^{BA} \delta_{\beta'}^\beta \delta_A^{A'} \quad (6.13)$$

We refer to them as the scalar, vector and versor components of the spinor connection respectively. Let us exploit the idea from general relativity that forces essentially arise from the components of the connection, however we expect more than just gravitational forces are being described here.

6.6 The gravitational connection

Hawthorn [1] uses the equation $\nabla_k(T_{i\alpha}^\beta) = 0$ to show that

$$\Gamma_{ki}^j = \partial_k(T_{i\alpha}^\beta) T_{y\beta}^\alpha g^{yj} + G_k^y T_{yi}^j$$

and

$$G_k^t = \frac{1}{6} \left(\Gamma_{ki}^j T_{mj}^i g^{mt} - \partial_k(T_{i\alpha}^\beta) T_{y\beta}^\alpha g^{yj} T_{mj}^i g^{mt} \right)$$

We see that the connection Γ_{ij}^k determines G_j^i and vice versa. Thus G_j^i must describe forces which arise from the curvature of the manifold i.e. gravity.

Accordingly we refer to G_j^i as the *gravitational connection*.

6.7 The electromagnetic connection

The scalar component of the connection is A_i and we have already said that it transforms according to the equation

$$A_{i'} = \delta_{i'}^i A_i - \frac{1}{4} \delta_{i'}^i \delta_{\alpha'}^{\alpha} \frac{\partial}{\partial x^i} (\delta_{\alpha}^{\alpha'})$$

If we assume that the change of basis matrix for spinors is not a function of position on the manifold then the last term will be zero. In such a case A_i will transform like a tensor.

In the case where A_i is the only component present in the spinor connection, the Dirac equation will take the form

$$(\partial^i + A^i) T_i^{(*)} (\psi^{\alpha}) = \lambda \psi^{\alpha} \quad (6.14)$$

It is now evident that A^i appears in the Dirac equations precisely as the electromagnetic potential should. We therefore identify A_i as the *electromagnetic potential* on our manifold from which the electromagnetic forces arise.

The stark difference between A_i and the usual electromagnetic potential of relativity is that A_i is 10-dimensional. Whatever the extra six Lorentz components may be, we expect them only to provide an $O(\frac{1}{r})$ perturbation to electromagnetism given that they are coefficients of matrices preceded by a factor of $\frac{1}{r}$ in equation 6.14.

In light of these connections we tentatively identify N_i^A with the strong and weak nuclear forces.

Chapter 7

Electromagnetism on the manifold

In usual electromagnetic theory the fields can be constructed from the potentials. Indeed in relativity the field tensor consists purely of derivatives of the electromagnetic 4-potential. We should therefore expect to be able to construct Maxwell's equations from the electromagnetic 10-potential A_i , identified in the previous chapter. In this chapter we attempt to do just that, however we find that things don't work properly. In particular we shall find that assumption 4.9 leads to an identity (theorem 7.2) which removes the vital terms from the field tensor. We then explore how this issue can be resolved.

7.1 Maxwell's equations - a first attempt

While seeking an electromagnetic theory for an ADS manifold we have already seen it appropriate to extend the definition of the electromagnetic potential from four components to ten. We shall identify the first four components of A_i with those of the ordinary 4-potential, the extra six components are at this stage unidentified although we presume they will provide only small correction terms to the electromagnetic forces. Redefining A_i automatically redefines F_{ij} to a tensor with one-hundred components (we must now use the ten component operator ∇_i . The antisymmetry of F_{ij} means it has only 45

independent components. In order to keep things consistent with the way we have hereto extended equations 2.11 - 2.14, we define a ten component current-density vector J_i , where the first four components are those of the usual current-density. An important difference to note is the extra torsion term in the field tensor

$$F_{ij} = \nabla_i A_j - \nabla_j A_i = \partial_i A_j - \partial_j A_i + T_{ij}^k A_k$$

This arises because we have not assumed the symmetry $\Gamma_{ij}^k = \Gamma_{ji}^k$ as is done in usual relativity. We shall refer to equations 7.1 - 7.3 as the **extended Maxwell equations** (expressed in natural units).

$$F_{ij} = \nabla_i(A_j) - \nabla_j(A_i) \quad \text{Definition of the field tensor.} \quad (7.1)$$

$$g^{jl} \nabla_l(F_{jk}) = J_k \quad \text{Source equation.} \quad (7.2)$$

$$\nabla_k(F_{ij}) \stackrel{(ijk)}{=} 0 \quad \text{Faraday-Gauss equation.} \quad (7.3)$$

$$\nabla_i(J^i) = 0 \quad \text{Continuity equation.} \quad (7.4)$$

$$\nabla_i(A^i) = 0 \quad \text{Gauge condition.} \quad (7.5)$$

7.2 Constraining the potential

Proposition 7.1 *If $\nabla_k(s_{\alpha\beta}) = 0$, then $\nabla_k(s^{\alpha\beta}) = 0$.*

Proof.

$$\begin{aligned} 0 &= \nabla_k(1_\alpha^\beta) = \nabla_k(s_{\alpha\lambda} s^{\beta\lambda}) \\ &= \nabla_k(s_{\alpha\lambda}) s^{\beta\lambda} + s_{\alpha\lambda} \nabla_k(s^{\beta\lambda}) \\ &= s_{\alpha\lambda} \nabla_k(s^{\beta\lambda}) \end{aligned}$$

but $s_{\alpha\lambda} \neq 0$, hence the result follows. \square

Given that $\nabla_k(s_{\alpha\beta}) = \partial_k(s_{\alpha\beta}) + \Gamma_k^*(s_{\alpha\beta}) = 0$, we establish the following

$$\begin{aligned}
\partial_k s_{\alpha\beta} &= \Gamma_{k\alpha}^\lambda s_{\lambda\beta} + \Gamma_{k\beta}^\lambda s_{\alpha\lambda} \\
&= (A_k 1_\alpha^\lambda + G_k^m T_{m\alpha}^\lambda + N_k^A T_{A\alpha}^\lambda) s_{\lambda\beta} + (A_k 1_\beta^\lambda + G_k^m T_{m\beta}^\lambda + N_k^A T_{A\beta}^\lambda) s_{\alpha\lambda} \\
&= A_k (1_\alpha^\lambda s_{\lambda\beta} + 1_\beta^\lambda s_{\alpha\lambda}) + G_k^m (T_{m\alpha}^\lambda s_{\lambda\beta} + T_{m\beta}^\lambda s_{\alpha\lambda}) + N_k^A (T_{A\alpha}^\lambda s_{\lambda\beta} + T_{A\beta}^\lambda s_{\alpha\lambda}) \\
&= 2A_k s_{\alpha\beta} + G_k^m (T_{m\alpha}^\lambda s_{\lambda\beta} - T_{m\beta}^\lambda s_{\lambda\alpha}) + N_k^A (T_{A\alpha}^\lambda s_{\lambda\beta} - T_{A\beta}^\lambda s_{\lambda\alpha}) \\
&= 2A_k s_{\alpha\beta} + G_k^m (T_{m\alpha}^\lambda s_{\lambda\beta} - T_{m\alpha}^\lambda s_{\lambda\beta}) + N_k^A (T_{A\alpha}^\lambda s_{\lambda\beta} + T_{A\alpha}^\lambda s_{\lambda\beta}) \\
&= 2A_k s_{\alpha\beta} + 2N_k^A T_{A\alpha}^\lambda s_{\lambda\beta}
\end{aligned}$$

Contracting this result with $s^{\alpha\beta}$ gives

$$\begin{aligned}
\partial_k(s_{\alpha\beta})s^{\alpha\beta} &= 2A_k s_{\alpha\beta} s^{\alpha\beta} + 2N_k^A T_{A\alpha}^\lambda s_{\lambda\beta} s^{\alpha\beta} \\
&= 2A_k 1_\alpha^\alpha + 2N_k^A T_{A\alpha}^\lambda 1_\lambda^\alpha \\
&= 8A_k + 2N_k^A T_{A\alpha}^\alpha \\
&= 8A_k
\end{aligned}$$

and in a like manner

$$\partial_k(s^{\alpha\beta})s_{\alpha\beta} = -8A_k$$

Theorem 7.2 $\partial_i A_j = \partial_j A_i$

Proof. First we establish a useful result from two simple facts: $\partial_k(1_\beta^\alpha) = 0$ and $1_\beta^\alpha = s^{\alpha\mu} s_{\beta\mu}$. Combining these

$$\begin{aligned}
\partial_k(s^{\alpha\mu})s_{\beta\mu} + s^{\alpha\mu}\partial_k(s_{\beta\mu}) &= 0 \\
\Rightarrow \partial_k(s^{\alpha\mu})s_{\beta\mu}s^{\beta\lambda} &= -s^{\alpha\mu}\partial_k(s_{\beta\mu})s^{\beta\lambda} \\
\Rightarrow \partial_k(s^{\alpha\mu})1_\mu^\lambda &= -s^{\alpha\mu}\partial_k(s_{\beta\mu})s^{\beta\lambda} \\
\Rightarrow \partial_k(s^{\alpha\lambda}) &= -s^{\alpha\mu}\partial_k(s_{\beta\mu})s^{\beta\lambda}
\end{aligned}$$

This provides us with a way to raise/lower indices of $s_{\alpha\beta}$ when it is being

operated on by a partial derivative. Consider the following

$$\begin{aligned}
& 8(\partial_i A_j - \partial_j A_i) \\
&= \partial_i(\partial_j(s_{\alpha\beta})s^{\alpha\beta}) - \partial_j(\partial_i(s_{\alpha\beta})s^{\alpha\beta}) \\
&= \partial_i\partial_j(s_{\alpha\beta})s^{\alpha\beta} + \partial_j(s_{\alpha\beta})\partial_i(s^{\alpha\beta}) - \partial_j\partial_i(s_{\alpha\beta})s^{\alpha\beta} - \partial_i(s_{\alpha\beta})\partial_j(s^{\alpha\beta}) \\
&= \partial_j(s_{\alpha\beta})\partial_i(s^{\alpha\beta}) - \partial_i(s_{\alpha\beta})\partial_j(s^{\alpha\beta}) \\
&= -\partial_j(s_{\alpha\beta})s^{\alpha\mu}\partial_i(s_{\lambda\mu})s^{\lambda\beta} + \partial_i(s_{\alpha\beta})s^{\alpha\mu}\partial_j(s_{\lambda\mu})s^{\lambda\beta}
\end{aligned}$$

After relabelling the dummy indices we see that these terms are in fact equal and opposite. Hence $\partial_i A_j = \partial_j A_i$. \square

Following theorem 7.2, the field tensor obtains the elegant form

$$F_{ij} = A_{j;i} - A_{i;j} = \partial_i A_j - \partial_j A_i + T_{ij}^m A_m = T_{ij}^m A_m$$

However the left-hand-side of the source equation is

$$\begin{aligned}
\nabla^j(F_{jk}) &= \nabla^j T_{jk}^m A_m \\
&= g^{ij}\nabla_i T_{jk}^m A_m \\
&= g^{ij}T_{jk}^m \nabla_i A_m \\
&= -g^{mj}T_{jk}^i \nabla_i A_m \\
&= -g^{ij}T_{jk}^m \nabla_m A_i \quad (\text{renaming summed over indices})
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \nabla^j(F_{jk}) &= \frac{1}{2}(g^{ij}T_{jk}^m \nabla_i A_m - g^{ij}T_{jk}^m \nabla_m A_i) \\
&= \frac{1}{2}g^{ij}T_{jk}^m (\nabla_i A_m - \nabla_m A_i) \\
&= \frac{1}{2}g^{ij}T_{jk}^m F_{im} \\
&= \frac{1}{2}g^{ij}T_{jk}^m T_{im}^p A_p \\
&= \frac{1}{2}(6_k^p)A_p \quad (\text{By equation 5.9.}) \\
&= 3A_k
\end{aligned}$$

But the right-hand-side of the source equation is J_k , this would imply $3A_k = J_k$, viz. the potential is proportional to the current-density. This is not right, in fact we should not really be comfortable with the result $\partial_i A_j = \partial_j A_i$ since if this were true in normal relativity (which should be an approximation of our theory) then $F_{ij} = 0$ everywhere, i.e. there exists only a trivial solution.

7.3 The resolution

This untenable result follows from theorem 7.2 which is based on the assumption that $s_{\alpha\beta}$ is globally invariant. It seems that we are being forced to forego this stipulation. However since we have identified $s_{\alpha\beta}$ as an intertwining map, abandoning assumption 4.9 raises a problem. This is because the global invariance of intertwining maps asserts the equivalence of some component of a representation to another representation. This is expressed in equations 4.12-4.13. The covariant derivatives of these expressions are

$$\begin{aligned}\nabla_k(x^\Sigma) &= \nabla_k(x^{\alpha_1\alpha_2\dots\alpha_n})s_{\alpha_1\alpha_2\dots\alpha_n}^\Sigma + x^{\alpha_1\alpha_2\dots\alpha_n}\nabla_k(s_{\alpha_1\alpha_2\dots\alpha_n}^\Sigma) \\ \nabla_k(x^{\alpha_1\alpha_2\dots\alpha_n}) &= \nabla_k(x^\Sigma)s_{\Sigma}^{\alpha_1\alpha_2\dots\alpha_n} + x^\Sigma\nabla_k(s_{\Sigma}^{\alpha_1\alpha_2\dots\alpha_n})\end{aligned}$$

Given that we do not expect the structure of the group to change under parallel transfer, nor the way it's representations relate to each other - which is expressed by the equations

$$\begin{aligned}\nabla_k(x^\Sigma) &= \nabla_k(x^{\alpha_1\alpha_2\dots\alpha_n})s_{\alpha_1\alpha_2\dots\alpha_n}^\Sigma \\ \nabla_k(x^{\alpha_1\alpha_2\dots\alpha_n}) &= \nabla_k(x^\Sigma)s_{\Sigma}^{\alpha_1\alpha_2\dots\alpha_n}\end{aligned}$$

it necessarily follows that

$$\nabla_k(s_{\alpha_1\dots\alpha_n}^\Sigma) = 0 \quad \text{and} \quad \nabla_k(s_{\Sigma}^{\alpha_1\dots\alpha_n}) = 0$$

The problem is that we have identified the 1-dimensional trivial component of the decomposition of the space $\{X^{\alpha\beta}\}$ with scalars (which is reasonable). This meant we were able to choose $s_{\alpha\beta} = s_{\alpha\beta}^\circ$. To resolve this contradiction we must no longer insist that we are dealing with a scalar representation. Referring to

this representation with a different index \bullet (instead of \circ) we hope to show that $s_{\alpha\beta}$ parallel transports differently to $s_{\alpha\beta}^\bullet$ (although they may be locally the same). Let us see how covariant derivatives of these quantities are related.

$$0 = \nabla_k(s_{\alpha\beta}^\bullet) = \partial_k(s_{\alpha\beta}^\bullet) + \Gamma_k^{\bullet(*)}(s_{\alpha\beta}^\bullet) \quad (7.6)$$

$$= \partial_k(s_{\alpha\beta}^\bullet) + \Gamma_{k\bullet}^\bullet s_{\alpha\beta}^\bullet - \Gamma_{k\alpha}^\lambda s_{\lambda\beta}^\bullet - \Gamma_{k\beta}^\lambda s_{\alpha\lambda}^\bullet \quad (7.7)$$

and

$$\nabla_k(s_{\alpha\beta}) = \partial_k(s_{\alpha\beta}) + \Gamma_k^{(*)}(s_{\alpha\beta}) \quad (7.8)$$

$$= \partial_k(s_{\alpha\beta}) - \Gamma_{k\alpha}^\lambda s_{\lambda\beta} - \Gamma_{k\beta}^\lambda s_{\alpha\lambda} \quad (7.9)$$

Hence if the components of $s_{\alpha\beta}^\bullet = s_{\alpha\beta}$

$$\nabla_k(s_{\alpha\beta}) = -\Gamma_{k\bullet}^\bullet s_{\alpha\beta}$$

We now see that assumption 4.9 is equivalent to the claim $\Gamma_{k\bullet}^\bullet = 0$. If this is no longer held to be true, then $\nabla_k(s_{\alpha\beta}^\bullet) = 0$ and $\nabla_k(s_{\alpha\beta}) \neq 0$ can be simultaneously true. That means we must permit the existence of scalar-like quantities which have non-trivial parallel transport. Although the property of parallel transport is not commonly attributed to scalars, there is no reason for us to assert the non-existence of such scalar-like entities. Our conclusion is that the 1-dimensional irreducible component of the space of two component spinors cannot parallel transport trivially. The paradox is thus resolved: while the components of $s_{\alpha\beta}^\bullet$ might equal $s_{\alpha\beta}$ in one frame, the ways in which each of these tensors transform are not in fact equivalent. Since $\nabla_k(s_{\alpha\beta}) \neq 0$, theorem 7.2 can no longer disallow Maxwell's equations.

This process has dispelled assumption 4.9 (section 4.10). We shall therefore have to revisit all the areas of our model which depended on this fact and see instead what is the case if we use the less stringent condition $\nabla_k(s_{\alpha\beta}^\bullet) = 0$.

Chapter 8

Revising The Hawthorn Model

Hitherto the difficulties encountered in the previous chapter, the Hawthorn model included assumption 4.9. This meant it was sufficient for us to use $s_{\alpha\beta}$ to raise and lower spinor indices. (We may still use it to do so, but there is now no guarantee that the resulting equation will remain true after parallel transport, i.e. it will not be globally invariant.) In this chapter we shall give consideration to the scalar-like quantities mentioned at the end of chapter 7 and revise any results from the Hawthorn model which depended on assumption 4.9. This material parallels [2]. We shall then make a second (and more successful) attempt at putting Maxwell's equations on an ADS manifold.

8.1 Bullet scalars

In addition to usual scalars - which parallel transport trivially, we now need to define quantities on our manifold which we shall refer to as **bullet scalars**, or b-scalars, denoted by f^\bullet or f_\bullet . These are essentially scalars with a non-trivial parallel transport property.

In an obvious manner, b-scalars can either be contravariant or covariant and of any rank. It should be noted that any b-scalar of mixed rank is equal to a b-scalar without mixed rank since b-indices \bullet automatically contract with each other, e.g. $a^{\bullet\bullet} = a_\bullet$. As a result we need only a single integer to describe the rank of a b-scalar. We let positive integers refer to contravariant indices

and negative integers refer to covariant indices, e.g. $v_{\bullet\bullet}$ has b-rank (or b-index) of -2 .

The local action on b-scalars is trivial: $T_{i\bullet}^\bullet = 0$. But as this is the same as the local action on scalars, we conclude that the local action does not uniquely determine the global action. Since $s_{\alpha\beta}^\bullet$ is locally invariant, and $T_{i\bullet}^\bullet = 0$ it follows that

$$T_{i\alpha}^\lambda s_{\lambda\beta}^\bullet + T_{i\beta}^\lambda s_{\alpha\lambda}^\bullet = 0$$

thus $s_{\alpha\beta}^\bullet$ is a locally invariant symplectic form.

Consider the b-scalar which takes the value 1 at every point, denoted by 1^\bullet . Tensor product with 1^\bullet raises b-index while tensor product with its dual 1_\bullet lowers b-index. These maps will alter the global action. The set $\{1^\bullet\}$ is a basis for the 1-dimensional trivial representation. We can transform to and from another basis $\{1^{\bullet'}\}$ using non-singular change of basis matrices $\delta_{\bullet'}^\bullet$ and $\delta_{\bullet}^{\bullet'}$. Thus $\delta_{\bullet'}^\bullet = k$ for some scalar k , hence

$$\begin{aligned} s_{\alpha\beta}^{\bullet'} &= \delta_{\bullet'}^\bullet \cdot s_{\alpha\beta}^\bullet \\ s_{\alpha\beta}^{\bullet'} &= k \cdot s_{\alpha\beta}^\bullet \end{aligned}$$

so transforming b-basis is equivalent to picking a different symplectic form.

8.2 The (new) reduced curvature tensor

Now that we do not maintain $s_{\alpha\beta}$ is globally invariant, we are no longer at liberty to use equation 4.2 to conclude $R_{ij} \binom{*}{*} (s_{\alpha\beta}) = 0$. However, it does follow from equation 4.2 that

$$R_{ij} \binom{*}{*} (s_{\alpha\beta}^\bullet) = 0$$

Consequently we must update theorem 5.1.

Theorem 8.1 *There exists a tensor R_{ij}^s such that $R_{ij\alpha}^\beta = R_{ij}^s T_{s\alpha}^\beta + \frac{1}{2} R_{ij\bullet}^\bullet 1_\alpha^\beta$.*

Proof. Since $R_{ij} \binom{*}{*} (s_{\alpha\beta}^\bullet) = 0$

$$\begin{aligned}
R_{ij} (*)(s_{\alpha\beta}^\bullet) &= 0 \\
\Rightarrow -R_{ij\alpha}^\lambda s_{\lambda\beta}^\bullet - R_{ij\beta}^\lambda s_{\alpha\lambda}^\bullet + R_{ij\bullet}^\bullet s_{\alpha\beta} &= 0 \\
\Rightarrow -R_{ij\alpha}^\lambda s_{\lambda\beta} - R_{ij\beta}^\lambda s_{\alpha\lambda} + R_{ij\bullet}^\bullet s_{\alpha\beta} &= 0 \\
\Rightarrow \left[R_{ij} (*) - \frac{1}{2} R_{ij\bullet}^\bullet 1 (*) \right] s_{\alpha\beta} &= 0
\end{aligned}$$

this means that $R_{ij\alpha}^\beta - \frac{1}{2} R_{ij\bullet}^\bullet 1_\alpha^\beta$ is a matrix in $sp(4, \mathbb{R})$, for fixed i and j .

Therefore we can write it as a linear combination of basis elements of $sp(4, \mathbb{R})$.

Hence

$$R_{ij\alpha}^\beta - \frac{1}{2} R_{ij\bullet}^\bullet 1_\alpha^\beta = R_{ij}^s T_{s\alpha}^\beta \quad (8.1)$$

□

It turns out that the result theorem 5.2 remains true, though the argument must be altered somewhat.

Theorem 8.2 *If R_{ij}^s is defined as above then $R_{ijk}^t = R_{ij}^s T_{sk}^t$*

Proof. $R_{ij} (*)(T_{k\alpha}^\beta) = 0$ hence it follows that

$$\begin{aligned}
R_{ijk}^t T_{t\alpha}^\beta &= R_{ij\lambda}^\beta T_{k\alpha}^\lambda - R_{ij\alpha}^\lambda T_{k\lambda}^\beta \\
&= \left(R_{ij}^s T_{s\lambda}^\beta + \frac{1}{2} R_{ij\bullet}^\bullet 1_\lambda^\beta \right) T_{k\alpha}^\lambda - \left(R_{ij}^s T_{s\alpha}^\lambda + \frac{1}{2} R_{ij\bullet}^\bullet 1_\alpha^\lambda \right) T_{k\lambda}^\beta \\
&= R_{ij}^s \left(T_{s\lambda}^\beta T_{k\alpha}^\lambda - T_{s\alpha}^\lambda T_{k\lambda}^\beta \right) \\
&= R_{ij}^s T_{sk}^t T_{t\alpha}^\beta
\end{aligned}$$

Since $T_{t\alpha}^\beta$ is in general non-zero, we conclude that $R_{ijk}^t = R_{ij}^s T_{sk}^t$. □

8.3 Maxwell's equations - a second attempt

Now that we have refined our model we wish to once again consider the extended Maxwell equations. The simplest test would be to compare these to the Maxwell equations in flat spacetime. The first thing we expect is for any

additional terms in the extended version of Maxwell's equations 2.1-2.5 to be of order $1/r$ (at least). This means they should reduce correctly in the limit $r \rightarrow \infty$. Secondly, given that we have an additional six components in both our electromagnetic potential and current-density vectors, we shall obtain an extra set of equations. We seek to obtain relationships which these extra components obey.

Let us refer to the vector whose components are A_i as \mathcal{A} . We introduce the notation $\mathcal{A} = (\phi, \mathbf{A}, \mathbf{P}, \mathbf{M})$, ϕ is a scalar and \mathbf{A} , \mathbf{P} and \mathbf{M} are 3-vectors. In a similar manner the charge-density 10-vector $\mathcal{J} = (\rho, \mathbf{J}, \vec{\mathbf{J}}, \mathring{\mathbf{J}})$ has components J_i .

8.4 The source equations

Expanding the LHS of equation 7.2

$$\begin{aligned}
\nabla^j(F_{jk}) &= g^{ij}\nabla_i(F_{jk}) \\
&= g^{ij}\nabla_i(\nabla_j(A_k) - \nabla_k(A_j)) \\
&= g^{ij}[\partial_i\partial_j A_k - \partial_i\partial_k A_j - \Gamma_{ij}^l\partial_l A_k - \Gamma_{ik}^l\partial_j A_l \\
&\quad + \Gamma_{ik}^l\partial_l A_j + \Gamma_{ij}^l\partial_k A_l + T_{jk}^p\partial_i A_p - T_{jk}^p\Gamma_{ip}^l A_l] \quad (8.2)
\end{aligned}$$

We wish to consider the extended Maxwell equations for flat space i.e. in the absence of curvature. In usual relativity the condition $\Gamma_{ij}^k = 0$ is sufficient to ensure the (4-dimensional) curvature tensor vanishes and Einstein's equations reduce to $R_{ij} = 0$. The Minkowski metric is a solution to this form of Einstein's equations.

On an ADS manifold the connection Γ_{ij}^k has antisymmetric components which we are not free to make zero since this would remove the Lie structure T_{ij}^k . If we use

$$\Gamma_{ij}^k = -\frac{1}{2}T_{ij}^k \quad (8.3)$$

as our flat space condition then the curvature tensor reduces to $R_{ijk}^l = \frac{1}{4}T_{ij}^x T_{kx}^l$. Contraction yields the Ricci tensor $R_{ik} = -\frac{1}{4}g_{ik}$. Across the spacetime dimensions this is Einstein's equation for empty space with non-zero cosmological constant and has as its solution the anti de Sitter metric given in equation 3.5. Flat space for an ADS manifold looks like anti de Sitter spacetime.

Using condition 8.3, the Casimir identity 5.9 and the identity $g^{ij}T_{ik}^l = -g^{il}T_{ik}^j$ (a consequence of the local invariance of g_{ij}), the LHS of equation 7.2 reduces to

$$\nabla^j(F_{jk}) = g^{ij}[\partial_i\partial_j A_k - \partial_i\partial_k A_j + 2T_{ik}^l\partial_j A_l] + 3A_l 1_k^l$$

We have therefore simplified our source equation for flat space

$$g^{ij}\partial_i\partial_j A_k - g^{ij}\partial_i\partial_k A_j + g^{ij}2T_{ik}^l\partial_j A_l + 3A_k = J_k \quad (8.4)$$

As there is only one free index in this expression we have a total of ten equations. It will be clearer for us to use the notation $\nabla = (\partial_X, \partial_Y, \partial_Z)$, $\vec{\nabla} = (\partial_A, \partial_B, \partial_C)$, $\overset{\circ}{\nabla} = (\partial_I, \partial_J, \partial_K)$. The first equation, corresponding to the index value $k = 0$ (k , the free index from 8.4) is then (in natural units)

$$\begin{aligned} \nabla \cdot \nabla\phi + \vec{\nabla} \cdot \vec{\nabla}\phi - \overset{\circ}{\nabla} \cdot \overset{\circ}{\nabla}\phi - \nabla \cdot \partial_T \mathbf{A} - \vec{\nabla} \cdot \partial_T \mathbf{P} + \overset{\circ}{\nabla} \cdot \partial_T \mathbf{M} \\ - 2\nabla \cdot \mathbf{P} + 2\vec{\nabla} \cdot \mathbf{A} + 3\phi = \rho \end{aligned} \quad (8.5)$$

We have chosen to work with the basis where the components of the torsion tensor are given by the matrices in appendix A. Hence the metric will be diagonal. In flat space we can establish the following proposition.

Proposition 8.3 *Partial derivatives transform as tensors provided the assumption (8.3) holds.*

Proof. According to (8.3)

$$\nabla_i = \partial_i - \frac{1}{2}T_i^{\ast} \quad \Rightarrow \quad \partial_i = \nabla_i - \frac{1}{2}T_i^{\ast}$$

Since ∇_i and T_i^{\ast} are tensors and the difference of any two tensors is again a tensor, it follows that ∂_i is a tensor too. \square

Up until now we have assumed ourselves to be working in natural units. One natural unit of time is equal to r ordinary units of time: $T = r\bar{T}$. Similar relations hold for space: $X = rc\bar{X}$, Lorentz boost: $A = c\bar{A}$, and rotation: $I = \bar{I}$. Using these relations we deduce that the change of basis matrix which transforms from natural coordinates x^i and ordinary coordinates \bar{x}^j is

$$\delta_j^{i'} = \frac{\partial \bar{x}^i}{\partial x^j} = \begin{cases} 0 & i \neq j \\ 1/r & i = j = 0 \\ 1/(rc) & i = j = 1, 2, 3 \\ 1/c & i = j = 4, 5, 6 \\ 1 & i = j = 7, 8, 9 \end{cases}$$

We know what the matrix g_{ij} and the matrices T_{ij}^k look like in natural units. We are interested in expressing A_i and the derivative operators in ordinary units.

$$A_j = \delta_j^{i'} A_{i'} \quad \text{and} \quad \frac{\partial}{\partial x^j} = \delta_j^{i'} \frac{\partial}{\partial x^{i'}}$$

Hence in ordinary units (8.5) becomes

$$\begin{aligned} r^3 c^2 \nabla \cdot \nabla \phi + r c^2 \vec{\nabla} \cdot \vec{\nabla} \phi - r \overset{\circ}{\nabla} \cdot \overset{\circ}{\nabla} \phi - r^3 c^2 \nabla \cdot \partial_T \mathbf{A} - r c^2 \vec{\nabla} \cdot \partial_T \mathbf{P} \\ + r \overset{\circ}{\nabla} \cdot \partial_T \mathbf{M} - 2r c^2 \nabla \cdot \mathbf{P} + 2r c^2 \vec{\nabla} \cdot \mathbf{A} + 3r \phi = r \rho \end{aligned} \quad (8.6)$$

We now introduce constants to allow us to adjust the units of the components of \mathcal{A} and \mathcal{J} (to e.g. SI units).

$$\begin{aligned} \phi &\rightarrow k_\phi \phi & \mathbf{A} &\rightarrow k_A \mathbf{A} & \mathbf{P} &\rightarrow k_P \mathbf{P} & \mathbf{M} &\rightarrow k_M \mathbf{M} \\ \rho &\rightarrow k_\rho \rho & \mathbf{J} &\rightarrow k_J \mathbf{J} & \vec{\mathbf{J}} &\rightarrow k_{\vec{\mathbf{J}}} \vec{\mathbf{J}} & \overset{\circ}{\mathbf{J}} &\rightarrow k_{\overset{\circ}{\mathbf{J}}} \overset{\circ}{\mathbf{J}} \end{aligned}$$

Accordingly (8.6) is

$$\begin{aligned} k_\phi c^2 \nabla \cdot \nabla \phi - k_A c^2 \nabla \cdot \partial_T \mathbf{A} + \frac{1}{r^2} \left(k_\phi c^2 \vec{\nabla} \cdot \vec{\nabla} \phi - k_\phi \overset{\circ}{\nabla} \cdot \overset{\circ}{\nabla} \phi \right. \\ \left. - k_P c^2 \vec{\nabla} \cdot \partial_T \mathbf{P} + k_M \overset{\circ}{\nabla} \cdot \partial_T \mathbf{M} - 2k_P c^2 \nabla \cdot \mathbf{P} + 2k_A c^2 \vec{\nabla} \cdot \mathbf{A} + 3k_\phi \phi \right) = \frac{k_\rho \rho}{r^2} \end{aligned} \quad (8.7)$$

If we assume $k_\phi(r) \propto k_A(r)$, $k_P/k_A \propto r^n$ and $k_M/k_A \propto r^n$ where $n \leq 1$, we may neglect the second order $\frac{1}{r}$ terms on the left hand side of equation 8.7

when r is large. In accord with the assumption that r is large we shall drop the second order terms from the left hand side

$$k_\phi c^2 \nabla \cdot \nabla \phi - k_A c^2 \nabla \cdot \partial_T \mathbf{A} = \frac{k_\rho \rho}{r^2} \quad (8.8)$$

Assuming $k_A = -k_\phi$ allows us to factorise

$$k_A c^2 \nabla \cdot (-\nabla \phi - \partial_T \mathbf{A}) = \frac{k_\rho \rho}{r^2} \quad (8.9)$$

Substituting the electromagnetic fields in terms of their potentials using equations 2.8 and 2.9 makes equation 8.9 equivalent to

$$k_A c^2 \nabla \cdot \mathbf{E} = \frac{k_\rho \rho}{r^2} \quad (8.10)$$

which is precisely Gauss's Law (equation 2.1) for the distribution of electric charge in SI units provided

$$k_A = -k_\phi \quad k_\rho = \frac{k_A r^2 c^2}{\epsilon_0}$$

In a similar manner equation 8.4 also yields the Ampere-Maxwell equation (equation 2.4)

$$k_A \partial_T \mathbf{E} - k_A c^2 \nabla \times \mathbf{B} = \frac{k_J \mathbf{J}}{r^2} \quad (8.11)$$

where

$$k_J = -k_A \mu_0 r^2 c^2 \quad \Rightarrow \quad k_J = -\frac{k_\rho}{c}$$

We also obtain two new equations

$$k_P c (-\partial_T^2 + c^2 \nabla^2) \mathbf{P} - 2k_A c \mathbf{E} = \frac{k_J c}{r^2} \mathbf{J} \quad (8.12)$$

$$k_M (-\partial_T^2 + c^2 \nabla^2) \mathbf{M} - 2k_A c^2 \mathbf{B} = \frac{k_J \mathbf{J}}{r^2} \quad (8.13)$$

If k_P/k_A and k_M/k_A are proportional to r then the \mathbf{E} and \mathbf{B} terms will also disappear in the limit $r \rightarrow \infty$.

8.5 The Faraday-Gauss equations

We expand the Faraday-Gauss equation 7.3

$$\begin{aligned} \partial_k \partial_i A_j - \partial_k \partial_j A_i - \Gamma_{ki}^x (\partial_x A_j - \partial_j A_x) - \Gamma_{kj}^y (\partial_i A_y - \partial_y A_i) \\ + T_{ij}^l \partial_k A_l - T_{ij}^l \Gamma_{kl}^n A_n \stackrel{ijk}{=} 0 \end{aligned}$$

Making our usual assumption that $T_{ij}^k = -2\Gamma_{ij}^k$, simplifies this

$$\begin{aligned} \partial_k \partial_i A_j - \partial_k \partial_j A_i + \frac{1}{2} T_{ki}^x (\partial_x A_j - \partial_j A_x) + \frac{1}{2} T_{kj}^y (\partial_i A_y - \partial_y A_i) \\ + T_{ij}^l \partial_k A_l - T_{ij}^l \frac{1}{2} T_{kl}^n A_n \stackrel{ijk}{=} 0 \end{aligned}$$

The last term is the Jacobi identity when cycled through ijk . The other torsion terms can also be simplified.

$$\partial_k \partial_i A_j - \partial_k \partial_j A_i + T_{ki}^x \partial_x A_j \stackrel{ijk}{=} 0 \quad (8.14)$$

which provides us with so many equations that writing them all down will not be an insightful exercise. However we shall write down the ones arising from only considering the cases in which i, j, k are space or time indices. In the limit $r \rightarrow \infty$ we obtain the following

$$\nabla \cdot \mathbf{B} = 0 \quad (8.15)$$

and

$$\partial_T(\mathbf{B}) + \nabla \times \mathbf{E} = 0 \quad (8.16)$$

which are precisely equations 2.2 and 2.3.

8.6 Maxwell's equations - a final adjustment

In order for two tensor quantities to be equivalent both their components and transformation properties must be identical. Consider now $s_{\alpha\beta}^\bullet$, which has components equal to the bilinear form $s_{\alpha\beta}$. Hence the expression $s_{\alpha\beta}^\bullet s_{\bullet}^{\alpha\lambda}$ has components equal to those of $s_{\alpha\beta} s^{\alpha\lambda} = 1_\beta^\lambda$. Furthermore the indices \bullet and α are summed over thus these two expressions have identical transformation properties. Therefore they are the same quantity. This essentially means that the process of e.g. raising a spinor index, though it leaves behind a bullet index, can be undone by lowering the spinor index whereby the bullet index will be cancelled out. We have defined $A_k = \frac{1}{4} \Gamma_{k\alpha}^\alpha$ and we shall now proceed

to see how it is related to $D_k \equiv \Gamma_{k\bullet}^\bullet$.

$$\begin{aligned}
0 &= \nabla_k (s_{\alpha\beta}^\bullet) s_{\bullet}^{\alpha\beta} = \partial_k (s_{\alpha\beta}^\bullet) s_{\bullet}^{\alpha\beta} + \Gamma_{k\bullet}^\bullet s_{\alpha\beta}^\bullet s_{\bullet}^{\alpha\beta} - \Gamma_{k\alpha}^\lambda s_{\lambda\beta}^\bullet s_{\bullet}^{\alpha\beta} - \Gamma_{k\beta}^\lambda s_{\alpha\lambda}^\bullet s_{\bullet}^{\alpha\beta} \\
&= \partial_k (s_{\alpha\beta}^\bullet) s_{\bullet}^{\alpha\beta} + \Gamma_{k\bullet}^\bullet 1_\alpha^\alpha - \Gamma_{k\alpha}^\lambda 1_\lambda^\alpha - \Gamma_{k\beta}^\lambda 1_\lambda^\beta \\
&= \partial_k (s_{\alpha\beta}^\bullet) s_{\bullet}^{\alpha\beta} + 4D_k - 2A_k
\end{aligned}$$

If we now consider contractions of the tensors $R_{ij\bullet}^\bullet$ and $R_{ij\alpha}^\beta$ obtained from the action of $R_{ij}^{\left(\ast\right)}$ on a bullet scalar and a spinor respectively. We find that

$$R_{ij\bullet}^\bullet = \partial_i (D_j) - \partial_j (D_i) \quad (8.17)$$

$$R_{ij\alpha}^\alpha = \partial_i (A_j) - \partial_j (A_i) \quad (8.18)$$

Taking the trace of the Greek indices in equation 8.1 it follows that

$$\partial_i A_j - \partial_j A_i = 2(\partial_i D_j - \partial_j D_i)$$

We have chosen to use $\partial_i A_j - \partial_j A_i + T_{ij}^k A_k$ as our Electromagnetic field tensor, however 8.18 indicates that $\partial_i A_j - \partial_j A_i$ might be a more natural choice as it would indicate that the electromagnetic force arises from the presence of curvature. Nevertheless let us recall the Bianchi identity 5.10 obtained from applying the Jacobi identity to spinors

$$T_{ij}^l R_{lk\alpha}^\beta - \nabla_k (R_{ij\alpha}^\beta) \stackrel{(ijk)}{=} 0 \quad (8.19)$$

It will constrain $R_{ij\alpha}^\alpha$ and hence the extended Maxwell equations in some way. Substituting equation 8.18 into the contracted version of equation 8.19 and using the fact that we can permute the i, j, k indices of any term without altering the equation yields

$$-\partial_k (\partial_i A_j - \partial_j A_i) \stackrel{(ijk)}{=} 0 \quad (8.20)$$

as identically true on an ADS manifold.

This means the extended Maxwell equations (as they stand) are inconsistent with the geometry of an ADS manifold since they permit the torsion term in

equation 8.14 which is a contradiction of identity 8.20. In order to make the unwanted term disappear, we make a fundamental redefinition of the electromagnetic field tensor as follows

$$\boxed{\hat{F}_{ij} \equiv \partial_i A_j - \partial_j A_i} \quad (8.21)$$

According to equation 8.18 this actually appears to be a more natural definition anyway. Aside from the fact that $R_{ij\alpha}^\alpha$ is a tensor, \hat{F}_{ij} is clearly a tensor since $\nabla_i A_j - \nabla_j A_i - T_{ij}^k A_k$ is a tensor. Such a redefinition will not alter the approximations 8.15 and 8.16 obtained in the limit $r \rightarrow \infty$.

This brings us to the realisation that the Faraday-Gauss equation 7.3 is simply a consequence of the Bianchi identity 8.19. Equation 7.3 no longer needs to be postulated independently but follows from the geometry of an ADS manifold.

8.7 Consequences for the source equation

Using \hat{F}_{ij} instead of F_{ij} will also simplify the LHS of the source equation 8.4

$$g^{ij} (\partial_i \partial_j A_k - \partial_i \partial_k A_j + T_{ik}^l \partial_j A_l) = J_k \quad (8.22)$$

(We are using condition 8.3.) The only difference this will make to the approximations 8.10-8.13 will be losing the factor of 2 in front of the \mathbf{E} and \mathbf{B} terms from equations 8.12 and 8.13 respectively.

It remains to interpret the quantities \mathbf{P} , \mathbf{M} , $\vec{\mathbf{J}}$ and $\overset{\circ}{\mathbf{J}}$. The type of particles which carry charge - electrons and protons - also possess spin. In order to describe the electromagnetic evolution of a distribution of charged particles more accurately it would make sense to also take into account the spin-density¹ of the distribution (although for practical purposes this addition may often-times be negligible). We therefore predict the six components $\vec{\mathbf{J}}$ and $\overset{\circ}{\mathbf{J}}$ are related to the spin-density of a charge distribution, indeed the total angular

¹The intrinsic angular momentum of a charged particle gives rise to a magnetic moment.

momentum tensor of relativity is antisymmetric, hence it has six independent components, see p. 157-159 [11]. Thus we predict the components \mathbf{P} and \mathbf{M} of the 10-potential should give rise to fields which exert forces on test particles possessing an intrinsic angular momentum.

Chapter 9

Conclusion

We have explored how the Poincaré Lie algebra approximates the Lie algebra $so(2, 3)$ meaning that they may both be used to describe local spacetime symmetry for classical physics. We found reason to choose $so(2, 3)$ as our local symmetry group and it is upon this assumption that the Hawthorn model has been constructed. The axioms involved were clearly stated before we moved on to show that the Dirac equation fits very nicely on an ADS manifold. From the covariant derivative in the Dirac equation arises what we have called a spinor connection. This connection decomposes into three terms, one of which we identify as the electromagnetic potential. Using this potential to construct the extended Maxwell equations (an appropriate generalisation of Maxwell's equations on the ADS manifold) we discover a new result, that the assumption $\nabla_k(s_{\alpha\beta}) = 0$ essentially ensures their non-existence. To relinquish this assumption we are forced to accept the existence of the so-called *bullet scalars* on the ADS manifold. We then reconstructed the extended Maxwell equations in a way that is consistent with Maxwell's equations in the limit $r \rightarrow \infty$. In the process we obtain new equations pertaining to the extra components of the potential \mathcal{A} and current-density \mathcal{J} . We then identified a relationship between the electromagnetic field tensor and the trace of the spinor curvature tensor. As the curvature tensor must obey the Bianchi identities it turns out that this condition contradicts the Faraday-Gauss equation 8.14 unless we re-

define our field tensor. This redefinition means the electromagnetic field tensor arises from the trace of the spinor components of the spinor curvature tensor. Furthermore we have found that the Faraday-Gauss equation 7.3 is purely a consequence of the Bianchi identity 8.19 i.e. the geometry of spacetime, and does not need to be postulated independently.

$$\text{ad}_X = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{ad}_Y = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{ad}_Z = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{ad}_A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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