Perceiving Necessity

Abstract: There are many examples of diagrams in which one seems to perceive necessity – one sees not only that something *is* so, but that it *must be* so. That conflicts with some well-known philosophical theses, inherited from Hume, according to which there cannot be any "necessary connections between distinct existences" to be perceived; and even if there were, perception would not be capable of gaining access to them. We defend the perception of necessity, and explain why Hume fails to show its impossibility.

1. Examples of Diagrams in Which Necessity is Perceived

We may gain knowledge that $2 \times 3 = 3 \times 2$ not by rote but by looking at the diagram:

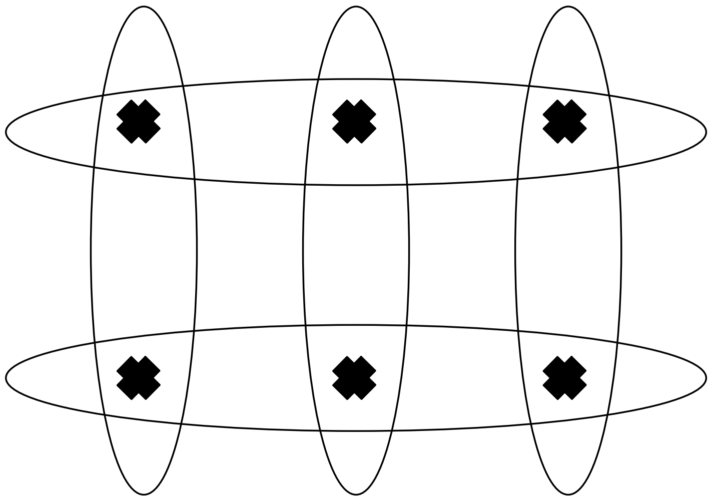


Fig 1. Why $2 \times 3 = 3 \times 2$

What is being perceived in this case? We are not just perceiving that 2×3 *is* 3×2, but that 2×3 *must be* 3×2. It is clear that to try to create another option—such as $2 \times 3 = 3 \times 3$—would be futile. It is as though we can see the truth and at the same time the reason why it must be true.[1] It therefore appears that *we may perceive necessary truth*.

What has just been said typically attracts many objections. What does it mean to *see* a reason? Surely reasons are more to be *stated* or *understood* than seen. Didn't key cases in nineteenth century mathematics show that deriving mathematical results from visual intuition risks wholescale error?[2] While mathematics as an *a priori* science does seem to possess a particular perspicuousness, couldn't that rather be because it involves, for example, reasoning using unusually explicit chains of inference, and particularly compelling premises? In any case, would not perception of necessity involve seeing necessary connections between distinct existences, something shown impossible by Hume? And surely, even if there were such necessary connections to be known, perception is too simple (or, frankly, too stupid) a faculty to perceive them? And is not any "must-detecting" faculty (Blackburn 1986) incompatible with naturalism?

Before considering those objections, let us look at a slightly more complex example, in which the role of visual perception in seeing a figure as divided into parts in two ways is exceptionally lucid. It is a remarkable fact of elementary mathematics that the sum of the first *n* odd numbers is always a square:

$1 + 3 = 4 = 2^2$

$1 + 3 + 5 = 9 = 3^2$

$1 + 3 + 5 + 7 = 16 = 4^2$

and so on …

Why? The reason is clear in the following diagram:

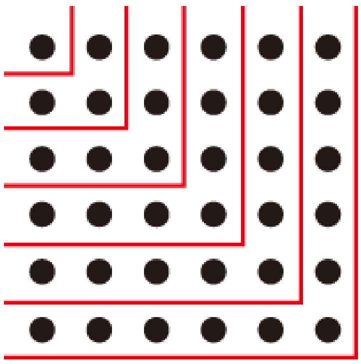


Fig 2. Why $1 + 3 + 5 + 7 + 9 + 11 = 6\times6$ [3]

Of special interest are the dashed L-shaped lines: plainly, their purpose is to guide visual perception in dividing the array of dots. They show how to divide the array into the sum of the odd numbers; while perception also, separately, recognizes that the array is a 6×6 square. Hence the result, that $1 + 3 + 5 + 7 + 9 + 11$ *must* equal 6^2. What is the chain of inference here? There is none. The equivalence of the square of L-shaped arrays of dots and the 'squared number' is directly apprehended.

The conclusion is strengthened if we ask how the reasoning could be translated into symbolic form, without reliance on the diagram. There exist several relatively simple stepwise symbolic proofs of the result, such as a proof by mathematical induction. However, they do not give the same immediate insight, nor are they translations of the visual insight in any sense of ‘translation’ meaningful in this context.

One might ask whether mathematical necessity is only perceivable for relatively simple *arithmetic* truths concerning discrete, countable numbers of things easily gathered and inspected, as above. The answer is not strictly relevant to our thesis, which is merely that at least some mathematical diagrams allow the perception of necessity. However, a brief survey will provisionally indicate a number of other areas of mathematics where necessity may be directly perceived.

Geometry traffics in irrational quantities, so one might worry that at least some of its necessary truths concern distinctions too fine for the naked eye to discern (for example that the ratio of a circle’s circumference and diameter is π). However necessity may also be perceived in geometry. Giaquinto gives this simple example, similar to that in Plato’s *Meno*. The inside square, in diamond orientation, has sides that connect the midpoints of the sides of the larger square. How much is the area of the outside, compared to the inside square? If we imagine folding in the four outer triangles, we see that they would cover the inner square exactly—therefore the larger square is twice its area (Giaquinto 2007 fig 4.1).

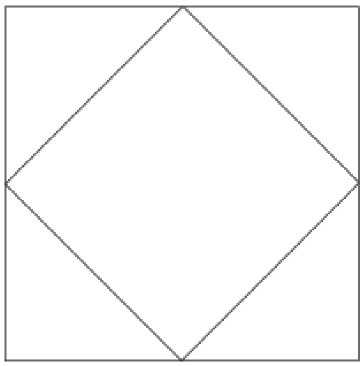


Fig 3. Why the inside square is half the outside square

Giaquinto (2007, pp. 28-29) argues—and we agree—that perception of this geometrical necessity does not depend on any infinitely precise perception of line-straightness or point-positioning. To use instruments to precisely measure sides and areas in order to 'verify' this proof would be beside the point.[4] The visual sense is able to 'rectify' an approximate diagram and thereby see what must be so in the relations of two exact squares positioned as shown. We want to say something like this: the diagram, at the same time as being a particular, inexactly-drawn set of marks, also shows a necessary truth which is fully exact. Once again, some element of 'seeing the reason why it must be so' seems to guide this rectification process.

What about very abstract areas of mathematics such as higher analysis and transfinite mathematics? It might be supposed that here one must visualise infinitely large sets, which is impossible. However, certain examples suggest that it is even possible to perceive necessary truths that 'go to infinity'. Consider the following diagram, which illustrates that the set of rational numbers has the same cardinality as the (countable) natural numbers:

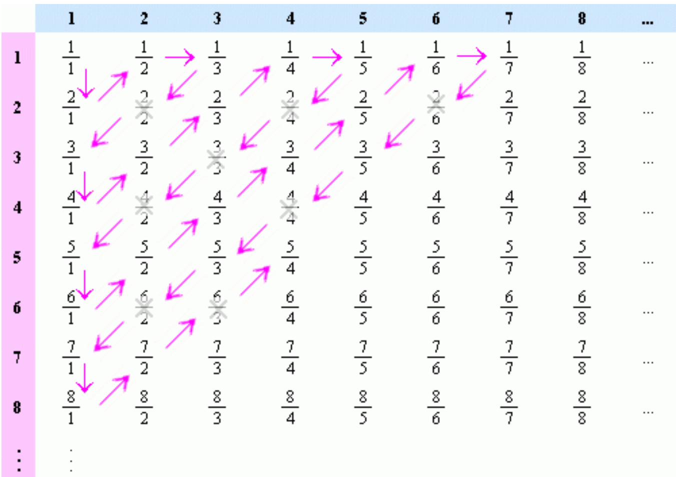


Fig 4. Why the set of rational numbers is countable.[5]

This result was enormously counterintuitive, since the rational numbers are so thickly distributed on the real line. Cantor, who first proved it, famously wrote in a letter to Dedekind, of a closely related result, 'I see it but I don't believe it!'[6] Note how naturally Cantor reaches for visual language to express such crucial moments.

Specifically, by means of the clever organization of the fractions in *Fig. 4* one can see that a progression through them has been defined so systematically that every rational number has a unique well-defined place in the sequence. This means that every whole number may be assigned one corresponding rational number, and *vice versa*. One is not viewing the entire infinite sequence of natural—rational pairings, but *one doesn't* *need to*. One can already see that the correspondence has to be one-one, by seeing 'how it works' in some very general sense.[7]

Our project connects with a recent upsurge of interest in the role of diagrams in necessary reasoning: both in mathematics (Brown 1999; Franklin 2000; Giaquinto 2007; Sherry 2009a; Mumma 2010; Catton and Montelle 2012: many of these authors inspired by Manders 2008) and in logic (Shin 2002; Macbeth 2005 & 2009; Legg 2012). This has been prompted by growing awareness that diagrams frequently serve as more than just a *post hoc* illustration of necessary reasoning which has already taken place in some more 'serious' nondiagrammatic form. This movement has brought fresh perspectives to philosophy of mathematics and philosophical logic. However we believe that, even more importantly, it holds profound implications for general epistemology which are currently blocked by fealty to a certain picture of perception descending from Hume. We here aim to challenge that picture and open up that wider epistemological debate.

The opposition which Hume's legacy offers to acceptance of our thesis includes ontological and epistemological dimensions which are deeply related. The ontological dimension will be examined in the next section. This is Hume's thesis that 'there are no necessary connections between distinct existences', which has been taken to imply that there are no such necessities 'out there', hence no possibility of perceiving them. The epistemological dimension, examined in sections 3 and 4, includes a thesis that perception is both *passive*—involving a 'registering' of the impact of actual individual physical objects on the sense organs—and *atomistic* such that ideas are only considered to be distinct—or even *distinguishable*—if fully separable in the imagination. Together these claims are taken to imply the impossibility of perceiving anything modal, such as necessities. The final sections of this paper provisionally indicate significant empirical inadequacy in this understanding of perception (section 5), and show how the examples above escape it (section 6), arguing that Hume's dictum is either trivially true but lacks sufficient content to refute our examples, or else begs the question against them. In conclusion (section 7) we briefly explore the idea that what Hume's maxim most deeply begs the question against is the possibility of real universals.

Before beginning to discuss Hume, we will briefly clarify what we mean by 'perception' and 'necessity' in our claim that necessity can be perceived. We intend a minimal and uncontroversial reading of each term. By perception we mean the acquiring of beliefs about reality through veridical operations of the sense organs without the assistance of deliberate mental actions such as imagining or explicit reasoning. By necessity we understand just the strong kind of necessity evident in the basic truths of mathematics; we do not commit ourselves to any particular analysis of that necessity, such as that it is logical or formal.

2. 'No Necessary Connections Between Distinct Existences': The Claim and what it Means

Hume asserts:

> No connexions between distinct existences are ever discoverable among human understanding. (*Treatise*, Appendix)

And again:

> There is no object, which implies the existence of any other if we consider these objects in themselves. (*Treatise*, 1, III, vi)

And more expansively:

> Any thing may produce any thing. Creation, annihilation, motion, reason, volition; all these may arise from one another, or from any other object we can imagine … no objects are contrary to each other, but existence and non-existence. (*Treatise*, 1, III, xv)

This conclusion that no objects are contrary but existence and non-existence is quite remarkable. Does Hume really mean it? How about the 'objects' black and white? Or more relevantly to our present purpose, equal and not equal?

The meaning of these claims clearly depends crucially on what is meant by 'existences', or 'objects'. If Hume means non-overlapping physical substances, such as tables in a room, that is one claim, for which it seems good arguments could be mounted. But he does not mean that, nor do his modern followers. 'Creation', 'non-existence' and so on are not physical substances. And in Hume's most famous application of his principle, to causality, he does not mean physical substances either. In the famous scene where one billiard ball strikes another, all we strictly see, Hume argues, is that the first ball moves towards the second, *and then* touches it briefly, *and then* the second ball moves away from the first:

Comment [YUN1]: This passage already sufficiently blurs the distinction between '[physical] objects' and 'ideas' in Hume's use of the maxim, to my mind to enable us to be calling that whole distinction into question in our challenge to Hume.

> I turn my eye to two objects suppos'd to be plac'd in that relation [of cause and effect] … I immediately perceive, that they are contiguous in time and place, and that the object we call cause precedes the other we call effect. In no instance can

> I go any farther, nor is it possible for me to discover any third relation betwixt these objects. (*Treatise,* 1, III, xiv)

Here Hume speaks as if the 'objects' are just the billiard balls themselves, but he is actually talking about their *motions*. He is claiming that the motion of the first could exist without the motion of the second, so there is no necessary connection between them. Motions might be construed as properties of the balls, or perhaps as series of events, but they are not physical substances.

What, then, does count as 'objects' or 'existences' in the dictum: 'There are no necessary connections between distinct existences'? Properties and events must be allowed, if motions are to count. But then, what about the properties 'black' and 'white'? The events of 2 and 3 hours passing? Or, to return to mathematics, the 'objects' $\emptyset$ and $\{\emptyset\}$? Even if we restrict ourselves to the motions of billiard balls, it seems there *are* necessary connections between the three 'existences':

1. The ball's being at position 0 at time 0
2. The ball's being at position 1 at time 1
3. The ball's having velocity 1 throughout the time interval (0,1)

Namely, 1 and 3 imply 2; 2 and 3 imply 1; 3 implies (1 if and only if 2).[8] Such examples suggest that an inquisitive eye be turned also on the notion of 'distinct'. Are the empty set and its singleton 'distinct'? They are not identical, but the former is an element of the latter, so it is not surprising that the latter cannot exist without it (if of course the two can be said to exist).

What then *does* Hume mean by 'distinct existences'? In an intricate and searching paper Jessica Wilson has pointed out that it is very difficult to define an unambiguous version of Hume's dictum that is simultaneously not subject to obvious counterexample, reasonably in accord with intuition, and non-trivial, and which of these failings it ends up exemplifying depends a great deal on the interpretation given to 'distinct'. She explores *numerical distinctness*, *weak modal distinctness*, *spatiotemporal distinctness*, *mereological distinctnes* and *strong modal distinctness* (Wilson 2010, pp. 5-11). Considering many different permutations of these definitions with different interpretations of 'necessary' (and also 'intrinsic property'), she concludes overall that weaker versions of Hume's dictum are analytically true and unsuitable for doing any metaphysical work, while stronger versions are highly questionable. However we believe that what Hume means by 'distinct' cannot be fully understood apart from his theory of perception and associated epistemology, and to this we now turn.

3. Humean Theory of Perception: Passive

Yablo (1993) sketches the kind of 'big-picture' counterargument commonly given against the perceivability of necessity:

> … perception itself brings word of sensory mechanisms seemingly hard at work monitoring external conditions. By contrast, "we do not understand our own must-detecting faculty." Not only are we *aware* of no bodily mechanism attuned to modal aspects, it is unclear how such a mechanism could work even in principle. (Yablo 1993, pp. 3-4)[9]

The talk of 'mechanism', 'monitoring' and 'attunement' indicates a certain view of perception, as a passive and mechanical process something like a thermometer's response to temperature. That is not the only possible approach. Older Aristotelian and Rationalist traditions took a more active view of perception (Ebert 1983; Spruit 2008; Hatfield 2007); so does much modern perceptual psychology, as we will see later. The view Yablo describes descends from Hume.[10]

Hume's account of perception begins with causal contact between our senses and objects both 'internal' and 'external' which somehow generates *impressions* and *ideas.* The latter are merely pale copies of the former. Impressions may spring from 'outside the mind' (impressions of *sensation*) or 'inside the mind' (impressions of *reflexion*), but the latter consist solely in combining previous impressions of sensation, which are the building blocks of all thought. To emphasise the mechanical and passive nature of reflexion, Hume likens it too to perception—remembering is essentially a perception of ideas that are 'weaker' and 'less vivid'. In fact Hume notes (following Locke, Essay, IV.i.2-3) that there is an important sense in which *all* mental activity is perception: 'To hate, to love, to think, to feel, to see; all this is nothing but to perceive.' (*Treatise* 1, II, vi)[11]

Hume's commitment to epistemic passivity also emerges in his denial of *abstract ideas*. He defines these as ideas which are *general* in that at least some of their determinable properties lack determination (a 'general triangle' is neither isosceles or scalene, a 'general man' has no particular height or age). Allowing abstract ideas would render the mind active since it would need to choose which determinables to abstract from. Hume sees their denial as a properly naturalistic position to take against scholastic obscurantism, since a major plank of pre-modern epistemology was the mind's grasp of Aristotelian real essences, such as 'man in general'.

Thus Hume extravagantly praises Berkeley's claim that there are no general ideas, only particular ideas used in a general way. That an idea might be not entirely determinate in all its determinables is something Hume simply denies. When we appear to be reasoning in a general manner—for instance proving properties of the triangle that apply equally to equilateral, isosceles and scalene, though each triangle idea can only be one or the other—we are merely drawing on a number of different, entirely determinate ideas.[12]

The widespread philosophical influence of Hume's view of perception is nicely illustrated by a contemporary argument in the philosophy of mathematics concerning whether sets can be perceived. Maddy proposed that, when I open a refrigerator and look at three eggs, I not only perceive the curved white surfaces, I perceive that the surfaces form three eggs, and in so doing I perceive a set of three eggs. Balaguer counter-argues as follows:

> ... we cannot perceive these sets. I begin by asking whether we can perceive the structural difference between an aggregate and a set. That is, when we look into the egg carton, can we see the aggregate and the set? ... Since the set and the aggregate are made of the same matter, both lead to the same retinal stimulation ... But ... then the perceptual data about the set is identical to the perceptual data about the aggregate. Thus, we cannot perceive the difference between the aggregate and the set. But since it is pretty obvious that we can perceive the aggregate, and since there is a difference between the aggregate and the set, it follows that we cannot perceive the set. (Balaguer 1994, p. 104; Maddy's original argument in Maddy 1990, pp. 60-61)

The move from 'we receive only one retinal stimulation' (from both the aggregate and the set) to 'we cannot perceive the difference between the aggregate and the set' assumes that perception is a function of single sensory inputs transformed without residue into single ideas. That expresses a passive view of perception—for instance, it rules out the possibility that a relatively low-level visual process might register the egg aggregate and a higher level, more active, process recognise the set. (This is arguably an empirical description of the mind's forming an abstract idea.)

4. Humean Theory of Perception: Atomist

Hume's theory of perception is also importantly atomistic in that any distinguishable ideas effectively constitute separate objects. In other words, distinguishability must mean more than, '… that distinction of reason, which is so much talked of, and is so little understood, in the schools.' (*Treatise*, 1, I, vii). For instance, when we distinguish shape from colour in an object such as a white cube, as noted, it is not that we examine the white cube and use reason to distinguish its colour and shape *as abstract ideas*. This process was sometimes referred to in the medieval period as 'prescinding', 'prescission' or 'abstraction' of qualities from a given object. (Weinberg 1965, part 1). Rather, what we do is imagine *black cubes* and *white globes.* Without such a literal, quasi-perceptual forcing apart of ideas, Hume claims, we cannot distinguish them – though we might think we can, a cause of much confusion and wasted time in philosophy:

> …if the figure be different from the body, their ideas must be separable as well as distinguishable; if they be not different, their ideas can neither be separable nor distinguishable. What then is meant by a distinction of reason, since it implies neither a difference nor separation? (*Treatise*, 1, I, vii)

We now have a clear criterion of distinctness to use in evaluating Hume's maxim. Let's call it Hume's *Separate Imaginability Criterion of Distinctness*. It essentially consists in the denial that, when distinguishing ideas and objects, one might *prescind without separating*. This is crucial.

Let us return to Hume's remark '... no objects are contrary to each other, but existence and non-existence'. This stark claim bears much examination. We will refer to it as *Modal Combinatorialism*. We name the claim thus as it holds that 'objects' are all compossible, in other words, they have no properties which might generate impossibilities in combining the objects with one another in any way. Here it is important to distinguish two definitions of combinatorialism and indicate which is meant. The first, 'top-down' definition claims that any whole can be decomposed into some given set of atomic parts.[13] The second, 'bottom-up' definition holds that given some set of atomic parts, any combination of them is possible. We suggest that Hume is committed to the second kind of combinatorialism.

It should be noted that Robert Fogelin has raised a potential objection to this claim. In (Fogelin, 1984), he writes that we should not understand Hume as holding a theory of perception that is atomistic in the purely combinatorial sense sketched above given what Hume also says about the missing shade of blue, since:

> …an atomist in perceptual theory would deny the existence of any structure below the lowest level of the perceptual ontology and thus would hold that each simple impression is a pure content standing in no systematic relationship to any other simple impressions except for being qualitatively identical with it or simply qualitatively different from it...Hume's discussion of the missing shade of blue shows that he does not accept such a theory of perception. (*get page no*)

This is an acute observation. If we are able to imagine the missing shade of blue given its darker and lighter neighbours on the colour spectrum then it seems that we are able to decompose the seen blues into more fundamental concepts (such as a hue and a darkness) with their own logic – albeit primitive – such that for instance it is not possible for blue A to be darker than blue B, blue B to be darker than blue C, and blue A not to be darker than blue C[14]. **[PLEASE CHECK FOOTNOTE]** This violates the pure mutual compossibility outlined above. But it also violates Hume's stated dictum that "…nothing is contrary but existence and non-existence". We suggest that what Hume says about the missing shade of blue is notoriously ambiguous and inconclusive **[JIM CAN YOU ADD ANYTHING HERE?]** and that Hume's views at this point suffer some overall inconsistency. (It's worth noting that Fogelin also finishes the remark above by noting "…it is not easy to find a clear statement of [Hume's] positive views on these matters.")

If we accept that Hume is a combinatorialist in our second sense, then, an arguable upshot of the view is Hume's Fork (in the terminology of Flew 1961, p. 53): his strict division of knowledge into *relations between ideas* which are determined a priori, and *matters of fact* which can only be learned through experience. To show this, imagine a simple physical example: if any set of 'objects' of simple mass and motion may – *as far as the objects themselves are concerned* – be combined with any other, in any way, then a law of nature such as **F = ma** is no longer strictly *necessary*, only an extremely widespread regularity.[15] This entire lack of necessity on the factual side means that Hume must equate knowledge of necessary connexion solely with 'relations between ideas'. He does not exactly conclude that all necessity is analytic in the sense of a trivial consequence of words such as 'all bachelors are unmarried.' Nevertheless he claims that all relations of ideas are 'discoverable by the mere operation of thought' (*Enquiry* 4.1) and the tradition of philosophy since Kant has overwhelmingly tended to assimilate his 'relations of ideas' to analytic truths uncoverable merely by inspecting the meanings of words (Dicker 1991; Backhaus 1994). Thus was Quine able to so famously opine,

"...necessity resides in the way we talk about things, not in the things we talk about" (Quine 1976, p. 176).

The Fork has been most problematic for mathematics, where the general agreement that mathematics does deal with relations of ideas, discoverable *a priori*, combines with the Fork to produce the conclusion that mathematics is entirely disconnected from real-world observation. As Einstein put the received opinion, 'As far as the propositions of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.' (Einstein 1954, p. 233)[16] Correspondingly, our mathematical examples can serve as a particularly clear stumbling block to the Fork's overwhelming influence on modern epistemology, if rendered immune to Humean challenges. In the next section we step back and look at how Hume's theory of perception outlined in the last two sections is undermined by recent empirical research.

5. Cognitive Psychology's Counterexamples to Humean Theories of Perception

In using recent scientific results to bolster philosophical argument one risks falling hostage to empirical fortune. Nevertheless, particularly given that Hume himself drew so explicitly on the naturalism of his day to justify his views, it is worth pausing and checking what current research in cognitive psychology says about them.

In fact there is no lack of contemporary research which undermines both the passivity and atomism of Hume's view of perception. With respect to the passivity, it has become clear that the brain's perceptual functioning is far from Hume's simple copying of impressions into ideas. Rather the brain is continually comparing its input against complex sets of prior (possibly learned) expectations, using mismatches for purposes of self-correction. Gregory (1980) advanced the idea that *all* perceptions should in fact be understood as *hypotheses*[17], and this idea has lately been gaining ground (e.g. Hohwy 2010; Clark 2012).

With respect to the atomism, current consensus appears to be that most key aspects of visual perception are attained not only—as a Humean would expect—via 'bottom-up' accumulation of masses of low-level visual impressions, but also by directly registering high-level structural properties. This applies at least to colour, contour, size, and relative motion (Kaufman 1974). Moreover, those researchers who understand perception as hypothesis have shown that not only is the mind organised sufficiently holistically to support ‘top-down predictions’, it operates this way at a number of distinct functional levels which are richly inter-related, producing ‘… a cascade of cortical processing events in which higher-level systems attempt to predict the inputs to lower level ones on the basis of their own emerging models of the causal structure of the world.’ (Clark 2012).

A particularly nice anti-atomistic example is the perception of *symmetry*. This is a high-level structural property. In fact the necessary truth perceivable in *Fig 1* draws directly on symmetry-perception in recognising the sameness of the rows and columns, while the symmetry of the square and of the diamond plays a key role in recognising the necessary truth shown in *Fig 3*.

As an example of the many scientific results regarding symmetry perception, we take the differences between perception of bilateral symmetry about a vertical axis and other symmetries (such as symmetries about other axes, or repetitions of shapes in friezes). Vertical bilateral symmetry is perceived much more rapidly. It is said to be ‘preattentive’ in that it can be perceived when the stimulus array is presented for less than 160 milliseconds—less than the time the brain takes to attend to anything. It is possible to perceive vertical bilateral symmetry in simple random shapes presented for only 25 milliseconds (Wagemans 2002). These precise measurements confirm experimentally what is obvious perceptually in figures such as *Fig 5*: that vertical bilateral symmetry has an immediate and salient ‘look’ compared to symmetry about an oblique axis.

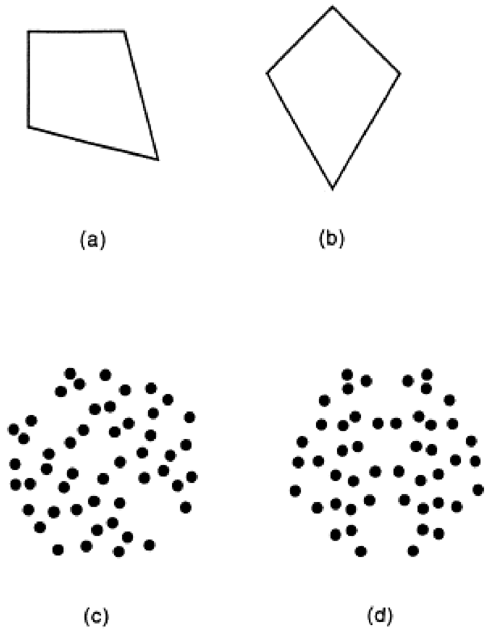


Fig 5. Vertical-axis symmetry (b and d) contrasted with other axes (a and c)[18]

This effect is what underlies the fact that a square is perceived as a different shape from a diamond, as in *Fig. 3* above. Though the two are congruent, their symmetries are differently related to the environmental horizontal and vertical axes.

These results show that symmetry perception combines what Hume alleges to be impossible. Symmetry is a global property of an array: it is a relation between parts, not a property of any one part. So perception of it cannot be atomic. Nor can it be passive, since activity is needed to make the comparison between parts. Yet it is immediate—as immediate as any perception can be, at least in the case of vertical bilateral symmetry—and unquestionably pre-reflective, as the perception occurs below the timescale on which reflection operates. The deeply automatic and pre-reflective nature of symmetry perception is confirmed by its appearance in animals, including very simple ones. Perception of symmetry has been demonstrated in apes, dolphins and birds; it is possible to train bees to prefer either symmetrical or asymmetrical patterns, but the preference for symmetry comes more naturally to them (Giurfa et al 1996). Bees are animals innocent of reflection, and it is hard to believe they deal in 'relations of ideas', in Hume's sense. They just see.

6. How our Examples Escape the Humean Theses

Let us return to our examples of section 1 – *prima facie* clear cases of perceiving necessity – and reexamine them in the light of what we have learned about Hume's maxim, and his theory of perception. We begin with *Fig. 1*:

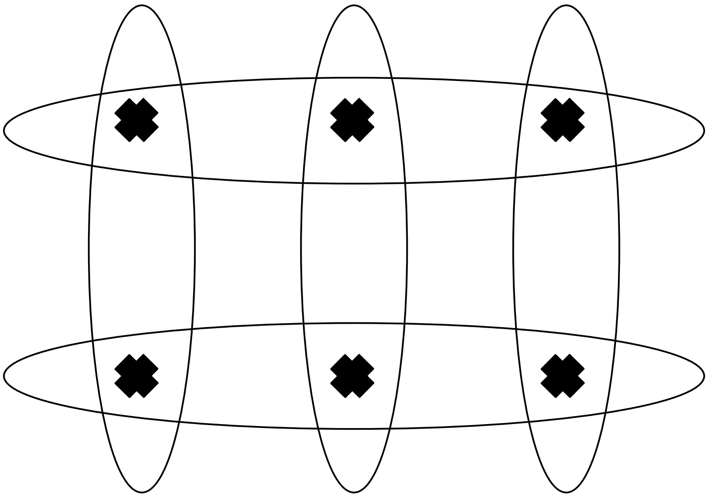


Fig 1. Why $2 \times 3 = 3 \times 2$

This diagram demonstrates a necessary truth. Does it demonstrate a necessary connection between distinct existences? Well, what are the 'existences' here? What are the perceived objects? We will suggest that the Humean has three choices, and argue that all of them are problematic. We will then sketch a fourth, non-Humean, view which we favour.

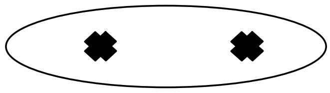
i) **Physical Mark View**: Here we might understand the relevant objects/existences as the 6 stars and 5 ovals (and further combinations of these, for example the 5 oval + star combinations). This appears to be a natural choice of objects in terms of the organisation of our visual field when regarding the page. Recall the Humean Separate Imaginability Criterion of Distinctness: we can imagine each shape existing on its own on the page. But then it is false that there are no necessary connections between these objects as positioned in *Fig. 1*. For instance, one cannot change the number of stars in the vertical ovals without changing the number of stars in the horizontal ovals. Interpreted thus, then, Hume's maxim is simply incorrect.

ii) **Abstract Object View**: On the other hand, we might take the necessary connection just noted as a sign that our diagram does not display a truth about physical marks but about something more purely mathematical or 'abstract'. Under this interpretation, the relevant objects/existences are (three) '2s' and (two) '3s'. These objects are arguably not distinct from one another. For instance we might understand 2 as made up of 'two ones' and 3 as made up of 'three ones', and thus see 2 as a proper part of 3. At this point, then, Hume might defend his maxim by stating that *Fig 1* solely expresses relations between ideas, and he never meant to claim that *ideas* were distinct existences (his 'Fork' is precisely meant to teach this), so it is not a counterexample to his maxim.

But there is something unsatisfying here. It seems puzzling to claim that we can gain mathematical knowledge, as we clearly can, by examining *Fig. 1*, and yet that mathematical objects are *entirely* separate from perceptual experience. Furthermore, now Hume's claim that there are no necessary connections between distinct existences seems to beg the question. He seems to be arbitrarily ruling out that we perceive the kinds of existences between which necessary connections demonstrably hold, by labelling them as 'mere ideas'. He seems happy to apply a Separate Imaginability Criterion of distinctness in the causal case (the billiard balls), where it rules that the motions of the two balls are distinct since one can be imagined apart from the other. But in *Fig. 1* one may equally imagine each oval + star object existing on its own and apply the criterion to infer that these are distinct existences. Yet as noted, these objects as assembled in *Fig. 1* do seem to have necessary connections between them. If Hume then argues that the stars and ovals cannot be distinct existences precisely because of these necessary connections, his maxim effectively becomes: 'there are no necessary connections between distinct existences, which are those existences between which there are no necessary connexions'. The maxim now appears devoid of philosophical content.

iii) **'Both' view:** Given these problems with both the Physical Mark view and the Abstract Object view, a Humean might attempt to compromise by combining the two, and stating: the best account of the objects represented by *Fig. 1* is that there are ovals and stars *and* 2s and 3s. However such a compendium raises tricky questions of the relationship between the physical marks and the abstract objects. Specifically, the view seems to treat the physical marks and the abstract objects as existences entirely distinct from each other, in which case, according to the Humean, they must be separable. In that case, why include the stars and ovals in the diagram at all? Why not lose the physical marks, keep the twos and threes and draw the mathematical moral straight from them? This is obviously impossible in the case of *Fig. 1*, thus the view itself risks incoherence.

This final remark points the way to a fourth interpretation which we believe is closest to the truth. This view is '*hybrid* (between the Physical Mark and the Abstract Object views)...*but not both*'. Rather than understanding physical marks and abstract objects as separate objects, it is more accurate to understand abstraction as a mental process whereby we perceive in the diagram as it exists on the page certain *partial identities*. What does this mean? Just that 'twoness' may be *prescinded* from this:



while precisely *not* being separable from it. This kind of 'distinction of reason' is of course exactly what Hume's epistemology rules out as impossible. Yet prescinding without separation is essential for all structural reasoning, and this is surely a significant part of mathematics. One might even argue that all necessary reasoning is structural (*author reference*).

Of course, perception of the structural *partial identities* described above is necessary but not sufficient for perceiving the necessity of 2 × 3 = 3 × 2. But there is only one further thing that needs to be recognised in this case. This is the *full-fledged identity* between the two prescindable but not separable structures realised by the diagram: namely 2 × 3 (two rows of three stars) and 3 × 2 (three columns of two stars). In other words, once the two-row/three-column structure is perceived in *Fig 1*, all that is needed to perceive 2 × 3 = 3 × 2 is the recognition that these are two decompositions of *the same whole*. There is no difficulty in perceiving that. The six stars and their two decompositions are plain for the eye to see.[19] Thus we see that the two sides of the equation are equal, and that variants such as 2 × 3 = 3 × 3 are impossible.

The analysis of *Fig. 2* presents no new issues. As explained in section 1, it relies, like *Fig. 1*, on the perception of a whole as divided into parts in two different ways. The geometrical *Fig. 3*, however, requires attention.

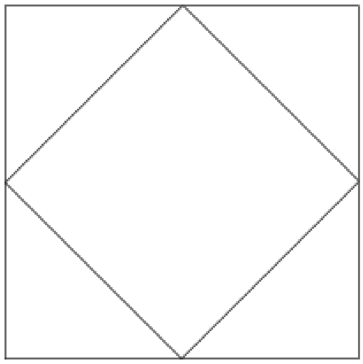


Fig 3. Why the inside square is half the outside square

Here the result that the diamond is half the area of the square comes from recognizing that the area of each quarter of the diamond equals the area of the triangle outside it. That is dramatized by the act of mentally folding the triangle ‘flap’ in to cover the quarter-diamond, but all that is required for this is perception of the symmetry of the figure: that the triangle is equal to the quarter-diamond by symmetry. What are this diagram’s relevant objects? Again, we might attempt to define a Physical Mark view, stating that the relevant objects are 1 square, 1 diamond and 4 triangles, as inscribed on the page. However here in the non-discrete field of geometry we can see that this interpretation is not right. We noted before that diagrams as actually instantiated frequently contain imperfections which perception ‘rectifies’. Thus the necessary truth perceived here (that the diamond is half the area of the square) might not even be true of *Fig. 3*, if its shapes were carefully measured.[20]

An abstract object view on the other hand would say that the real objects of the diagram are some kind of *general* square, diamond and triangle. And yet despite that, somehow, viewing the physical marks helps us to perceive the necessary truth. Why is this? Once again, our favoured account rests on the perceiving mind’s ability to prescind from the diagram (but also at the same time appropriately reunite mentally) *structural features* sufficiently general to rectify minor diagrammatic inaccuracies where necessary.

Fig. 4 concerns transfinite arithmetic. Hume is famously dismissive of infinity, but, as usual, his reasons involve reading epistemology into ontology: because our ideas of quantity cannot be divided indefinitely, he argues, neither can quantity itself.[21]Arguably Hume is correct that our ideas are not infinite. And yet this proof in transfinite arithmetic *works*, and its work is heavily dependent on perception. Our diagram shows in some general sense how one-one matching between natural and rational numbers can proceed across an infinite series, without showing the entire series. Although we do not perceive the series, we see enough recursive structure to be sure the proof is correct. We saw that Hume claimed that any supposed abstract idea is nothing but a particular, entirely determinate idea that annexes itself to a general term, and associates with other particular ideas which it resembles in relevant ways. But that story clearly does not fit this case, since it would require either a particular, determinate idea which is infinitely complex, or an infinite number of them.

7. Conclusion

We can perceive necessary truths. In such perception the mind is not passive and atomistic as Hume supposes, but active and integrated – simultaneously prescinding, and recognising identities between, structural features of what is perceived in ways that confer real insight. If we work out the consequences of this for epistemology, where would it take us? We suggest that at this point it could be extremely helpful to look back to rationalist and scholastic views largely left behind by Anglo-American philosophy, which has so admired Hume. Such views taught that certain knowledge may be attained from ideas that are sufficiently *clear and distinct*. Our examples arguably show clear and distinct perceptions giving rise to knowledge of mathematical necessities. The onus is on contemporary epistemology to assimilate these examples to its received ideas, or to explain them away.

The last few decades have seen an outpouring of work in the metaphysics of modality remarkably mismatched by relatively slight investigation into its epistemology. We suggest that the strictures imposed by Hume's relegation of necessary truths to mere 'relations between ideas', and his horror of empirically investigating the mind's capacity for abstract thought (as an 'anti-naturalistic', inevitably confused inquiry) have left the latter investigation on relatively barren ground. Recent attempts to address the mismatch again largely follow Hume in pursuing the idea that conceivability is our guide to possibility (Yablo 1993, Gendler and Hawthorne, 2002).[22] Yet a clear, principled account of what exactly makes certain states of affairs 'inconceivable', and therefore impossible, still seems lacking.

Hume's Modal Combinatorialism has had enormous downstream philosophical influence. Mill treated it as a law of logic.[23] It is also found in Wittgenstein's *Tractatus*[24], which strongly influenced Carnap's treatment of necessity as truth in all 'state-descriptions': complete permutations of truth-values across a given language's atomic propositions (Carnap 1956). From there, the development of possible worlds semantics in logic inspired an arguably 'un-Humean' leap to replacing state descriptions conceived of as linguistic entities with analogous metaphysical entities (of a variety of kinds) which were thought to 'truth-make' modal claims. Yet even in Lewis' most extreme form of modal realism (Lewis 1986), Modal Combinatorialism played a powerful role in his influential ('Humean') analysis of laws of nature as mere patterns of regularity across subsets of possible worlds. At that point it had enormous influence on analyses of physicalism and supervenience through the '70s and '80s in terms of patterns (of a variety of kinds) of covariance of properties across possible worlds. Armstrong (1989, especially pp. 116-8) also made use of Modal Combinatorialism to motivate an explicitly combinatorial account of modality, while at the same time Lewis used it against Armstrong's states of affairs (Lewis 1992). Amidst this brandishing of Hume's maxim on all sides in recent analytic philosophy, it has even played a significant role in recent meta-ethics (Smith 1994).

Overall we believe that these aspects of Hume's thought can be usefully summed up as a *syntactic approach to modality*. To argue this, let us attempt to frame the issue in maximally general terms. Consider a world consisting of 4 'idea / objects' (**a**, **b**, **c** and **d**) – conceived of as particulars which may combine to make larger states of affairs. Imagine that these idea / objects are all *distinguishable*. Then according to Hume they must be *separable*. Let us now imagine a toy universe in which, as Hume suggests, the only contraries are existence and non-existence. In such a universe ontologically there may exist – and epistemologically we may imagine – all possible 'combinations' of the objects. Here just one two-way combination is illustrated for purposes of simplicity, represented by the names of our objects appearing to the left or right of each other:

ab ac ad ba bc bd ca cb cd da db dc…

Let us now imagine a toy universe in which Modal Combinatorialism is false. This just means that not all combinations are realisable. Here is just one example:

ab ad ba bd ca cb cd da db…

This toy universe is missing **ac**, **bc** and **dc** (for some reason, let us imagine it is to do with the nature of **c**). An intelligent mind inspecting the world above might think to summarise the combinations missing from it in a simple statement – something like, '**c** can never come last in combination'. This statement is obviously a rudimentary law, or universal.

Our point is now merely that the second scenario is not incoherent. It is not analytically false to conceive constraints on the happy combination of any conceivable object with any other conceivable object (bearing in mind that of course these objects will have *natures*). Mathematics, as we have seen, rules out such combinations regularly: for example, the combination of 2×3 with $3 \times x$, for any x other than 2. Hume's Modal Combinatorialism is therefore an underhanded way of killing off a kind of realism about universals.[25] Hume rules out such constraints by fiat, not by argument – but necessary mathematical truths should trump plausible philosophical overgeneralisations.

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[1] A claim somewhat reminiscent of Aristotle in the *Posterior Analytics*, in particular bk II, ch. 2.
[2] An example is the claim that a function which is continuous is differentiable almost everywhere: visual intuition suggested this was true, but it is false. This example is nicely discussed in Giaquinto 2007, pp. 3-4, and Mumma 2010, pp. 3-4.
[3] Figure from http://www.ndl.go.jp/math/e/s1/c6.html
[4] That would be to agree with the naïve view that technical drawing is the same subject as geometry, but more accurate. David Sherry notes, "… more sophisticated empirical observations, such as measurement, play no role in diagrammatic reasoning–even if they can be useful for suggesting theorems" (Sherry, 2009a, p. 62). He points out that under the right circumstances a diagram may even be used to prove of a mathematical object properties which are clearly *false* of the diagram.
[5] Diagram from http://www.homeschoolmath.net/teaching/rational-numbers-countable.php

[6] Cantor to Dedekind, June 29, 1877. The result referred to is the sameness of cardinality of the unit interval and the unit square, of which Cantor gives a partly visual proof: Gouvêa 2011.
[7] Further on seeing truths in diagrams that continue to infinity in Feferman 1998 and Feferman 2012.
[8] Advanced players will notice that the three distinct implications just listed also have necessary connections between them. Many more complex mathematical necessities about billiard ball trajectories are listed at http://mathworld.wolfram.com/Billiards.html .
[9] The internal quote is Wright citing Blackburn 1986, p. 52; Yablo is actually considering the relation between conceivability and possibility, but the issue of naturalism and modality is the same as in our case.
[10] There is some ancestry in earlier empiricism: "… the reception of the *Ideas* of light, roundness, and heat, wherein I am not active but barely passive, and cannot in that position of my Eyes, or Body, avoid receiving them." (Locke, *Essay* 2.21.72).
[11] See also *Enquiry*, p. 152.
[12] 'Thus shou'd we mention the word, triangle, and form the idea of a particular equilateral one to correspond to it, and shou'd we afterwards assert, *that the three angles of a triangle are equal to each other*, the other individuals of a scalenum and isoceles, which we overlook'd at first, immediately crowd in upon us, and make us perceive the falshood of this proposition, tho' it be true with relation to that idea, which we had form'd.' (Hume, *Treatise*, 1, I, vii)
[13] This seems to be what is meant by 'combinatorialism' in philosophy of language and related disciplines.
[14] It was noted above that Hume largely follows Locke in his passive theory of perception and in seeking to construct 'an entirely perception-driven epistemology'. Locke also tackles the issue of 'colour logic', and where in such a framework, it might come from. He writes: "For when we know that white isn't black, what do we perceive other than that these two ideas don't agree?" (Essay, IV.i.2) But here the phrase 'don't agree' hides a crucial elision – from 'black and white are *different*' to 'black and white are *contrary*' – to which Locke is arguably not entitled. The authors are grateful to Max Cresswell for pressing us to work out this point.
[15] This view has not been found by metaphysicians to be entirely satisfactory.
[16] It is clear in the context that by 'certain' Einstein meant 'necessary'. For some arguments against this claim, see Franklin 1989. Once again Sherry writes with useful originality on this issue, noting that many inferences in applied mathematics 'proceed…from an observational premise to a factual conclusion in accordance with a mathematical rule. Thus, they are probable inferences that draw their force from the understanding as much as from experience. Had Hume paid attention to such cases, they could have had an enormous impact on the doctrine of the *Enquiry*.' (Sherry 2009b, p. 70). A good example is: 'A weighs the same as B, B weighs the same as C, therefore A weighs the same as C'.
[17] See also Gregory 1970 and Gregory, 1998. Gregory attributes the original idea to Helmholtz. Influential intermediary work was done by MacKay 1956 and Neisser 1967.
[18] Figure from Wenderoth 1996; the effect is demonstrated in 4-month-olds: Bornstein and Krinsky 1985.
[19] When this material has been presented publically attempts have been made to argue that insofar as the *partial identities* must be prescinded and therefore perceived separately (consist in'seeing as', in Wittgensteinian terms), this means that the *full-fledged* identity cannot be accessed via pure perception but

must rely on memory or inference to at least some degree. We reject this criticism on phenomenological grounds.

[20] Again, Sherry 2009a is useful here, noting that in fact geometrical diagrams function via *stipulation* as much as by *observation*. In *Fig. 3* the equality of the four triangles, the perfect straightness of the lines, and other matters are taken to be implicitly stipulated.

[21] Hume, *Treatise*, Book I, Part II, section 5, p. 52; Hume has been roundly criticised for this, e.g. Flew 1976; Fogelin 1985; Franklin 1994; Jacquette 2002. For a more sympathetic reading, see De Pierris 2012.

[22] Vaidya 2007 distinguishes "conceivability-based accounts" of the epistemology of modality from "understanding-based accounts" according to which, "our natural capacity to use concepts, understand statements involving them, and understand relations between them as a guide to what is possible and necessary", and accounts according to which, "the general procedure for arriving at justified beliefs about metaphysical modality is through working out counterfactual conditionals in imagination". But the difference of the latter two accounts from the first is not entirely clear to the authors.

[23] 'No thing or attribute is such that it can be said to be both wholly itself but also necessarily connected to something other than itself: each thing or attribute is logically and ontologically independent of every other thing or attribute' (Mill, quoted in Wilson 2007).

[24] §1.12: 'Each item can be the case or not the case while everything else remains the same' (Wittgenstein 1961).

[25] This broad-brush characterisation of realism about universals might usefully be compared with a distinction that descends from Wilfrid Sellars through Robert Brandom of 'material' as opposed to 'formal' inference.