

Deepening students' understanding of multiplication and division by exploring divisibility by nine

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This paper explores how a focus on understanding divisibility rules can be used to help deepen students' understanding of multiplication and division with whole numbers. It is based on research with seven Year 7–8 teachers who were observed teaching a group of students a rule for divisibility by nine. As part of the lesson, students were shown a way of proving why the divisibility rule for nine works, using materials grouped in tens and hundreds. After the lesson, students' understanding of multiplication and division was considerably deepened.

Introduction

Understanding multiplication and division is an important part of the mathematics curriculum. The importance of multiplicative thinking for understanding later mathematics, e.g., algebra, is also well established (Baek, 2008; Brown & Quinn, 2006).

The term *multiplicative thinking* refers to a particular type of thinking that is used to solve a variety of problems—multiplication, division, fraction, ratio and other mathematical concepts—that involve multiplication and division. Multiplicative thinking has been defined in a variety of ways. New Zealand's numeracy initiative (Numeracy Development Projects [NDP]) has defined multiplicative thinking as:

The construction and manipulation of factors (the numbers being multiplied) in response to a variety of contexts; [and] deriving unknown results from known facts using the properties of multiplication and division [e.g., commutative, associative, distributive, inverse] (Ministry of Education, 2008, p. 3).

What these words might mean in practice can be seen in the following example that one of the authors recently experienced while working with a group of six-year-olds. The children had been given the problem 6×7 by their teacher, and asked to model the problem using linkable cubes. Most of the children constructed six towers, each made up of seven cubes linked together. It was noticed that many of the children initially counted

the blocks in groups of two (skip counting) to work out their answers of 42 (keeping track of the twos was a challenge that led one boy to an answer of 41). It was suggested to the children that they laid their six towers down on the table, and then broke each one into a tower of five (below) and a tower of two (above). They were then asked whether it might be helpful to think first about the six towers of five blocks. Quickly, one bright spark suggested that, “Six times five is 30, and six times two is 12, so the answer must be 42.” The others in the group immediately recognised this as a helpful way of working out the answer to the problem of 6×7 .

Structure and pattern are at the heart of learning multiplication and division, with multiplication involving groups of groups (Mulligan & Mitchelmore, 1997, 2009). Students’ appreciation of structure and pattern is very important for their understanding of mathematics. It is vital that teachers draw students’ attention to structure and pattern to help develop their mathematical understanding and learning, particularly that of low achievers who don’t always notice structural features spontaneously.

The development of multiplicative thinking

Initially, students may count by ones to solve a simple problem such as two biscuits on each of five plates. Children who can skip count by twos can count the five groups of two, as in: “two, four, six, eight, ten”. Skip counting links to repeated addition $2 + 2 + 2 + 2 + 2 = 10$. Eventually students come to realise that multiplication can effectively shorten the repeated addition process to $5 \times 2 = 10$ (5 groups of 2 equal 10). This understanding forms the foundation for students to be able to apply simple multiplicative part-whole strategies to combine (multiplication) or partition (division) whole numbers. Eventually students’ knowledge of basic facts and understanding of partitioning strategies enable them to choose flexibly from a broad range of different part-whole strategies to find answers to multiplication and division problems. This level of understanding is referred to in the New Zealand Number Framework as advanced multiplicative thinking (Ministry of Education, 2007). It is expected that most students will have this understanding by the end of Year 8 (12 and 13-year-olds). The Australian mathematics curriculum document mentions multiplication and division in Year 2 under the Number and Place Value section of the content strand Number and Algebra and thinking becomes progressively more sophisticated (Australian Curriculum, Assessment and Reporting Authority, 2011).

Advanced multiplicative thinkers can partition a dividend in various ways and use known multiplication/division facts to work out which parts comprise the quotient (the result of division) (Young-Loveridge, 2011). For example, in solving $72 \div 4$, they might halve the 72 and work out that each part of 36 consists of 9 groups of 4, then double the 9 to get the final quotient of 18. Alternatively, they might split the 72 into 40 and 32 to work out that 10 groups of 4 plus 8 groups of 4 give the final quotient of 18. Other possible ways to partition 72 include 48 and 24, 60 and 12, or 64 and 8. By rounding 72 up to 80, then taking 2 groups of 4 (i.e., 8) away from the 20 groups of 4 ($20 - 2$), the final quotient of 18 can be found using a rounding and compensation strategy. Another possible strategy is to use repeated halving, first halving the 72 to get 36, and then halving the 36 to get the quotient of 18.

Understanding division is critical for work with rational number. Work with whole-number division provides an important foundation for work with

rational number. However, we have noticed in our work in schools that many teachers spend far more time on multiplication than on division. Although division problems can be solved using a reversibility strategy, building up the groups using multiplication, a deep understanding of division concepts themselves is necessary to work flexibly with division strategies.

Mathematics has many little tricks that can be used to work things out, such as whether or not a large number is divisible by single-digit values. Divisibility by nine can be determined by adding up the digits in a multi-digit number to check whether the sum is nine, or a multiple of nine. The lesson that is the focus here helps the students learn to prove why the divisibility rule for nine works. Their understanding of multiplication and division is deepened in the process.

The divisibility lesson

We observed seven teachers working with students at the Year 7–8 level (11 to 13-year-olds). Each teacher chose a group of students to work with on enhancing multiplicative thinking. We first observed a lesson on multi-digit multiplication, and a week later, a lesson on the divisibility rule for nine. All teachers used the lesson *Nines and threes* described in *NDP Book 6: Teaching multiplication and division* outlined below (Ministry of Education, 2008, pp. 70–72).

Initially students were presented with multiples of nine from the “times nine” ($\times 9$) table (e.g., 18, 27, 36, 54, 81), and were asked what these numbers had in common. They were then asked to make the number 27 using plastic beans, some of which were grouped in tens inside translucent film canisters. The teacher asked them how many groups of nine are in the number 27, and how they worked out their answers.

Next students were asked to make 45 with the beans and canisters. Again, they were questioned about how many groups of nine, and their solution strategy. The teacher explored whether the rule or method used to work out the nines in 27 was the same rule as used for 45. Attention was drawn to each group of ten beans, and the group of nine, plus one leftover “one” in each ten. Students could then see that when the two leftover “ones” from each “ten” in 27 were combined with the seven single beans, the total formed a further group of nine. Likewise, with 45, the four leftover “ones” from 40 together with the five single “ones” beans make another group of nine.

Students were then asked to make the number 32 (a number not divisible by nine). This time there were some beans left over (3 leftover “ones” from the “tens” plus 2 “ones” totalling 5) and the teacher discussed with students why this happened and how it was connected to the fact that the sum of the digits ($3 + 2$) did not equal 9.

The next number to be made with the beans was 135. Figure 1 shows the way the canisters of ten beans could be used to make the number 135, with ten canisters of ten beans on one ten-frame to show 100, three canisters of ten beans on the second ten-frame to show 30, and the 5 loose beans on the third ten-frame.



Figure 1. Representation of the number 135 using ten-frames to show 10 tens (100), 3 tens (30), and 5 single beans.

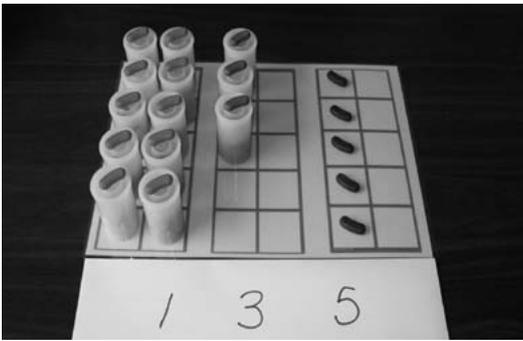


Figure 2. For each of the groups of ten beans in a canister, one bean is removed and placed on top of the container, leaving a group of nine beans in each canister.



Figure 3. An eleventh group of nine can be made out of the 100 beans, leaving one bean on top of a canister to show the leftover "one" after the 11 groups of 9 (99) have been made out of 100.



Figure 4. The four leftover "ones" have been moved from on top of the canisters and combined with the five single beans in the right-hand ten-frame to show one further group of nine that can be made.



Figure 5. The eleventh group of nine from 100 has been placed in a canister and all the canisters of nine have been labelled with the digit 9

The next step involved taking the leftover "one" out of each canister of ten beans to leave nine beans in each canister. The tenth bean was placed on top of the canister, as shown in Figure 2.

There were then now 10 single beans on top of the ten canisters of ten beans representing 100. Nine of those beans were put together in a group beside the canisters to make an eleventh group of nine beans (99 beans altogether in 11 groups of nine), and with one leftover bean on top of the group of ten canisters (see Figure 3). Students' attention was drawn to the way that the leftover beans on top of the canisters corresponded directly to the digits in the number itself; that is, 1 for 100, 3 for 30, and 5 for the five single beans.

The leftover beans on top of the canisters were then placed with the five "ones" on the right-hand ten-frame to form another group of nine, which corresponded to the sum of the digits in the number 135 ($1 + 3 + 5 = 9$) (see Figure 4).

In Figure 5, the eleventh group of nine is in a canister and each group labelled with the digit 9.

The lesson concluded by asking students to state the divisibility rule for nine: If the digits add to nine (e.g., 324) or a multiple of nine (e.g. 1467), then the number is divisible by nine.

Why the divisibility rule works

For every 100 beans, there are 11 groups of 9 beans (99) and one bean left over. For every 10 beans there is one group of 9, and one bean left over. If the leftover "ones" are added to the single beans ($1 + 3 + 5$), it is evident that there are nine beans leftover, which makes one more group of nine. Hence the digit in any position in a multi-digit number from the "tens" upwards tells how many leftover "ones" there are after the groups of nine are made. For example, in 30, there are 3 nines and 3 leftover "ones". In 100, there are 11 nines and one leftover "one". In the number 200, there are 22 nines and two leftover "ones". In the number 1000, there are 111 nines and one leftover "one", and so on.

There is scope for extending the lesson to explore divisibility by three, using numbers that are divisible by three but not nine (e.g., 132) but none of the teachers got to that point during our time with them. The links between divisibility by nine and divisibility by three can be explored as part of this process.

Observations of the lessons

Most of the teachers used grouped materials (i.e., canisters of ten beans) to show the groups of nine and leftover “ones” in numbers up to 81, and some used ten-frames to show the structure of 10 tens in 100 for three-digit numbers.

Some teachers drew diagrams in the group’s recording book to show the leftover “ones” coming out of the groups of ten to create the groups of nine. By recording the digits representing the number below the diagram, students could see how the leftover “ones” correspond to the digit in the original number (see Figure 6).

Figure 6 presents a copy of the diagram that Ann recorded in her group’s workbook showing the process of combining the 3 leftover “ones” from the tens with the 6 original “ones” to make a total of 9. At the top of the diagram, she had written: “How many groups of 9 in 10?” Each drawing of a canister has the digit 9 inside it, and the digit 1 is written above each leftover “one” coming out of the canisters of ten to leave 9.

Figure 7 shows a diagram for the representation of 135.

Some teachers told their students that it didn’t matter how many groups of nine there were (because it was not mentioned in the lesson description in Book 6). However, Ann, our most experienced teacher, made a point of drawing her students’ attention to the number of groups of nine that were made from the canisters as well as the leftover “ones”. We think that understanding about the groups of nine is important for the students and helps to deepen their understanding of multiplication as well as division. Interviews with the students in Ann’s group revealed how delighted they were at the end of the lesson, having understood how to prove the divisibility rule for nine.

Not all of the teachers in our study taught the lesson successfully. Two teachers (out of seven) had not spent time prior to the lesson thoroughly familiarising themselves with the details of the lesson, possibly because they mistakenly assumed that simply following the description in Book 6 while teaching would be sufficient preparation for the lesson. Another teacher had clearly spent time preparing for teaching the lesson with numbers less than 100, but did not realise that the shift from two- to three-digit numbers involves an additional challenge. Because the number 100 involves making

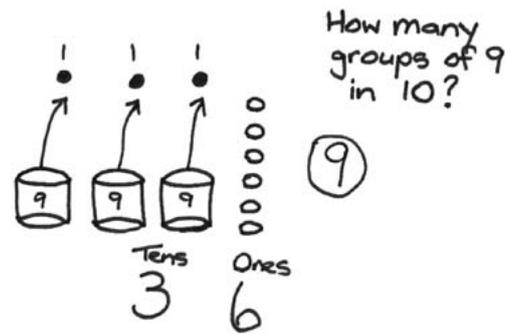


Figure 6. A copy of Ann’s diagram in the group workbook showing a leftover “one” coming out of each group of 10, leaving a group of 9 in each canister, and the 3 leftover “ones” plus 6 “ones” totalling 9

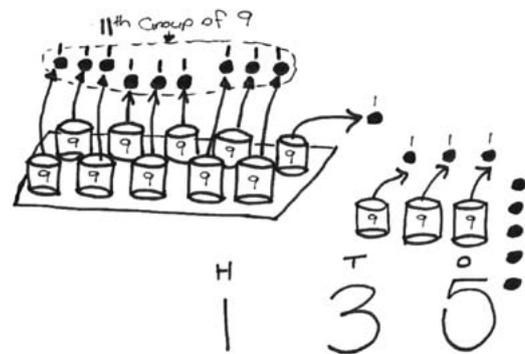


Figure 7. A diagram showing the representation of 135, with 10 groups of 9 in canisters, an 11th group of 9 made up of 9 leftover “ones”, one leftover “one” for 100, 3 leftover “ones” for 30, and 5 “ones” for 5

an additional group of nine from the ten leftover ones on top of the ten canisters, we think that it is vital that teachers make sure that they understand the lesson fully, including how to use the grouped materials for three-digit numbers, before they try it with their students (NB: older students might be intrigued to discover that 1000 has 111 groups of nine, 10,000 has 1111 nines, etc.).

Conclusions

This lesson contains far more deep learning about multiplication and division than it may first appear. Unpacking an easy trick with numbers such as the divisibility rule for nine offers the opportunity to further develop students' understanding of multiplication and division. Divisibility rules for other numbers may also be explored to further develop this understanding.

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