The Number Of (0,1)-Matrices With Fixed Row and Column Sums.

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Abstract

Let R and S be non-negative and non-increasing vectors of order m and n respectively. We consider the set $\mathcal{A}(R, S)$ of all $m \times n$ matrices with entries restricted to $\{0, 1\}$. We give an alternative proof of the Gale-Ryser theorem, which determines when $\mathcal{A}(R, S)$ is non-empty. We show conditions for R and S so that $|\mathcal{A}(R, S)| \in \{1, n!\}$. We also examine the case where $|\mathcal{A}(R, S)| = 2$ and describe the structure of those matrices. We show that for each positive integer k, there is a possible choice of R and S so that $|\mathcal{A}(R, S)| = k$. Furthermore, we explore $g_{m,n}(\mathbf{x}; \mathbf{y})$, the generating function for the cardinality $|\mathcal{A}(R, S)|$ of all possible combinations of R and S.

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Contents

Chapter 1

Introduction

Simply put, in mathematics, a $(0,1)$-*matrix* is a matrix whose entries are restrictred to be either 0 or 1. Such a matrix, can be a representation of different things so, is also known as the *logical matrix, binary matrix, relation matrix* and also as the *Boolean matrix* depending on the context of its use.

In this thesis, where possible, we keep the notation consistent with [4] for convenience sake.

Now, let A be a $(0,1)$-matrix of m rows and n columns. Let the sum of row i of A be denoted by r_i $(i = 1, \ . \ . \ ., \ m)$ and let the sum of column j of A be denoted by s_j $(j = 1, \ . \ . \ ., \ n)$. It is clear that if τ denotes the total number of 1's in A, then

$$\tau = \sum_{i=1}^{m} r_i = \sum_{j=1}^{n} s_j. \tag{1.1}$$

We associate with the (0,1)-matrix A the *row sum vector* $R = (r_1, \ . \ . \ ., r_m)$, where r_i gives the sum of row i of A. Similarly, the *column sum vector* S is denoted by

$$S = (s_1, \ . \ . \ ., \ s_n).$$

We denote by

$$\mathcal{A}(R, S)$$

the *class* [1] *of all (0,1)-matrices with row sum vector* R *and column sum vector* S.

Example 1.1

Let

$$M_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \textit{and} \quad M_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Then M_1 *and* M_2 *are* 3×4 $(0,1)$*-matrices and* $M_1, M_2 \in \mathcal{A}(R,S)$ *with*

$$R = (4,\ 2,\ 1) \ \textit{and} \ S = (2,\ 2,\ 2,\ 1).$$

Also,

$$\tau = \sum_{i=1}^{3} r_i = 4+2+1 = 7 = 2+2+2+1 = \sum_{j=1}^{4} s_j.$$

The class $\mathcal{A}(R,S)$ is a classic object in different branches of mathematics. In combinatorics, $(0,1)$-matrices with prescribed row and column sums encode hypergraphs with prescribed degrees of vertices and related structures, as seen in [27]. In algebra, for example see Chapter 1 of [28], certain structural constants in the ring of symmetric functions and in the representation theory of the symmetric and general linear groups are expressed as numbers of $(0,1)$-matrices with prescribed row and column sums. In statistics, $(0,1)$-matrices with prescribed row and column sums are known as *binary contingency tables*, see [11]. Modern foundations of $(0,1)$-matrix reconstruction can be traced to Herb Ryser, Delbert Ray Fulkerson, and Richard Brualdi: [33], [14], [15], [5], [16], [17].

In Chapter 2 we define the concept of *majorization* (strictly on vectors, for majorization can be extended to larger structures such as matrices). This concept was introduced by Muirhead [30] and later developed by Hardy, Littlewood and Polya in their study of symmetric means [24]. This concept of vector majorization is required in determining

[1] Ryser began the tradition of referring to $\mathcal{A}(R,S)$ as a class, rather than a set, of matrices.

simple arithmetic condition(s) for the construction of a $(0,1)$-matrix with prescribed row and column sum vectors. In studying the property of vector majorization, we define and study the structure of a *maximal matrix* by defining the *conjugate* of a given vector. We also discuss how majorization requires vectors involved to be in non-increasing order, hence, making it a process of comparing partial sums.

As we want to find more about the cardinality of the set $\mathcal{A}(R,S)$ we ask the question: What condition guarantees $\mathcal{A}(R,S)$ to be non-empty? We give an answer to this question particularly in Chapter 3 by proving the well-known result originally shown independently by both Gale [20] and Ryser [32], namely the *Gale-Ryser Theorem*. Such a result does tell us something about the number of matrices in $\mathcal{A}(R,S)$, but it does not enumerate them exactly. It just ensures that there exists a matrix in $\mathcal{A}(R,S)$ whenever S is majorized by R^* the conjugate of R.

In Chapter 4 we consider the case where $|\mathcal{A}(R,S)| > 1$. Given that $A, B \in \mathcal{A}(R,S)$, we discuss the process of how we can transform A into B (or vice-versa) by the concept of *interchanges*. We then answer the question: When is there only one element of $\mathcal{A}(R,S)$? We also show that when $|\mathcal{A}(R,S)| = 2$, each of the matrices in $\mathcal{A}(R,S)$ contains exactly one interchange.

We define and study the *structure matrix* $T(R,S)$ for a given $(0,1)$-matrix $A \in \mathcal{A}(R,S)$ in Chapter 5. We prove the assertion by Ford and Fulkerson [13] that it is necessary and sufficient for the structure matrix $T(R,S)$ to be a non-negative matrix for $\mathcal{A}(R,S)$ to be non-empty. We also discuss a combinatorial interpretation of the structure matrix. This becomes useful in studying the structure of matrices in $\mathcal{A}(R,S)$ by discussing the properties of *invariant sets* in Chapter 6.

In Chapter 7 we answer the following: Given a positive integer k, can we construct a $(0,1)$-matrix so that it has exactly k number of completions? Or in other words, with a given k can we find R and S so that $|\mathcal{A}(R,S)| = k$? We also determine the number of $(0,1)$-matrices of order n, for $n \geq 2$, with exactly one 1-entry in each row and column. We state and give a straightforward proof of a generating function for $|\mathcal{A}(R,S)|$ in Chapter 8. We also briefly explain the limitation of such function with regards to our purpose.

Finally in conclusion, we discuss possible areas of what to do next in future study as a continuation of this thesis. We state a few open problems and conjectures regarding the aspects of $\mathcal{A}(R,S)$ covered in this thesis. Finally, we give a summary of new material.

Chapter 2

Maximal Matrix and Majorization

Consider any given non-increasing vector of non-negative integers

$$R = (r_1, r_2, \ . \ . \ . \ , \ r_m).$$

Let

$$\delta_i = [1, \ . \ . \ . \ , \ 1, \ 0, \ . \ . \ . \ , \ 0]$$

be a vector of r_1 components with 1's in the first r_i positions, and 0's elsewhere. We can visualize R using

$$\bar{A} = \begin{bmatrix} \delta_1 \\ . \\ . \\ . \\ \delta_m \end{bmatrix}.$$

Here, $\bar{A} = [\bar{a}_{ij}]$ is a (0, 1)-matrix of size m by r_1 with row sum vector R and with the property:

$$\text{if } \bar{a}_{ij} = 0 \text{ then } \bar{a}_{ik} = 0 \text{ for all } k \geq j.$$

We call such (0,1)-matrix $\bar{A}$, the *Ferrers matrix* of R [22].

We define *the conjugate of* R to be the column sum vector of the Ferrers matrix of R, denoted by R^*. Note that $\bar{A}^T$, the transpose of $\bar{A}$, is the Ferrers matrix of R^*, hence,

R^* is the row sum vector of $\bar{A}^T$ with R as the column sum vector, so $(R^*)^* = R$. Also, observe that R and R^* are conjugate partitions of the same positive integer.

Example 2.1

For $R = (4, 2, 1)$,

$$\bar{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \textit{and} \quad \bar{A}^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

So, $R^* = (3, 2, 1, 1)$, *with* $(4, 2, 1)$ *and* $(3, 2, 1, 1)$ *as conjugate partitions of 7.*

Equivalently, for

$$R = (r_1, r_2, ..., r_m),$$

the conjugate of R is given by

$$R^* = (r_1^*, r_2^*, ..., r_{r_1}^*)$$

where

$$r_i^* = \{|r_j| \ : \ r_j \geq i, \ 1 \leq j \leq m\} \quad (1 \leq i \leq r_1).$$

Hence, r_i^* is the count of how many numbers in R that are greater than or equal to i.

A $(0,1)$-matrix whose row sum and column sum vectors are conjugates is called *maximal*. Thus, a Ferrers matrix is a maximal matrix.

Cavenagh gives an equivalent pictorial version of a maximal matrix in [10] as a $(0,1)$-matrix whose rows and columns are arranged so that a line of non-decreasing gradient can be drawn with only 0's below the line and only 1's above the line or vice-versa.

It is clear to see that the row sum vector R determines the Ferrers matrix $\bar{A}$ and also

R^*, which makes $\bar{A}$ maximal. So, the maximal matrix $\bar{A}$ may be obtained from any matrix $A \in \mathcal{A}(R, S)$ by a rearrangement of the 1's in the rows of A. Also, by inverse row rearrangements one may construct the given A from $\bar{A}$. This shows that we can get from R^* to S and vice-versa *(similarly from R to S^* and vice-versa)* by exchanging 1's and 0's within the same row in certain rows.

Example 2.2

Let $R = (4, 2, 1)$ and $S = (2, 2, 2, 1)$. Then we have

$$\bar{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad \textit{and} \qquad A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

where $\bar{A}$ is maximal since $\bar{A} \in \mathcal{A}(R, R^*)$ and $A \in \mathcal{A}(R, S)$. It is clear to see that we can get from $\bar{A}$ to A and vice-versa by rearranging entries in row 2. In row 2, we exchange the 1 from the first column for the 0 from the third column. So we get from $R^* = (3, 2, 1, 1)$ to $S = (2, 2, 2, 1)$ by deducting 1 from r_1^* and adding 1 to r_3^*.

From here onwards, we will use $\bar{A} = [\bar{a}_{ij}]$ to denote the maximal matrix with row sum vector R and column sum vector R^*, the conjugate of R.

Let $X = (x_1,\ x_2,\ \ldots\ ,\ x_n)$ and $Y = (y_1,\ y_2,\ \ldots\ ,\ y_n)$ be two partitions of a positive integer τ. That is, X and Y are two non-increasing n-vectors of non-negative integers with:

$$x_1 \geq x_2 \geq \ldots \geq x_n \text{ and } y_1 \geq y_2 \geq \ldots \geq y_n$$

where

$$\tau = \sum_{i=1}^{n} x_i = \sum_{j=1}^{n} y_j.$$

If the following inequality holds

$$\sum_{i=1}^{k} x_i \leq \sum_{j=1}^{k} y_j \qquad (1 \leq k \leq n)$$

with equality for $k = n$; equivalently

$$\sum_{i=k+1}^{n} x_i \geq \sum_{j=k+1}^{n} y_j \qquad (0 \leq k \leq n-1)$$

with equality for $k = 0$, then we say that X *is majorized by* Y, denoted by $X \preceq Y$.

Theorem 2.3 *Let $X = (x_1, x_2, ..., x_n)$ and $Y = (y_1, y_2, ..., y_n)$ be non-increasing vectors of positive integers. Then $X \preceq Y$ implies $Y^* \preceq X^*$.*

The proof provided below is an adaption of Brualdi's proof in [4].

Proof.

$$\begin{aligned} X \preceq Y \;\Leftrightarrow\; & \sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i \quad (1 \leq k \leq n) \\ \Leftrightarrow\; & \sum_{i=k+1}^{n} y_i \leq \sum_{i=k+1}^{n} x_i \quad (0 \leq k \leq n-1) \qquad (*) \end{aligned}$$

Recall: $y_i^* = \{|y_j| \;:\; y_j \geq i,\; 1 \leq j \leq n\}$ and $Y^* = (y_1^*, ..., y_{y_1}^*)$.

Let p be a positive number. It then follows that for $1 \leq p \leq y_1$,

$\sum_{i=1}^{p} y_i^* = \sum_{j=1}^{n} \min\{y_j,\ p\}$.

Now, let k be the largest index such that $p \leq x_k$. Then we have,

$$\begin{aligned} \sum_{i=1}^{p} y_i^* &= \sum_{j=1}^{n} \min\{y_j,\ p\} \\ &\leq kp + \sum_{j=k+1}^{n} y_j \\ &\leq kp + \sum_{j=k+1}^{n} x_j \qquad \text{from } (*) \\ &= \sum_{j=1}^{n} \min\{x_j,\ p\} \\ &= \sum_{i=1}^{p} x_i^*. \end{aligned}$$

Hence, $Y^* \preceq X^*$. $\square$

Example 2.4

Let $X = (3, 2, 2, 1)$ *and* $Y = (4, 2, 2, 0)$.

So, $X^* = (4, 3, 1, 0)$, $Y^* = (3, 3, 1, 1)$ *with* $n = 4$.

Since $y_1 = 4$, *let* p *be a positive number such that* $1 \leq p \leq 4$.

We exhibit the case $p = 3$.

Observe,

$$\sum_{i=1}^{p} y_i^* = 3 + 3 + 1 = 7 = 3 + 2 + 2 + 0 = \sum_{j=1}^{n} min\{y_j, p\}.$$

Now, letting k *be the largest index such that* $p \leq y_k$, *we have* $k = 1$ *and*

$$kp + \sum_{j=k+1}^{n} y_j = 3 + (2 + 2 + 0) = 7,$$

while

$$kp + \sum_{j=k+1}^{n} x_j = 3 + (2 + 2 + 1) = \sum_{j=1}^{n} \ min\{x_j, \ p\} = 8 = 4 + 3 + 1 = \sum_{i=1}^{p} x_i^*.$$

So we have

$$\begin{aligned} \sum_{i=1}^{p} y_i^* &= 7 \\ &\leq 8 \\ &= \sum_{i=1}^{p} x_i^*. \end{aligned}$$

We can still compare X and Y for majorization if they have different sizes. Without loss of generality, let us assume that X has dimension m while Y has dimension n with $m < n$, then we can append $(n - m)$ 0's to X so as to obtain n-vectors. Because of this, there is a certain arbitrariness in the size of the vectors we compare. So, we can append 0's to R, R^* and S if needed be to make them have the same size.

Example 2.5

Let $R = (4, \ 2, \ 1)$ *and* $S = (2, \ 2, \ 2, \ 1)$.

Appending one more 0 to R *makes* $R = (4, \ 2, \ 1, \ 0)$ *with* $R^* = (3, \ 2, \ 1, \ 1)$ *unchanged still,* *but at least R, R^*, S are all 4-vectors now. In likely manner we have $S^* = (4,\ 3,\ 0,\ 0)$. So in this particular instance, $S \preceq R^*$ and also $R \preceq S^*$.*

We can also do away with the assumption that X and Y are non-increasing vectors when comparing partial sums for majorization, provided we first reorder the components of the vectors to be in non-increasing order. Hence, if given two n-vectors $X = (x_1, x_2, ..., x_n)$ and $Y = (y_1, y_2, ..., y_n)$ and we obtain $X' = (x_{[1]}, x_{[2]}, ..., x_{[n]})$ and $Y' = (y_{[1]}, y_{[2]}, ..., y_{[n]})$ from X and Y, respectively, by rearranging components in non-increasing order, then we say that $X \preceq Y$ provided that $X' \preceq Y'$. Note that X' and Y' are both partitions of the same positive integer.

Example 2.6

Let $X = (2, 3, 4, 3, 1)$ and $Y = (1, 2, 4, 5, 1)$. After having the components reordered in non-increasing order, we obtain $X' = (4, 3, 3, 2, 1)$ from X and $Y' = (5, 4, 2, 1, 1)$ from Y. So, since X' is majorized by Y', we say that X is majorized by Y.

Now it is safe to say that majorization is not a partial order since $X \preceq Y$ and $Y \preceq X$ are only implications that X and Y are rearrangements of one another.

Majorization, however, has a nice characterization for integral vectors (such as S and R) in terms of some *transfers*. Let us consider $Y = (y_1, y_2, ..., y_n)$ and $X = (x_1, x_2, ..., x_n)$ to be integral vectors. Assume that $y_i > y_j$ for some i, j such that $1 \leq i < j \leq n$; we define

$$x_i = y_i - 1, \quad x_j = y_j + 1 \quad \text{ with } x_k = y_k \ \ \forall k \neq i, j.$$

We say that X is obtained from Y by a transfer from i to j. Clearly $X \preceq Y$. This idea of a transfer is implicit in [26].

Theorem 2.7 *Let $X = (x_1, x_2,\ \ldots, x_n)$ and $Y = (y_1, y_2,\ \ldots, y_n)$ be integral vectors. If $X \preceq Y$, then X can be obtained from Y by a finite sequence of transfers.*

Proof.

Let i be minimal such that $x_i < y_i$, and let j be minimal such that $x_j > y_j$. Then obviously $i < j$ since $X \preceq Y$.

Now, define

$$Y^{(1)} = (y_1^{(1)}, y_2^{(1)}, \ . \ . \ . \ , y_n^{(1)})$$

to be an integral vector obtained from Y by a *transfer* from i to j (i.e. we define $y_i^{(1)} = y_i - 1,\ y_j^{(1)} = y_j + 1$ with $y_k^{(1)} = y_k \ \ for\ all\ k \neq i, j)$.

Clearly we have $X \preceq Y^{(1)} \preceq Y$ as $y_s^{(1)} \geq x_s \ \ for\ all\ s < j$, and since

$$\sum_{l=1}^{t} x_l \leq \sum_{l=1}^{t} y_l = \sum_{l=1}^{t} y_l^{(1)} \ \ for\ all\ t \geq j.$$

Similarly, repeating the whole transfer process, by letting i be minimal such that $x_i < y_i^{(1)}$, and letting j be minimal such that $x_j > y_j^{(1)}$, again $i < j$ since $X \preceq Y^{(1)}$. Then a transfer from i to j yields a new increasing vector $Y^{(2)}$ such that $X \preceq Y^{(2)} \preceq Y^{(1)} \preceq Y$.

Hence, after repeating the process for some t finite times, we have $X = Y^{(t)} \preceq Y^{(t-1)} \preceq \ . \ . \ . \ \preceq Y^{(2)} \preceq Y^{(1)} \preceq Y$. $\square$

Example 2.8

Let $X = (4, 3, 3, 2, 1)$ *and* $Y = (5, 4, 2, 1, 1)$ *so that* X *is* majorized *by* Y. *We have,*

$$Y^{(1)} = (4, 4, 3, 1, 1) \ \ (with\ i = 1, j = 3)$$

$$Y^{(2)} = (4, 3, 3, 2, 1) \ \ (with\ i = 2, j = 4)$$

Hence, $X = Y^{(2)} \preceq Y^{(1)} \preceq Y$.

Lemma 2.9 *Without loss of generality, let* R, X, Y *be partitions of a positive integer* τ *with* $R = (r_1, r_2, \ . \ . \ . \ , r_n), X = (x_1, x_2, \ . \ . \ . \ , x_n)$ *and* $Y = (y_1, y_2, \ . \ . \ . \ , y_n)$. *If* $\mathcal{A}(R, Y)$ *is non-empty and* X *is obtained from* Y *by a transfer from* i *to* j, *then* $\mathcal{A}(R, X)$ *is also non-empty.*

Proof.

Since $\mathcal{A}(R,Y)$ is non-empty, there exists a matrix

$$A = [a_{ij}] \in \mathcal{A}(R,Y).$$

Letting Y and X be as claimed, we have $y_i = y_j + 1$ $(1 \leq i < j \leq n)$, so there is a row k in A $(1 \leq k \leq n)$ where $a_{ki} = 1$ and $a_{kj} = 0$ (since y_i is the sum of the i^{th} column and y_j is the sum of the j^{th} column in A). Let A' be obtained from A by letting $a'_{ki} = 0$ and $a'_{kj} = 1$ and leaving all other entries of A' being equal to those of A. Hence, $A' \in \mathcal{A}(R,X)$ and we are done. $\square$

Example 2.10

Let R, Y, X be partitions of 7 with $R = (3,\ 2,\ 2,\ 0)$, $\ Y = (3,\ 2,\ 1,\ 1)$ and X is obtained from Y by a transfer from $i = 1$ to $j = 3$ and let A be a matrix in $\mathcal{A}(R,Y)$, say

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So we have, $X = (2,2,2,1)$ and $y_1 = 3 > y_3 = 1$. In matrix A, we can see in the second row that $a_{21} = 1$ and $a_{23} = 0$. Now, let A' be obtained from A by letting $a'_{21} = 0$ and $a'_{23} = 1$ while entries of A' in all other positions being equal to those of A. Then we have

$$A' = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and A' is a matrix in the class $\mathcal{A}(R,X)$.

We are now in a position to look at the condition(s) under which the class $\mathcal{A}(R,S)$ is non-empty.

Chapter 3

Basic Existence Theorem

If we are given any two vectors $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ as partitions of some positive integer τ, then the fundamental equation (1.1) holds. However, it is not sufficient for the existence of a (0,1)-matrix A with R and S as its row and column sum vectors respectively. In other words, we can find different partitions of a particular positive integer that can never be row and column sum vectors of a (0,1)-matrix. For instance, it is impossible to construct a (0,1)-matrix with row sum vector $R = (3, 3, 2, 1)$ and column sum vector $S = (4, 4, 1)$. Hence, $\mathcal{A}((3, 3, 2, 1), (4, 4, 1))$ is an empty set.

In this section, we are going to lay our focus on a very well-known and important theorem called the *Gale-Ryser Theorem*. This is a basic theorem for the existence of a (0,1)-matrix with row-sum vector R and column-sum vector S. In other words, this theorem gives the condition that ensures that the class $\mathcal{A}(R, S)$ is non-empty. Ryser in [32] used induction and direct combinatorial reasoning whereas Gale in [20] used the theory of network flows as they both independently proved this theorem. This theorem is also derived in [9] using network flows and is also derived in [5]. Brualdi in [4] recites the inductive proof he derived in [6]. Krause also has provided, what he called, a simple proof of this theorem in [26] along the same reasoning provided by Ryser with use of ordinary Euclidean norm.

Even though the Gale-Ryser theorem does not go as far as enumerating the $(0, 1)$-matrices in $\mathcal{A}(R,S)$, it gives a nice characterization of the existence of a (0,1)-matrix in $\mathcal{A}(R,S)$, in terms of majorization. Our proof here takes elements from different proofs in the literature. It is partially based on the proof in [26], see also [27], and it incorporates the theory of majorization.

Theorem 3.1 (Gale-Ryser, [20, 32]) *Let $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ be non-increasing vectors of non-negative integers. Then $\mathcal{A}(R,S)$ is non-empty if and only if S is majorized by the conjugate R^* of R, that is,*

$$S \preceq R^*. \tag{3.1}$$

Proof.

Let us look at the necessity of the majorization condition. It follows by looking at the maximal matrix $\bar{A} = [\bar{a}_{ij}]$. Let us assume that $\mathcal{A}(R,S)$ is non-empty with

$$A = [a_{ij}] \in \mathcal{A}(R,S).$$

If all the 1's in each row of A are left-justified, that is, there are no i, j, k with $i \le m,\ j < k \le n$ such that $a_{ij} = 0$ but $a_{ik} = 1$, then A is a Ferrers matrix of R, hence $A = \bar{A}$. So, $S = R^*$ and (3.1) holds.

If A is not maximal otherwise, then there exist i, j, k with $i \le m,\ j < k \le n$ such that $a_{ij} = 0$ but $a_{ik} = 1$. So, A has at most as many 1's in the first k columns as $\bar{A}$ has, that is,

$$\sum_{j=1}^{k} s_j = \sum_{i=1}^{m}\sum_{j=1}^{k} a_{ij} \le \sum_{i=1}^{m}\sum_{j=1}^{k} \bar{a}_{ij} = \sum_{j=1}^{k} r_j^*.$$

So, (3.1) holds.

Now, to prove the converse, let us assume that the majorization condition (3.1) holds. Then by Theorem 2.7, S can be obtained by R^* by a sequence of transfers, namely

$$S = S^{(0)} \preceq S^{(1)} \preceq \ \ldots \ \preceq S^{(t-1)} \preceq S^{(t)} = R^*$$

where $S^{(i)}$ is obtained from $S^{(i+1)}$ by a transfer $(0 \leq i \leq t-1)$.

Since the Ferrers matrix of R uniquely determines R^* and the maximal matrix $\bar{A}$, the class $\mathcal{A}(R, R^*)$ is therefore non-empty. So by Lemma 2.9, it follows inductively that the class $\mathcal{A}(R, S)$ is also non-empty.

Thus, we have proven the theorem. □

Example 3.2

(i) *Let* $R = (3,1,1)$ *and* $S = (2,2,1)$*, then* S *is majorized by* $R^* = (3,1,1)$*. Hence, there exists a matrix* $M \in \mathcal{A}(R,S)$ *by Gale-Ryser. So, we have* A *the Ferrers matrix for* R *to begin with,*

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and we can construct the following two elements of $\mathcal{A}((3,1,1),(2,2,1))$*:*

$$M_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad M_2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Note that $R = (3,1,1)$ *is also majorized by* $S^* = (3,2)$*.*

(ii) *By the Gale-Ryser theorem,* $\mathcal{A}((4,1),(3,2))$ *is empty.*

(iii) *There exists* $M \in \mathcal{A}((4),(1,1,1,1))$ *and* M *is unique.*

Note, $R^* = S \;\Leftrightarrow\; S^* = R$*.*

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}.$$

From here onwards we assume $\mathcal{A}(R, S)$ to be non-empty unless otherwise stated.

As mentioned before, Theorem 3.1 gives a nice characterization of the existence of a (0,1)-matrix in $\mathcal{A}(R,S)$, in terms of majorization. So, it gives a simple necessary and sufficient condition (3.1) for the non-emptiness of $\mathcal{A}(R,S)$. This condition requires only the checking of $(n-1)$ inequalities and one equality. A different condition was obtained by Ford and Fulkerson in [13] by using a theorem in network flows. While such condition requires more inequalities than those in (3.1), it is still worthwhile for it involves a certain property associated with $\mathcal{A}(R,S)$. We will discuss that property in a later chapter, but for now let us show how starting from any single matrix in $\mathcal{A}(R,S)$, the entire class $\mathcal{A}(R,S)$ can be generated by simple transformations.

Chapter 4

Interchanges

Let us first look at some simple ideas of graph theory. This discussion of some notions of graph theory is not intended to be complete, hence, readers can find more details in books like [38]. Let D be a directed graph[1]. A *directed cycle* γ (of *length* p) of D is a sequence $(v_1, v_2, ..., v_p, v_1)$ of vertices of D such that $v_1, v_2, ..., v_p$ are distinct and $\alpha(\gamma) = \{(v_1, v_2), (v_2, v_3), ..., (v_{p-1}, v_p), (v_p, v_1)\}$ is a set of arcs of D. The directed graph D is defined to be *balanced* if it has the property that for each vertex v the number of arcs entering v (the *indegree* of v) equals the number of arcs exiting v (the *outdegree* of v).

Lemma 4.1 *Let D be a directed graph in which every vertex has outdegree ≥ 1. Then D contains a directed cycle.*

Proof.

> Starting a walk at an arbitrary vertex v_0, and at each step, continue from the vertex v_i along an arbitrary arc with tail v_i *(which is possible since each vertex has outdegree ≥ 1)* until a vertex is repeated. At this point, we have a directed cycle.

□

Now let $X = \{x_1, x_2, ..., x_m\}$ and $Y = \{y_1, y_2, ..., y_n\}$ be disjoint sets of m and n elements, respectively. Let $C = [c_{ij}]$ be a $(0, \pm 1)$-matrix (that is the entries of C are restricted to

[1] In Graph Theory, a *directed graph* or a *digraph* is a graph where the arcs (or the edges) have a direction associated with them.

the set $\{-1, 0, 1\}$) of size m by n. We define a *directed bipartite graph* $\Gamma(C)$ with vertices $X \cup Y (= \{x_1, x_2, ..., x_m, y_1, y_2, ..., y_n\})$ as follows. In $\Gamma(C)$ there is an arc (x_i, y_j) from x_i to y_j if $c_{ij} = 1$, and an arc (y_j, x_i) from y_j to x_i if $c_{ij} = -1$. If each row and each column sum of C equals 0, then we say that the matrix C is *balanced.* If C is balanced then it follows that $\Gamma(C)$ is a balanced directed bipartite graph.

Lemma 4.2 *Let D be a balanced directed graph. Then there exists a list of directed cycles $\gamma_1, \gamma_2, ...\gamma_q$ so that every arc appears in exactly one.*

Proof.

Choose a maximal list of cycles $\gamma_1, \gamma_2, ...\gamma_q$ so that every arc appears in at most one. Suppose that there is an arc not included in any cycle γ_i for $i = \{1, 2, ..., q\}$. Let H be a component of $D \setminus \cup_{i=1}^{q} \alpha(\gamma_i)$ which contains an arc. Now, H must be a balanced directed graph since every vertex v in H satisfies

$$deg_H^+(v) = deg_H^-(v) \neq 0.$$

So by Lemma 4.1, there is a directed cycle γ_{q+1} in H. But then γ_{q+1} may be appended to the list of cycles $\gamma_1, \gamma_2, ...\gamma_q$. This contradicts the maximality of the list $\gamma_1, \gamma_2, ...\gamma_q$. Hence, there are directed cyles $\gamma_1, \gamma_2, ...\gamma_q$ that partition the arcs of D. □

A *minimal balanced matrix* is a non-zero balanced $(0, \pm 1)$-matrix where each non-zero line contains exactly two non-zero elements (namely a 1 and a -1). By definition, it follows that a minimal balanced matrix has the property that its rows and columns can be rearranged so that, for some integer $k \geq 2$, the resulting matrix $D = [d_{ij}]$ satisfies $d_{11} = d_{22} = ... = d_{kk} = 1$, and $d_{12} = d_{23} = ... = d_{k-1,k} = d_{k1} = -1$, with all other d_{ij} equal to 0.

Let C be a non-zero balanced $(0, \pm 1)$-matrix of size m by n. Then a directed cycle γ of $\Gamma(C)$ is a balanced directed bipartite graph, with the same set of vertices as $\Gamma(C)$, corresponding to a m by n minimal balanced matrix C_γ. In fact, by Lemma 4.2, there are directed cycles $\gamma_1, \gamma_2, ...\gamma_q$ that partition the arcs of $\Gamma(C)$. By letting $C_i = C_{\gamma_i}$, $(i = 1, 2, ..., q)$, as Brualdi showed in Section 3.2 of [4], we can decompose C, which is a non-zero balanced matrix, into minimal balanced matrices:

$$C = C_1 + C_2 + ... + C_q \qquad \textit{(for some positive integer } q\textit{)} \tag{4.1}$$

where C_i is a minimal balanced matrix and for $i \neq j$, the set of positions of the non-zero elements of C_i is disjoint from the set of positions of the non-zero elements of C_j. Such a decomposition (4.1) is called *minimal balanced decomposition* of C.

Example 4.3

This example shows that a minimal balanced decomposition of a balanced matrix does not have to be unique. Consider the balanced matrix

$$C = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}.$$

Then

$$C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

is a minimal balanced decomposition of C. *Also*

$$C = C_1 + C_2 + C_3 + C_4$$

is a minimal balanced decomposition, where

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$C_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}, \quad C_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}.$$

Lemma 4.4 *If $A = [a_{ij}]$ and $B = [b_{ij}]$ are m by n matrices in $\mathcal{A}(R, S)$ with $A \neq B$, then $C = A - B$ is a balanced $(0, \pm 1)$-matrix.*

Proof.

Since A and B are both members of $\mathcal{A}(R, S)$, A and B have the same number of 1's and 0's in each line (but not necessarily in the same positions since $A \neq B$). Therefore, the entries of C is restricted to $\{-1, 0, 1\}$ and the sum of a line of A equals the sum of the same line of B for each and every line. It follows that the sum of each line of C is 0. □

So, if A and B are m by n matrices in $\mathcal{A}(R, S)$, then from the previous discussion and by Lemma 4.4, we can conclude that there exist minimal balanced matrices $C_1, C_2, ..., C_q$ whose non-zero elements are in pairwise disjoint positions, such that

$$A = B + C_1 + C_2 + ... + C_q.$$

A non-zero balanced matrix contains at least four elements. A balanced matrix $C = [c_{ij}]$ of size m by n with exactly four non-zero elements is obtained as follows:

(i) choose distinct integers p and q with $1 \leq p < q \leq m$;

(ii) choose distinct integers k and l with $1 \leq k < l \leq n$;

(iii) let $c_{pk} = c_{ql} = 1$,

(iv) let $c_{pl} = c_{qk} = -1$,

(v) let all other $c_{ij} = 0$.

We call a balanced matrix with exactly four non-zero elements an *interchange matrix*. In the previous example, the matrices C_1, C_2, C_3, C_4 are interchange matrices of order 4. Clearly, the negative of an interchange matrix is also an interchange matrix.

Let A and B be matrices in $\mathcal{A}(R,S)$ such that $A = B + C$ where C is an interchange matrix. Then A is obtained from B by replacing a submatrix

$$A_1[\{p,q\},\{k,l\}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

of B of order 2 with

$$A_2[\{p,q\},\{k,l\}] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

or vice-versa. Ryser [33] has defined this transformation between A_1 and A_2 as an *interchange*; or more precisely we say that A is obtained from B by a $(p,q;k,l)-interchange$. If A is obtained from B by a $(p,q;k,l)$-interchange, then B is also obtained from A by a $(p,q;k,l)$-interchange. Note that $A_1 = A_2 + C_1$ and $A_2 = A_1 + C_2$ where C_1 and C_2 are the following interchange matrices:

$$C_1 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Clearly an interchange (and hence any finite sequence of interchanges) does not change the row and column sum vectors of a matrix and therefore transforms a matrix in $\mathcal{A}(R,S)$ into another matrix in $\mathcal{A}(R,S)$. Ryser [32, 33] had proved the converse; that is, given $A, B \in \mathcal{A}(R,S)$, there is a sequence of interchanges which transforms B into A; that is, $A = B+C_1+C_2+...+C_q$ where $C_1, C_2, ..., C_q$ are interchange matrices and $B+C_1+...+C_k$ is in $\mathcal{A}(R,S)$ for $k = 1, 2, ..., q$; these interchange matrices may have non-zero elements in overlapping positions.

This discussion implies a characterization of those pairs R and S for which $\mathcal{A}(R,S)$ contains an unique matrix [5, 32, 39].

Theorem 4.5 *Let $R = (r_1, r_2, ..., r_m)$ and $S = (s_1, s_2, ..., s_n)$ be non-increasing, non-negative integral vectors for which $\mathcal{A}(R,S)$ is non-empty. Then there is a unique matrix in $\mathcal{A}(R,S)$ if and only if $S = R^*$ (where R^* is the conjugate of R).*

We follow the proof in [4].

Proof.

Suppose that $S = R^*$.

Using the result and notation in Chapter 2, we have $\mathcal{A}(R, S) = \mathcal{A}(R, R^*)$ containing the maximal matrix $\bar{A}$. Existence of another matrix in $\mathcal{A}(R, S)$ suggests that an interchange (or a sequence of interchanges) can be applied to $\bar{A}$. So, let us assume an interchange can be applied to $\bar{A}$.

Then there is a row i in $\bar{A}$ where the j^{th}-entry is a 1 and the k^{th}-entry is a 0 (with $j < k$) and we must have a row i' in $\bar{A}$ (with $i \neq i'$) where a 0 in the j^{th} position precedes a 1 in the k^{th} position.

This contradicts $\bar{A}$ being maximal, hence no interchange can be applied to $\bar{A}$. So, there can be no other matrix in $\mathcal{A}(R, S)$ but $\bar{A}$.

Now assume that there is a unique matrix $A = [a_{ij}]$ in $\mathcal{A}(R, S)$.

Suppose in row p of A a 0 precedes a 1 $\;(1 \leq p \leq m)$.

Let $a_{pk} = 0$ and $a_{pl} = 1 \quad (1 \leq k < l \leq n)$.

Since by definition $s_k \geq s_l$, there must be a $q \neq p$ such that $a_{qk} = 1$ and $a_{ql} = 0$. Now a $(p, q; k, l)$-interchange gives another matrix in $\mathcal{A}(R, S)$ contradicting the uniqueness of A. Hence A has no row where a 0 precedes a 1. So the 1's in each row of A are in the leftmost positions implying that A is maximal.

This implies that $S = R^*$ and we are done. $\square$

There is enough motivation to suggest that if $A \in \mathcal{A}(R, S)$ and exactly one interchange can be applied to A, then there are exactly two members of the class $\mathcal{A}(R, S)$. In this case, A contains exactly one submatrix of type

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ or } A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Without loss of generality, suppose A contains a submatrix of type A_1, then an interchange will result in another member of $\mathcal{A}(R, S)$, say B. So, exactly one interchange can be applied to B and B contains exactly one submatrix of type A_2. Applying an interchange to B will only result with A. So, there can be no other matrix in $\mathcal{A}(R,S)$.

Theorem 4.6 *Let $\mathcal{A}(R,S)$ be non-empty. Then $|\mathcal{A}(R,S)| = 2$ if and only if each matrix in $\mathcal{A}(R,S)$ contains exactly one submatrix of type*

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ or } A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Proof.

Suppose $|\mathcal{A}(R,S)| = 2$. Let $A, B \in \mathcal{A}(R,S)$ with $A \neq B$.
Let $C_1, C_2, ..., C_q$ be the minimal list of interchange matrices such that
$A = B + C_1 + C_2 + ... + C_q$, with $q \geq 2$ and $C_i \neq C_j$ whenever $i \neq j$ $(1 \leq i, j \leq q)$.
Then $B + C_i$ where $i = \{1, 2, ..., q\}$ is also a matrix in $\mathcal{A}(R,S)$.
Without loss of generality, suppose $B + C_1 \in \mathcal{A}(R,S)$. Since $|\mathcal{A}(R,S)| = 2$, this implies that $A = B + C_1$ which implies that $C_2 + ... + C_q = 0$.
This contradicts the minimality of q, hence, $q = 1$.

Let $\mathcal{A}(R,S)$ be non-empty with each matrix in $\mathcal{A}(R,S)$ containing exactly one submatrix of type A_1 or A_2 but not both. Then exactly one interchange can be applied to each matrix in $\mathcal{A}(R,S)$ that results in another matrix in $\mathcal{A}(R,S)$.
Suppose $|\mathcal{A}(R,S)| > 2$ with M_1, M_2, M_3 are distinct matrices in $\mathcal{A}(R,S)$, say.
Then without loss of generality, let $M_2 = M_1 + C_1$ where C_1 is an interchange matrix. Then C_1 is unique in being the only interchange matrix that can be added to M_1 to obtain another member of $\mathcal{A}(R,S)$. This therefore implies that $M_1 = M_2 + C_2$ where C_2 is the negative of C_1, and C_2 is the only interchange matrix that can be added to M_2 to obtain another matrix in $\mathcal{A}(R,S)$. It follows that either $M_3 = M_1$ or $M_3 = M_2$, which is a contradiction. $\square$

More generally, we can say that if an element of $\mathcal{A}(R,S)$ contains k disjoint submatrices of type A_1 or A_2, then there are 2^k matrices in $\mathcal{A}(R,S)$.

Chapter 5

The Structure Matrix $T(R, S)$

Let $A \in \mathcal{A}(R, S)$. We can assume without loss of generality that R and S satisfy:

$$r_1 \geq r_2 \geq \ \ldots \ \geq r_m > 0 \tag{5.1}$$

and

$$s_1 \geq s_2 \geq \ \ldots \ \geq s_n > 0. \tag{5.2}$$

This means that there is no zero rows and no zero columns in A, and the rows and columns had been permuted so that R and S are non-increasing.

A non-empty $\mathcal{A}(R, S)$ satisfying (5.1) and (5.2) is called *normalized.* In this chapter we assume $\mathcal{A}(R, S)$ to be normalized.

We partition A as follows:

$$A = \left[\begin{array}{c|c} A_1 & X \\ \hline Y & A_2 \end{array}\right] \tag{5.3}$$

where A_1 is of size $k \times l$ $\ (0 \leq k \leq m; \ \ 0 \leq l \leq n)$.

Let M be a (0,1)-matrix and let $N_0(M)$ denote the number of zeros in M and let $N_1(M)$ denote the number of ones in M. Now let

$$t_{kl} = N_0(A_1) + N_1(A_2) \quad \textit{with}\ (k = 0, 1, 2, \ \ldots, m; \ \ l = 0, 1, \ \ldots, n). \tag{5.4}$$

We call the $m+1$ by $n+1$ matrix

$$T = T(R,S) = [t_{kl}] \tag{5.5}$$

the *structure matrix* of the normalized class $\mathcal{A}(R,S)$

(the structure matrix associated with R and S).

By calculation, one can see that

$$kl = N_0(A_1) + N_1(A_1),$$

$$\sum_{i=k+1}^{m} r_i = N_1(Y) + N_1(A_2)$$

and

$$\sum_{j=1}^{l} s_j = N_1(A_1) + N_1(Y).$$

So,

$$\begin{aligned} kl - \sum_{j=1}^{l} s_j &= (N_0(A_1) + N_1(A_1)) - (N_1(A_1) + N_1(Y)) \\ &= N_0(A_1) - N_1(Y). \end{aligned}$$

and

$$\begin{aligned} kl - \sum_{j=1}^{l} s_j + \sum_{i=k+1}^{m} r_i &= (N_0(A_1) - N_1(Y)) + (N_1(Y) + N_1(A_2)) \\ &= N_0(A_1) + N_1(A_2). \end{aligned}$$

Thus,

$$t_{kl} = kl + \sum_{i=k+1}^{m} r_i - \sum_{j=1}^{l} s_j. \tag{5.6}$$

So, $T = [t_{kl}]$ does not rely on our particular choice of the element A of $\mathcal{A}(R,S)$.

For covenience of notation, let us number the rows of T from 0 through m and its columns from 0 through n. Let τ denote the total number of ones in $A \in \mathcal{A}(R,S)$. Then from (5.6), we have the following elements of T:

$$t_{00} = \tau; \quad t_{0n} = 0; \quad t_{1n} = n - r_1; \quad t_{m1} = m - s_1; \quad \text{and} \quad t_{mn} = mn - \tau.$$

More generally, it follows from (5.6) that the entries of row 0 and column 0 of T are:

$$t_{0l} = \tau - \sum_{j=1}^{l} s_j \quad \text{and} \quad t_{k0} = \sum_{i=k+1}^{m} r_i.$$

So, with $(k = 0, 1, \ . \ . \ . \ , m; \ \ l = 0, 1, \ . \ . \ . \ , n)$ we have

$$\begin{aligned} t_{kl} &= kl + t_{k0} - (\tau - t_{0l}) \\ &= kl + t_{k0} + t_{0l} - t_{00}. \end{aligned} \tag{5.7}$$

Equation (5.7) shows how to construct T from the $(m + n + 1)$ elements in row 0 and column 0.

Example 5.1

For $R = (3, 2, 2, 1) = S$,

$$T = \begin{bmatrix} 8 & 5 & 3 & 1 & 0 \\ 5 & 3 & 2 & 1 & 1 \\ 3 & 2 & 2 & 2 & 3 \\ 1 & 1 & 2 & 3 & 5 \\ 0 & 1 & 3 & 5 & 8 \end{bmatrix}.$$

For R = (3,1) and $S = (2, 2)$,

$$T = \begin{bmatrix} 4 & 2 & 0 \\ 1 & 0 & -1 \\ 0 & -2 & 0 \end{bmatrix}.$$

Note that by Theorem 3.1, $\mathcal{A}((3, 1), (2, 2))$ is empty.

Theorem 5.2 (Ford-Fulkerson, [13])

Let $R = (r_1, r_2, \ . \ . \ . \ , r_m)$ and $S = (s_1, s_2, \ . \ . \ . \ , s_n)$ be non-increasing vectors of non-negative integers. Then $\mathcal{A}(R, S)$ is non-empty if and only if

$$t_{kl} \geq 0 \quad (k = 0, 1, \ . \ . \ . \ , m; \ \ l = 0, 1, \ . \ . \ . \ , n);$$

that is, the structure matrix T is a non-negative matrix.

Ford and Fulkerson, who introduced the concept of the structure matrix, used the theory of network flows in [13] to prove this theorem. Brualdi also derived this theorem in [4] following the proof technique he used in [6]. Here we provide a proof that relies on the result of Theorem 3.1.

Proof. Using equation (5.6), and the definition of the conjugate sequence, we have:

$$\begin{aligned} t_{r_j^*,j} &= r_j^*.j + \sum_{i=r_j^*+1}^{m} r_i - \sum_{i=1}^{j} s_i \\ &= \sum_{i=1}^{j} r_i^* - \sum_{i=1}^{j} s_i. \end{aligned}$$

This implies that:

$$t_{r_j^*,j} \geq 0 \textit{ (for all j), if and only if } S \preceq R^*.$$

That is, T is non-negative whenever equation (3.1) holds and vice-versa.

So, by Theorem 3.1, our proof is done. □

The structure matrix may be given a combinatorial interpretation. Let $A \in \mathcal{A}(R, S)$, satisfying (5.1) and (5.2) and we partition A as was in (5.3). Thus, as we can see in the equation (5.4), t_{kl} counts something: the number of zeros in A_1 (which has size $k \times l$) plus the number of ones in A_2. This reveals something about the combinatorial meaning of the structure matrix T, and with this interpretation, the necessity of the above theorem is obvious. That is, if A is an existing element of $\mathcal{A}(R, S)$ then the structure matrix T of A is non-negative.

Interestingly enough, as we have seen in equation (5.7), that all other entries of T are determined by the entries in the first (actually, zeroth) row and column. Indeed, as we will discuss further in the next chapter, the structure matrix T does reveal a lot about the structure of matrices in $\mathcal{A}(R, S)$. For instance, assume that $t_{kl} = 0$ for some k, l with $k, l \geq 1$. By the combinatorial interpretaion $t_{kl} = N_0(A_1) + N_1(A_2)$, this means that every matrix $A \in \mathcal{A}(R, S)$ satisfies

$$\begin{aligned} a_{ij} &= 1 \quad (1 \leq i \leq k,\ 1 \leq j \leq l) \\ a_{ij} &= 0 \quad (k+1 \leq i \leq m,\ l+1 \leq j \leq n). \end{aligned}$$

Chapter 6

Invariant Sets For The Class $\mathcal{A}(R, S)$

Let $\mathcal{A}(R,S)$ be non-empty with $|\mathcal{A}(R,S)| > 1$. Let $I \subseteq \{1,2,...,m\}$ and $J \subseteq \{1,2,...,n\}$ with $\bar{I} = \{1,2,...,m\} - I$ and $\bar{J} = \{1,2,...,n\} - J$ being the complements of I and J respectively. Let $A \in \mathcal{A}(R,S)$ and $A[I,J]$ be the submatrix of A with rows indexed by I and columns indexed by J; also, let $A(I,J)$ be the submatrix of A with rows indexed by $\bar{I}$ and columns indexed by $\bar{J}$.

We say that $I \times J$ is an *invariant set* for $\mathcal{A}(R,S)$ if for any two matrices A and B in $\mathcal{A}(R,S)$, $N_1(A[I,J]) = N_1(B[I,J])$ with $A \neq B$. Clearly $I \times J$ is an invariant set for $\mathcal{A}(R,S)$ whenever one of the following holds:

$$I = \emptyset, \qquad I = \{1,2,...,m\}, \qquad J = \emptyset, \qquad J = \{1,2,...,n\}.$$

We refer to such invariant sets as *trivial*; all other invariant sets are *non-trivial.*

Since

$$N_1(A(I,J)) = N_1(A[I,J]) + \sum_{i \in \bar{I}} r_i - \sum_{j \in J} s_j,$$

it follows that if $I \times J$ is an invariant set for $\mathcal{A}(R,S)$ then so is $\bar{I} \times \bar{J}$. Similarly, since

$$N_1(A[I,\bar{J}]) = \sum_{i \in I} r_i - N_1(A[I,J]),$$

and

$$N_1(A[\bar{I}, J]) = \sum_{j \in J} s_j - N_1(A[I, J]),$$

$I \times \bar{J}$ and $\bar{I} \times J$ are invariant sets whenever $I \times J$ is.

An *invariant position* is an invariant set of cardinality one. Thus the position (i, j) is invariant provided that all elements of $\mathcal{A}(R, S)$ have the same value for their (i, j)-entry. That is, suppose $\{A_1, A_2, ..., A_k\} = \mathcal{A}(R, S)$, then (i, j) is an invariant position if either the (i, j)-entry of A_p is 1 for all $p = \{1, 2, ..., k\}$, or the (i, j)-entry of A_p is 0 for all $p = \{1, 2, ..., k\}$.

Ryser's *invariant* 1 [32] is an invariant position (i, j) for which the (i, j)-entry of each matrix in $\mathcal{A}(R, S)$ equals 1. In other words, for $A \in \mathcal{A}(R, S)$, an element $a_{ij} = 1$ of A is an invariant 1 provided that no sequence of interchanges applied to A replaces $a_{ij} = 1$ by 0.

Theorem 6.1 (Ryser, [32, 33]) *Let A be a $(0, 1)$-matrix of size m by n in $\mathcal{A}(R, S)$, then the following are equivalent:*

(i) *A has an invariant position.*

(ii) *There exist integers k and l satisfying*

$$0 \leq k \leq m \qquad 0 \leq l \leq n$$

such that $t_{kl} = 0$ with $(k, l) \neq (0, n), (m, 0)$.

(iii) *There exist integers k and l satisfying*

$$0 \leq k \leq m \qquad 0 \leq l \leq n$$

where $(k, l) \neq (0, n), (m, 0)$ such that each of the positions in $\{(i, j) : 1 \leq i \leq k, 1 \leq j \leq l\}$ is an invariant 1-position of $\mathcal{A}(R, S)$ and each of the positions in $\{(i, j) : k + 1 \leq i \leq m, l + 1 \leq j \leq n\}$ is an invariant 0-position.

The equivalence of the statements $(i), (ii)$ and (iii) had been proved by Ryser in [32, 33] and also by Haber in [23].

It follows from Theorem 6.1 that $\mathcal{A}(R,S)$ does not have any invariant positions if the only zeros in the structure matrix $T(R,S)$ are t_{0n} and t_{m0}. Otherwise, if there are other zeros in T apart from t_{0n} and t_{m0} then $\mathcal{A}(R,S)$ has at least one invariant position.

Example 6.2

Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

So $A \in \mathcal{A}((3,2,2,1),(3,2,2,1))$ and, as we can see, each entry of A associates with an interchange (or a submatrix of type A_1 or A_2). We saw in Example 5.1 that the only zeros in the structure matrix for A are $t_{0,4}$ and $t_{4,0}$. So, A has no invariant positions.

Moreover, as we briefly mentioned at the end of the previous chapter, it follows from Theorem 6.1 that if $t_{kl} = 0$ for non-negative integers k and l with $(k,l) \neq (0,n),(m,0)$, then each matrix $A \in \mathcal{A}(R,S)$ has a decomposition of the form

$$A = \left[\begin{array}{c|c} J_{k,l} & X \\ \hline Y & O_{m-k,n-l} \end{array}\right]. \tag{6.1}$$

$J_{k,l}$ is of the specified size $k \times l$ $(1 \leq k \leq m,\ 1 \leq l \leq n)$ with all entries as invariant 1's and $O_{m-k,n-l}$ is a $(m-k)\times(n-l)$ zero matrix.

Let us now consider our structure matrix T and how it reveals properties of the class $\mathcal{A}(R,S)$. Let the 0's in T occupy the positions

$$(0,n) = (i_0,j_0),(i_1,j_1),...,(i_p,j_p) = (m,0)$$

where

$$i_0 \leq i_1 \leq ... \leq i_{p-1} \leq i_p, \quad j_0 \geq j_1 \geq ... \geq j_{p-1} \geq j_p,$$

and $(i_{t-1},j_{t-1}) \neq (i_t,j_t)$ for $t = 1,2,...,p$. In fact, each (i_t,j_t) gives appropriate integers k and l for the decomposition (6.1).

Let

$$I_t = \{i_{t-1}+1, i_{t-1}+2, ..., i_t\} \quad (t = 1, 2, ..., p)$$

and

$$J_t = \{j_t+1, j_t+2, ..., j_{t-1}\} \quad (t = 1, 2, ..., p).$$

Then $I_1, I_2, ..., I_p$ are pairwise disjoint sets satisfying

$$I_1 \cup I_2 \cup ... \cup I_p = \{1, 2, ..., m\}.$$

Similarly $J_1, J_2, ..., J_p$ are pairwise disjoint sets satisfying

$$J_1 \cup J_2 \cup ... \cup J_p = \{1, 2, ..., n\}.$$

One but not both of the sets in each pair I_t, J_t may be empty. So, we have

$$A[I_k, J_l] = \begin{cases} J, & \text{if} \quad l > k \\ O, & \text{if} \quad k > l \end{cases} \qquad (k, l = 1, 2, ..., t).$$

Let the matrix $A[I_t, J_t]$ have row-sum vector R^t and column-sum vector S^t $(t = 1, 2, ..., p)$. Then since $A[I_t, J_t]$ is in $\mathcal{A}(R^t, S^t)$, $\mathcal{A}(R^t, S^t)$ is non-empty and has no invariant positions, and any invariant position of $\mathcal{A}(R^t, S^t)$ is also an invariant position of $\mathcal{A}(R, S)$. Thus, the non-invariant positions of $\mathcal{A}(R, S)$ are precisely the positions occupied by the t matrices $A[I_1, J_1], A[I_2, J_2], ..., A[I_t, J_t]$.

For the case where the submatrices $A[I_1, J_1], A[I_2, J_2], ..., A[I_t, J_t]$ are smallest possible; that is each of them equal to

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

we have

$$\left|\mathcal{A}(R^i, S^i)\right| = 2 \quad \textit{for all} \quad i = \{1, 2, ..., t\}.$$

Hence,

$$|\mathcal{A}(R, S)| = \prod_{i=1}^{t} \left|\mathcal{A}(R^i, S^i)\right| = 2^t.$$

Let each $A[I_t, J_t]$ represents a $(u, v; w, z)$-interchange where $I_t = \{u, v\}$ and $J_t = \{w, z\}$. So, the set of entries $\{a_{uw}, a_{uz}, a_{vw}, a_{vz}\}$ in A is a non-invariant set and by observation, in the structure matrix T of A, we have $t_{v,w-1} = 0$ and $t_{u-1,z} = 0$.

Example 6.3

For $R = S = (3, 3, 1, 1)$, *we have*

$$T = \begin{bmatrix} 8 & 5 & 2 & 1 & 0 \\ 5 & 3 & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 3 & 5 \\ 0 & 1 & 2 & 5 & 8 \end{bmatrix}.$$

The 0*'s in* T *are in positions* $(0, 4), (2, 2)$ *and* $(4, 0)$. *The pairs* (I_t, J_t) *both of whose sets are non-empty:*

$$(I_1, J_1) \quad \textit{where} \quad I_1 = \{1, 2\}, \;\; J_1 = \{3, 4\}$$

and

$$(I_2, J_2) \quad \textit{where} \quad I_2 = \{3, 4\}, \;\; J_2 = \{1, 2\}.$$

Thus, each matrix A *in the class* $\mathcal{A}((3, 3, 1, 1), (3, 3, 1, 1))$ *has the form*

$$A = \left[\begin{array}{cc|cc} 1 & 1 & * & * \\ 1 & 1 & * & * \\ \hline * & * & 0 & 0 \\ * & * & 0 & 0 \end{array}\right]$$

where each of the submatrices $A[\{1, 2\}, \{3, 4\}] = A[I_1, J_1]$ *and* $A[\{3, 4\}, \{1, 2\}] = A[I_2, J_2]$ *are either of the form*

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \textit{or} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

There are $2^2 = 4$ *matrices in this class* $\mathcal{A}((3, 3, 1, 1), (3, 3, 1, 1))$.

Consider the non-invariant set $A[\{1, 2\}, \{3, 4\}] = A[I_1, J_1]$, *and let* $u = 1, v = 2, w = 3$

and $z = 4$. This implies that $t_{2,2} = 0$ and $t_{0,4} = 0$. Similarly by considering the non-invariant set $A[I_2, J_2]$, we have $t_{4,0} = 0$ and $t_{2,2} = 0$.

Now we take a look at the structure of matrices in $\mathcal{A}(R,S)$ where $|\mathcal{A}(R,S)| = 2$ in light of Theorem 4.6.

Theorem 6.4 *Let $R = (r_1, r_2, ..., r_m)$ and $S = (s_1, s_2, ..., s_n)$ be two positive non-increasing integral vectors, where $m \le n$ and $m, n > 2$. Let $A = [a_{ij}] \in \mathcal{A}(R,S)$ with $|\mathcal{A}(R,S)| = 2$. Let $A[\{p, p+1; k, k+1\}]$ represents a $(p, p+1; k, k+1)$-interchange in A where $1 \le p < m,\ 1 \le k < n$. Then the structure of A is as the following:*

$$A = \left[\begin{array}{c|c|c} \mathbf{1} & \mathbf{1} & M_1 \\ \hline \mathbf{1} & A[\{p,p+1;k,k+1\}] & \mathbf{0} \\ \hline M_2 & \mathbf{0} & \mathbf{0} \end{array}\right]$$

where $\mathbf{1}$ is a matrix of 1's, and $\mathbf{0}$ is a matrix of 0's, and M_1 and M_2 are maximal.

Proof.

Obviously, from Theorem 4.6, matrix A contains exactly one $(p, q; k, l)$-interchange where $1 \le p < q \le m,\ 1 \le k < l \le n$. That is, either

$$\begin{bmatrix} a_{pk} & a_{pl} \\ a_{qk} & a_{ql} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

or

$$\begin{bmatrix} a_{pk} & a_{pl} \\ a_{qk} & a_{ql} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Now, let $T = [t_{ij}]$ be the structure matrix for A.

Since R and S are non-increasing, it is apparent that $q = p+1$ and $l = k+1$.

The non-invariant positions of $\mathcal{A}(R,S)$ are precisely the positions occupied by the matrix $A[\{p, q; k, l\}]$. Hence, it follows from our discussion of non-invariant sets that $t_{q,k-1} = 0 = t_{p-1,l}$. So, from the combinatorial interpretation of T and since $t_{p-1,l} = 0$, we can partition A as follows:

$$A = \left[\begin{array}{c|c|c} \mathbf{1} & \mathbf{1} & * \\ \hline * & A[\{p,q;k,l\}] & \mathbf{0} \\ \hline * & * & \mathbf{0} \end{array}\right].$$

Similary, since $t_{q,k-1} = 0$, we have the following partitioning of A:

$$A = \left[\begin{array}{c|c|c} \mathbf{1} & * & * \\ \hline \mathbf{1} & A[\{p,q;k,l\}] & * \\ \hline * & \mathbf{0} & \mathbf{0} \end{array}\right].$$

Combining the two ways of partitioning A above we have

$$A = \left[\begin{array}{c|c|c} \mathbf{1} & \mathbf{1} & M_1 \\ \hline \mathbf{1} & A[\{p,p+1;k,k+1\}] & \mathbf{0} \\ \hline M_2 & \mathbf{0} & \mathbf{0} \end{array}\right].$$

Since A contains exactly one interchange, both M_1 and M_2 cannot be of the form

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad or \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix};$$

and by the property of invariant sets, there is no restriction on the entries of both M_1 and M_2 to be only of one value (but they can be). Hence, we conclude that they are maximal matrices. □

To conclude this section, it is worth noting that an invariant position gives a non-trivial invariant set of cardinality one. Brualdi and Ross in [8] proved that if $\mathcal{A}(R,S)$ has an invariant position then it has a non-trivial invariant set.

Chapter 7

Spectrum Of Sizes For $\mathcal{A}(R, S)$

In this chapter, we show the following:

Theorem 7.1

Let k be a given positive integer. Then there exist R and S such that $|\mathcal{A}(R, S)| = k$.

Beginning with $k = 1$; for any row sum vector R, let the column sum vector S be the conjugate of R (that is, $S = R^*$), then $|\mathcal{A}(R, S)| = 1$ and the only element of $\mathcal{A}(R, S)$ is the Ferrers matrix of R as asserted by Theorem 4.5. For instance, if $R = (2, 1)$ and $S = (2, 1)$, then there is only one matrix that can be uniquely reconstructed from R as its row sum vector and S as its column sum vector:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

We also have looked at the case where the set $\mathcal{A}(R, S)$ has exactly two elements. We only need to construct a $(0, 1)$-matrix A with exactly one submatrix of type

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{or} \qquad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

but not both. In other words, $|\mathcal{A}(R, S)| = 2$ when each of the matrices in $\mathcal{A}(R, S)$ contains only one interchange exactly. Since we obtain A_1 from A_2 (or vice-versa) by exactly one interchange, and their row and column sum vectors are both equal to (1, 1), we have $|\mathcal{A}((1, 1), (1, 1))| = 2$.

Now, let A be a $2 \times n$ $(0,1)$-matrix with row sum vector $R = (a, (n-a))$ and column sum vector $S = (1, 1, ..., 1)$, where a, n are positive integers ($a \geq 1$; $n \geq 2$). Since each 1-entry in the first row of A corresponds to a 0-entry in the second row of A (and vice-versa), the number of ways we can reconstruct A by the given R and S is the same as the number of ways of selecting a (or $n-a$) elements from a set of n elements:

$$\binom{n}{a} = \frac{n!}{a! \cdot (n-a)!} = \binom{n}{n-a}.$$

Example 7.2

How many $(0,1)$*-matrices in the class* $\mathcal{A}((2,2),(1,1,1,1))$*? Here we have* $n = 4$ *and* $a = 2$*. So, since*

$$\binom{4}{2} = \frac{4!}{2! \cdot 2!} = 6,$$

we conclude that there are six 2×4 $(0,1)$*-matrices in the class* $\mathcal{A}((2,2),(1,1,1,1))$*. Here they are:*

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

It follows that, since $\binom{k}{1} = k$ for $k \geq 2$, there are k different ways of constructing a $2 \times k$ $(0,1)$-matrix with row sum vector $(1, k-1)$ and column sum vector $(1, 1, ..., 1)$. Hence, for any given integer $k \geq 2$, we have $|\mathcal{A}((1, k-1), (1, 1, ..., 1))| = k$ and thus proven Theorem 7.1.

Example 7.3

Let $k = 5$. There are five $(0, 1)$-matrices in the set $\mathcal{A}((1, 4), (1, 1, 1, 1, 1))$ and they are:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

A non-negative square matrix $A = [a_{ij}]$ of order n is called *doubly stochastic* provided each of its row and column sums equals 1:

$$\sum_{j=1}^{n} a_{ij} = 1 \quad (i = 1, 2, ..., n), \quad \text{and} \quad \sum_{i=1}^{n} a_{ij} = 1 \quad (j = 1, 2, ..., n).$$

A doubly stochastic $(0, 1)$-matrix is known as a *permutation matrix.*

A permutation matrix of order n is therefore a matrix obtained by permuting the rows of an $n \times n$ identity matrix according to some permutation of the numbers 1 to n. In other words, a permutation matrix of order n represents a specific permutation of n elements.

Theorem 7.4

Let n be a given positive integer. Then there are $n!$ permutation matrices of order n.

Proof.

We are trying to answer the question: How many different ways can we reconsruct a permutation matrix of order n? For each row/column, we have n available positions for our 1-entry. Once we decide on the position of the 1 in the first row/column, the second row/column has now only $n - 1$ available positions for its 1-entry. After deciding the position of the 1 for the second row/column, only $n - 2$ available positions for the 1-entry of the third row/column. As we come to the n^{th} row/column, only one position left for the 1-entry.

Evidently, what we are trying to do is to find out the number of ways of permuting n elements. So, by multiplication rule, there are $n!$ permutation matrices of order n. $\square$

Let $\mathcal{A}(n,k)$ denote the set of (0,1)-matrices of order n with exactly k 1's in each row and column. By letting $a_{n,k} = |\mathcal{A}(n,k)|$, we have $a_{n,1} = a_{n,n-1} = n!$. In general, $a_{n,k} = a_{n,n-k}$ holds since a $(0,1)$-matrix obviously has only two symbols as its entries (namely 0 and 1).

Example 7.5

By letting $n = 3$ *and* $k = 1$*, there are* $3! = 6$ *matrices in* $\mathcal{A}(3,1)$ *and they are the permutation matrices of order* 3*:*

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}, \quad \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}, \quad \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix},$$

$$\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}, \quad \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}, \quad \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}.$$

Similarly, there are also 6 *matrices in* $\mathcal{A}(3,2)$ *(as we note* $n-k = 3-1 = 2$*), and they are obtained from the* 6 *matrices above by exchanging the zeros and the ones:*

$$\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}, \quad \begin{bmatrix}0&1&1\\1&1&0\\1&0&1\end{bmatrix}, \quad \begin{bmatrix}1&0&1\\0&1&1\\1&1&0\end{bmatrix},$$

$$\begin{bmatrix}1&0&1\\1&1&0\\0&1&1\end{bmatrix}, \quad \begin{bmatrix}1&1&0\\1&0&1\\0&1&1\end{bmatrix}, \quad \begin{bmatrix}1&1&0\\0&1&1\\1&0&1\end{bmatrix}.$$

So, we have $a_{3,0} = a_{3,3} = 1,\ a_{3,1} = a_{3,2} = 3! = 6$.

Moreover, for the case of $n = 2$*, we have* $a_{2,0} = a_{2,2} = 1,\ a_{2,1} = 2! = 2$.

In the next chapter, we state the generating function for $|\mathcal{A}(R, S)|$. Brualdi in Chapter 4 of [4] suggested that one can deduce a formula for $a_{n,2}$ and $a_{n,3}$ from such function, building upon ideas presented in [1] and pages 235 and 236 of [12].

Chapter 8

Generating Function For $|\mathcal{A}(R, S)|$

Our primary focus in this chapter is on the evaluation of the number of (0,1)-matrices with prescribed row and column sum vectors R and S respectively. In particular, there has been a considerable amount of study of integer matrices with a prescribed row and column sum.

Let us assume $\mathcal{A}(R, S)$ to be normalized. Let $x_1, x_2, ..., x_m, y_1, y_2, ..., y_n$ be $m+n$ variables with $\mathbf{x} = (x_1, x_2, ..., x_m)$ and $\mathbf{y} = (y_1, y_2, ..., y_n)$. By the Gale-Ryser Theorem of Chapter 3, $|\mathcal{A}(R, S)| \neq 0$ if and only if the non-increasing rearrangement of S is majorized by the conjugate R^* of R. We can write the *generating function for the numbers* $|\mathcal{A}(R, S)|$ as

$$g_{m,n}(\mathbf{x}; \mathbf{y}) = g_{m,n}(x_1, x_2, ..., x_m; y_1, y_2, ..., y_n)$$

where

$$g_{m,n}(\mathbf{x}; \mathbf{y}) = \sum_{R,S} |\mathcal{A}(R, S)|\, x_1^{r_1} x_2^{r_2} ... x_m^{r_m} y_1^{s_1} y_2^{s_2} ... y_n^{s_n}.$$

The following theorem has been proved in [12] and in [41]. It is also implied with not much details in Knuth's derivation of identities of Littlewood [25] and is used in [19] and [36] in showing a remarkable correspondence between $(0, 1)$-matrices and pairs of combinatorial constructs known as Young tableaux.

Theorem 8.1 *Let m and n be positive integers. Then*

$$g_{m,n}(\mathbf{x};\mathbf{y}) = \prod_{i=1}^{m}\prod_{j=1}^{n}(1+x_iy_j).$$

Here, we follow Brualdi's proof in [4].

Proof.

We have

$$\prod_{i=1}^{m}\prod_{j=1}^{n}(1+x_iy_j) = \sum_{\{a_{ij}=0 \text{ or } 1:\ 1\le i\le m,\ 1\le j\le n\}}\prod_{i=1}^{m}\prod_{j=1}^{n}(x_iy_j)^{a_{ij}} \tag{8.1}$$

where the summation extends over all 2^{mn} $(0,1)$-matrices $A=[a_{ij}]$ of size $m\times n$. By expansion and simplification then collecting like terms, we see that the coefficient of

$$x_1^{r_1}x_2^{r_2}...x_m^{r_m}y_1^{s_1}y_2^{s_2}...y_n^{s_n}$$

on the right side of (8.1) equals $|\mathcal{A}(R,S)|$. $\square$

Example 8.2

Let $m=2=n$ with $\mathbf{x}=(x_1,x_2)$ and $\mathbf{y}=(y_1,y_2)$. We consider the complete set of 2×2 matrices with entries restricted to $\{0,1\}$; that is all the classes $\mathcal{A}(R,S)$ with $R=(r_1,r_2)$ and $S=(s_1,s_2)$. There are $2^{2\times 2}=16$ of them. So, our generating function for 2×2 $(0,1)$-matrices is as follows:

$$\begin{aligned}
g_{2,2}(\mathbf{x};\mathbf{y}) = &\ \prod_{i=1}^{2}\prod_{j=1}^{2}(1+x_iy_j)\\
= &\ (1+x_1y_1)(1+x_1y_2)(1+x_2y_1)(1+x_2y_2)\\
= &\ (1+x_1y_1+x_1y_2+x_1^2y_1y_2)(1+x_2y_1)(1+x_2y_2)\\
= &\ (1+x_1y_1+x_1y_2+x_2y_1+x_1^2y_1y_2+x_1x_2y_1^2+x_1x_2y_1y_2+x_1^2x_2y_1^2y_2)(1+x_2y_2)\\
= &\ 1+x_1y_1+x_1y_2+x_2y_1+x_1^2y_1y_2+x_1x_2y_1^2+x_1x_2y_1y_2+x_1^2x_2y_1^2y_2+\\
&\ x_2y_2+x_1y_1x_2y_2+x_1x_2y_2^2+x_2^2y_1y_2+x_1^2x_2y_1y_2^2+x_1x_2^2y_1^2y_2+x_1x_2^2y_1y_2^2+x_1^2x_2^2y_1^2y_2^2.
\end{aligned}$$

and by collecting like terms, we have

$$\begin{aligned}
g_{2,2}(\mathbf{x};\mathbf{y}) = &\ 1+x_1y_1+x_1y_2+x_2y_1+x_2y_2+2(x_1x_2y_1y_2)+x_1^2y_1y_2+x_1x_2y_1^2+x_1x_2y_2^2\\
&+x_2^2y_1y_2+x_1^2x_2y_1^2y_2+x_1^2x_2y_1y_2^2+x_1x_2^2y_1^2y_2+x_1x_2^2y_1y_2^2+x_1^2x_2^2y_1^2y_2^2.
\end{aligned}$$

Hence, by considering the indices and coefficient of each term, we conclude that

$$|\mathcal{A}((0,0),(0,0))| = 1,$$

$$|\mathcal{A}((1,0),(1,0))| = |\mathcal{A}((1,0),(0,1))| = |\mathcal{A}((0,1),(1,0))| = |\mathcal{A}((0,1),(0,1))| = 1,$$

$$|\mathcal{A}((1,1),(1,1))| = 2,$$

$$|\mathcal{A}((2,0),(1,1))| = |\mathcal{A}((1,1),(2,0))| = |\mathcal{A}((1,1),(0,2))| = |\mathcal{A}((0,2),(1,1))| = 1,$$

$$|\mathcal{A}((2,1),(2,1))| = |\mathcal{A}((2,1),(1,2))| = |\mathcal{A}((1,2),(2,1))| = |\mathcal{A}((1,2),(1,2))| = 1,$$

$$|\mathcal{A}((2,2),(2,2))| = 1.$$

By replacing some of the terms $(1 + x_i y_j)$ in (8.1) with a 1 or with $x_i y_j$'s we get the generating function for the number of matrices in $\mathcal{A}(R, S)$ having 0's and 1's in prescribed places.

Example 8.3

Let us consider the class $\mathcal{A}((2,2,1),(2,2,1))$. *In the expansion of the generating function, we are looking for the different combination of* $x_i y_j$ $(1 \leq i, j \leq 3)$ *that will generate the* $x_1^2 x_2^2 x_3 y_1^2 y_2^2 y_3$*-term. So, we are allowed to use* x_1 *and* x_2 *twice but* x_3 *only once; and similarly* y_1 *and* y_2 *are used twice with* y_3 *only once. By replacing some of the terms* $(1 + x_i y_j)$ *in (8.1) with a* 1 *or with* $x_i y_j$*'s in corresponding to having* 0*'s and* 1*'s in prescribed places, here are the different* $x_i y_j$ *product combinations that will give an* $x_1^2 x_2^2 x_3 y_1^2 y_2^2 y_3$*-term:*

$$(x_1 y_1)(x_1 y_2)(x_2 y_1)(x_2 y_2)(x_3 y_3),$$

$$(x_1 y_1)(x_1 y_2)(x_2 y_1)(x_2 y_3)(x_3 y_2),$$

$$(x_1 y_1)(x_1 y_2)(x_2 y_2)(x_2 y_3)(x_3 y_1),$$

$$(x_1 y_1)(x_1 y_3)(x_2 y_1)(x_2 y_2)(x_3 y_2),$$

$$(x_1 y_3)(x_1 y_2)(x_2 y_1)(x_2 y_2)(x_3 y_1).$$

So, the coefficient of the $x_1^2x_2^2x_3y_1^2y_2^2y_3$-term is 5*, hence,* $|\mathcal{A}((2,2,1),(2,2,1)| = 5$.
Here are the five matrices in $\mathcal{A}((2,2,1),(2,2,1)$*:*

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The stated generating function $g_{m,n}(\mathbf{x};\mathbf{y})$ is exhaustive in giving the cardinality of the set $\mathcal{A}(R,S)$ for all possible combinations of R and S of order m and n respectively. However, the calculation is clearly cumbersome especially when m and n are very big. It is difficult to evaluate the cardinality of $\mathcal{A}(R,S)$ for prescribed R and S by going through all of the 2^{mn} $(0,1)$-matrices of size $m \times n$. Nevertheless, $g_{m,n}(\mathbf{x};\mathbf{y})$ can be used in deriving formulas for the cardinality of $\mathcal{A}(R,S)$ in special cases of specified R and S.

Recall, we defined $a_{n,k}$ in the previous chapter as the number of (0,1)-matrices of order n with exactly k 1's in each row and column. Brualdi [4] evaluates $a_{n,2}$ in the next corollary, based on ideas from [1] and [12, p. 235, 236].

Corollary 8.4 (Brualdi [4])

For $n \geq 2$ *we have*

$$a_{n,2} = \frac{1}{4^n}\sum_{j=0}^{n}(-1)^j(2n-2j)!j!\begin{pmatrix} n \\ j \end{pmatrix}^2 2^j.$$

Proof of the above corollary can be found in Section 4.1 of [4].

Example 8.5

Let $n = 4$. Then

$$\begin{aligned}
a_{4,2} = & \frac{1}{4^4}\sum_{j=0}^{4}(-1)^j(2(4)-2j)!j!\begin{pmatrix}4\\j\end{pmatrix}^2 2^j \\
= & \frac{1}{256}\left[8! - 6!(16)(2) + 4!(2)(36)(4) - 2(3!)(16)(8) + 4!(16)\right] \\
= & \frac{1}{256}\left[23040\right] \\
= & 90.
\end{aligned}$$

Thus, we have $a_{4,0} = a_{4,4} = 1$; $a_{4,1} = a_{4,3} = 4! = 24$; and $a_{4,2} = 90$.

In case of $n = 5$, we have

$$\begin{aligned}
a_{5,2} = & \frac{1}{4^5}\sum_{j=0}^{5}(-1)^j(2(5)-2j)!j!\begin{pmatrix}5\\j\end{pmatrix}^2 2^j \\
= & \frac{1}{1024}\left[3628800 - 2016000 + 576000 - 115200 + 19200 - 3840\right] \\
= & \frac{1}{1024}\left[2088960\right] \\
= & 2040.
\end{aligned}$$

Thus, we have $a_{5,0} = a_{5,5} = 1$; $a_{5,1} = a_{5,4} = 5! = 120$; and $a_{5,2} = a_{5,3} = 2040$.

Brualdi [4] also noted that Corollary 8.4 implies that the exponential-like generating function

$$g(t) = \sum_{n\geq 0} a_{n,2}\frac{t^n}{n!^2}$$

satisfies

$$g(t) = \frac{e^{-t/2}}{\sqrt{1-t}}.$$

Chapter 9

Conclusion

Our main aim in this thesis was to explore the cardinality of $(0,1)$-matrix classes with fixed row and column sums, that is $|\mathcal{A}(R,S)|$. Our first task was to give a proof of the Gale-Ryser Theorem, which describes a condition for $\mathcal{A}(R,S)$ to be non-empty based on majorization. We gave a proof using the concept of some transfers together with different ideas used in various proofs of the Gale-Ryser Theorem in the literature. We showed the link between the Gale-Ryser Theorem and the result shown by Ford and Fulkerson, which is based on the flows of network using the concept of a structure matrix. We also classified the case where there is only one unique matrix in $\mathcal{A}(R,S)$.

The concept of interchanges shows that the class $\mathcal{A}(R,S)$ can have have more than one matrix as its elements. We showed how the combinatorial interpretation of a structure matrix T of a $(0,1)$-matrix A precisely determines the non-invariant positions in A. This aided us in exploring not only the number of matrices in $\mathcal{A}(R,S)$ but also their structure.

Ideally we would like to establish a formula for $a_{n,k}$ with $(2 < k \leq \lfloor \frac{n}{2} \rfloor)$. However, this may not be possible; at least, without giving a function which overly intricate and difficult to compute. It is known that the number of $(0,1)$-matrices with m rows and n columns uniquely reconstructible from their row and column sums are the poly-Bernoulli numbers of negative index, $B_m^{(-n)}$; see [3]. There is some interest, as a continuation of this thesis, to explore the relationship between the poly-Bernoulli formula and $g_{m,n}(\mathbf{x};\mathbf{y})$ the generating function for the cardinality of $\mathcal{A}(R,S)$ in Chapter 8; since the former considers a specific prescribed R and S while the later deals with all possible combinations of R and S of order m and n respectively. Another possible area of continuation would be to write programs that would assist in the process of vector-majorization, or even a program that would take two vectors as R and S and determine if $\mathcal{A}(R,S)$ is non-empty which would be a good foundation in working towards a program for the enumeration of $(0,1)$-matrices.

Consider square matrices of even order. We conjecture the following:

Conjecture 9.1 *For a positive integer m, let R and S be non-negative vectors of order $2m$. Then $|\mathcal{A}(R,S)|$ is maximized when $r_i = m = s_j$ $(i,j = 1,2,...,2m)$.*

In other words, we conjecture that the number of matrices in $\mathcal{A}(R,S)$ is maximized when, for each matrix $A \in \mathcal{A}(R,S)$, half of the entries of each line of A are ones and the other half as zeros. How about the case where R and S are of order n where n is odd? Well, in surveying the cases of $n \in \{2,3,4,5\}$ and observing all matrices of specified sizes, we extend the above conjecture as follows:

Conjecture 9.2 *Let R and S be non-negative vectors of order n with n a positive integer. Then $|\mathcal{A}(R,S)|$ is maximized when $r_i = \left\lfloor \frac{n}{2} \right\rfloor = s_j$ $(i,j = 1,2,...,n)$.*

Enumerating all $n \times n$ matrices for $n \in \{2,3,4,5\}$ by hand, together with examples we have encountered in this thesis, the above conjecture is true when $n \in \{2,3,4,5\}$. The list of matrices becomes extremely enormous for large n, hence to explore these conjectures further, we need to establish formulas of enumeration that are not as cumbersome as the generating function of Chapter 8.

The main results that are new in this thesis are:

(1) A description of the structure of matrices in $\mathcal{A}(R,S)$ when $|\mathcal{A}(R,S)| = 2$ (Theorem 6.4),

and

(2) Showing that for each positive integer k, there exist row and column sum vectors R and S such that $|\mathcal{A}(R, S)| = k$
(Theorem 7.1).

We conclude with the following open question: *For a given k, how small can R and S be?*

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