Investigating Gravity and Electromagnetism on a 10 Dimensional Manifold with Local Symmetry $so(2,3)$

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Abstract

The Hawthorn model [1] is built upon the idea that the Lie algebra $so(2,3)$ is a more natural description of the local structure of space-time than the Poincare Lie algebra, with the former contracting to the latter in the limit of the contraction parameter r tending to infinity. This notion is explored in the context of a 10-dimensional space-time referred to as an *ADS manifold*. Here we build on the work of Crump [2] and try to incorporate field equations for gravity into the model. We derive two apparently different equations describing gravitational phenomena, demonstrate an intimate connection between gravity and electromagnetism and provide a first estimate as to the value of the contraction parameter r.

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Contents

Chapter 1

Introduction

This thesis seeks to incorporate a description of gravity into the Hawthorn model [1]. This involves seeking an appropriate form of Einstein's field equations such that they arise naturally in the context of a 10-dimensional ADS manifold. In the process we develop geometric derivations of the 10-dimensional Ampere-Gauss equation, the 10-dimensional Einstein field equations and also a new, seemingly independent, equation offering up a new constraint on gravitational phenomena. In deriving these equations we also demonstrate connection between electromagnetic and gravitational fields and we produce the models first experimental prediction.

1.1 History of the Hawthorn Model

The Hawthorn model originally arose in an attempt to answer the question as to whether the local symmetry of space-time is better described by the Anti-deSitter Lie algebra as opposed to the Poincare Lie algebra. The initial inspiration stemming from the observation that the Dirac equation is more conveniently described with this symmetry group. Initial work by Hawthorn ([1]) developed the mathematical formalism of the model emphasising its utility with regard to the Dirac equation, however issues were run into when attempting to describe electromagnetism. It was observed that along with Maxwell's equations came an extra constraint that implied only trivial EM phenomena

could exist on the manifold. It was this issue that Crump ([2]) set out to resolve for his Master's thesis. Crump was successful in resurrecting electromagnetism but this success came at the expense of the assumption regarding the invariance of the spinor bilinear form $s_{\alpha\beta}$ and necessitated the introduction of bullet scalars. However, once remedied, Crump not only demonstrated non-trivial electromagnetic phenomena was permissible but in fact that the Faraday-Gauss equation was a geometric identity of the manifold. The combination of Hawthorn and Crump's work is found in [1] and represents the most up to date version of the model and the starting point of this thesis.

1.2 Thesis Overview

Chapter 2 In this chapter we give outlines of electromagnetism and gravity including a discussion of attempts to unite the two forces followed by a brief outline of Weyl's attempt at geometric unification. This is followed by a summary of the Dirac equation including its derivation, simple solutions and interaction terms. Relevant references are [2], [6], [7], [8], [9], [11], [12], [13], [15], [16], [17], [19], [25], [27], [30], and [31].

Chapter 3 Here we construct a basis for what we call the *canonical representation* of the Lie algebra $so(2,3)$. We then go on to demonstrate how the Anti-deSitter (AdS) Lie algebra contracts to the Poincare Lie algebra in the limit of the contraction parameter r tending to infinity. The chapter finishes off with discussion of why the AdS Lie algebra is a better description of the local symmetry of space-time than the Poincare Lie algebra. Relevant references for this chapter are [1], [2], [3], [4], and [6].

Chapter 4 In this chapter we concern ourselves with developing the fundamental mathematical tools needed to make sense of the idea of local AdS symmetry. Relevant references are [1] and [2].

Chapter 5 This chapter follows the previous chapter as narrowing down of our focus and application of the results of the last chapter to specific low dimensional representations. Relevant references are [1], [2], and [4].

Chapter 6 Here investigate the properties and characteristics of the model with regard to the global action: ∇. We study the spinor connection in depth and introduce the notion that the fundamental forces manifest themselves as curvatures of the manifold. This chapter also generalises the differential operators of gradient, divergence and curl to 10 dimensions. Important results from this chapter include the geometric derivation of the Faraday-Gauss equation and the divergence free Einstein tensor plus a non-zero cosmological constant. Relevant references are [1] and[2].

Chapter 7 This chapter develops the Dirac equation in the context of the model. It demonstrates that the equation arises simply as the action of the generalised curl operator on a spinor. The chapter ends with discussion of the clarity the model affords us when dealing with the Dirac equation. Namely it allows us to interpret the charge as eigenvalues of the intrinsic time operator and the velocity fluctuation associated with the zitterbewegung to be fluctuation of intrinsic velocity. Relevant references are [1], [2] and [9].

Chapter 8 This chapter demonstrates how electromagnetism fits into the model. It is essentially a review of the work done in [2]. In it we demonstrate the initial problems associated with trying to accommodate electromagnetism and the subsequent resolution of these problems. This is followed by the demonstration of Crump's main result found in [2], namely the geometric derivation of the Faraday-Gauss equations as a necessary condition of the manifold. The relevant reference is [2].

Chapter 9 This chapter is concerned with the attempt to develop a theory of gravity on the manifold and represents the main contribution of

this thesis to the model. In it we attempt to develop equations linking gravity and electromagnetism from a variational approach. This is followed by a geometric proof of the subsequent field equations. Having done this, the chapter then goes on to try and link the equation relating to gravity to Einstein's field equation. However it is determined that equations relating to Einstein's field equations with a non-zero cosmological term already exist in the model. As a result we interpret our extra gravitational equation as an extra condition on gravity. It then goes on to develop implications of these equations including constraints on the unknown divergence-less tensors that arise in the equations. The final section of this chapter determines a lower bound on the contraction parameter r. Relevant references for this chapter are [6], [11] and [14].

Chapter 10 This chapter discusses the results of the previous chapter and lists future research avenues for the model including new questions raised by this thesis and long standing issues with the model.

Chapter 2

The Classical Forces

The goal of this model is to demonstrate that the physical laws are more conveniently described with an Anti-deSitter symmetry group. To do this it is necessary to formulate the laws of physics in this format. Thus, it is worthwhile to give a review of the physics that we wish to describe in the Hawthorn model as it is described in standard physics. In this chapter we will be looking at the classical forces: electromagnetism and gravity and review some attempts at unifying them. We will also give a brief summary of the Dirac equation.

2.1 Electromagnetism

Electromagnetism, as the name suggests, gives a unified account of the behaviour of electric and magnetic fields. Here we follow [2] and [8].

2.1.1 Maxwell's Equations

Let $\nabla = (\partial_x, \partial_y, \partial_z)$, then a complete description of electromagnetic phenomena in a vacuum is given by Maxwell's equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{2.1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.3}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t} \tag{2.4}$$

Where $\mathbf{E}$ is the electric field intensity, $\mathbf{B}$ is the magnetic field density, ρ is the electric charge density, ϵ_0 and μ_0 the permittivity and permeability of free space, $\mathbf{J}$ is current density and $\cdot$ and $\times$ are the standard dot and cross products. Equations 2.1, 2.3, and 2.4 are called Gauss', Faraday's, and Ampere's laws, respectively and equation 2.2 tells us that there are no magnetic monopoles. It can be shown that the current density $\mathbf{J}$ and the charge density ρ satisfy the continuity equation:

$$\partial_t(\rho) + \nabla \cdot \mathbf{J} = 0 \tag{2.5}$$

If we consider that $\mathbf{J} = \rho \mathbf{v}$, then we can rewrite the continuity equation as

$$\partial_i J^i = 0 \tag{2.6}$$

Where $\partial_i = (c^{-1}\partial_t, \nabla)$ and $J^i = (c\rho, \mathbf{J})$.

We can reformulate Maxwell's equations in a more elegant way if we consider the identifications:

$$\mathbf{E} = -\nabla\phi - \partial_t(\mathbf{A}) \tag{2.7}$$

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{2.8}$$

Where ϕ is called the electric scalar potential and $\mathbf{A}$ is called the magnetic vector potential.

As an aside we note that these potentials are not unique. Considering equations 2.7 and 2.8 it's not difficult to realise that letting $\mathbf{A} \rightarrow \mathbf{A} + \nabla\chi$ and $\phi \rightarrow \phi - \partial_t\chi$, where χ is some scalar field, results in the same $\mathbf{E}$ and $\mathbf{B}$ fields. A transformation of this sort is called a *gauge transformation.*

Going back to our ϕ and $\mathbf{A}$, we may define a 4-vector $A^i = (c^{-1}\phi, \mathbf{A})$, called a 4-potential. With this 4-potential we can form the electromagnetic field tensor F_{ij},

$$F_{ij} = \partial_i A_j - \partial_j A_i$$

Using the electromagnetic field tensor, Maxwell's equations reduce to two equations:

$$\partial^i F_{ij} = \mu_0 J_j \tag{2.9}$$

$$\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0 \tag{2.10}$$

Equation 2.9 is called the Ampere-Gauss equation and equation 2.10 is called the Faraday-Gauss equation.

2.2 Gravity

In this section we will go over the modern theory gravity, i.e. Einstein's General Relativity. We will discuss two derivations: Einstein's derivation and Hilbert's derivation. We will also briefly touch on gauge formulations of gravity.

2.2.1 Einstein's Equations

Here we give a brief construction of Einstein's equations in rough accordance with the path Einstein took towards them. We follow [11].

In his theory of gravitation, Einstein desired to conflate the presence of a gravitational field with the curvature of a Riemannian manifold. The mathematical manifestation of this idea being an equation relating the non-flatness of the metric g_{ij} and the source of the gravitational field, the energy-momentum tensor: τ_{ij} (with indices running from 0-3). As both the metric and the energy-momentum tensor are divergence-less, it is tempting to try to equate these two. This however raises problems when considering empty space and when attempting reduce down to Newton's theory. Hence, relating non-flatness in the metric with energy density requires going through the curvature tensor, see [12]. As the Riemannian curvature tensor, R^k_{itj}, is rank 4 and the energy-momentum tensor rank 2 we may either contract two of the curvature tensor's indices or make it proportional to a quadratic form of the energy-momentum tensor. Keeping in mind that Einstein's theory needs to collapse to Newton's theory in the limiting case it can be seen that the latter of these two possibilities presents difficulties. Thus Einstein made the identification

$$R_{ij} = \kappa \tau_{ij}$$

Where R_{ij} is the Ricci tensor and κ is a constant of proportionality given by $\kappa = -\frac{8\pi G}{c^2}$, where G is Newton's gravitational constant and c is the speed of light. This, however, is problematic as the Ricci tensor is not divergence-less and the energy-momentum tensor is. In order to make the left hand side divergence-less it must be modified by introducing the term $-\frac{1}{2}g_{ij}R$, where R is the curvature scalar. This combination of the Ricci tensor and the Ricci scalar is called Einstein's tensor. Taking this modification into consideration, Einstein's field equations are:

$$R_{ij} - \frac{1}{2}g_{ij}R = \kappa \tau_{ij} \tag{2.11}$$

2.2.2 The Hilbert Action

Now we turn to Hilbert's variational derivation. Here we follow related sections in [11] and [7].

The derivation of the field equations for gravity using a variational principle was first done by David Hilbert in 1915, [30]. In his formulation he put forward three axioms from which he expected the field equations to arise. These axioms and their justifications are given as follows:

Axiom 1. The field equations should be derived from a variational technique where the components of the metric tensor form the independent variables of the action integral

Justification: The first part is the underlying assumption of the approach. The second part however is specific to Hilbert's approach. The action need not be varied with respect to components of the metric (we could, if we wanted, use the connection components), for derivations that forgo this approach see [14].

Axiom 2. The action functional should be a scalar.

Justification: This follows from the fact that if we want the integral to be a tensorial quantity the integrand must be a scalar.

Axiom 3. The equations of motion must be differential equations of second order in g_{ij}

Justification: This arises from the fact that the Poisson equation

$$\nabla^2\Phi = 4\pi G\rho$$

where Φ is the field potential and ρ is the mass density, should result as limiting case of the equations. As the Poisson equation is second order, we expect the field equations of gravity to be second order as well.

From the three axioms, we can conclude three things:

1. The action integral should take the form

$$I(g) = \int_{\Omega_4} \mathcal{L} d^4x$$

Where Ω_4 is a volume element of space-time on the boundary of which $\delta g_{ij} = 0$, and the function $\mathcal{L}$ is dependent on g_{ij} and derivatives of g_{ij} ($g_{ij,k}$, $g_{ij,kl}$, *etc.*).

2. For the integral to be a scalar, as d^4x is a scalar density of weight 1 $\mathcal{L}$ must be a scalar density of weight -1. The simplest scalar density of weight -1 is $\sqrt{-g}$ thus $\mathcal{L} = \sqrt{-g}L$ where L is a proper scalar function.

3. As we want the equations of motion to be second order, and the E.L. equations give equations of motion that are of twice the order of the highest derivative appearing in $\mathcal{L}$, we would like $\mathcal{L}$ to be a function of g_{ij} and $g_{ij,k}$ only. It proves difficult to construct a non-trivial scalar just using g_{ij} and $g_{ij,k}$, however it can be noted that if $\mathcal{L}$ does contain higher order derivatives of the metric their contribution to the field equations may be ignored if they can be collected into a divergence term that vanishes at the boundary of the volume. We may only do this if $\mathcal{L}$ is linear in these higher order derivatives. So we are looking for a scalar that is linear in higher order derivative of g_{ij} and $g_{ij,k}$, and we may find that R-the curvature scalar fits the bill.

Therefore the Einstein-Hilbert action is:

$$I_{EH} = \int_{\Omega_4} R\sqrt{-g} d^4x$$

To produce the source free field equation we must vary the action with respect to the metric and set the whole thing to zero.

$$\delta I = \int_{\Omega_4} \delta(R\sqrt{-g}) d^4x = 0$$

Examining the variation of the integrand

$$\begin{aligned}\delta(R\sqrt{-g}) &= \delta(R)\sqrt{-g} + R\delta(\sqrt{-g}) \\ &= \delta(g^{ij}R_{ij})\sqrt{-g} + R\delta(\sqrt{-g}) \\ &= \delta(g^{ij})R_{ij}\sqrt{-g} + g^{ij}\delta(R_{ij})\sqrt{-g} + R\delta(\sqrt{-g}) \qquad (2.12)\end{aligned}$$

Considering Proposition 7.2 on pg 298 of [7]

a) $\delta g^{ij} = -g^{il}g^{kj}\delta g_{lk}$

b) $\delta\sqrt{|g|} = \frac{1}{2}\sqrt{|g|}g^{lk}\delta g_{lk}$

c) $\delta R_{ij} = \nabla_k \delta\Gamma^k_{ji} - \nabla_j \delta\Gamma^k_{ki}$

Substituting these into (1) we see

$$\begin{aligned}\delta(R\sqrt{-g}) &= \delta(g^{ij})R_{ij}\sqrt{-g} + g^{ij}\delta(R_{ij})\sqrt{-g} + R\delta(\sqrt{-g}) \\ &= -g^{il}g^{kj}\delta g_{lk}R_{ij}\sqrt{-g} + g^{ij}(\nabla_k\delta\Gamma^k_{ji} - \nabla_j\delta\Gamma^k_{ki})\sqrt{-g} + R\frac{1}{2}\sqrt{-g}g^{lk}\delta g_{lk} \\ &= (-g^{il}g^{kj}R_{ij} + \frac{1}{2}g^{lk}R)\sqrt{-g}\delta g_{lk} + g^{ij}(\nabla_k\delta\Gamma^k_{ji} - \nabla_j\delta\Gamma^k_{ki})\sqrt{-g} \\ &= -(R^{lk} - \frac{1}{2}g^{lk}R)\sqrt{-g}\delta g_{lk} + g^{ij}(\nabla_k\delta\Gamma^k_{ji} - \nabla_j\delta\Gamma^k_{ki})\sqrt{-g}\end{aligned}$$

The second term on the right is a total divergence hence it does not contribute to the variation. Substituting this back into the integral we get

$$\delta I = -\int_{\Omega_4}(R^{lk} - \frac{1}{2}g^{lk}R)\sqrt{-g}\delta g_{lk}d^4x = 0$$

Thus obtaining Einstein's source free field equation

$$G^{lk} = R^{lk} - \frac{1}{2}g^{lk}R = 0$$

In the presence of matter this equation is modified thusly:

$$G^{lk} = R^{lk} - \frac{1}{2}g^{lk}R = \kappa\tau^{lk}$$

Where τ^{lk} is the energy-momentum tensor and κ is the constant of proportionality from before. Thus, once again we have Einstein's equations.

2.2.3 A Note on the Cosmological Constant

The equations we have derived from both Einstein's and Hilbert's approaches are not the most general formulation. In full generality we must also add an extra term: $g^{lk}\Lambda$, where Λ is called the cosmological constant. This arises if we consider the fact that the addition of a constant, divergence-less tensor to Einstein's tensor does not modify the divergence equation. Hence there is a degree of freedom regarding the addition of a divergence-less constant to Einstein's equations. The effect of this constant is to govern the evolution of the universe (its value distinguishes between an expanding, shrinking or static universe). It should be noted though that its presence prevents Einstein's equations from reducing to Newtonian gravity in the weak field limit unless it is very small and in fact its absolute value is constrained to have an upper limit of no more than 10^{-50}cm^{-2} (p. 145, [11]).

2.2.4 Gravity as a Gauge Theory

Having considered both Einstein's and Hilbert's derivations of gravity, it is also worthwhile to take note of another formulation-that of a gauge formulation.

A gauge formulation is similar to some extent to Hilbert's derivation, in that it arises from the invariance of a Lagrangian. Here the Lagrangian is invariant with respect to a group of transformations that form a semi-simple Lie group. These transformations become gauge transformations if the elements of the Lie algebra have a coordinate dependence. The invariance of the Lagrangian under this type of transformation is a gauge invariance. What follows is a qualitative assessment closely related to the intro of [25].

After the successful gauge theoretic treatment of nuclear forces by Yang and

Mills ([15]) it was asked whether gravity would admit such a formulation. It turns out the answer to this question is yes: gravity may be formulated as a gauge theory of the Poincare group, the first formulations of which were done by Utiyama, Sciama and Kibble ([16], [17], and [19], respectively). One deviation from standard GR though, is that to account for spin interactions, torsion must be non-zero, [17]. Hence gauge theories of gravity require space-time be a Riemann-Cartan space-time as opposed to just a Riemann space-time. This however does not lead to any contradictions with what we observe (see [6]) and it has been suggested that a non-zero torsion may account for some of the phenomena we typically associate with dark matter, [26].

Since the initial attempts at a gauge theory of gravity, this approach has been an active field of research see for example [23], [24], [35] and need not be restricted to the Poincare group: [20], [21], [22].

2.3 Attempts at Unification

The goal of physics is ultimately to describe all the forces of nature as facets of one thing. This was done by Maxwell in 1865 for electricity and magnetism and later on for electromagnetism and the weak force by Abdus Salam, Sheldon Glashow and Steven Weinberg in 1968. Prior to the discovery of the nuclear forces however, there were only the classical forces: gravity and electromagnetism, and many attempts were (and still are) made to unite the two (see [27] for a list and description of many of these attempts including. Einstein's and Schroedinger's, and see [28] and [29] for some modern attempts).

The pursuit of a unified theory of gravity and electromagnetism has spawned many influential models, for example the Kaluza-Klein model (for references see [2][6][11]) and Weyl's model (put forward in [31], for rough method and historical context see [27]). The Kaluza-Klein model introduced the notion

of a curled up dimension which is famously exploited in String Theory and Weyl's model laid much of the groundwork for what would become Gauge Theory. What follows is a brief synopsis of Weyl's attempt at unifying gravity and electromagnetism geometrically.

2.3.1 Weyl's Model

The first prominent attempt (if not the first attempt) at the geometric unification of gravity and electromagnetism was done by the German mathematician, Hermann Weyl in 1918, [31]. Here we give a brief summary of his strategy, based off of the review found in [6].

In his model Weyl sought to make an analogy with the typical approach to electrodynamics in Riemannian space, described by the action:

$$I_V = \int b(-aR + \alpha G_{\mu\nu}G^{\mu\nu})d^4x$$

Here $b(=\sqrt{-g})$ is a scalar density, a and α are proportionality constants, and R and $G_{\mu\nu}$ are the Ricci scalar and Electromagnetic field tensor, respectively. Here the V subscript represents the Riemannian back drop.

Weyl modified this action so that the action did not vary in Riemann space, but in *Weyl space*. If we note that a Riemann space is a general affinely-connected space with the non-metricity condition and zero torsion:

$$\nabla^x g_{ij} = T^x_{ij} = 0$$

then a Weyl space is similar to a Riemann space except instead of the non-metricity condition we have the *semi-metricity* condition:

$$\nabla^k g_{ij} = \psi^k g_{ij}$$

Where ψ^k is a vector field. By letting ψ^k be the electromagnetic vector potential, he was able to geometrize the electromagnetic field.

Thus the action in Weyl space is given by:

$$I_W = \int d^4xb(-R^2 + \beta F_{\mu\nu}F^{\mu\nu})$$

Where $F_{\mu\nu} = \partial_\mu\psi_\nu - \partial_\nu\psi_\mu$.

Varying this equation we get:

$$\delta I_W = \int d^4x[-2bR\delta R - \delta bR^2 + \beta\delta(bF^2)] = 0$$

Now, introducing a scale of length such that $R = \lambda$ (and λ represents the cosmological constant) we can re-express the above equation as:

$$\delta I_W = \delta \int d^4xb(-R + \frac{\beta}{2\lambda}F^2 + \frac{\lambda}{2}) = 0$$

If we consider the semi-metricity condition, solving for the connection allows us to find a relationship between the Weyl curvature scalar and the Riemann curvature scalar:

$$R_W = R_V - \frac{3}{2}\psi_\mu\psi^\mu + 3\nabla_\mu\psi^\mu$$

When we substitute this expression back into the Weyl action the last term can be discarded as a surface term, thus the we get:

$$I_W = \int d^4xb[-R_V + \frac{1}{2}\beta\overline{F}^2 + \lambda(\frac{1}{2} + \frac{3}{2}\overline{\psi}_\mu\overline{\psi}^\mu)]$$

Where the bar represents a denominator of $\sqrt{\lambda}$. The first two terms give the Riemannian action I_V and the remaining terms give a correction due to the cosmological constant.

Therefore we can see that Weyl's theory unites gravity and electricity geometrically.

However, the theory is not without it's problems. While the theory is a good approximation to free electrodynamics, when interactions are introduced it breaks down. For instance, the theory is unable to distinguish between particles and antiparticles and thus predicts identical behaviour for electrons and positrons. As a result the theory was scrapped as a unification for gravity and electromagnetism, however this approach was greatly influential in both the search for a unified field theory and (as previously mentioned) in the development of gauge theories.

2.4 The Dirac Equation

In this section we will develop the Dirac equation as it was derived by Dirac in 1928. This derivation can be found in all good texts on Relativistic Quantum Mechanics, here we specifically follow [9] a brief but mathematically thorough dealing may also be found in [13].

2.4.1 The Equation

The Dirac equation originally arose from the desire for a relativistically covariant equation that satisfied the time dependent Schroedinger equation with positive definite probability density. In this pursuit Dirac required a Hamiltonian operator with the properties

$$i\hbar\partial_t\psi = \hat{H}\psi \tag{2.13}$$

$$\hat{H}\hat{H} = \hat{p}^2c^2 + m^2c^4 \tag{2.14}$$

Noting that letting $\hat{H} = \sqrt{\hat{p}^2c^2 + m^2c^4}$ raises more problems than it solves, Dirac required the $\hat{H}$ to take the form

$$\hat{H} = \alpha^i \hat{p}_i c + \beta mc^2 \tag{2.15}$$

In order for 2.15 to satisfy 2.14 Dirac determined that α and β must be matrices satisfying the following conditions

i. $\alpha_i\alpha_k + \alpha_k\alpha_i = 2\delta_{ik}$

ii. $\alpha_i\beta + \beta\alpha_i = 0$

iii. $\alpha_i^2 = \beta^2 = 1$

Thus finding the desired equation is reduced to finding the smallest dimension matrices that satisfy the above conditions. Observe that as $\hat{H}$ must be hermitian the matrices α_i and β must also be hermitian. Also we note that from (iii.) we require α_i and β to have eigenvalues of ± 1 and from (ii.) that their traces are each zero. As the trace is the sum of the eigenvalues, α_i and β must be even dimensional. It turns out that the smallest dimension of the matrices necessary to satisfy these conditions is 4, therefore we may find an explicit form for α_i and β:

$$\alpha_i = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix} \tag{2.16}$$

$$\beta = \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix} \tag{2.17}$$

Where σ_i are the 2×2 Pauli spin matrices and I_2 are 2×2 identity matrices.

Thus the Dirac equation is

$$(\gamma^i \hat{p}_i - mc)\psi = 0, \ i = 0, ..., 3 \tag{2.18}$$

Where the $\gamma^0 = \beta$ and $\gamma^a = \beta\alpha_a$ $(a = 1, 2, 3)$.

2.4.2 The Solutions and Electromagnetic Interaction

2.4.2.1 Solutions

Here, briefly, we will state the solutions of the Dirac equation for a free electron at rest. We also endeavour to provide an interpretation to these solutions, however for brevity these interpretations will merely be stated further justification may be found in chapters 3 and 5 of [9].

The Dirac equation for a free electron at rest is

$$i\hbar\gamma^0\partial_t\psi = mc^2\psi \tag{2.19}$$

Using 2.17 for our representation of β we find four solutions

$$\psi^1 = e^{-(imc^2/\hbar)t}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad \psi^2 = e^{-(imc^2/\hbar)t}\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$$

$$\psi^3 = e^{(imc^2/\hbar)t}\begin{bmatrix}0\\0\\1\\0\end{bmatrix} \quad \psi^4 = e^{(imc^2/\hbar)t}\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

Thus the solutions are four component bispinors with positive energy eigenvalues for the first two and negative energy eigenvalues for the second two. The first two solutions are easily identified as electrons, the negative energy solutions are not so readily identifiable. The resolution to the problem posed by these negative energy solutions can be found in the notion of antiparticles. Hence if we identify the negative energy solutions as positively charged elec-

trons or, as they are properly known: positrons, then we may resolve the issue of negative energy.

2.4.2.2 Electromagnetic Interaction

Electromagnetic interaction may be introduced into the Dirac equation by means of the minimal substitution $p_i \rightarrow p_i - \frac{e}{c}A_i$, where A_i is a four potential with components (Φ, A_a). Thus the Dirac equation for a particle interacting with an electromagnetic field is given by

$$i\hbar\partial_t\psi = \left(c\alpha^a(p_a - \frac{e}{c}A_a) + \beta mc^2 + e\Phi\right)\psi \tag{2.20}$$

Chapter 3

The Lie Algebra so(2,3)

Physics may be studied by considering the symmetries of space-time. Pre-relativity physics operated under the assumption that the symmetry group of space-time was the Galilean group. However, with the advent of special relativity this assumption had to be reconsidered and the group was extended to the Poincare group: $ISO(1,3)$, which collapses to the Galilean group in the limit of the speed of light approaching infinity. It is of interest to note that the Poincare group may also be viewed as the limiting case of de-Sitter groups: $SO(1,4)$, $SO(2,3)$, [5]. Here we wish to examine the case when the Poincare group is the limiting case of $SO(2,3)$, referred to as the Anti-deSitter group. What follows is closely related to the relevant sections in [1] and [2].

3.1 The Lie Algebra so(2,3)

Our primary hypothesis is that the local symmetry group of space-time is $SO(2,3)$. To explore this notion let us consider a bilinear form

$$(\ , \ ) : \mathbb{R}^5 \to \mathbb{R} \tag{3.1}$$

operating on vectors $x, y \in \mathbb{R}^5$, explicitly:

$$(x, y) = x_0 y_0 + x_1 y_1 - x_2 y_2 - x_3 y_3 - x_4 y_4 \tag{3.2}$$

We may define elements of $SO(2,3)$ as 5×5 real matrices which preserve this bilinear form. Considering the matrix

$$F = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{3.3}$$

we may rewrite (x, y) as

$$(x, y) = x^T F y \tag{3.4}$$

Thus, for a matrix G to be an element of $SO(2,3)$ it must satisfy the condition

$$\begin{aligned} (Gx, Gy) &= (x, y) \\ x^T G^T F G y &= x^T F y \end{aligned}$$

As $x, y \in \mathbb{R}^5$ are arbitrary, this implies

$$G^T F G = F \tag{3.5}$$

Therefore if we wish to establish that a matrix is in the group $SO(2,3)$ all we must do is show that it satisfies this relationship. However, as we are dealing with a matrix Lie group (p5, [3]) we may also study the Lie algebra of the group, denoted $so(2,3)$ (this will play a primary role in what's to come as it is the Lie algebra that describes local symmetry on the manifold). Define the matrix exponential as:

$$e^{\theta X} = \sum_{m=0}^{\infty} \frac{(\theta X)^m}{m!} \tag{3.6}$$

Where θ some real number, we say that X is an element of the Lie algebra if the matrix exponential is in the group for all values of θ. From this we may recover the general form of elements of the Lie algebra. As 3.6 is true for all

real values of θ, it is true for small values and we may therefore consider the group action about a point: $G = I + \theta X$. Substituting this expression for G back into equation 3.5 and keeping only first order terms we get:

$$\begin{aligned} G^T F G &= (I + \theta X)^T F (I + \theta X) \\ &= F + \theta(X^T F + F X) \end{aligned} \tag{3.7}$$

This must be equal to F, thus the second term on the right must be zero and therefore:

$$X^T F = -FX \tag{3.8}$$

Exploring the guts of this relationship, we let

$$F = \begin{pmatrix} I_2 & 0 \\ 0 & -I_3 \end{pmatrix} \text{ and } X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{3.9}$$

Where I_n is an $n \times n$ identity matrix and block elements in X have the same dimensions as their corresponding block elements in F. Thus equation 3.8 may be rewritten as

$$\begin{aligned} \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix} \begin{pmatrix} I_2 & 0 \\ 0 & -I_3 \end{pmatrix} &= -\begin{pmatrix} I_2 & 0 \\ 0 & -I_3 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \\ \begin{pmatrix} A^T & -C^T \\ B^T & -D^T \end{pmatrix} &= \begin{pmatrix} -A & -B \\ C & D \end{pmatrix} \end{aligned} \tag{3.10}$$

From this we may infer

$$A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$$

$$B = C^T = \begin{pmatrix} b & c & d \\ e & f & g \end{pmatrix}$$

and

$$D = \begin{pmatrix} 0 & h & i \\ -h & 0 & j \\ -i & -j & 0 \end{pmatrix}$$

and thus a general element of $so(2,3)$ has the form:

$$X = \begin{pmatrix} 0 & a & b & c & d \\ -a & 0 & e & f & g \\ b & e & 0 & h & i \\ c & d & -h & 0 & j \\ d & g & -i & -j & 0 \end{pmatrix}$$

As can be seen this matrix consists of 10 independent components. Thus all elements of $so(2,3)$ may generated from a 10 basis of matrices given in Figure 3.1.

$$T = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$X = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$I = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad J = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad K = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Figure 3.1 Canonical representation of $so(2,3)$

This will be referred to henceforth as the natural representation. Considering these matrices and equation 3.6 we can therefore recover our group elements.

We note here that, the natural representation is not the only representation

of $so(2,3)$, or indeed the only representation to feature in this model. Naturally, we can construct another representation by considering the structure coefficients associated with the commutation relations, we can use these to define the elements of the adjoint representation. It can be demonstrated, also, that the natural representation is isomorphic to the canonical representation of $sp(4,\mathbb{R})$. These are the matrices that preserve a fixed antisymmetric bilinear form on $\mathbb{R}^4$. This representation, it will be demonstrated, is responsible for spinors in the model. These representations along with one other and the general theory of representations are examined in Appendix A.

3.2 Anti-deSitter Space-time

As we posit that the local symmetry of space-time is described by the symmetry group $SO(2,3)$ it is worth investigating how we may recover the Poincare regime.

It is in this way that we are led to consider the 4D invariant hypersphere H_4 embedded in $\mathbb{R}^5$. Considering the coordinates $\lambda, t, x, y, z \in \mathbb{R}^5$ and the quadratic form: $\lambda^2 + t^2 - x^2 - y^2 - z^2$, H_4 is the invariant sub-manifold associated with:

$$\lambda^2 + t^2 - x^2 - y^2 - z^2 = a^2 \tag{3.11}$$

Where a is the radius of our hypersphere and in keeping with the literature ([1], [2], [6]) will be called the radius of the universe. Now consider an invariant interval on the manifold:

$$\begin{aligned} ds^2 &= d\lambda^2 + dt^2 - dx^2 - dy^2 - dz^2 \\ &= d\lambda^2 + \eta_{ij} x^i x^j \end{aligned} \tag{3.12}$$

Here η_{ij} is the Minkowski metric and the indices run over the space-time coordinates. We can rearrange equation 3.11 to get an expression for λ in

terms of the other coordinates:

$$\lambda = \sqrt{a^2 - \eta_{ij}x^i x^j} \tag{3.13}$$

As a result it is possible to find an expression for the element $d\lambda$ in terms of the other coordinates, doing this we find:

$$d\lambda = -\eta_{ij}x^i dx^j/\lambda \tag{3.14}$$

Using this expression for the element $d\lambda$ and the previous expression for λ in the invariant interval we get:

$$ds^2 = \eta_{ij}dx^i dx^j + \frac{(\eta_{ij}x^i dx^j)^2}{a^2 - \eta_{ab}x^a x^b} \tag{3.15}$$

If we consider this interval in the neighbourhood of $\lambda = a$, then this interval takes on the form:

$$ds^2 = g_{ij}dx^i dx^j \tag{3.16}$$

Where g_{ij} is the metric given by the expression:

$$g_{ij} = \eta_{ij} + \frac{x^i x^j}{a^2} \tag{3.17}$$

Therefore we can see that an invariant interval on the hypersphere H_4 is given by equation 3.16, which if we let $a \to \infty$ *looks* like an interval in flat Minkowski space. Note that H_4 is endowed with a natural unit of distance: a, which combined with the natural unit of velocity: c, gives a natural unit of time which will be denoted: r, and in keeping with Hawthorn and Crump ([1], [2]), will be called the *radius of the universe in seconds*. If considered in natural units these are all set to unity ($a = c = r = 1$).

Now let's consider how the elements of $SO(2,3)$ act on this space. Considering once again the neighbourhood of the point $\lambda = a$ we let group element $e^{\theta T}$ act

on coordinate vector $v = (a, t, x, y, z)^T$, where θ, t, x, y, z are assumed small. Explicitly, this is:

$$\begin{aligned} e^{\theta T} v &= (I + \theta T) v \\ &= \begin{pmatrix} 1 & -\theta & 0 & 0 & 0 \\ \theta & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ t \\ x \\ y \\ z \end{pmatrix} \\ &= \begin{pmatrix} a - \theta t \\ t + \theta \\ x \\ y \\ z \end{pmatrix} \end{aligned} \tag{3.18}$$

As both θ and t are small this just becomes $(a, t + \theta, x, y, z)^T$. Therefore we can see that T is the transformation that generates translations in time and repeating this procedure for the other nine matrices we find that X, Y, and Z are translation operators in x, y, and z coordinates; A, B, and C represent Lorentz boost operators and I, J, and K are rotation operators.

Using natural units ($a = c = r = 1$) the Lie algebra basis elements in *ordinary units* are $\{\frac{1}{r}T, \frac{1}{rc}X, \frac{1}{rc}Y, \frac{1}{rc}Z, \frac{1}{c}A, \frac{1}{c}B, \frac{1}{c}C, I, J, K\}$ (thus a translation of one ordinary time unit is equal to a translation of $\frac{1}{r}$ natural time units). Defined as it is acting on H_4, the group $SO(2,3)$, is called the Anti-deSitter group. A table of commutation relations for the elements of $so(2,3)$ is given in Figure 3.2. Considering this table in ordinary units, we see the translation generators (T, X, Y, and Z) all have factors of $\frac{1}{r}$. Thus we can see that their commutators should have factors $\frac{1}{r^2}$, therefore in the limit of $r \to \infty$ space-time translations commute and the table reduces to that of the Poincare

algebra and the AdS group (in the language of [5]) is said to *contract* to the Poincare group, and r is termed the *contraction parameter*.

Thus we can see that in the limit of $r \to \infty$, the AdS group and the Poincare group are indiscernible. However what we are particularly interested in is the case where r is large but not, for all practical purposes, infinitely large. In this regime we would expect phenomena that deviate from the predicted Poincare model and one of the purposes of this thesis is to predict how these deviations may manifest themselves in the physical laws. Naturally, the ability to determine a value for r is a fundamental concern for the model as if it is too big then it will have no measurable effect in the universe, making the model redundant and if it's too small then its effects would be too large making the model just plain wrong. This condition on r will be called the **Goldilocks condition** [39]. This concern will be addressed later on when we explore gravity.

	T	X	Y	Z	A	B	C	I	J	K
T	0	A	B	C	-X	-Y	-Z	0	0	0
X	-A	0	-K	J	-T	0	0	0	Z	-Y
Y	-B	K	0	-I	0	-T	0	-Z	0	X
Z	-C	-J	I	0	0	0	-T	Y	-X	0
A	X	T	0	0	0	-K	J	0	C	-B
B	Y	0	T	0	K	0	-I	-C	0	A
C	Z	0	0	T	-J	I	0	B	-A	0
I	0	0	Z	-Y	0	C	-B	0	K	-J
J	0	-Z	0	X	-C	0	A	-K	0	I
K	0	Y	-X	0	B	-A	0	J	-I	0

Figure 3.2 Commutation relations for $so(2,3)$.

3.3 Advantages of $so(2,3)$

Thus far the majority this work has discussed the properties of the Lie group $SO(2,3)$ and its associated Lie algebra $so(2,3)$ with no real explanation of why we may want to do this. In this section we will cover some of the advantages of assuming that the local symmetry of space-time is described by $SO(2,3)$ and not $ISO(1,3)$.

The idea of using $SO(2,3)$ to describe symmetries in physics is not a new one. It has long been known that the Poincare group is a limiting case of the de-Sitter groups [5] and numerous authors have published physical models based around de Sitter/Anti-deSitter symmetries. In particular it is common to see the dS/AdS groups mentioned in papers on gauge theories of gravity([20], [21], [22], [33], [34]) and String Theory/CFT, for example the space $AdS_m \times S^n$ is currently popular amongst string theorists (m and n are usually various combinations of 5, 4 and 3)([36], [37]).

It's demonstrated in section 3.2 that if a is very large the AdS metric approaches the Minkowski metric. Noting that $a = rc$, this is equivalent to the same condition on r. Therefore locally for large r the two are indistinguishable, so in the very least no contradiction arises from the assumption that locally the universe is described by $SO(2,3)$ and not $ISO(1,3)$. It should be noted here that this is not the case if we assume a *global* symmetry group of $SO(2,3)$. In this model causality violating time-like loops arise and this is in direct conflict with observation. However we do not take the stance that $SO(2,3)$ represents the global symmetry group of the universe we only require that space-time has local $SO(2,3)$ symmetry (i.e. the symmetries of space-time are properly described by the Lie algebra $so(2,3)$), so we may avoid the problems inherent in a typical AdS cosmological model. The notion of describing local symmetry will be more thoroughly developed in the next chapter.

Assuming we restrict our investigation to local symmetry, why do we not just employ Occam's Razor and stick with the Poincare group? For a start we may consider the closing remarks of [18] in which the author reviews the attempt at a unifying gravity and electromagnetism found in the Einstein-Schroedinger model: "...if unification is desired as well as geometrization, a new group will be required.". In an attempt to show that this "new group" should be $SO(2,3)$ we are led to consider the action of both of these groups on quantum mechanical wave functions.

Experiments in the first half of the last century demonstrated that the rotation operator in Quantum Mechanics has two actions: an *extrinsic* action which has eigenvalues of angular momentum and an *intrinsic* action with eigenvalues of spin. This second action was not predicted and came as a surprise. What makes this equally odd is the fact that this action is not extended to the rest of the group. This may be seen if we consider T and I, these commute and thus should be simultaneously observable. However, under the Poincare group T is non-compact and therefore has a continuous spectrum and we do not observe any continuous intrinsic quantities that we may associate with eigenvalues of an intrinsic T operator. Thus we conclude T must not act intrinsically. If we consider this same problem from the perspective of $so(2,3)$ we do not hit the same snag.

In the four dimensional representation of $so(2,3)$ (see Appendix A) the operator T is compact and therefore has discrete eigenvalues $\pm\frac{1}{2}$. Again, T and I commute thus we may hypothesise that they represent simultaneously observable intrinsic quantities. The question now posed is: what might the eigenvalues of T represent? If we look to the solutions of the Dirac equation we recognise that solutions are characterised by discrete spin and charge. This is highly suggestive of a link between intrinsic energy eigenvalues and charge. Thus, we are led to the interpretation that intrinsic energy is charge.

Therefore we may see that by taking $so(2,3)$ to describe local symmetry we may resolve the problem of extrinsic/intrinsic action.

On top of this, the Dirac operator arises very naturally when considering differential operators on the manifold (p.66, [1] and it may be shown that the Faraday-Gauss equation arises as a geometric property of a manifold with local $so(2,3)$ symmetry [2]. It will also be shown in this work that using geometric properties only we may also produce the Ampere-Gauss equation and Einstein's equation. Thus we put forward our fundamental assumption:

Assumption. The local symmetry of the universe is described by the group $SO(2,3)$.

Chapter 4

The Hawthorn Model

4.1 Mathematical Tool Kit

Here we wish to investigate the notion that $so(2,3)$ describes the local symmetry of space-time. To do this we need to forge some mathematical tools by which we may make sense of this proposition. The following is closely related to relevant sections in [1] and [2].

4.1.1 Tensor Derivations and the Covariant Derivative

We start off with a few definitions and propositions, the proofs of which are relegated to Appendix B.

Definition 4.1 *Consider a mapping D on a manifold $\mathcal{M}$, that takes tensors onto tensors:*

$$D : tensors \rightarrow tensors$$

The mapping D is called a ***tensor derivation*** *if it satisfies the following properties:*

- *Linearity*
- *Leibnitz condition on tensor products*
- *Commutes with contraction*

Proposition 4.1 *The following holds for all tensor derivations:*

i) If D and E are tensor derivations then so is [D,E].

ii) Every tensor derivation has a rank $\binom{i}{j}$ *and maps tensors of rank* $\binom{k}{l}$ *to tensors of rank* $\binom{k+i}{l+j}$.

iii) If D is a tensor derivation and S any tensor, then $S \otimes D$ *is a tensor derivation where* $(S \otimes D)(T) = S \otimes D(T)$

We define an **ordinary derivation** as that which maps scalar functions to scalars, thus we may think of it as a tensor derivation of rank $\binom{0}{0}$ that acts on components. We denote this derivation as $a^i \frac{\partial}{\partial x^i}$.

We can view every tensor derivation of rank $\binom{0}{0}$ acting on functions as an ordinary derivation. Thus we can establish an equivalence between rank $\binom{0}{0}$ tensor derivations and tangent vector fields written with respect to some coordinate system: $D(f) = a^i \frac{\partial}{\partial x^i}(f)$. In order to satisfy this identification all we must do is show that the two agree in their action on functions. Thus, by linearity, $D - a^i \frac{\partial}{\partial x^i}$ is also a tensor derivation of rank $\binom{0}{0}$ which maps all functions onto the zero function.

Proposition 4.2 *If E is a tensor derivation of rank* $\binom{0}{0}$ *with* $E(f) = 0$ *for all functions* f *on* $\mathcal{M}$*, then there exists a tensor* Γ^i_j *of rank* $\binom{1}{1}$ *so that*

$$E(X^{\alpha_1\alpha_2\ldots\alpha_m}_{\beta_1\beta_2\ldots\beta_n}) = \sum_s \Gamma^{\alpha_s}_{\hat{\alpha}_s} X^{\alpha_1\ldots\hat{\alpha}_s\ldots\alpha_m}_{\beta_1\beta_2\ldots\beta_n} - \sum_t \Gamma^{\hat{\beta}_t}_{\beta_t} X^{\alpha_1\alpha_2\ldots\alpha_m}_{\beta_1\ldots\hat{\beta}_t\ldots\beta_n}$$

Alternatively we may define a tensor derivation of rank $\binom{1}{1}$ that acts on $X^{\alpha_1\alpha_2\ldots\alpha_m}_{\beta_1\beta_2\ldots\beta_n}$ in the same way as E.

$$\Gamma\tbinom{*}{*}(X^{\alpha_1\alpha_2\ldots\alpha_m}_{\beta_1\beta_2\ldots\beta_n}) = \sum_s \Gamma^{\alpha_s}_{\hat{\alpha}_s} X^{\alpha_1\ldots\hat{\alpha}_s\ldots\alpha_m}_{\beta_1\beta_2\ldots\beta_n} - \sum_t \Gamma^{\hat{\beta}_t}_{\beta_t} X^{\alpha_1\alpha_2\ldots\alpha_m}_{\beta_1\ldots\hat{\beta}_t\ldots\beta_n}$$

Thus if we define E to be a tensor derivation of rank $\binom{0}{0}$ that maps functions onto zero we see:

$$E = D - a^i \frac{\partial}{\partial x^i} = \Gamma \binom{*}{*}$$

Thus all rank $\binom{0}{0}$ tensor derivations may be written in the form:

$$D = a^i \frac{\partial}{\partial x^i} + \Gamma \binom{*}{*}$$

If we consider now tensor derivations of rank $\binom{m}{n}$, $D^{\lambda_1 \ldots \lambda_m}_{\mu_1 \ldots \mu_n}$, and consider them contracting with tensor components, we find that by fixing indices λ_i and μ_j each of the operators obtained is a derivation of rank $\binom{0}{0}$. Now consider the following:

Proposition 4.3 *Every tensor derivation of rank* $\binom{m}{n}$ *takes the form:*

$$D^{\lambda_1 \ldots \lambda_m}_{\mu_1 \ldots \mu_n} = (a^{\lambda_1 \ldots \lambda_m}_{\mu_1 \ldots \mu_n})^i \frac{\partial}{\partial x^i} + \Gamma^{\lambda_1 \ldots \lambda_m}_{\mu_1 \ldots \mu_n} \binom{*}{*}$$

where

$$\Gamma^{\lambda_1 \ldots \lambda_m}_{\mu_1 \ldots \mu_n} \binom{*}{*} (T^{\alpha_1 \alpha_2 \ldots \alpha_m}_{\beta_1 \beta_2 \ldots \beta_n}) = \sum_s (\Gamma^{\lambda_1 \ldots \lambda_m}_{\mu_1 \ldots \mu_n})^{\alpha_s}_{\hat{\alpha}_s} X^{\alpha_1 \ldots \hat{\alpha}_s \ldots \alpha_m}_{\beta_1 \beta_2 \ldots \beta_n} - \sum_t (\Gamma^{\lambda_1 \ldots \lambda_m}_{\mu_1 \ldots \mu_n})^{\hat{\beta}_t}_{\beta_t} T^{\alpha_1 \alpha_2 \ldots \alpha_m}_{\beta_1 \ldots \hat{\beta}_t \ldots \beta_n}$$

Of particular interest are the derivations of rank $\binom{0}{1}$ where $a^t_i = 1^t_i$. Explicitly:

$$D_i = \frac{\partial}{\partial x^i} + \Gamma_i \binom{*}{*}$$

Derivations of this form are called **covariant derivatives**, denoted ∇_i.

4.1.2 Torsion, Curvature and Bianchi Identities

Consider a manifold $\mathcal{M}$ and the commutator of two covariant derivatives as defined in the previous section:

$$[\nabla_i, \nabla_j]$$

As covariant derivatives are tensor derivations, their commutator bracket is also a tensor derivation of rank $\binom{0}{2}$ and therefore by proposition 4.3 takes on the form:

$$[\nabla_i, \nabla_j] = T^k_{ij}\frac{\partial}{\partial x_k} + K_{ij}\binom{*}{*} \tag{4.1}$$

Applying this to a scalar function f and recalling that tensor derivations act on scalar functions as ordinary derivations we observe

$$\begin{aligned}
[\nabla_i, \nabla_j]f &= T^k_{ij}\frac{\partial f}{\partial x_k} + K_{ij}\binom{*}{*}f \\
\left(\frac{\partial}{\partial x_i} + \Gamma_i\binom{*}{*}\right)\frac{\partial f}{\partial x_j} - \left(\frac{\partial}{\partial x_j} + \Gamma_j\binom{*}{*}\right)\frac{\partial f}{\partial x_i} &= T^k_{ij}\frac{\partial f}{\partial x_k} \\
\frac{\partial^2 f}{\partial x_i \partial x_j} - \Gamma^k_{ij}\frac{\partial f}{\partial x_k} - \frac{\partial^2 f}{\partial x_j \partial x_i} + \Gamma^k_{ji}\frac{\partial f}{\partial x_k} &= T^k_{ij}\frac{\partial f}{\partial x_k} \\
-\Gamma^k_{ij}\frac{\partial f}{\partial x_k} + \Gamma^k_{ji}\frac{\partial f}{\partial x_k} &= T^k_{ij}\frac{\partial f}{\partial x_k}
\end{aligned} \tag{4.2}$$

We thus make the identification

$$T^k_{ij} = -(\Gamma^k_{ij} - \Gamma^k_{ji}) \tag{4.3}$$

It can be noted that the tensor on the right is the negative of the usual definition of the torsion, thus the definition of torsion in the Hawthorn model is the negative of the usual definition. As it arises naturally this way and doesn't cause any problems later on we maintain 4.3 as our definition of the torsion:

Definition 4.2 *The torsion, T^k_{ij}, is defined as*

$$T^k_{ij} = -(\Gamma^k_{ij} - \Gamma^k_{ji})$$

The structure of $K_{ij}\binom{*}{*}$ may be determined by applying the commutator to a vector field and ignoring the terms containing partials of the vector field. If

we do this we get for $K_{ij}\binom{*}{*}$ the expression

$$K^y_{ijx} = [\partial_i \Gamma^k_{jx} - \partial_j \Gamma^k_{ix}] + [\Gamma^k_{it}\Gamma^t_{jx} - \Gamma^k_{jt}\Gamma^t_{ix}] + T^t_{ij}\Gamma^k_{tx} \tag{4.4}$$

Therefore we see that $K_{ij}\binom{*}{*}$ is essentially the standard Riemann tensor with an extra torsion term attached to it. Letting the commutator act on a vector field and associating the extra torsion term from $K_{ij}\binom{*}{*}$ with the partial derivative results in

$$\begin{aligned}
[\nabla_i, \nabla_j]v_x &= T^k_{ij}\partial_k v_x - [\partial_i \Gamma^k_{jx} - \partial_j \Gamma^k_{ix}]v_k - [\Gamma^k_{it}\Gamma^t_{jx} - \Gamma^k_{jt}\Gamma^t_{ix}]v_k - T^t_{ij}\Gamma^k_{tx}v_k \\
&= T^k_{ij}[\partial_k v_x - \Gamma^k_{tx}v_k] - R^k_{ijx}v_k \\
&= T^k_{ij}\nabla_k v_x - R^k_{ijx}v_k
\end{aligned} \tag{4.5}$$

Thus 4.5 gives us a much more useful expression for the commutator of two covariant derivatives:

$$[\nabla_i, \nabla_j] = T^k_{ij}\nabla_k + R_{ij}\binom{*}{*} \tag{4.6}$$

As the covariant derivative lies in the Lie algebra of derivations we note that the commutator of covariant derivatives obeys the Jacobi identity

$$[[\nabla_i, \nabla_j], \nabla_k] + [[\nabla_j, \nabla_k], \nabla_i] + [[\nabla_k, \nabla_i], \nabla_j] = 0 \tag{4.7}$$

If we let 4.7 act on a vector field v^x and use 4.6 for the commutators we can separate the terms involving covariant derivatives of v^x and terms without covariant derivatives. Thus we get a statement of the form

$$\big(T^s_{ij}T^t_{sk} - \nabla_k T^t_{ij} - R^t_{ijk}\big)\nabla_t(v^x) + \big(T^s_{ij}R^x_{skt} - \nabla_k R^x_{ijt}\big)v^t \overset{ijk}{=} 0 \tag{4.8}$$

Where the notation $\overset{ijk}{=}$ represents a cyclic permutation over those indices. As ∇v and v are linearly independent and in general non-zero, their coefficients

must be zero. Thus we get two identities:

$$R^t_{ijk} + T^t_{kx}T^x_{ij} + \nabla_k(T^t_{ij}) \overset{ijk}{=} 0 \tag{4.9}$$

and

$$\nabla_k(R^t_{ijs}) + T^x_{ij}R^t_{kxs} \overset{ijk}{=} 0 \tag{4.10}$$

Which are identifiable as the first and second Bianchi identities, respectively.

4.2 The Fundamental Conjecture

If we consider the manifold $\mathcal{M}$ to be a Lie group then the action of the Lie algebra on the Lie group naturally defines the covariant derivative. In that respect we can view the torsion as the structure coefficients arising from the commutator of elements in the Lie algebra and as such they obey the Jacobi bracket condition:

$$T^y_{ix}T^x_{jk} \overset{ijk}{=} 0 \tag{4.11}$$

And as the Lie structure should be the same everywhere, we observe global invariance on the torsion as well:

$$\nabla_x T^k_{ij} = 0 \tag{4.12}$$

These two conditions provide the defining conditions for what will be known from now on as **Local Lie Manifolds**.

Definition 4.3 *A Local Lie Manifold is a manifold $\mathcal{M}$ together with a covariant derivative ∇_k where*

1. $T^t_{ix}T^x_{jk} \overset{ijk}{=} 0$

2. $\nabla_k(T^t_{ij}) = 0$

The first part implies that the torsion defines a Lie structure and the second part tells us that this Lie structure is invariant.

Thus we are interested in manifolds that are locally similar to the group $SO(2,3)$ without necessarily having the global structure. We thus come to the fundamental conjecture of the model

Definition 4.4 *An **ADS manifold** is a local Lie manifold of $so(2,3)$*

Fundamental Conjecture. *Our universe is an ADS manifold.*

It is the purpose of this thesis to explore the consequences of this statement.

4.2.1 The Low Hanging Fruit

Given definition 4.4 and our fundamental conjecture it's worth investigating some of the implications, briefly (the main body of which is left to subsequent chapters).

Our first result for an ADS manifold may be obtained if we consider definition 4.3 applied to the first Bianchi, we see that the terms with torsion drop out and we get:

$$R^t_{ijk} \overset{ijk}{=} 0 \tag{4.13}$$

Another consequence that can be observed is that as we now have the torsion describing the Lie structure on each tangent space, we can also define the

Killing form of the Lie algebra

$$k_{ij} = T^y_{ix} T^x_{jy} \tag{4.14}$$

As $so(2,3)$ is semi-simple k_{ij} is non-degenerate and defines an invariant pseudo-metric on the manifold. As a result we can observe that an ADS manifold is endowed with a natural distance scale which the Poincare group lacks. It can be observed also that this pseudo-metric matches the Minkowski metric (up to a factor of a scalar) in the space-time dimensions:

$$\begin{aligned} k_{ij} &\propto \text{diag}(-1,1,1,1,-1,-1,-1,1,1,1) \\ &\propto \text{diag}(\eta_{ab}, -I_3, I_3) \end{aligned}$$

Where $a, b = 0 - 3$, the remaining components of the metric, we expect to relate to spin and helicity. Here we will set up the convention that whenever talking about the space-time coordinates t, x, y and z we will refer to them as Minkowski coordinates and the remaining six will be referred to as Lorentz coordinates. Also we observe that each representation may furnish us with an equally appropriate candidate for the metric, all being equivalent up to a scalar factor. We will find it most convenient in the future to define our metric with respect to the spinor representation (which will be introduced shortly), in which case the scaling factor for our Killing form turns out to be 6, e.g.:

$$k_{ij} = 6 g_{ij}$$

Where $g_{ij} = \text{diag}(\eta_{ab}, -I_3, I_3)$ is the metric defined with the spinor representation.

It is also worth noting that using the Lie algebra $so(2,3)$ to describe the local symmetry of the manifold implies that the Hawthorn universe is locally 10-dimensional. The Minkowski coordinates maintain their standard identifi-

cation, however we need to explain the other six. We may avoid any pathological explanations involving curled up dimensions etc. if we observe that the other six dimensions are rotation and boost dimensions. Thus if we identify the manifold with a manifold of inertial frames where an object has a position, orientation, and boost coordinate, we may avoid a conceptual nightmare. We therefore amend our fundamental conjecture to include this idea:

re-Fundamental Conjecture. *Our universe is a 10-dimensional ADS manifold of inertial frames*

4.3 Generalised Tensors

So far our mathematical tools have been focussed on developing objects that describe characteristics of the manifold (curvature, torsion etc.), these only deal with the geometry of the manifold. In order to be able to satisfiably incorporate what we know about physics into the model we must address the matter of matter on the manifold. This pursuit occupies this section and the remainder of this chapter.

The way matter is typically dealt with is, if I want to be able describe electrons for example on the manifold in typical space-time, I need to be able to associate each point of the manifold with the representation $sl(2,\mathbb{C})$. This essentially means that if we wish to describe particles as elements of certain representations, then we need to be able to map each point on the manifold into a vector space that the appropriate representation acts on. This is the approach we take in the Hawthorn model, but instead of dealing with representations of the $so(1,3)$ we need to tailor it for describing local $so(2,3)$ symmetry.

We therefore need a way of attaching vector spaces at each point on the manifold in a consistent and natural way that preserves our Lie structure. Hawthorn

([1]) approaches this by developing the notion of $\mathfrak{X}$-tensors. These are mappings from the manifold into a set of vector spaces denoted $\mathfrak{X}$, all of which may have $so(2,3)$ represented on them. This approach is a rigorous, ground up approach. In it Hawthorn ([1]) defines $\mathfrak{X}$-tensors and $\mathfrak{X}$-tensor derivations and proves several results associated with them, many of which follow their tensor analogues from the previous section and then goes on to refine certain conditions on the mappings. Such a development is discussed in more detail by Hawthorn and Crump in [1] and [2], we will, however, be eschewing such a rigorous development in favour of a more elegant and compact route derived from the main points distilled from the $\mathfrak{X}$-tensor approach. This approach is based on the defining conditions of what Hawthorn ([1]) calls *generalised tensors*.

In what follows when dealing with mappings from the manifold to vector spaces, of these mappings we will only be interested in ones that satisfy three conditions:

1. **Local Action Exists:** If we have mappings into a vector space V, then $T_k(^*_*)$ is defined on V and is a representation.

$$T_i(^*_*)T_j(^*_*) - T_j(^*_*)T_i(^*_*) = T^k_{ij}T_k(^*_*)$$

2. **Global Action Exists:** We have a connection $\Gamma_i(^*_*)$ which defines parallel transport of maps into V, and which globally represents the Lie algebra in the sense that

$$\nabla_i\nabla_j - \nabla_j\nabla_i = T^k_{ij}\nabla_k + R_{ij}(^*_*)$$

Where $R_{ij}(^*_*)$ is a point-wise, linear operation on V.

3. **Local and Global Actions Commute:** This is equivalent to the statement that the local action is globally invariant

$$\nabla_m(T_k\binom{*}{*}) = 0$$

If we take a moment to consider these conditions we can see that these are reasonable conditions to impose. We want to associate our manifold with $so(2,3)$ symmetry and we see that conditions 1 and 2 are the manifestations of that notion in terms of the local action and global action, where we identify the torsion as the structure constant of the Lie algebra. Condition 3 arises from considerations of a more physical nature. It can be demonstrated that this condition implies metric invariance and torsion invariance (the trace form of the local action defines the metric up to a factor of a scalar and we are able to rearrange condition 1 for the torsion). We note that metric invariance is a common fundamental property assumed in theories of gravity. It is commonly referred to the *metric postulate* or *metricity condition* and it is the main condition separating general affinely connected metric spaces, and manifolds that accommodate theories of gravity, see [14] or [6]. It implies that intervals are preserved under the global action, which is a reasonable condition to have. Similarly, invariance of the torsion implies the preservation of the Lie algebra at each point and is also a defining condition of a Local Lie Manifold. It is conceivable that we could drop condition 3 and assume the metricity condition and torsion invariance independently, however, a theory that makes less assumptions is naturally more agreeable than the alternative, thus we adopt condition 3.

With these conditions we will now demonstrate how spinors may be put on the manifold.

4.3.0.1 Spinors on the Manifold

We will associate spinors with the elements of the representation $sp(4,\mathbb{R}) \simeq so(2,3)$. Using Greek letters for spinor indices we define the local action on a spinor to be $T_i(^*_*)v^\alpha = T^\alpha_{i\beta}v^\beta$, where the set $\{T^\beta_{i\alpha}\}$ are basis elements of the canonical representation of $sp(4,\mathbb{R})$ (see Appendix A). As was mentioned before, it is with respect to this basis that we wish to define out metric, hence we make the identification:

$$T^\beta_{i\alpha}T^\alpha_{j\beta} = g_{ij} \tag{4.15}$$

Which we recall from our discussion of the Killing form is $\frac{1}{6}k_{ij}$.

As we've defined the local action as basis elements of the representation $sp(4,\mathbb{R})$, and as this is a representation of the Lie algebra $so(2,3)$ we can see automatically that the local action satisfies condition 1, where T^k_{ij} is an element of the adjoint representation. For condition 2, we assume we have a connection defining parallel transport on the manifold, such that it satisfies condition 2 and the justification for condition 3 follows from the physical arguments discussed previously. We therefore adopt the axiom:

$$\nabla_i(T^\beta_{j\alpha}) = 0$$

Thus, all three conditions are satisfied for the spinor representation and so, by construction, we may accommodate spinors on the manifold.

Chapter 5

Representations of Low Dimension

Here we would like to examine a few low dimensional representations that are pertinent to the model. These are the 1-dimensional, 5-dimensional and 10-dimensional representations. We consider these representations as they arise from the decomposition of transformations with two spinor indices which are the primary objects that the model concerns itself with. Other higher dimensional transformations and their decompositions are dealt with in both [1] and [2]. When dealing with several representations we will distinguish between them using index notation where elements of one representation are represented by one set of characters as indices and another representation with a different set of characters. Similar treatments may be found in [1] and [2].

5.1 Spinor Transformations

Consider a tensor with two spinor indices, denoted X_{α}^{β}. The indices run from 1-4 and therefore the space of these tensors is 16-dimensional. We may decompose this representation into a direct sum of irreducibles, resulting in 1, 5 and 10 dimensional components. We may define for each of these local irreducibles, idempotent projections maps: Π, the sum of which is simply the identity map.

5.1.1 1-Dimensional Component

The 1D component is spanned by $\{\frac{1}{2}1^{\beta}_{\alpha}\}$, which behaves as the trivial representation under the local action.

Consider the maps

$$X^{\beta}_{\alpha} \to \frac{1}{2}1^{\beta}_{\alpha}X^{\alpha}_{\beta} = x \tag{5.1}$$

$$x \to \frac{1}{2}1^{\beta}_{\alpha}.x = X^{\beta}_{\alpha} \tag{5.2}$$

These are the projection maps to and from this component (the factor of $\frac{1}{2}$ is necessary to satisfy idempotency).

Elements of this representation will be called **scalars** and we can define local and global actions on them such that these maps are totally invariant. We let the local action be the trivial action: $T_i\binom{*}{*}x = 0$ and let the global action be a standard derivative as defined on scalars: $\nabla_k(x) = \partial_k(x)$.

The idempotent projection map is given by:

$$\Pi : X^{\beta}_{\alpha} \to \frac{1}{4}1^{\beta}_{\alpha}X^{\lambda}_{\lambda} \tag{5.3}$$

5.1.2 10-Dimensional Component

The 10D component is spanned by tensors of the form $T^{\beta}_{i\alpha}$. Choosing $\{T^{\beta}_{i\alpha}\}$ as a basis we may define projection and injection mappings to and from this component

$$X^{\beta}_{\alpha} \to g^{ki}T^{\alpha}_{i\beta}X^{\beta}_{\alpha} = x^k \tag{5.4}$$

$$x^k \to T^{\beta}_{k\alpha}x^k = X^{\beta}_{\alpha} \tag{5.5}$$

We would like to have local and global actions on these mappings such that they are invariant. We can see though that as we have assumed the $T^{\beta}_{i\alpha}$ to be totally invariant, then it follows that both these mappings inherit that total invariance.

A 10-dimensional irreducible component associated with these mappings is called a **vector**, and will be denoted with Latin indices.

The idempotent map for vectors is:

$$\Pi : X^{\beta}_{\alpha} \to g^{ij} T^{\beta}_{i\alpha} T^{\beta'}_{j\alpha'} X^{\alpha'}_{\beta'} \tag{5.6}$$

5.1.3 5-Dimensional Component

The 5-dimensional component is spanned by the basis $\{T^{\beta}_{A\alpha}\}$. As these map trivially onto the trivial component and the vector component we have:

$$T^{\beta}_{A\beta} = 0 \tag{5.7}$$

$$T^{\beta}_{A\alpha} T^{\alpha}_{i\beta} = 0 \tag{5.8}$$

As with the vector components, we may define a trace form with the basis $\{T^{\beta}_{A\alpha}\}$:

$$g_{AB} = T^{\beta}_{A\alpha} T^{\alpha}_{B\beta} \tag{5.9}$$

We also have projection and injection maps to and from this component given by:

$$X^{A} \to T^{\beta}_{A\alpha} X^{A} = X^{\beta}_{\alpha} \tag{5.10}$$

$$X^{\beta}_{\alpha} \to S^{A\alpha}_{\beta} X^{\beta}_{\alpha} = X^{A} \tag{5.11}$$

By subtracting the idempotent projection maps of the trivial and vector representations from the identity map we can define the idempotent projection for this representation:

$$T^{\alpha}_{A\beta}S^{A\beta'}_{\alpha'}X^{\alpha'}_{\beta'} = (1^{\alpha}_{\alpha'}1^{\beta'}_{\beta} - g^{ij}T^{\beta}_{i\alpha}T^{\beta'}_{j\alpha'} - \frac{1}{4}1^{\alpha}_{\beta}1^{\beta'}_{\alpha'})X^{\alpha'}_{\beta'} \tag{5.12}$$

We denote the local action on this component by T^{A}_{iB}, and the equation

$$T_i(^{*}_{*})T^{\beta}_{A\alpha} = 0 \tag{5.13}$$

and define a connection Γ^{B}_{iA}, such that our basis is totally invariant:

$$\nabla_k(T^{\beta}_{A\alpha}) = 0 \tag{5.14}$$

A consequence of this is that the trace form defined with respect to this basis is totally invariant also.

Elements of this representation are called **versors** and are denoted with a capitalised Latin index: x^A. If we observe that the 5-dimensional representation is just the canonical representation of $so(2,3)$, then we can see that the form, g_{AB}, is simply the canonical metric.

5.2 Casimir Identities

The operators: $g^{ij}T_i(^{*}_{*})T_j(^{*}_{*})$, define an operator called the Casimir operator, (see section 1.3 of [1]). These operators are scalars in every irreducible representation and therefore provide us with some useful identities called **Casimir**

Identities:

$$g^{ij}T^{\beta}_{i\lambda}T^{\lambda}_{j\alpha} = \frac{5}{2}.1^{\beta}_{\alpha} \tag{5.15}$$

$$g^{ij}T^{B}_{iX}T^{X}_{jA} = 4.1^{B}_{A} \tag{5.16}$$

$$g^{ij}T^{b}_{ix}T^{x}_{ja} = 6.1^{b}_{a} \tag{5.17}$$

A Casimir identity exists for every irreducible finite dimensional representation and it should be noted that these identities do not hold on reducible representations.

5.3 Spinor Bilinear Form and Bullet Scalars

5.3.1 Spinor Bilinear Form

In the previous sections we saw that we could define bilinear forms g_{ij} and g_{AB} on 10D and 5D representations, respectively, via the trace form. We can see from their definitions that they are both totally invariant. We would like also to have a totally invariant bilinear form defined for spinors. The canonical representation of $sp(4,\mathbb{R})$ is characterised by the existence of a locally invariant antisymmetric bilinear form that is unique up to a scalar. Such a form is defined at each point of the manifold and is here denoted as $s_{\alpha\beta}$.

As we have mentioned, $s_{\alpha\beta}$ is unique up to a scalar. Thus if $s_{\alpha\beta}$ is one such bilinear form and f is a scalar field, then $t_{\alpha\beta}$ defined:

$$t_{\alpha\beta} = f.s_{\alpha\beta} \tag{5.18}$$

is an equally appropriate bilinear form.

We note that by definition such a bilinear form is locally invariant, what we would like to examine is whether it may be chosen such that it also possesses

global invariance. Observe that local invariance on $s_{\alpha\beta}$ implies

$$T^{\lambda}_{i\alpha}\nabla_k(s_{\lambda\beta}) + T^{\lambda}_{i\beta}\nabla_k(s_{\alpha\lambda}) = 0 \tag{5.19}$$

Thus we can see that for each k, $\nabla_k(s_{\alpha\beta})$ is also a locally invariant bilinear form. Hence, we may associate with $s_{\alpha\beta}$ a vector A_k, such that

$$\nabla_k(s_{\alpha\beta}) = -A_k s_{\alpha\beta} \tag{5.20}$$

This vector A_k is dependent on our choice of $s_{\alpha\beta}$. Now, as global invariance implies $A_k = 0$, the question of the total invariance of $s_{\alpha\beta}$ is reduced to the question of whether or not we may satisfy this condition by some choice of $s_{\alpha\beta}$. Consider $t_{\alpha\beta}$ from before and assume f is positive then we have

$$t_{\alpha\beta} = e^f s_{\alpha\beta}$$

Applying a covariant derivative to this and letting the vector associated with $t_{\alpha\beta}$ be B_k, we get

$$\begin{aligned} \nabla_k(t_{\alpha\beta}) &= (\nabla_k(f) - A_k)t_{\alpha\beta} \\ B_k t_{\alpha\beta} &= (A_k - \nabla_k(f))t_{\alpha\beta} \end{aligned} \tag{5.21}$$

Thus we conclude that A_k and B_k are related by

$$B_k = A_k - \nabla_k(f)$$

This implies that if we wish to impose global invariance on $s_{\alpha\beta}$, then the vector A_k must be the gradient of some scalar field

$$\nabla_k(s_{\alpha\beta}) = 0 \rightarrow A_k = \nabla_k(f)$$

It should be noted that this condition on the vector A_k is not something we can prove given the assumptions of the model. Consequently, if we wish to have global invariance for $s_{\alpha\beta}$, then this identification for A_k must be added in as an additional assumption. We therefore do not assume that $s_{\alpha\beta}$ is globally invariant and in fact, it will be demonstrated later on that such an assumption would prove to be undesirable if we wish to have non-trivial EM phenomena on the manifold.

5.3.2 Bullet Scalars

Consider the space $\{X^{\alpha\beta}\}$, this space is 16-dimensional and decomposes in to irreducibles of dimension 1,5 and 10.

We wish to consider elements of the trivial representation thus we pick an element $s^{\alpha\beta}_{\bullet}$, where the bullet index is the 'alphabet' we will use to denote an element of the trivial representation. Let $s^{\bullet}_{\alpha\beta}$ represent a projection map onto this component and choose local and global actions such that it is totally invariant. If we consider the local action on this component we have

$$T^{\lambda}_{i\alpha}(s^{\bullet}_{\lambda\beta}) + T^{\lambda}_{i\beta}(s^{\bullet}_{\alpha\lambda}) = 0 \tag{5.22}$$

(As $T^{\bullet}_{i\bullet} = 0$). We therefore see that $s_{\alpha\beta} = s^{\bullet}_{\alpha\beta}$ is a locally invariant symplectic form.

If we consider now the global action on both of these: from the last section we saw that $\nabla_k(s_{\alpha\beta}) = -A_k.s_{\alpha\beta}$ and we've defined $s^{\bullet}_{\alpha\beta}$ as globally invariant, hence $\nabla_k(s^{\bullet}_{\alpha\beta}) = 0$. This implies a non-zero connection associated with the bullet index: $\Gamma^{\bullet}_{k\bullet}$. If we note that $s_{\alpha\beta}$ and $s^{\bullet}_{\alpha\beta}$ locally equivalent, this implies that $\Gamma^{\bullet}_{k\bullet} = A_k$.

We conclude therefore that any quantity with a bullet index undergoes a non-

trivial parallel transport. As the bullet index is associated with the trivial representation it suggests the existence of scalar like quantities that behave locally like scalars but parallel transport non-trivially in contrast to a normal scalar. Scalars with bullet indices are called **bullet scalars**.

5.4 Raising and Lowering Indices

Each of the quantities described in the previous sections (vector, versor, spinor) come in covariant and contravariant forms. We note that the bilinear forms g_{ij}, g^{ij}, g_{AB}, and g^{AB} can be used raise and lower vector and versor indices and that they commute with local and global actions. For spinor indices however we need to use $s^{\bullet}_{\alpha\beta}$ and $s^{\alpha\beta}_{\bullet}$ and not $s_{\alpha\beta}$ and $s^{\alpha\beta}$ as the latter are not globally invariant.

As the symplectic bilinear forms are antisymmetric with respect to the spinor indices we need to be careful about how we go about raising and lowering spinor indices as it can introduce a negative sign. Thus we follow the convention mentioned in [2], we **L**ower on the left and **R**aise on the right:

$$s^{\bullet}_{\alpha\lambda}v^{\lambda} = v^{\bullet}_{\alpha}$$

$$v_{\lambda}s^{\lambda\alpha}_{\bullet} = v^{\alpha}_{\bullet}$$

Also we see that the raising and lowering operation leaves a bullet behind.

Chapter 6

Global Structure

The previous chapters have furnished us with the mathematical tools necessary to build the Hawthorn model. We've seen that given a transformation with two spinor indices we can decompose it into contributions from the trivial, vector and versor representations. We now wish to examine this property with regards to the global action, ∇, looking specifically at the spinor connection $\Gamma^{\beta}_{i\alpha}$. Here we will examine how each of the representations mentioned in the previous chapter contributes to the model.

6.1 The Connection

We can define change of basis matrices: $\delta^{i'}_{j}$, $\delta^{\alpha}_{\beta'}$ etc., at every point of the manifold and denote the new bases with primed indices. If we consider a change of basis for the spinor connection $\Gamma^{\beta}_{i\alpha}$, we get

$$\begin{aligned}
\nabla_i \psi^{\beta} = \partial_i \psi^{\beta} + \Gamma^{\beta}_{i\alpha}\psi^{\alpha} &= \delta^{i'}_{i}\delta^{\beta}_{\beta'}\nabla_{i'}\psi^{\beta'} \\
\delta^{i'}_{i}\partial_{i'}(\delta^{\beta}_{\beta'}\psi^{\beta'}) + \Gamma^{\beta}_{i\alpha}\psi^{\alpha} &= \delta^{i'}_{i}\delta^{\beta}_{\beta'}\partial_{i'}\psi^{\beta'} + \delta^{i'}_{i}\delta^{\beta}_{\beta'}\Gamma^{\beta'}_{i'\alpha'}\psi^{\alpha'} \\
\delta^{i'}_{i}\delta^{\beta}_{\beta'}\partial_{i'}\psi^{\beta'} + \delta^{i'}_{i}\partial_{i'}(\delta^{\beta}_{\beta'})\psi^{\beta'} + \Gamma^{\beta}_{i\alpha}\psi^{\alpha} &= \delta^{i'}_{i}\delta^{\beta}_{\beta'}\partial_{i'}\psi^{\beta'} + \delta^{i'}_{i}\delta^{\beta}_{\beta'}\Gamma^{\beta'}_{i'\alpha'}\psi^{\alpha'} \quad (6.1)
\end{aligned}$$

We can see that the first term on the left is common to both sides of the equation thus we can cancel it. Rearranging for $\Gamma^{\beta}_{i\alpha}$, we can see that the

spinor connection transforms as

$$\Gamma^{\beta}_{i\alpha} = \delta^{i'}_{i}\delta^{\beta}_{\beta'}\delta^{\alpha'}_{\alpha}\Gamma^{\beta'}_{i'\alpha'} - \delta^{i'}_{i}\delta^{\beta}_{\alpha}\partial_{i'}(\delta^{\beta}_{\beta'}) \tag{6.2}$$

Hence, the spinor connection is not a tensor. Note that the second term on the right implies that this is a result of a gauge dependence on the choice of spinor basis. Thus, if we fix the spinor basis we may treat the connection as a tensor.

We can treat the connection (for each fixed k) as a linear transformation on spinor indices defined at each point. Therefore we may decompose it into contributions of irreducibles:

$$\Gamma^{\beta}_{i\alpha} = A_i.1^{\beta}_{\alpha} + G^{k}_{i}T^{\beta}_{k\alpha} + N^{A}_{i}T^{\beta}_{A\alpha} \tag{6.3}$$

The coefficients A_i, G^k_i, and N^A_i are the scalar, vector, and versor components of the connection and are called the connection coefficients. If we consider the connection transformation rule and use the projection maps onto the scalar, vector, and versor components individually, then we can resolve the transformation rules for the connection coefficients:

$$\begin{aligned}
A_{i'} &= \delta^{i}_{i'}A_i - \frac{1}{4}\delta^{i}_{i'}\delta^{\alpha}_{\alpha'}\partial_i(\delta^{\alpha'}_{\alpha}) \\
G^{k'}_{i'} &= \delta^{i}_{i'}\delta^{k'}_{k}G^{k}_{i} - \delta^{i}_{i'}\delta^{k'}_{k}\delta^{\mu}_{\alpha'}\partial_k(\delta^{\alpha}_{\lambda})T^{\lambda}_{t\mu}g^{tj} \\
N^{A'}_{i'} &= \delta^{i}_{i'}\delta^{A'}_{A}N^{A}_{i} - \delta^{i}_{i'}\delta^{A'}_{A}\delta^{\mu}_{\alpha'}\partial_A(\delta^{\alpha}_{\lambda})T^{\lambda}_{B\mu}g^{AB}
\end{aligned}$$

Which again we see unless we fix a basis for the spinors, are not tensors either.

Considering the projection maps onto the respective representations, the con-

nection coefficients are expressible as:

$$A_i = \frac{1}{4}\Gamma^{\alpha}_{i\alpha} \tag{6.4}$$

$$G^k_i = \Gamma^{\beta}_{i\alpha} T^{\alpha}_{m\beta} g^{mk} \tag{6.5}$$

$$N^A_i = \Gamma^{\beta}_{i\alpha} T^{\alpha}_{B\beta} g^{BA} \tag{6.6}$$

We can also find expressions for the vector and versor connections: Γ^k_{ij} and Γ^A_{iB}. Consider the fact that the local spinor action is globally invariant:

$$\nabla_i T^{\beta}_{j\alpha} = \partial_i T^{\beta}_{j\alpha} - \Gamma^k_{ij} T^{\beta}_{k\alpha} - \Gamma^{\sigma}_{i\alpha} T^{\beta}_{j\sigma} + \Gamma^{\beta}_{i\sigma} T^{\sigma}_{j\alpha} = 0$$

Rearranging gives:

$$\begin{aligned}
\Gamma^k_{ij} T^{\beta}_{k\alpha} &= \partial_i T^{\beta}_{j\alpha} - \Gamma^{\sigma}_{i\alpha} T^{\beta}_{j\sigma} + \Gamma^{\beta}_{i\sigma} T^{\sigma}_{j\alpha} \\
\Gamma^k_{ij} g_{km} &= \partial_i T^{\beta}_{j\alpha} T^{\alpha}_{m\beta} - \Gamma^{\sigma}_{i\alpha} T^{\beta}_{j\sigma} T^{\alpha}_{m\beta} + \Gamma^{\beta}_{i\sigma} T^{\sigma}_{j\alpha} T^{\alpha}_{m\beta} \\
\Gamma^t_{ij} &= \partial_i (T^{\beta}_{j\alpha}) T^{\alpha}_{m\beta} g^{mt} - \Gamma^{\sigma}_{i\alpha} T^{\beta}_{j\sigma} T^{\alpha}_{m\beta} g^{mt} + \Gamma^{\beta}_{i\sigma} T^{\sigma}_{j\alpha} T^{\alpha}_{m\beta} g^{mt} \\
\Gamma^t_{ij} &= \partial_i (T^{\beta}_{j\alpha}) T^{\alpha}_{m\beta} g^{mt} + \Gamma^{\beta}_{i\sigma} (T^{\sigma}_{j\alpha} T^{\alpha}_{m\beta} - T^{\sigma}_{m\alpha} T^{\alpha}_{j\beta}) g^{mt} \\
\Gamma^t_{ij} &= \partial_i (T^{\beta}_{j\alpha}) T^{\alpha}_{m\beta} g^{mt} + G^s_i T^t_{sj}
\end{aligned} \tag{6.7}$$

In an identical fashion it can be shown:

$$\Gamma^D_{iA} = \partial_i (T^{\beta}_{A\alpha}) T^{\alpha}_{C\beta} g^{CD} + G^s_i g_{sk} g^{CD} T^k_{AC} \tag{6.8}$$

It should be pointed out here that unlike the connection of General Relativity, the vector connection here cannot be symmetric. A symmetric connection implies a trivial Lie structure at each point which is something we wish to avoid. In that respect we can consider the vector connection as the sum of a symmetric component and a non-symmetric component

$$\Gamma^k_{ij} = \Gamma^k_{(ij)} + \Gamma^k_{[ij]} \tag{6.9}$$

Where round braces denote the symmetry of the indices and square braces denote the antisymmetry. If we consider the definition of the torsion then we can see that the anti-symmetric component of the connection must be $\Gamma^k_{[ij]} = -\frac{1}{2}T^k_{ij}$.

6.2 The Curvature Tensor

Let's examine the curvature tensor: $R^\beta_{ij\alpha}$, defined as

$$\begin{aligned} R^\beta_{ij\alpha} &= \partial_i\Gamma^\beta_{j\alpha} - \partial_j\Gamma^\beta_{i\alpha} + \Gamma^\beta_{i\lambda}\Gamma^\lambda_{j\alpha} - \Gamma^\beta_{j\lambda}\Gamma^\lambda_{i\alpha} \\ &= \nabla_i\Gamma^\beta_{j\alpha} - \nabla_j\Gamma^\beta_{i\alpha} + \Gamma^\beta_{i\lambda}\Gamma^\lambda_{j\alpha} - \Gamma^\beta_{j\lambda}\Gamma^\lambda_{i\alpha} - T^k_{ij}\Gamma^\beta_{k\alpha} \end{aligned} \tag{6.10}$$

Proposition 6.1 *If we consider the curvature $R^\beta_{ij\alpha}$ to be a set of spinor transformations indexed by i and j, then we may decompose it into contributions of irreducibles, having an explicit form:*

$$R^\beta_{ij\alpha} = F_{ij}1^\beta_\alpha + R^k_{ij}T^\beta_{k\alpha} \tag{6.11}$$

Where

$$F_{ij} = \nabla_i A_j - \nabla_j A_i - T^k_{ij}A_k \tag{6.12}$$

$$R^k_{ij} = \nabla_i G^k_j - \nabla_j G^k_i - G^x_i G^y_j T^k_{xy} - N^A_i N^B_j T^k_{AB} - G^k_m T^m_{ij} \tag{6.13}$$

Proof. First, let $M\binom{*}{*}$ be any operator defined on both spinors and bullet scalars. If $M\binom{*}{*}(s^\bullet_{\alpha\beta}) = 0$, then

$$M^\beta_\alpha = M^k T^\beta_{k\alpha} + M.1^\beta_\alpha$$

Where M^k is some vector and $M = \frac{1}{2}M^\bullet_\bullet$. To prove this consider:

$$\begin{aligned} M\binom{*}{*}s^\bullet_{\alpha\beta} = M\binom{*}{*}(1^\bullet)s_{\alpha\beta} + 1^\bullet M\binom{*}{*}s_{\alpha\beta} &= 0 \\ 1^\bullet\big(M^\bullet_\bullet s_{\alpha\beta} + M\binom{*}{*}s_{\alpha\beta}\big) &= 0 \end{aligned} \tag{6.14}$$

As $1\binom{*}{*}s_{\alpha\beta} = -2s_{\alpha\beta}$, this is equivalent to

$$1^{\bullet}\big(M\binom{*}{*} - \frac{1}{2}M^{\bullet}_{\bullet}1\binom{*}{*}\big)(s_{\alpha\beta}) = 0$$

From this we see that the symplectic form is invariant under this operation, hence the operation lies in the Lie algebra of $sp(4,\mathbb{R})$. Therefore we see

$$M\binom{*}{*} - \frac{1}{2}M^{\bullet}_{\bullet}1\binom{*}{*} = M^{k}T_{k}\binom{*}{*}$$

Which rearranges to give our identity.

Now, to prove proposition 6.1 we just need to observe that $R_{ij}\binom{*}{*}(s^{\bullet}_{\alpha\beta}) = 0$, in which case the curvature may be decomposed as:

$$R^{\beta}_{ij\alpha} = R^{t}_{ij}T^{\beta}_{t\alpha} + F_{ij}.1^{\beta}_{\alpha}$$

Where $F_{ij} = \frac{1}{2}R^{\bullet}_{ij\bullet} = \frac{1}{4}R^{\alpha}_{ij\alpha}$. Thus the result is proven. To prove equations 6.12 and 6.13, we need simply to decompose the connections in the definition of the curvature and equate the coefficients of 1^{β}_{α} and $T^{\beta}_{k\alpha}$ with F_{ij} and R^{t}_{ij}.

It is important to note that the versor representation makes no contribution to the curvature. Later on we will interpret this to mean that there is no field associated with the versor representation. However, that is not to say that it has no effect. If we note the definition of the reduced curvature tensor we will see that a quadratic term associated with the versor connection coefficient: N^{A}_{i}, contributes to the reduced curvature tensor. We will again see, later on, that this implies that the versor representation makes a contribution to the gravitational field.

Proposition 6.2 *The Riemannian curvature tensor, R^t_{ijk}, is related to the reduced curvature tensor via the equation:*

$$R^t_{ijk} = R^s_{ij} T^t_{sk} \tag{6.15}$$

That is, the curvature of the manifold depends only on the reduced curvature tensor.

Proof. Consider the fact that the local action on spinors is totally invariant, thus

$$[\nabla_i, \nabla_j] T^\beta_{k\alpha} = (T^s_{ij} \nabla_s + R_{ij}(^*_*)) T^\beta_{k\alpha} = 0$$

Implies $R_{ij}(^*_*) T^\beta_{k\alpha} = 0$. Expanding this expression out, contracting with $T^\alpha_{m\beta}$, and rearranging we find

$$\begin{aligned} R^t_{ijk} g_{tm} &= R^\lambda_{ij\alpha} T^\beta_{k\lambda} T^\alpha_{m\beta} - R^\beta_{ij\lambda} T^\lambda_{k\alpha} T^\alpha_{m\beta} \\ &= R^\mu_{ij\nu} T^x_{km} T^\nu_{x\mu} \\ &= (F_{ij} 1^\mu_\nu + R^y_{ij} T^\mu_{y\nu}) T^x_{km} T^\nu_{x\mu} \end{aligned}$$

Expanding this last expression out we find that as the $T^\mu_{i\nu}$ are traceless, the component of the trivial representation drops out and we are left with

$$\begin{aligned} R^t_{ijk} g_{tm} &= R^y_{ij} T^x_{km} g_{yx} \\ &= -R^y_{ij} T^x_{ky} g_{mx} \\ &= R^y_{ij} T^t_{yk} g_{tm} \end{aligned} \tag{6.16}$$

Therefore $R^t_{ijk} = R^y_{ij} T^t_{yk}$.

□

If we consider also that the versor basis is totally invariant, then using a similar proof but replacing vector indices with versor indices where appropriate

we can produce the result: $R^B_{ijA} = R^k_{ij} T^B_{kA}$ (it's worthwhile to note that typically when trying to construct gauge theories for gravity for the Anti-deSitter group, authors use this curvature tensor [21], [22], [35], etc., though they do not necessarily relate it to a reduced curvature tensor like we have).

It should also be pointed out at this point that the Riemannian curvature tensor we use: $R^t_{ijk} = R^x_{ij} T^t_{xk}$, is not the same as the curvature tensor used in General Relativity. This is in part due to the fact that the vector indices range from 1-10, but also because our connection is not symmetric like the one used in GR. Instead, as mentioned before, the vector connection: Γ^k_{ij}, is the sum of a symmetric component and the torsion which is antisymmetric. It is not difficult to prove that the symmetric component of the connection is the Christoffel symbol as generated by the metric g_{ij} on our 10-D manifold.

Proposition 6.3 *The vector connection decomposes into the sum of a symmetric component and an antisymmetric component. The symmetric component is the Christoffel symbol and the antisymmetric component is the torsion.*

Proof. We'll first prove that the antisymmetric component is the torsion, consider the decomposed connection:

$$\Gamma^k_{ij} = \Gamma^k_{(ij)} + \Gamma^k_{[ij]}$$

By definition the torsion is given by:

$$\begin{aligned} T^k_{ij} &= -(\Gamma^k_{ij} - \Gamma^k_{ji}) \\ &= -\Gamma^k_{(ij)} - \Gamma^k_{[ij]} + \Gamma^k_{(ji)} + \Gamma^k_{[ji]} \\ &= -2\Gamma^k_{[ij]} \end{aligned} \tag{6.17}$$

Therefore $\Gamma^k_{[ij]} = -\frac{1}{2} T^k_{ij}$.

And now to prove that the symmetric component is the Christoffel symbol

observe:

$$\begin{aligned}
\nabla_i g_{jk} + \nabla_j g_{ik} - \nabla_k g_{ij} &= 0 \\
\partial_i g_{jk} - \Gamma^s_{ij} g_{sk} - \Gamma^s_{ik} g_{js} + \partial_j g_{ik} - \Gamma^s_{ji} g_{sk} - \Gamma^s_{jk} g_{is} - \partial_k g_{ij} + \Gamma^s_{ki} g_{sj} + \Gamma^s_{kj} g_{is} &= 0 \\
\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij} - (\Gamma^s_{ji} + \Gamma^s_{ij}) g_{sk} + (\Gamma^s_{ki} - \Gamma^s_{ik}) g_{js} + (\Gamma^s_{kj} - \Gamma^s_{jk}) g_{is} &= 0 \\
\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij} - 2\Gamma^s_{(ij)} g_{sk} + T^s_{ik} g_{js} + T^s_{jk} g_{is} &= 0 \\
\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij} - 2\Gamma^s_{(ij)} g_{sk} &= 0
\end{aligned}$$

Rearranging, dividing by two and contracting with g^{kt} gives the result

$$\Gamma^t_{(ij)} = \frac{1}{2} g^{kt} (\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})$$

Which is the definition of the Christoffel symbol.

□

Considering this decomposition it is possible to separate the curvature tensor out into components with torsion dependence and without torsion dependence, thus giving us a relationship between our curvature tensor and the curvature tensor of Relativity. Consider the definition of the curvature tensor in light of our decomposition of the connection:

$$\begin{aligned}
R^t_{ijk} &= \partial_i \Gamma^t_{jk} - \partial_j \Gamma^t_{ik} + \Gamma^t_{ix} \Gamma^x_{jk} - \Gamma^t_{jx} \Gamma^x_{ik} \\
&= \partial_i (\Gamma^t_{(jk)} - \frac{1}{2} T^t_{jk}) - \partial_j (\Gamma^t_{(ik)} - \frac{1}{2} T^t_{ik}) + (\Gamma^t_{(ix)} - \frac{1}{2} T^t_{ix})(\Gamma^x_{(jk)} - \frac{1}{2} T^x_{jk}) \\
&\quad -(\Gamma^t_{(jx)} - \frac{1}{2} T^t_{jx})(\Gamma^x_{(ik)} - \frac{1}{2} T^x_{ik}) \\
&= \partial_i \Gamma^t_{(jk)} - \partial_j \Gamma^t_{(ik)} + \Gamma^t_{(ix)} \Gamma^x_{(jk)} - \Gamma^t_{(jx)} \Gamma^x_{(ik)} + X^t_{ijk}(T) \\
&= \hat{R}^t_{ijk} + X^t_{ijk}(T) \qquad (6.18)
\end{aligned}$$

Where $\hat{R}^t_{ijk}$ restricted to the Minkowski coordinates is the curvature tensor of Relativity and $X^t_{ijk}(T)$ is the torsion dependent contribution to the curvature

given by the expression:

$$\begin{aligned} X^t_{ijk} &= \frac{1}{2}(\Gamma^t_{(jx)}T^x_{ik} - \Gamma^t_{(ix)}T^x_{jk}) + \frac{1}{2}(T^t_{jx}\Gamma^x_{(ik)} - T^t_{ix}\Gamma^x_{(jk)}) \\ &\quad +(\partial_j T^t_{ik} - \partial_i T^t_{jk}) + \frac{1}{4}T^x_{ij}T^t_{xk} \\ &= \frac{1}{4}T^x_{ij}T^t_{xk} \end{aligned} \tag{6.19}$$

Equation 6.18 provides a useful expression for comparing results on an ADS manifold with their Einsteinian analogues. For example we may use this identity to determine another form of the reduced curvature tensor:

$$\begin{aligned} R^x_{ij}T^t_{xk} &= \hat{R}^t_{ijk} + \frac{1}{4}T^x_{ij}T^t_{xk} \\ 6R^x_{ij}g_{xb} &= \hat{R}^t_{ijk}T^k_{bt} + \frac{1}{4}T^x_{ij}T^t_{xk}T^k_{bt} \\ R^y_{ij} &= \frac{1}{6}\hat{R}^t_{ijk}T^k_{bt}g^{yb} + \frac{1}{4}T^y_{ij} \end{aligned} \tag{6.20}$$

6.3 Bianchi Identities, Contractions and Invariant Operators

Now that we have examined the connection and the curvature and their decomposition into irreducibles, we now turn our attention to establishing identities using the decompositions. As the title suggests, these identities will broadly fall under the categories of Bianchi identities, contractions of the curvature tensor and invariant operators. Most of what will be stated here will for the sake of brevity, be stated without proof, for proofs see [1].

6.3.1 Bianchi Identities

Consider the Bianchi identity

$$R^t_{ijk} \overset{ijk}{=} 0$$

We can express this in terms of the reduced curvature tensor:

$$R^s_{ij}T^t_{sk} \overset{ijk}{=} 0 \tag{6.21}$$

Now consider the second Bianchi with two spinor indices

$$R^\beta_{is\alpha}T^s_{jk} + \nabla_i(R^\beta_{jk\alpha}) \overset{ijk}{=} 0$$

We can decompose the curvature tensor into contributions of irreducibles, $R^\beta_{ij\alpha} = F_{ij}1^\beta_\alpha + R^s_{ij}T^\beta_{s\alpha}$. By linear independence of the basis elements, the components must each separately satisfy the second Bianchi, hence:

$$F_{is}T^s_{jk} + \nabla_i F_{jk} \overset{ijk}{=} 0 \tag{6.22}$$

$$R^s_{im}T^m_{jk} + \nabla_i R^s_{jk} \overset{ijk}{=} 0 \tag{6.23}$$

Considering equation 6.22 we can prove an important result regarding the behaviour of the field tensor, F_{ij}:

Proposition 6.4 *The field tensor satisfies the Faraday-Gauss equation*

Proof. Consider the second Bianchi for the field tensor and expand the covariant out:

$$\begin{aligned} F_{is}T^s_{jk} + \nabla_i F_{jk} &\overset{ijk}{=} 0 \\ F_{is}T^s_{jk} + \partial_i F_{jk} - \Gamma^x_{ij}F_{xk} - \Gamma^x_{ik}F_{jx} &\overset{ijk}{=} 0 \end{aligned} \tag{6.24}$$

As this is permuted over i, j, k we can swap these indices around without changing the identity (provided we maintain the order of the indices). Therefore this becomes

$$\begin{aligned} F_{is}T^s_{jk} + \partial_i F_{jk} + \Gamma^x_{jk}F_{ix} - \Gamma^x_{kj}F_{ix} &\overset{ijk}{=} 0 \\ F_{is}T^s_{jk} + \partial_i F_{jk} - T^x_{jk}F_{ix} &\overset{ijk}{=} 0 \\ \partial_i F_{jk} &\overset{ijk}{=} 0 \end{aligned} \tag{6.25}$$

Looking at this last equation, we can see that if we equate the first four components of the field tensor with the electromagnetic field tensor then we have the Faraday-Gauss equation.

□

6.3.2 Contractions of the Curvature

Considering contractions of the reduced curvature tensor, we can define three quantities:

Curvature vector: $R_k = R^x_{kx}$

Ricci tensor: $R_{ij} = R^x_{iy} T^y_{jx}$

Curvature scalar: $R = g^{ij} R_{ij}$

Note that the definition of the Ricci tensor implies the relationship:

$$R^j_{ijk} = -R_{ik}$$

With these quantities defined we now consider contractions of the Bianchi identities (see [1] for proof):

Proposition 6.5

$$R_{ij} = R_{ji} \tag{6.26}$$

$$R_s T^s_{ij} + \nabla_i R_j - \nabla_j R_i + \nabla_k (R^k_{ij}) = 0 \tag{6.27}$$

$$\nabla_k (R) = 2\nabla^t (R_{tk}) - 6R_k \tag{6.28}$$

Equation 6.28 turns out to be of prime importance when we consider the link between gravity and electromagnetism, for this reason we will give its proof here. The proof of this equation follows from the contraction of 6.23 with

$g^{ix}T^j_{xs}$, observe:

$$\begin{aligned} g^{ix}T^j_{xs}(R^s_{im}T^m_{jk} + R^s_{jm}T^m_{ki} + R^s_{km}T^m_{ij} + \nabla_i R^s_{jk} + \nabla_j R^s_{ki} + \nabla_k R^s_{ij}) &= 0 \\ g^{jx}R^s_{mi}T^i_{xs}T^m_{jk} + -g^{ix}R^s_{mj}T^j_{xs}T^m_{ki} + R^s_{km}(g^{ix}T^j_{xs}T^m_{ij}) - \nabla^x(R^s_{kj}T^j_{xs}) & \\ -g^{ij}\nabla_j(R^s_{ki}T^i_{xs}) + g^{ix}\nabla_k(R^s_{ij}T^j_{xs}) &= 0 \\ 2g^{jx}R_{mx}T^m_{jk} + R^s_{km}(6.1^m_s) - 2\nabla^x R_{kx} + \nabla_k R &= 0 \end{aligned}$$

If we observe that the first term on the left is the product of a term symmetric in indices x and m (R_{mx}) and a term antisymmetric in x and m ($g^{jx}T^m_{jk}$), hence it must be zero. Thus we are left with

$$6R_k - 2\nabla^x R_{kx} + \nabla_k R = 0$$

Which upon moving all but the $\nabla_k R$ to the right establishes the identity.

6.3.3 Invariant Operators

A local Lie manifold is equipped with a metric, g_{ij} and a local Lie structure, T^k_{ij}. The metric allows us to define the inner product on the manifold and the Lie structure allows us to define the cross product. Combining these with covariant derivatives allows us to generalise differential operators such as the curl and the divergence to local Lie manifolds of arbitrary dimension. Here we will define some of these operators for a local ADS manifold and state a few identities associated with them.

Gradient Operator.

$$\blacktriangledown(X) = \nabla^k(X)$$

Where X is some tensor. The gradient operator is a map from tensors of rank $\binom{n,k}{m,l}$ to tensors of rank $\binom{n+1,k}{m,l}$.

Divergence Operator.

$$\blacktriangledown\bullet\,(v) = g_{ij}\nabla^i(v^j)$$

Where v is some vector field. This is a map from vector fields to scalar fields.

Curl Operator.

$$\blacktriangledown\mathbf{x}\,(X) = \nabla^i(T_i(^*_*)(X))$$

Where X is some tensor. This is a map of tensors of rank $\binom{n,k}{m,l}$ to tensors of rank $\binom{n,k}{m,l}$.

We can also construct second order operators like the generalised Laplacian in the obvious way.

We may associate with these operators certain identities, of which we will state two here without proof.

Proposition. *For any scalar field f and vector field v^k*

$$\blacktriangledown\mathbf{x}\,(\blacktriangledown f) = -3\blacktriangledown f \tag{6.29}$$

and

$$\blacktriangledown\bullet\,(\blacktriangledown\mathbf{x}\,v) = -3\blacktriangledown\bullet\,v + 6R_k v^k \tag{6.30}$$

For proof see [1]. These two identities allow us to prove an important result

Proposition 6.6 *The curvature vector is zero, $R_k = 0$*

Proof. Consider 6.30 in the event that $v^k = \blacktriangledown f$. In this circumstance we may use the result of 6.29,

$$\begin{aligned} \blacktriangledown\bullet\,(\blacktriangledown\mathbf{x}\blacktriangledown f) &= -3\blacktriangledown\bullet\,(\blacktriangledown f) + 6R_k\blacktriangledown f \\ \blacktriangledown\bullet\,(-3\blacktriangledown f) &= -3\blacktriangledown\bullet\,(\blacktriangledown f) + 6R_k\blacktriangledown f \end{aligned} \tag{6.31}$$

Thus $R_k\blacktriangledown f = 0$. As $\blacktriangledown f$ is in general non-zero we must have $R_k = 0$.

□

In light of this result proposition 6.5 can be rewritten:

Proposition 6.7

$$\begin{aligned} R_k &= 0 && (6.32)\\ \nabla_k(R^k_{ij}) &= 0 && (6.33)\\ \nabla_k(R) &= 2\nabla^t(R_{tk}) && (6.34) \end{aligned}$$

This proposition allows us to deduce a couple more identities, for example taking the trace of the reduced curvature tensor in equation 6.20 in light of proposition 6.7 implies:

$$\hat{R}^t_{ijk}T^k_{bt}g^{jb} = 0$$

The most significant result that follows from this though, is that the 10-dimensional Einstein tensor is divergence-less:

$$\begin{aligned} \nabla_k(R) &= 2\nabla^t(R_{tk})\\ \frac{1}{2}g_{tk}\nabla^t(R) &= \nabla^t(R_{tk})\\ \nabla^t(R_{tk} - \frac{1}{2}g_{tk}R) &= 0 && (6.35) \end{aligned}$$

This result is significant because if we hope to reproduce Einstein's theory of gravity on the ADS manifold it will be much easier if the analogous quantities on the manifold have the same properties as the quantities in standard General Relativity.

Once again, however, it must be pointed out that though we call the divergence-less tensor in the last equation the Einstein tensor, due to the asymmetry of the connection this is not the same as the Einstein tensor of Relativity, even when restricted to the Minkowski dimensions. In order to see how this expression relates to the proper Einstein tensor we must consider equation 6.18.

Contracting the j and the t in equation 6.18 we get

$$\begin{aligned} R^{j}_{ijk} &= \hat{R}^{j}_{ijk} + X^{j}_{ijk} \\ -R_{ik} &= -\hat{R}_{ik} - \frac{1}{4} T^{x}_{ij} T^{j}_{kx} \\ &= \hat{R}_{ik} + \frac{3}{2} g_{ik} \end{aligned} \tag{6.36}$$

Contracting through with g^{ik} we get that the curvature scalar is

$$R = \hat{R} + 15 \tag{6.37}$$

Combining these equations we get the expression:

$$R_{ik} - \frac{1}{2} g_{ik} R = \hat{R}_{ik} - \frac{1}{2} g_{ik} \hat{R} - 6 g_{ik} \tag{6.38}$$

Which, restricted to the Minkowski coordinates reads

$$R_{ik} - \frac{1}{2} g_{ik} R = G_{ik} - 6 g_{ik}$$

Where G_{ik} is the proper Einstein tensor of Relativity. In 10 dimensions, we'll call 6.38 the **Einstein-Hawthorn tensor**. We observe that this is the 10D version of Einstein's tensor with a non-zero cosmological term given by $-6g_{ik}$. Note that this equation combined with equation 6.35 implies the proper Einstein tensor is divergence free.

At this point it is worthwhile to reflect on this result. Noting that the curvature vector is zero, we observe that the Einstein-Hawthorn tensor follows simply as a contraction of the Bianchi identity:

$$R^{s}_{im} T^{m}_{jk} + \nabla_i R^{s}_{jk} \overset{ijk}{=} 0$$

Which itself follows from a decomposition of the equivalent Bianchi identity for the spinor curvature:

$$R^{\beta}_{is\alpha}T^{s}_{jk} + \nabla_i(R^{\beta}_{jk\alpha}) \overset{ijk}{=} 0$$

If we note that the scalar component of this decomposition gives us the Faraday-Gauss equation then we observe that the spinor identity has built into it both the Faraday-Gauss equation and the Einstein-Hawthorn tensor. This result is of particular importance to the model as it demonstrates fundamental relationships of electromagnetism and gravity arising as separate components of a single tensor identity, revealing an intimate connection between the two fields.

6.4 Summary

We have seen in this chapter that we can decompose both the connection and the curvature into contributions of irreducibles. After subjecting these decompositions to the Bianchi identities we have been able to produce identities for these components.

Subjecting these results to examination it can be shown that the Riemannian curvature of the manifold depends solely on the reduced curvature tensor and that the scalar contribution to the curvature satisfies the Faraday-Gauss equation. These observations encourage the association of the forces of nature with the different representations, such that we hypothesize that electromagnetism is associated with the trivial representation, gravity is associated with the vector representation and the remaining forces (that we will refer to as 'Nuclear') are associated with the versor representation. Upon making these identifications we also observe that the spinor Bianchi identity encodes important information about electromagnetism and gravity, demonstrating a link between the two forces.

Making these associations we therefore conclude that the connection coefficients must relate to the potentials of the field.

Intriguingly, we find that the versor component of the curvature is zero, but there is a contribution to the curvature associated with the 'Nuclear' field as its potential N contributes to the reduced curvature tensor. While this may seem a bit odd, it is perhaps apt to point out that of the three forces we know of, the nuclear forces are the only ones that have a finite range. That is to say that objects with gravitational or electromagnetic fields can be 'felt' (at least in theory) by a test particle from any distance, but to feel the nuclear force you must be within a very short distance of the source. Perhaps the behaviour of the versor curvature is saying something like this.

Chapter 7

The Dirac Equation

In order to satisfy ourselves that the universe is best described with an ADS manifold, it is necessary to show that the behaviour of matter in this environment matches what is already observed. Hence, our goal is to show that the equations describing behaviour of the fundamental constituents of matter survive in this environment. In fact we wish to go further than this: we want to show that these equations actually arise naturally from the mathematics. As an example of what we mean, we wish to investigate the Dirac equation which is the subject of this chapter.

7.1 Dirac's Equation on an ADS manifold

Having given a very brief run through of the Dirac equation in chapter 2, we would like now to turn our attention to its formulation in the Hawthorn model and see how the Dirac equation appears on an ADS manifold.

7.1.1 Hawthorn's Derivation

In a somewhat backwards fashion, we will start off with a proposed expression of the Dirac equation on an ADS manifold and then seek to justify the proposition and identify what conditions need to be met in order for the identification to be correct. Thus we start off with the a proposition

Proposition 7.1 *The Dirac equation on an ADS manifold is of the form*

$$\blacktriangledown_{\mathbf{x}}\psi = \lambda\psi \tag{7.1}$$

Where $\psi = \psi^{\alpha}$ is a four-vector, $\blacktriangledown_{\mathbf{x}}$ is the invariant curl operator and λ is a constant.

Using the definition of the curl operator, and ignoring curvature for the time begin this may be expanded out as

$$(-T\partial_t + X\partial_x + Y\partial_y + Z\partial_z + A\partial_a + B\partial_b + C\partial_c - I\partial_i - J\partial_j - K\partial_k)\psi^{\alpha} = \lambda\psi^{\alpha} \tag{7.2}$$

Converting to ordinary units we see

$$\begin{aligned} T\partial_t &\rightarrow rT\partial_t \\ (X\partial_x, Y\partial_y, Z\partial_z) &\rightarrow rc(X\partial_x, Y\partial_y, Z\partial_z) \\ (A\partial_a, B\partial_b, C\partial_c) &\rightarrow c(A\partial_a, B\partial_b, C\partial_c) \\ (I\partial_i, J\partial_j, K\partial_k) &\rightarrow (I\partial_i, J\partial_j, K\partial_k) \end{aligned}$$

If we divide through by rc we get

$$(-\frac{1}{c}T\partial_t + X\partial_x + Y\partial_y + Z\partial_z + O(\frac{1}{r}))\psi^{\alpha} = \frac{\lambda}{rc}\psi^{\alpha} \tag{7.3}$$

Which if $r \rightarrow \infty$ as we typically assume, then this equation is simply

$$(-\frac{1}{c}T\partial_t + X\partial_x + Y\partial_y + Z\partial_z)\psi^{\alpha} = \frac{\lambda}{rc}\psi^{\alpha} \tag{7.4}$$

If we compare this to the orthodox Dirac equation we see if we wish to equate the two that the (T, X, Y, Z) must be related to the gamma matrices somehow. In fact, is not difficult to demonstrate that if (T, X, Y, Z) are multiplied by $2i$ we do get the gamma matrices found in the Dirac equation. Thus eqn 7.4

becomes

$$(-\frac{1}{c}\gamma^0\partial_0 + \gamma^1\partial_1 + \gamma^2\partial_2 + \gamma^3\partial_3)\psi^\alpha = \frac{2i\lambda}{rc} \tag{7.5}$$

Which is the Dirac equation provided

$$\lambda = \frac{mc^2 r}{\hbar}$$

7.1.2 Adding Curvature

Initially in Hawthorn's derivation of the Dirac equation curvature was ignored, thus the covariant derivative in the curl operator reduced to the partial derivative. We would now like to reintroduce curvature back into the picture and examine the resulting equations. Letting $\partial \rightarrow \nabla$ we may express 7.1 explicitly

$$\begin{aligned} \nabla \mathbf{x}\, \psi^\alpha &= T^\alpha_{i\sigma}\nabla^i\psi^\sigma \\ &= g^{ij}T^\alpha_{i\sigma}(\partial_j\psi^\sigma + \Gamma^\sigma_{j\rho}\psi^\rho) \end{aligned} \tag{7.6}$$

If we expand the connection coefficient out, equation 7.6 becomes

$$\begin{aligned} g^{ij}T^\alpha_{i\sigma}(\partial_j\psi^\sigma + \Gamma^\sigma_{j\rho}\psi^\rho) &= g^{ij}T^\alpha_{i\sigma}[\partial_j\psi^\sigma + \\ &\quad (A_j 1^\sigma_\rho + G^k_j T^\sigma_{k\rho} + N^A_j T^\sigma_{A\rho})\psi^\rho] \end{aligned} \tag{7.7}$$

Now equating 7.7 with the right hand side of 7.1 (this time ignoring all but the scalar component of the curvature) we get

$$g^{ij}T^\alpha_{i\sigma}(\partial_j + A_j)\psi^\sigma = \lambda\psi^\alpha \tag{7.8}$$

If compare equation 7.8 with the Dirac equation with interaction terms we can see that the position A_j inhabits in Hawthorn's Dirac equation is identical to that of the Electromagnetic interaction term in the standard Dirac equation. This motivates us to interpret the scalar component of the curvature as the

Electromagnetic field vector.

Comparing 7.8 and an interaction Dirac equation therefore reinforces our interpretation that the observable forces are the result of the curvature of the underlying manifold, and we conclude that inclusion of the other curvature components in the equation should tell us how a Dirac particle interacts with Gravitational field and a Nuclear field.

7.1.3 Hawthorn's approach vs. Dirac's approach

There are several reasons as to why Hawthorn's approach may be viewed more favourably than Dirac's initial derivation. For example, if we consider the previous section we see that Hawthorn's approach naturally introduces interaction terms-there's no need to make any minimal coupling substitutions, interactions are a consequence of the mathematics. Also by virtue of being built from tensorial quantities, it follows almost trivially that the equation is invariant with respect to the group transformations (contrasting this with the standard Dirac equation, [9] set aside a whole chapter to demonstrate this fact).

Perhaps though, the most significant advantage the Hawthorn approach has over the Dirac approach is that all the quantities in the Hawthorn's derivation are known quantities with distinct physical interpretations. We consider the matrices T, X, Y, and Z and observe that they represent intrinsic translations in time and space. Therefore their eigenvalues should represent intrinsic energy and momentum, which relate to normal energy and momentum in a similar fashion as spin to angular momentum.

One of the immediate results of being able to directly and unambiguously pin physical interpretations on the matrices T,X,Y, and Z arises when we consider the operator $(\gamma_0)^{-1}\gamma^i$. If we consider the correspondence between these gamma matrices and the matrices T, X, Y, and Z we can see this is essentially

the operator $T^{-1}X$. The orthodox approach tells us that this combination of gamma matrices corresponds to the velocity operator acting on spinors, the eigenvalues of which take on the values $\pm c$. This is somewhat problematic as it asserts that the velocity of a massive fermion like an electron to be either +c or -c (this is the issue at the heart of the alleged phenomenon called the *Zitterbewegung*, an excellent in depth discussion about this may found in chapter 6 of [2]). The obvious problem with this is that this interpretation appears to be in violation of the special principle of relativity.

If we consider now again the equivalent combination of T and X from the Hawthorn model, it is clear that this operator should have eigenvalues of ± 1 in natural units which correspond to $\pm c$ in ordinary units. The difference comes in that though we also interpret this combination as a velocity operator, as T and X in this circumstance represent *intrinsic* quantities then this operator is *intrinsic* velocity. Using this interpretation we find that we can avoid a confrontation with SR, as intrinsic velocity has no relation to extrinsic velocity, in the same way that spin is independent of angular momentum.

The Hawthorn model also provides us with a compelling account of charge. Observe that operators T and I form a Cartan sub-algebra. As a result if we consider them in a quantum mechanical setting we would expect them to correlate to simultaneous observables, I corresponding to spin and T to some form of intrinsic energy. As mentioned earlier for the Poincare group this is a problem as T is non-compact thus has a continuous spectrum that cannot be reconciled with an intrinsic property as there are no known continuous intrinsic properties. In the Hawthorn model however, T is compact thus we should be able to associate it with a discrete intrinsic property simultaneously observable with spin. If we consider this in the context of the Dirac equation, we note that solutions to the Dirac equation are characterised by two properties with two distinct states: charge and spin. As spin is described by the eigenvalues

of the I matrix, we therefore associate charge with the eigenvalues of the T matrix. Thus we are led to identify charge with intrinsic energy. The link between an intrinsic time operator and charge seems very natural if we consider Feynmann's interpretation of the positron as begin a negatively charged electron travelling backwards in time, [32].

Clearly, the Hawthorn model has much to offer with the regards to the Dirac equation and indeed the physics becomes much more transparent when we take this approach.

Chapter 8

Electromagnetism

In this section we investigate Crump's ([2]) contribution to the Hawthorn model, namely, the Faraday-Gauss equations. We will examine the initial formulation, the problems inherent in it, the solution and its implications for the model as a whole. The work on electromagnetism provided the motivation for introducing bullet scalars and the modification of the invariant bilinear form $s_{\alpha\beta}$. For that reason initially we revert back to the assumption $\nabla_i s_{\alpha\beta} = 0$, and work up to the introduction of bullet scalars. As a result this section does contain repetitions of previously derived results. The following may be found in chapters 7 and 8 of [2].

8.1 Initial Attempt

As previously stated the Hawthorn model asserts that the fundamental forces on the manifold may be associated with the curvature of the manifold. Decomposing the 16-dimensional connection $\Gamma^{\beta}_{i\alpha}$ into irreducibles we associate each representation with a force:

$$\Gamma^{\beta}_{i\alpha} = A_i 1^{\beta}_{\alpha} + G^k_i T^{\beta}_{i\alpha} + N^A_i T^{\beta}_{A\alpha}$$

Thus, we associate electromagnetism with the 1D representation i.e. we identify A_i with our electromagnetic potential. This potential contrasts with that

of classical theory in that it is a 10-potential as opposed to the standard 4-potential of classical E.M.-theory. We assume however that in the Minkowski dimensions A_i is identical to that of the standard E.M. potential and we leave the other six dimensions unidentified assuming that their contribution is of the order $\frac{1}{r}$. The relationship between the potential and field tensor is maintained, thus F_{ij} in the Hawthorn model is the same as that of classical theory with two variations:

1. We replace the partial with covariant derivatives

$$\partial_i A_j - \partial_j A_i \to \nabla_i A_j - \nabla_i A_j$$

2. Indices run from 1-10

The first difference is a standard redefinition required to make the field equations tensor equations, and the second difference follows from the fact that A is a 10-vector. It is worthwhile to note that a consequence of the first condition is

$$F_{ij} = \nabla_i A_j - \nabla_j A_i = \partial_i A_j - \partial_j A_i + T^k_{ij} A_k$$

That is, that there is now a torsion term in the field tensor that is not there in the classical theory. This arises from the fact that we do not assume the symmetry of the Γ^k_{ij}'s.

Though we have redefined the electromagnetic field tensor, we still require that it is consistent with the standard field tensor in the Minkowski coordinates as $r \to \infty$. This consistency can be shown by considering the fact that the torsion in the Minkowski coordinates disappears as $r \to \infty$.

This motivates the definition of the **extended Maxwell equations**

$$F_{ij} = \nabla_i A_j - \nabla_j A_i \tag{8.1}$$

$$\nabla^i F_{ij} = J_k \tag{8.2}$$

$$\nabla_i F_{jk} + \nabla_j F_{ki} + \nabla_k F_{ij} = 0 \tag{8.3}$$

$$\nabla_i J^i = 0 \tag{8.4}$$

$$\nabla_i A^i = 0 \tag{8.5}$$

Thus we see that equation 8.1 is the definition of the field tensor, equations 8.2-8.3 are the Ampere-Gauss and Faraday-Gauss equations, respectively, and equations 8.4 and 8.5 are the continuity equation and the gauge condition on the potential A.

While these extensions seem appropriate, the next section will demonstrate that, due to fundamental assumptions about the behaviour of the symplectic bilinear form $s_{\alpha\beta}$ prohibit the existence of non-trivial electromagnetic phenomena on an ADS manifold.

8.1.1 The Problem

Having defined the quantities with which to develop electromagnetism with, we will now show that the theory in this form is a non-starter. It will be proven that the invariance of $s_{\alpha\beta}$ implies that F_{ij} as defined *classically* must always be zero, thus prohibiting any non-trivial E.M. phenomenon on the manifold in the limiting case of $r \to \infty$.

Theorem 8.1 $F_{ij} = \partial_i A_j - \partial_j A_i = 0$, *always.*

Proof. Consider the identity $\nabla_k(s_{\alpha\beta}) = 0$, this implies:

$$\begin{aligned}\partial_k s_{\alpha\beta} &= \Gamma^{\lambda}_{k\alpha} s_{\lambda\beta} + \Gamma^{\lambda}_{k\beta} s_{\alpha\lambda} \\ &= (A_k 1^{\lambda}_{\alpha} + G^m_k T^{\lambda}_{m\alpha} + N^A_k T^{\lambda}_{A\alpha}) s_{\lambda\alpha} + (A_k 1^{\lambda}_{\beta} + G^m_k T^{\lambda}_{m\beta} + N^A_k T^{\lambda}_{A\beta}) s_{\alpha\lambda}\end{aligned}$$

Collecting like terms and simplifying results in

$$\partial_k s_{\alpha\beta} = 2A_k s_{\alpha\beta} + 2N^A_k T^{\lambda}_{A\alpha} s_{\lambda\beta}$$

Which upon contraction with $s^{\alpha\beta}$ gives

$$\partial_k(s_{\alpha\beta}) s^{\alpha\beta} = 8A_k \tag{8.6}$$

Considering Proposition 7.1 of [2] and following a similar method as above it may be shown that:

$$\partial_k(s^{\alpha\beta}) s_{\alpha\beta} = -8A_k \tag{8.7}$$

Next consider the combination $8(\partial_i A_j - \partial_j A_i)$ in light of equations 8.6 and 8.7:

$$\begin{aligned}8(\partial_i A_j - \partial_j A_i) &= \partial_i(\partial_j(s_{\alpha\beta}) s^{\alpha\beta}) - \partial_j(\partial_i(s^{\alpha\beta}) s_{\alpha\beta}) \\ &= \partial_j(s_{\alpha\beta}) \partial_i(s^{\alpha\beta}) - \partial_i(s_{\alpha\beta}) \partial_j(s^{\alpha\beta}) \qquad (8.8)\end{aligned}$$

Making use of the identity

$$\partial_k(s^{\alpha\lambda}) = -s^{\alpha\mu} \partial_k(s_{\beta\mu}) s^{\beta\lambda}$$

Equation 8.8 becomes:

$$8(\partial_i A_j - \partial_j A_i) = -\partial_j(s_{\alpha\beta}) s^{\alpha\mu} \partial_i(s_{\lambda\mu}) s^{\lambda\beta} + \partial_i(s_{\alpha\beta}) s^{\alpha\mu} \partial_j(s_{\lambda\mu}) s^{\lambda\beta}$$

Relabelling the dummy indices it may be shown that the right hand side is the opposite of itself and hence 0. Thus $\partial_i A_j = \partial_j A_i$, finishing the proof.
□

8.1.2 The Fix

We have seen therefore that the first major result of Crump's work was to show that classical electromagnetism is forbidden on an ADS manifold given Hawthorn's initial assumptions. In order to fix this problem it is necessary to revise some of the assumptions in the foundation of the model.

In order to determine the condition or conditions that leads to a trivial electromagnetic field tensor it is necessary to examine the proof that demonstrates this. The only explicit assumption made was of the global invariance of the form $s_{\alpha\beta}$, and as mentioned in the first paragraph of 8.1.1, it is this assumption that the proof hangs on. Therefore the logical step would be to modify the assumption that $s_{\alpha\beta}$ is globally invariant, this however, is not an assumption to be abandoned lightly.

We may identify $s_{\alpha\beta}$ as an intertwining map of representations. The global invariance of $s_{\alpha\beta}$ therefore implies the equivalence of components in different representations. If we consider the equations

$$x^{\Sigma} = x^{\alpha_1\alpha_2\ldots\alpha_n} s^{\Sigma}_{\alpha_1\alpha_2\ldots\alpha_n}$$

$$x_{\Sigma} = x_{\alpha_1\alpha_2\ldots\alpha_n} s_{\Sigma}^{\alpha_1\alpha_2\ldots\alpha_n}$$

and their covariant derivatives

$$\nabla_k(x^{\Sigma}) = \nabla_k(x^{\alpha_1\alpha_2\ldots\alpha_n}) s^{\Sigma}_{\alpha_1\alpha_2\ldots\alpha_n} + x^{\alpha_1\alpha_2\ldots\alpha_n} \nabla_k(s^{\Sigma}_{\alpha_1\alpha_2\ldots\alpha_n})$$

$$\nabla_k(x_{\Sigma}) = \nabla_k(x_{\alpha_1\alpha_2\ldots\alpha_n}) s_{\Sigma}^{\alpha_1\alpha_2\ldots\alpha_n} + x^{\alpha_1\alpha_2\ldots\alpha_n} \nabla_k(s_{\Sigma}^{\alpha_1\alpha_2\ldots\alpha_n})$$

For the group structure and correspondence between representations to be preserved under parallel transport we require that $\nabla_k(s^{\Sigma}_{\alpha_1\alpha_2\ldots\alpha_n}) = 0$ and $\nabla_k(s_{\Sigma}^{\alpha_1\alpha_2\ldots\alpha_n}) = 0$.

Thus, we are adamant that $s_{\Sigma}^{\alpha_1\alpha_2\ldots\alpha_n}$ be globally invariant, this leaves us with very little wiggle room regarding the issue of electromagnetism. However, it can be observed that there is a subtle assumption made about the decomposition of the space $X^{\alpha\beta}$, namely that we have associated the 1-dimensional trivial component with scalars. It will be shown that in order rectify our situation we must abandon this assumption.

Denote this new non-scalar representation with a bullet index, $\bullet$. Thus the components of s become $s^{\bullet}_{\alpha\beta}$. Enforcing the condition that $s^{\bullet}_{\alpha\beta}$ be globally invariant, we see

$$\nabla_k(s^{\bullet}_{\alpha\beta}) = \partial_k(s^{\bullet}_{\alpha\beta}) + \Gamma^{\bullet}_{k\bullet}s^{\bullet}_{\alpha\beta} - \Gamma^{\lambda}_{k\alpha}s^{\bullet}_{\lambda\beta} - \Gamma^{\lambda}_{k\beta}s^{\bullet}_{\alpha\lambda} = 0 \tag{8.9}$$

Thus, equating the components $s^{\bullet}_{\alpha\beta} = s_{\alpha\beta}$ we can see the original assumption of $\nabla_k(s_{\alpha\beta}) = 0$, implied $\Gamma^{\bullet}_{k\bullet} = 0$. Therefore if we don't assume $\Gamma^{\bullet}_{k\bullet} = 0$, $s_{\alpha\beta}$ is not globally invariant hence the proof of a trivial electromagnetic field tensor is not valid and we escape the problem we encountered initially.

Thus in order to allow for non-trivial electromagnetic phenomena on the manifold the 1-dimensional irreducible representation of the space of two component spinors must be identified with scalar-like quantities that are not globally invariant and denoted with a bullet index, $\bullet$.

8.2 Maxwell's Equations on the Manifold

Having determined the conditions necessary to allow electromagnetic field on the manifold, Crump examined the implications of the extended Maxwell's

equations as they are stated in 8.1 (eqns. 8.1-8.5), and most importantly was able to prove that the Faraday-Gauss equation is a natural consequence of the Bianchi identities and hence a necessary quality of the manifold.

Following [2], section 8.3, we denote the vector with components A_i as $\mathcal{A}$, thus $\mathcal{A} = (\phi, \mathbf{A}, \mathbf{P}, \mathbf{M})$, where ϕ is a scalar and $\mathbf{A}$, $\mathbf{P}$, and $\mathbf{M}$ are 3-vectors. Likewise we denote the vector with components J_i as $\mathcal{J}$, where $\mathcal{J} = (\rho, \mathbf{J}, \vec{\mathbf{J}}, \dot{\mathbf{J}})$.

8.2.1 The Source Equation

Considering equation 8.2 and expanding out the covariant derivatives we get

$$\begin{aligned} \nabla^j F_{jk} &= g^{ij}(\partial_i\partial_j A_k - \partial_i\partial_k A_j - \Gamma^l_{ij}\partial_l A_k - \Gamma^l_{ik}\partial_j A_l \\ &\quad +\Gamma^l_{ik}\partial_l A_j + \Gamma^l_{ij}\partial_k A_l + T^p_{jk}\partial_i A_p - T^p_{ik}\Gamma^l_{ip}A_l) \end{aligned} \tag{8.10}$$

Imposing the flat space condition $\Gamma^k_{ij} = -\frac{1}{2}T^k_{ij}$, equation 8.10 reduces to

$$\nabla^j F_{jk} = g^{ij}(\partial_i\partial_j A_k - \partial_i\partial_k A_j + 2T^l_{ik}\partial_j A_l) + 3A_k \tag{8.11}$$

Therefore 8.2 can be rewritten as

$$g^{ij}(\partial_i\partial_j A_k - \partial_i\partial_k A_j + 2T^l_{ik}\partial_j A_l) + 3A_k = J_k \tag{8.12}$$

The only free index is k thus we may fix a value for k and investigate the resulting equation. Henceforth we'll just look at the case $k = 0$. Also we will take advantage of Proposition 8.3 of [2] (p. 76) and up until now we have been using natural units, we wish now to convert back into ordinary units. Conversion factors for natural units to ordinary units may be found on p.77 of [2]. Taking all of this into account eqn. 8.12 becomes

$$\begin{aligned} r^3c^2\nabla\cdot\nabla\phi &+ rc^2\vec{\nabla}\cdot\vec{\nabla}\phi - r\dot{\nabla}\cdot\dot{\nabla}\phi - r^3c^2\nabla\cdot\partial_T\mathbf{A} - rc^2\vec{\nabla}\cdot\partial_T\mathbf{P} \\ &\quad +r\dot{\nabla}\cdot\partial_T\mathbf{M} - 2rc^2\nabla\cdot\mathbf{P} + 2rc^2\vec{\nabla}\cdot\mathbf{A} + 3r\phi = r\rho \end{aligned} \tag{8.13}$$

Here $\nabla, \vec{\nabla}$, and $\dot{\nabla} = (\partial_X, \partial_Y, \partial_Z)$, $(\partial_A, \partial_B, \partial_C)$, and $(\partial_I, \partial_J, \partial_K)$, respectively. Dividing through by r^3c^2 and assuming $r \to \infty$ all terms on the left with a coefficient $\frac{1}{r}$ can be set to zero. We may also introduce constants which allow us to adjust the units of the components of $\mathcal{A}$ and $\mathcal{J}$, $\phi \to k_\phi\phi$, $\mathbf{A} \to k_A\mathbf{A}$ etc.thus eqn. 8.13 becomes

$$k_\phi \nabla \cdot \nabla\phi - k_A \nabla \cdot \partial_T \mathbf{A} = \frac{k_\rho \rho}{c^2 r^2} \tag{8.14}$$

If we assume $k_A = -k_\phi$ and $k_\rho = \frac{k_A r^2 c^2}{\epsilon_0}$, then the equation becomes

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

Which is Gauss' Equation.

Following similar procedure eqn. 8.12 also produces the Ampere-Maxwell equation

$$k_A \partial_T \mathbf{E} - k_A c^2 \nabla \times \mathbf{B} = \frac{k_J}{r^2}\mathbf{J} \tag{8.15}$$

Where $k_J = -k_A \mu_0 r^2 c^2 \Rightarrow -\frac{k_\rho}{c}$.

As expected the extended Maxwell's equations also produce extra equations, for the source equation these are

$$k_{PC}(-\partial_T^2 + c^2\nabla^2)\mathbf{P} - 2k_A c\mathbf{E} = \frac{k_{\vec{J}C}}{r^2}\vec{\mathbf{J}} \tag{8.16}$$

$$k_M(-\partial_T^2 + c^2\nabla^2)\mathbf{M} - 2k_A c\mathbf{B} = \frac{k_j}{r^2}\dot{\mathbf{J}} \tag{8.17}$$

It will be noted that the $\mathbf{E}$ and $\mathbf{B}$ terms drop out if the ratio of their k coefficients is proportional to r.

8.2.2 The Faraday-Gauss Equation

Here we will present Crump's most important result, namely the geometric necessity of the Faraday-Gauss equation. The first step will be to show that it is more desirable to define the electromagnetic field tensor as $\partial_i A_j - \partial_j A_i$. Once this is done it will be shown that the Faraday-Gauss equation follows from a straight forward application of the Bianchi identity.

Consider the tensor: $R^{\beta}_{ij\alpha}$. Expanding this out according to its definition we see:

$$R^{\beta}_{ij\alpha} = \partial_i \Gamma^{\beta}_{j\alpha} - \partial_j \Gamma^{\beta}_{i\alpha} + \Gamma^{\beta}_{i\sigma}\Gamma^{\sigma}_{j\alpha} - \Gamma^{\beta}_{j\sigma}\Gamma^{\sigma}_{i\alpha}$$

If we now contract the Greek indices the result is

$$R^{\alpha}_{ij\alpha} = \partial_i \Gamma^{\alpha}_{j\alpha} - \partial_j \Gamma^{\alpha}_{i\alpha} \tag{8.18}$$

Recalling that A_k is defined as $A_k = \frac{1}{4}\Gamma^{\alpha}_{k\alpha}$, 8.18 becomes

$$R^{\alpha}_{ij\alpha} = 4(\partial_i(A_j) - \partial_j(A_i))$$

Thus if we redefine the field tensor as $F_{ij} = \partial_i(A_j) - \partial_j(A_i)$, then it arises as a natural consequence of the curvature.

Consider now the Bianchi identity

$$T^{l}_{ij} R^{\beta}_{lk\alpha} - \nabla_k (R^{\beta}_{ij\alpha}) \overset{ijk}{=} 0$$

This is cycled over the Latin indices, leaving the Greek indices unperturbed. Thus the relationship holds even if we contract the Greek indices which we now choose to do:

$$T^{l}_{ij} R^{\alpha}_{lk\alpha} - \nabla_k (R^{\alpha}_{ij\alpha}) \overset{ijk}{=} 0$$

If we now express $R^{\alpha}_{ij\alpha}$ in terms of A_i, expanding the covariant derivative and simplifying results in

$$-\partial_k(\partial_i A_j - \partial_j A_i) \overset{ijk}{=} 0 \tag{8.19}$$

Which we may observe is the Faraday-Gauss equation.

Thus it is proven that if F_{ij} is defined as $\partial_i A_j - \partial_j A_i$ then the Faraday-Gauss equation is necessarily true on the manifold.

8.2.3 Consequences and Conclusions

8.2.3.1 Redefinition

The previous result is an amazing conclusion and it strongly motivates us to redefine the electromagnetic field tensor. If we use this definition in the source equations we find that it amounts to losing a factor of two out the front of the **E** and **B** in equations 8.16 and 8.17. Hence, we find no reason not to make this identification. Thus we redefine the electromagnetic field tensor as

$$F_{ij} = \partial_i A_j - \partial_j A_i \tag{8.20}$$

8.2.3.2 Conclusion

The end result of Crump's investigation was that in order to accommodate non-trivial EM phenomena on the manifold we must introduce quantities that behave locally like scalars but behave non-trivially under parallel transport. Crump demonstrated that if we consent to this, non-trivial EM phenomena is allowed and in fact the Faraday-Gauss equation follows as a geometric property of the ADS manifold. However, that does not completely wrap up electromagnetism. For starters the source equation is simply postulated. While it can be shown that its consequences on the manifold do comport with what we already know about the electromagnetic force, merely showing there is no con-

tradiction is not enough. In order to satisfy the fundamental hypothesis about of the model it is necessary to justify all the forces and their field equations geometrically.

Also the quantities $\mathbf{P}$, $\mathbf{M}$, $\vec{\mathbf{J}}$, and $\dot{\mathbf{J}}$ still require interpretations. That being said, we know that to fully describe the behaviour of charge carrying particles it is necessary to include spin interactions in the calculations. It is therefore postulated that $\vec{\mathbf{J}}$ and $\dot{\mathbf{J}}$ are somehow related to spin-density and it is predicted that the components of $\mathbf{P}$ and $\mathbf{M}$ are responsible for the fields that exert forces on particles possessing spin. In the limit of large r, these effects are expected to be small.

Chapter 9

Exploring Gravity and Electromagnetism

In this chapter we seek to incorporate the field equations for gravity into the model. General Relativity builds a model for gravity by relating curvature of the manifold to gravitational fields. As we have the appropriate 10-dimensional analogues of the components of Einstein's equations (bar the energy-momentum tensor) it is possible to simply recreate the equation and then go on to demonstrate it is a permissible relationship in the model. However, to simply impose the equations on the manifold would be contravariant to the philosophy of the model-they must be justified and what's more, will hopefully demonstrate a link with electromagnetism. Hence, our goal is twofold: we wish to demonstrate the equations for gravity arise naturally from the mathematics of the model and we seek to investigate the connection between the subsequent equations for gravity and the equations for electromagnetism. Our aim is to use a variational approach to suggest the form of such equations, the advantage of this approach is it allows us to build the EM/gravity link into our derivation of gravity.

9.1 Lagrangian Formalism

Of all the mathematical tools at the disposal of one who wants to study physical systems, one of most useful is that of the variational principle. The basic notion of this principle is there exists an action functional which exhibits explicit dependence on certain quantities, yet is invariant with respect to variation of these quantities. Theories developed from this principle may be found in all fields of physics.

9.1.1 General Theory

Development of the general theory of variational principles may be found to varying degrees in practically all graduate level texts on physics and mathematical principles of physics. Here we follow the general example set in [6], [14], and [10].

9.1.1.1 Invariance of the Action

Consider a field $\phi(x)$. Let us define its form variation (called functional variation in [14]) as:

$$\delta_0\phi(x) = \phi'(x) - \phi(x)$$

We distinguish this from its total variation:

$$\delta\phi = \phi'(x') - \phi(x)$$

We may establish a relationship between the total and form variation by expanding the first term on the right out as a first order Taylor series

$$\begin{aligned}
\delta\phi &= \phi'(x) + \delta x^i \partial_i \phi(x) - \phi(x) \\
&= \phi'(x) - \phi(x) + \delta x^i \partial_i \phi(x) \\
&= \delta_0\phi + \delta x^i \partial_i \phi(x)
\end{aligned}$$

Consider now an action integral over a general n-dimensional space-time region Ω,

$$I(\Omega) = \int_{\Omega} \mathcal{L}(\phi, \partial_k \phi; x) d^n x$$

We impose the condition that this integral is invariant under the space-time transformation: $x' = x + \epsilon(x)$.

$$\delta I = \int_{\Omega'} \mathcal{L}(\phi', \partial'_k \phi'; x') d^n x' - \int_{\Omega} \mathcal{L}(\phi, \partial_k \phi; x) d^n x$$

If we observe that to first order $d^n x' = (1 + \partial_a \epsilon^a) d^n x$, then this becomes

$$\begin{aligned} \delta I &= \int_{\Omega} \mathcal{L}(\phi', \partial'_k \phi'; x')(1 + \partial_a \epsilon^a) d^n x - \int_{\Omega} \mathcal{L}(\phi, \partial_k \phi; x) d^n x \\ &= \int_{\Omega} \delta\mathcal{L} + \mathcal{L}(\phi', \partial'_k \phi'; x') \partial_a \epsilon^a d^n x \end{aligned}$$

And if we just consider the integrand and keep only first order terms we get

$$\begin{aligned} \delta\mathcal{L} + \mathcal{L}(\phi', \partial'_k \phi'; x') \partial_a \epsilon^a &= \delta\mathcal{L} + \delta\mathcal{L} \partial_a \epsilon^a + \mathcal{L} \partial_a \epsilon^a \\ &= \delta_0 \mathcal{L} + \epsilon^a \partial_a \mathcal{L} + (\partial_a \epsilon^a) \mathcal{L} \end{aligned}$$

For the variation of our integral to be zero we require that

$$\delta_0 \mathcal{L} + \partial_a (\epsilon^a \mathcal{L}) = 0$$

9.1.1.2 Conserved Currents

We have established now, conditions on the Lagrangian that must be satisfied for the variation of the action integral to be zero. In its initial form this condition is not of much use to us, however it may be reworked to give something more useful.

Let's examine the first term: $\delta_0\mathcal{L}$.

$$\begin{aligned}\delta_0\mathcal{L} &= \mathcal{L}(\phi+\delta_0\phi,\partial_k\phi+\delta_0\partial_k\phi;x)-\mathcal{L}(\phi,\partial_k\phi;x)\\ &= \frac{\partial\mathcal{L}}{\partial\phi}\delta_0\phi+\frac{\partial\mathcal{L}}{\partial\phi_{,k}}\delta_0\phi_{,k}\\ &= \left(\frac{\partial\mathcal{L}}{\partial\phi}-\partial_k\left[\frac{\partial\mathcal{L}}{\partial\phi_{,k}}\right]\right)\delta_0\phi+\partial_k\left(\frac{\partial\mathcal{L}}{\partial_k\phi}\delta_0\phi\right)\end{aligned}$$

Substituting this back into the our original constraint equation we find

$$\left(\frac{\partial\mathcal{L}}{\partial\phi}-\partial_k\left[\frac{\partial\mathcal{L}}{\partial\phi_{,k}}\right]\right)\delta_0\phi+\partial_k\left(\frac{\partial\mathcal{L}}{\partial_k\phi}\delta_0\phi+\epsilon^k\mathcal{L}\right)=0 \tag{9.1}$$

The first term in brackets describes a system of equations called the Euler-Lagrange equations and by setting them equal to zero we may derive equations of motion for the system with the Lagrangian $\mathcal{L}$. The second term in brackets is commonly denoted J^k and if the the first term is zero, J^k is conserved. In the event that we specify a symmetry group for the transformations on space-time J^k represents the currents conserved under those transformations. For example if we specify the Poincare group as the group describing the symmetry transformations of space-time then

$$J^k=\frac{1}{2}\omega^{\nu\lambda}M^{\mu}_{\nu\lambda}-\varepsilon^{\nu}T^{\mu}_{\nu}$$

Where $M^{\mu}_{\nu\lambda}$ is the angular momentum tensor and T^{μ}_{ν} is the energy-momentum tensor. The currents defined by these tensors are invariant under rotations and translations respectively.

9.1.2 The Hawthorn Action

As we saw in the previous section there are three ingredients required in order to concoct a variational approach: an invariant volume element, a scalar action, and an independent variable (or variables) by which to vary our action with respect to.

From [14] we see that an invariant volume element for a d-dimensional manifold is given by: $\sqrt{|g|}d^dx$. As for our scalar action, we are looking for some scalar term which under the variation of some variable or variables produces physically meaningful equations regarding electromagnetism and gravity. It's worth mentioning the omission of 'nuclear' fields from the previous sentence: as we postulate that the fundamental forces are realisable as curvatures on the manifold and we observe that the curvature tensor $R^{\beta}_{ij\alpha}$ has no versor terms (hence no explicit 'nuclear' terms), we assume that the electromagnetism and gravity are more naturally unifiable. Therefore we ignore 'nuclear' fields for the most part and simply focus on linking EM and gravity.

It's known that the Ampere-Gauss equation can be derived by varying the Lagrangian: $-\frac{1}{4}F_{ij}F^{ij}$ with respect to A_i and $\partial_j A_i$. If we consider the term: $R^{\beta}_{ij\alpha}R^{\alpha}_{rx\beta}g^{ri}g^{rx}$, we can see it separates out into components:

$$R^{\beta}_{ij\alpha}R^{\alpha}_{rx\beta}g^{ri}g^{rx} = F_{ij}F^{ij} + R^{k}_{ij}R^{y}_{rx}g^{ri}g^{jx}g_{yk} \tag{9.2}$$

Which, as the second term on the right is independent of the scalar connection coefficient, we see under variation of A_i and $\partial_j A_i$ should give the Ampere-Gauss equation up to a constant factor. We therefore adopt $R^{\beta}_{ij\alpha}R^{\alpha}_{rx\beta}g^{ri}g^{rx}$ as our Lagrangian and as A and ∂A are connection coefficients we generalise this to Γ and $\partial\Gamma$. Thus we have:

$$\mathcal{L}(\Gamma, \partial\Gamma) = R^{\beta}_{ij\alpha}R^{\alpha}_{rx\beta}g^{ri}g^{rx}$$

We may now state the action that we wish to extremize with respect to the field strengths:

$$S[\Gamma, \partial\Gamma] = \int (F_{ij}F^{ij} + R^{k}_{ij}R^{y}_{rx}g^{ri}g^{xj}g_{ky})\sqrt{|g|}d^{10}x \tag{9.3}$$

By constructing our variational approach as a generalisation of the approach applied to classical electromagnetism, we expect that it will still obey the Euler-Lagrange equations of motion:

$$\frac{\partial \mathcal{L}}{\partial \Gamma} = \frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial(\partial \Gamma / \partial x^\mu)}$$

If we let Γ and $\partial\Gamma$ be A and ∂A then we get the Ampere-Gauss equation. Therefore, the analogous equation for gravity should be given by letting $\Gamma = G^t_s$ and $\partial\Gamma = \partial_p G^t_s$. Then the Euler-Lagrange equations become:

$$\frac{\partial \mathcal{L}}{\partial G^t_s} = \frac{\partial}{\partial x^p} \frac{\partial \mathcal{L}}{\partial(\partial G^t_s / \partial x^p)} \tag{9.4}$$

If we start with the left hand side we get:

$$\frac{\partial \mathcal{L}}{\partial G^t_s} = \left[\frac{\partial R^k_{ij}}{\partial G^t_s} R^y_{rx} + R^k_{ij} \frac{\partial R^y_{rx}}{\partial G^t_s} \right] g^{ir} g^{jx} g_{yk} \tag{9.5}$$

Recalling the definition of the reduced curvature tensor we get

$$\frac{\partial R^k_{ij}}{\partial G^t_s} = (1^s_i G^p_j - 1^s_j G^p_i) T^k_{tp} \tag{9.6}$$

and therefore

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial G^t_s} &= 2R^k_{ij}(g^{si} g^{xj} - g^{xi} g^{sj}) G^p_x T^y_{tp} g_{yk} \\ &= 4R^{sx}_y G^p_x T^y_{tp} \end{aligned}$$

Proceeding on to the right hand side, using Leibnitz' rule and the definition of the reduced curvature tensor we get

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial[\partial_p G^t_s]} &= [(1^p_i 1^s_j - 1^p_j 1^s_i) 1^k_t R^y_{rx} + R^k_{ij}(1^p_r 1^s_x - 1^p_x 1^s_r) 1^y_t] g^{ir} g^{jx} g_{gy} \\ &= 4R^{ps}_t \end{aligned} \tag{9.7}$$

where

$$R_t^{ps} = R_{ij}^k g^{ip} g^{js} g_{kt}$$

and

$$\frac{\partial R_{ij}^k}{\partial[\partial_p G_s^t]} = (1_i^p 1_j^s - 1_j^p 1_i^s) 1_t^k$$

thus

$$\partial_p \left(\frac{\partial \mathcal{L}}{\partial(\partial_p G_s^t)} \right) = 4 \partial_p R_t^{ps} \tag{9.8}$$

Plugging these results in to our E-L equations we get

$$\partial_p R_t^{ps} - R_y^{ps} G_p^x T_{xt}^y = 0 \tag{9.9}$$

Which by analogy with the Ampere-Gauss equation we are inclined to interpret as our source free field equations for gravity.

It doesn't take long however to find something unsatisfactory about this expression though: this is not a tensorial expression

$$\begin{aligned} \delta_s^{a'} \delta_{b'}^t (\partial_p R_t^{ps} - R_y^{ps} G_p^x T_{xt}^y) &= \delta_s^{a'} \delta_{b'}^t \partial_p R_t^{ps} - R_y^{pa'} G_p^x T_{xb'}^y \\ &\neq \partial_p R_{b'}^{pa'} - R_y^{pa'} G_p^x T_{xb'}^y \end{aligned}$$

As the deltas do not commute with the partial derivative. This is a concern as we require our equations of motion to be tensor equations.

In order to resolve this issue let's turn to the Dirac equation.

9.1.3 The Dirac Lagrangian

In the Hawthorn model the Dirac equation appears as:

$$g^{ij} T_{i\sigma}^{\alpha} \nabla_j \psi^{\sigma} = \lambda \psi^{\alpha}$$

and its Lagrangian is:

$$\mathcal{L}(\psi, \partial\psi) = s^{\bullet}_{\alpha\beta}\psi^{\beta} gij T^{\alpha}_{i\sigma} \nabla_j \psi^{\sigma} - \lambda s^{\bullet}_{\alpha\beta}\psi^{\beta}\psi^{\alpha}$$

If we equate $s^{\bullet}_{\alpha\beta}\psi^{\beta}$ with the adjoint field and note the antisymmetry of $s^{\bullet}_{\alpha\beta}$ we can see that this is the standard Lagrangian for the Dirac equation.

If our variational technique is correct then we should be able to reproduce the Dirac equation from it. An immediate problem arises in that as $s^{\bullet}_{\alpha\beta}$ is antisymmetric the last term is zero, this means that we will be unable to reproduce the Dirac equation unless we attach to our Lagrangian a field whose variation is equal to lambda, however this is an ad hoc solution and does not naturally arise from the mathematics. Another solution presents itself from the standard treatment. In the standard approach to variation of the Dirac Lagrangian the adjoint field and the field are treated as independent fields and this allows for the derivation of the equations of motion from the Lagrangian. We will take this approach but go a step further and instead of only nominally having the adjoint be independent we will actually let it be an independent field: ϕ. In this way we prevent our lambda term from disappearing. Thus our Dirac Lagrangian becomes:

$$\mathcal{L}(\psi, \partial\psi, \phi, \partial\phi) = s^{\bullet}_{\alpha\beta}\phi^{\beta} g^{ij} T^{\alpha}_{i\sigma} \nabla_j \psi^{\sigma} - \lambda s^{\bullet}_{\alpha\beta}\phi^{\beta}\psi^{\alpha}$$

If we calculate our Euler Lagrange equations from this we find:

$$\partial_s(s^{\bullet}_{\alpha\beta}\phi^{\beta} g^{is} T^{\alpha}_{i\nu}) = s^{\bullet}_{\alpha\beta}\phi^{\beta} g^{ij} T^{\alpha}_{i\sigma}\Gamma^{\sigma}_{j\nu} - \lambda s^{\bullet}_{\nu\beta}\phi^{\beta}$$

Here again we run into the problem of non-covariance due to a partial and connection coefficient. Due to the fact that our Lagrangian is the standard Dirac Lagrangian, we assume there is an error in our variational technique. The connection coefficient arises from expanding our covariant derivative out and

differentiating wrt ψ, if instead we assume our covariant derivatives and not the partial derivatives form an independent field we circumvent that problem. This in fact is consistent with the approach taken in the Hawthorn model where we replace the partial derivative in the standard Dirac equation with a covariant derivative and we assume the connection terms provide coupling terms to external fields. This, however, still leaves the partial derivative to deal with. If we take note that what we want is to reproduce our Dirac equation from the Lagrangian and that when we make the switch from $\partial\psi \rightarrow \nabla\psi$ we get the correct right hand side we note we may get the correct left hand side if we replace the partial derivative in the last step with a covariant derivative:

$$\partial_s \left(\frac{\partial \mathcal{L}}{\partial[\nabla_s \psi^\nu]} \right) \rightarrow \nabla_s \left(\frac{\partial \mathcal{L}}{\partial[\nabla_s \psi^\nu]} \right)$$

Thus our equation now becomes:

$$\nabla_s (s^\bullet_{\alpha\beta} \phi^\beta g^{is} T^\alpha_{i\nu}) = -\lambda s^\bullet_{\nu\beta} \phi^\beta$$

As all the contents of the bracket with exception to ϕ^β are zero under the covariant derivative this becomes:

$$s^\bullet_{\alpha\beta} g^{is} T^\alpha_{i\nu} \nabla_s \phi^\beta = -\lambda s^\bullet_{\nu\beta} \phi^\beta$$

Which can be written as:

$$g^{is} T^\alpha_{i\nu} \nabla_s (s^\bullet_{\alpha\beta} \phi^\beta) = -\lambda s^\bullet_{\nu\beta} \phi^\beta$$

or

$$s^\bullet_{\nu\alpha} g^{is} T^\alpha_{i\beta} \nabla_s \phi^\beta = s^\bullet_{\nu\alpha} \lambda \phi^\alpha$$

Which is our Dirac equation for $s^\bullet_{\alpha\beta}\phi^\beta$ and ϕ^β contracted with $s^\bullet_{\nu\alpha}$.

Therefore, in order to reproduce the Dirac equation using a variational ap-

proach we need to use modified Euler-Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial \psi^{\mu}} = \nabla_s \left(\frac{\partial \mathcal{L}}{\partial [\nabla_s \psi^{\mu}]} \right).$$

Having now fixed our variational approach with respect to the Dirac Lagrangian we wish to apply this 'fix' to our previous situation.

9.1.3.1 A Note on Gauge Theories

It is worth noting at this point the parallels between what we've done here and gauge theories.

In a gauge theory it is assumed that the Lagrangian is invariant under a set of transformations, which have an explicit coordinate dependence. Thus if partial derivatives are used in the equations of motion then the resulting equations are not tensorial (transformation matrices don't commute with the partial). The solution for this is to introduce a new derivative called the covariant derivative which does commute with the transformation matrices. This is done by adding to the partial derivative compensating fields or gauge fields (which inhabit the same position as the connection components in a typical covariant derivative) that counter the coordinate dependence in the transformation matrices. Therefore by replacing all partial derivatives by covariant derivatives in the equations of motion, the tensorial nature of the equations is restored.

This is identical to what we have done here, except we already had the so called compensating fields: they arise as the connections coefficients that satisfy the relationship:

$$\nabla_x T^{\alpha}_{i\beta} = 0$$

In that respect, we can view the condition that the global and local actions commute as a gauge condition.

It can therefore be argued that what we are doing with the variational approach is trying construct a gauge theory for gravity based off of the Anti-deSitter symmetry group.

For general theory regarding gauge invariant formulations consult [6] and [25]. Construction of gauge theories for gravity based off of the Poincare group are done by [16] and [19], for the Anti-deSitter group, [20], [21], and [22].

9.1.4 Redoing the Equations of Motion

Previously we only concerned ourselves with the finding the gravitational field equations, here we aim for more generality and we will only pull the specific forces out at the end. Recalling our original Lagrangian:

$$\mathcal{L} = R^{\beta}_{ij\alpha} R^{\alpha}_{rx\beta} g^{ri} g^{jx}$$

Our field equations should follow from the modified Euler Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial \psi} = \nabla_k \left[\frac{\partial \mathcal{L}}{\partial (\nabla_k \psi)} \right]$$

We identify Γ and $\nabla\Gamma$ as the fields we wish to vary. Thus the left hand side becomes:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \Gamma^{\sigma}_{s\rho}} &= \frac{\partial R^{\beta}_{ij\alpha}}{\partial \Gamma^{\sigma}_{s\rho}} R^{\alpha}_{rx\beta} g^{ir} g^{jx} + R^{\beta}_{ij\alpha} \frac{\partial R^{\alpha}_{rx\beta}}{\partial \Gamma^{\sigma}_{s\rho}} g^{ir} g^{jx} \\ &= 4 R^{\beta}_{ij\sigma} \Gamma^{\rho}_{r\alpha} g^{ir} g^{js} + 4 R^{\rho}_{ij\alpha} \Gamma^{\alpha}_{r\sigma} g^{is} g^{jx} - 2 R^{\rho}_{ij\sigma} T^{s}_{rx} g^{ir} g^{jx} \end{aligned} \tag{9.10}$$

and the right hand side:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial (\nabla_t \Gamma^{\sigma}_{s\rho})} &= \frac{\partial R^{\beta}_{ij\alpha}}{\partial (\nabla_t \Gamma^{\sigma}_{s\rho})} R^{\alpha}_{rx\beta} g^{ir} g^{jx} + R^{\beta}_{ij\alpha} \frac{\partial R^{\alpha}_{rx\beta}}{\partial (\nabla_t \Gamma^{\sigma}_{s\rho})} g^{ir} g^{jx} \\ &= 4 g^{ti} g^{js} R^{\rho}_{ij\sigma} \end{aligned} \tag{9.11}$$

After some index shuffling and contracting out the g^{js} we get the equations of motion as

$$R^{\alpha}_{it\sigma}\Gamma^{\rho}_{r\alpha}g^{ir} - R^{\rho}_{it\alpha}\Gamma^{\alpha}_{r\sigma}g^{ir} + \frac{1}{2}R^{\rho}_{ij\sigma}T^{j}_{rt}g^{ir} = \nabla^{i}(R^{\rho}_{ij\sigma}) \tag{9.12}$$

We note that every object with Greek indices may be decomposed into contributions of irreducibles. Doing this will allow us to determine the contributions from the various fields (EM, gravity and 'nuclear'). Thus we wish to separate out the different components. Recalling that $R^{\beta}_{ij\alpha} = F_{ij}1^{\beta}_{\alpha} + R^{k}_{ij}T^{\beta}_{k\alpha}$, the left hand side becomes

$$\begin{aligned} &= (F_{it}1^{\alpha}_{\sigma} + R^{k}_{it}T^{\alpha}_{k\sigma})\Gamma^{\rho}_{r\alpha}g^{ir} - (F_{it}1^{\rho}_{\alpha} + R^{k}_{it}T^{\rho}_{k\alpha})\Gamma^{\alpha}_{r\sigma}g^{ir} + \frac{1}{2}(F_{ij}1^{\rho}_{\sigma} + R^{k}_{ij}T^{\rho}_{k\sigma})T^{j}_{rt}g^{ir} \\ &= R^{k}_{it}(T^{\alpha}_{k\sigma}\Gamma^{\rho}_{r\alpha} - T^{\rho}_{k\alpha}\Gamma^{\alpha}_{r\sigma})g^{ir} + \frac{1}{2}(F_{ij}1^{\rho}_{\sigma} + R^{k}_{ij}T^{\rho}_{k\sigma})T^{j}_{rt}g^{ir} \end{aligned} \tag{9.13}$$

Decomposing Γ we get

$$= g^{ir}R^{k}_{it}G^{p}_{r}T^{s}_{pk}T^{\rho}_{s\sigma} - g^{ir}R^{k}_{it}N^{A}_{r}T^{B}_{kA}T^{\rho}_{B\sigma} + \frac{1}{2}(F_{ij}1^{\rho}_{\sigma} + R^{k}_{ij}T^{\rho}_{k\sigma})T^{j}_{rt}g^{ir}$$

Considering also,

$$\nabla^{i}R^{\rho}_{it\sigma} = \nabla^{i}F_{it}1^{\rho}_{\sigma} + \nabla^{i}R^{k}_{it}T^{\rho}_{k\sigma}$$

We may therefore equate coefficients of the basis elements $1^{\rho}_{\sigma}, T^{\rho}_{k\sigma},$ and $T^{\rho}_{B\sigma}$. This gives us three equations:

Electromagnetic Component.

$$\nabla^{i}F_{it} = \frac{1}{2}F_{ij}T^{j}_{rt}g^{ir}$$

Gravitational Component.

$$\nabla^{i}R^{k}_{it} = g^{ir}(R^{s}_{it}G^{p}_{r}T^{k}_{ps} + \frac{1}{2}R^{k}_{ij}T^{j}_{rt})$$

'Nuclear' Component

$$0 = g^{ir} N_r^A R_{itA}^B$$

Firstly and foremost we can observe that these equations are not tensorial: the presence of the connection coefficients in the gravitational and 'nuclear' components have a coordinate dependence. If we were to examine how these equations transform we'd see that this is because we are still using a coordinate dependent spinor basis $\{e_\alpha(x)\}$. If we fix the spinor basis (so that it is not dependent on the coordinate basis of the manifold) then these equations are tensorial, however this results in a constraint on the validity of the equations that we don't really want.

This coordinate dependence in the field equations should not actually come as a surprise. By varying with respect to a non-covariant quantity (the connection), we introduced a coordinate dependence into the equations of motion. In order to tackle this coordinate dependence we would have to go back and include a term accounting for the coordinate dependence i.e. we should use $\mathcal{L} = \mathcal{L}(\Gamma, \nabla\Gamma, x)$ not $\mathcal{L} = \mathcal{L}(\Gamma, \nabla\Gamma)$.

However, instead of going back and modifying our technique, we have reason to believe that the approach has succeeded (recalling we were only using to to *suggest* potential field equations). For instance, introduction of x-variation term into the variation of the Lagrangian shouldn't greatly modify the equations of motion as they arise as coefficients of the variation of the field variable (which in this case would be $\delta_0\Gamma$, see eqn (8.1) for context). Also if we consult [19], in section 2 the author constructs equations of motion from a Lagrangian that is quadratic in field strengths and finds that in order to get a conserved current it is necessary to add terms that are essentially identical to the non-covariant terms in the equations of motion given above-the result being conserved currents but non-covariant equations of motion.

Therefore we take the equations suggested by the our variational approach and seek to establish them geometrically.

9.2 Geometric Proof of the Field Equations

Having developed field equations from our variational approach, it would be nice if we could solidify the relationships with derivations solely based on the manifold's properties. It is this that will be attempted here.

Proposition 9.1 *The curvature tensor satisfies the relationship:*

$$\nabla^i R^\beta_{ij\alpha} = \frac{1}{2} g^{ir} T^m_{rj} R^\beta_{im\alpha} + K^\beta_{j\alpha}$$

Where $K^\beta_{j\alpha}$ is some divergence-less quantity. Furthermore, this equation may be decomposed into scalar contributions and vector contributions:

$$\nabla^i F_{ij} = \frac{1}{2} g^{ir} F_{im} T^m_{rj} + C_j$$

$$\nabla^i R^x_{ij} = \frac{1}{2} R^x_{im} T^m_{rj} + D^x_j$$

Proof. Consider the curvature tensor: $R^\beta_{ij\alpha}$, and the identity

$$\begin{aligned} g^{ir} g^{mj} [\nabla_m, \nabla_r] R^\beta_{ij\alpha} &= g^{ir} g^{mj} (\nabla_m \nabla_r - \nabla_r \nabla_m) R^\beta_{ij\alpha} \\ &= g^{ir} g^{mj} \nabla_m \nabla_r R^\beta_{ij\alpha} + g^{ir} g^{mj} \nabla_r \nabla_m R^\beta_{ji\alpha} \\ &= \nabla^j \nabla^i R^\beta_{ij\alpha} + \nabla^i \nabla^j R^\beta_{ji\alpha} \\ &= 2 \nabla^j \nabla^i R^\beta_{ij\alpha} \end{aligned} \tag{9.14}$$

As this is the commutator of two covariant derivatives it also must be equal to: $g^{ir} g^{mj} (T^k_{mr} \nabla_k + R_{mr}(^*_*)) R^\beta_{ij\alpha}$

$$2 \nabla^j \nabla^i R^\beta_{ij\alpha} = g^{ir} g^{mj} (T^k_{mr} \nabla_k + R_{mr}(^*_*)) R^\beta_{ij\alpha} \tag{9.15}$$

Examining the term: $R_{mr}(^{*}_{*})R^{\beta}_{ij\alpha}$

$$R_{mr}(^{*}_{*})R^{\beta}_{ij\alpha} = R^{\beta}_{mr\sigma}R^{\sigma}_{ij\alpha} - R^{\sigma}_{mr\alpha}R^{\beta}_{ij\sigma} - R^{s}_{mri}R^{\beta}_{sj\alpha} - R^{s}_{mrj}R^{\beta}_{is\alpha} \quad (9.16)$$

Using the fact that $R^{\beta}_{ij\alpha} = F_{ij}1^{\beta}_{\alpha} + R^{t}_{ij}T^{\beta}_{t\alpha}$, we see the first term

$$\begin{aligned} R^{\beta}_{mr\sigma}R^{\sigma}_{ij\alpha} &= (F_{mr}1^{\beta}_{\sigma} + R^{t}_{mr}T^{\beta}_{t\sigma})(F_{ij}1^{\sigma}_{\alpha} + R^{x}_{ij}T^{\sigma}_{x\alpha}) \\ &= F_{mr}F_{ij}1^{\beta}_{\alpha} + F_{mr}R^{x}_{ij}T^{\beta}_{x\alpha} + F_{ij}R^{t}_{mr}T^{\beta}_{t\alpha} \\ &\quad + R^{t}_{mr}R^{x}_{ij}T^{\sigma}_{x\alpha}T^{\beta}_{t\sigma} \end{aligned} \quad (9.17)$$

Similarly for the second term we find

$$\begin{aligned} R^{\sigma}_{mr\alpha}R^{\beta}_{ij\sigma} &= F_{mr}F_{ij}1^{\beta}_{\alpha} + F_{mr}R^{x}_{ij}T^{\beta}_{x\alpha} + F_{ij}R^{t}_{mr}T^{\beta}_{t\alpha} \\ &\quad + R^{t}_{mr}R^{x}_{ij}T^{\sigma}_{t\alpha}T^{\beta}_{x\sigma} \end{aligned} \quad (9.18)$$

As the first two terms in (9.16) = (9.17) − (9.18) we see the only term left is

$$R^{t}_{mr}R^{x}_{ij}(T^{\sigma}_{x\alpha}T^{\beta}_{t\sigma} - T^{\sigma}_{t\alpha}T^{\beta}_{x\sigma}) = R^{t}_{mr}R^{x}_{ij}T^{p}_{xt}T^{\beta}_{p\alpha} \quad (9.19)$$

Substituting this result into (9.16) we get

$$R_{mr}(^{*}_{*})R^{\beta}_{ij\alpha} = R^{t}_{mr}R^{x}_{ij}T^{p}_{xt}T^{\beta}_{p\alpha} - R^{s}_{mri}R^{\beta}_{sj\alpha} - R^{s}_{mrj}R^{\beta}_{is\alpha} \quad (9.20)$$

Further simplification may be found by considering 9.20 in the context of 9.15, we contract 9.20 with g^{ir} and g^{mj}:

$$\begin{aligned} g^{ir}g^{mj}R_{mr}(^{*}_{*})R^{\beta}_{ij\alpha} &= g^{ir}g^{mj}\big(R^{t}_{mr}R^{x}_{ij}T^{p}_{xt}T^{\beta}_{p\alpha} - R^{s}_{mri}R^{\beta}_{sj\alpha} \\ &\quad - R^{s}_{mrj}R^{\beta}_{is\alpha}\big) \end{aligned} \quad (9.21)$$

The first term is (ignoring $T^{\beta}_{p\alpha}$)

$$g^{ir}g^{mj}R^{t}_{mr}R^{x}_{ij}T^{p}_{xt} = g^{ir}g^{mj}R^{t}_{mr}R^{p}_{ijt} \quad (9.22)$$

Equivalently

$$\begin{aligned} g^{ir}g^{mj}R^{t}_{mr}R^{x}_{ij}T^{p}_{xt} &= -g^{ir}g^{mj}R^{t}_{mr}T^{p}_{tx}R^{x}_{ij} \\ &= -g^{ir}g^{mj}R^{p}_{mrx}R^{x}_{ij} \\ &= -g^{ir}g^{mj}R^{p}_{jix}R^{x}_{rm} \\ &= -g^{ir}g^{mj}R^{p}_{ijx}R^{x}_{mr} \end{aligned} \tag{9.23}$$

Relabelling $x \rightarrow t$ we can see that as (9.22) and (9.23) are equivalent, the term $g^{ir}g^{mj}R^{t}_{mr}R^{x}_{ij}T^{p}_{xt}$ must equal zero.

Examining the last two terms in (9.21) we see that

$$\begin{aligned} g^{ir}g^{mj}R^{s}_{mri}R^{\beta}_{sj\alpha} &= -g^{ir}g^{mj}R^{s}_{mrj}R^{\beta}_{si\alpha} \\ &= g^{ir}g^{mj}R^{s}_{mrj}R^{\beta}_{is\alpha} \end{aligned} \tag{9.24}$$

Thus the last two terms are in fact the same term, however upon further investigations we see that

$$\begin{aligned} g^{ir}g^{mj}R^{s}_{mrj}R^{\beta}_{is\alpha} &= g^{ir}g^{mj}R^{n}_{mr}T^{s}_{nj}R^{\beta}_{is\alpha} \\ &= -g^{ir}g^{sj}R^{n}_{mr}T^{m}_{nj}R^{\beta}_{is\alpha} \\ &= g^{ir}g^{sj}R^{n}_{rm}T^{m}_{nj}R^{\beta}_{is\alpha} \\ &= g^{ir}g^{sj}R_{rj}R^{\beta}_{is\alpha} \\ &= R^{is}R^{\beta}_{is\alpha} \end{aligned} \tag{9.25}$$

As the Ricci tensor is symmetric and the Curvature tensor is antisymmetric wrt to (i,s) this must also be zero.

Therefore we conclude that

$$g^{ir}g^{mj}R_{mr}\binom{*}{*}R^{\beta}_{ij\alpha} = 0 \tag{9.26}$$

The consequence of this is that equation (9.15) becomes

$$2\nabla^j\nabla^i R^\beta_{ij\alpha} = g^{ir}g^{mj}T^k_{mr}\nabla_k R^\beta_{ij\alpha} \tag{9.27}$$

Thus we enter the final step in the proof of the field equations, note the left hand side of (9.27):

$$\begin{aligned} g^{ir}g^{mj}T^k_{mr}\nabla_k R^\beta_{ij\alpha} &= -g^{ir}g^{mk}T^j_{mr}\nabla_k R^\beta_{ij\alpha} \\ &= -g^{ir}T^j_{mr}\nabla^m R^\beta_{ij\alpha} \\ &= -g^{ir}T^m_{jr}\nabla^j R^\beta_{im\alpha} \\ &= \nabla^j(g^{ir}T^m_{rj}R^\beta_{im\alpha}) \end{aligned} \tag{9.28}$$

Substituting this into (9.27) and taking all terms to the left we get

$$\nabla^j(\nabla^i R^\beta_{ij\alpha} - \frac{1}{2}g^{ir}T^m_{rj}R^\beta_{im\alpha}) = 0$$

Therefore

$$\nabla^i R^\beta_{ij\alpha} = \frac{1}{2}g^{ir}T^m_{rj}R^\beta_{im\alpha} + K^\beta_{j\alpha} \tag{9.29}$$

Where $K^\beta_{j\alpha}$ is some divergence-less quantity. Using the fact that $R^\beta_{ij\alpha} = F_{ij}1^\beta_\alpha + R^x_{ij}T^\beta_{x\alpha}$ and equating coefficients of 1^β_α and $T^\beta_{x\alpha}$ we get

$$\nabla^i F_{ij} = \frac{1}{2}g^{ir}F_{im}T^m_{rj} + C_j \tag{9.30}$$

$$\nabla^i R^x_{ij} = \frac{1}{2}g^{ir}R^x_{im}T^m_{rj} + D^x_j \tag{9.31}$$

Where C_j and D^x_j are divergence-less quantities defined as

$$C_j = \frac{1}{4}K^\alpha_{j\alpha} \tag{9.32}$$

$$D^x_j = K^\beta_{j\alpha}T^\alpha_{m\beta}g^{mx} \tag{9.33}$$

□

As F_{ij} is interpreted as the electromagnetic field tensor we can see that equation 9.30 is a modified version of the Ampere-Gauss equation. Analogously, we have interpreted the reduced curvature tensor to be some sort of field tensor for the gravitational field. Accordingly, we are inclined to view equation 9.31 as a potential source equation for gravity. While the divergence-less tensors associated with each equation lack any definitive physical interpretation, the way they arise and their position in the equations, especially with regards to the Ampere-Gauss equation, is highly suggestive of source terms and we do assume that they are related to the matter content of the manifold.

9.3 Examining the Field Equations

Having proven the relationships put forward in the proposition stated at the beginning of section 9.2, we now wish to see how they comport with what is currently known about the two forces.

9.3.1 Examining the Electromagnetic Equation

Our first task is to make sure the modified Ampere-Gauss equation is consistent with classical electromagnetism. To do this we must show that the divergence of our field tensor is (or in the limit of $r \to \infty$, approximately) a divergence-less quantity and that it satisfies the source free equation for the trivial field situations.

Consider the equation we have for our supposed EM source equation, ignoring C_t for the time being:

$$\nabla^i F_{it} = \frac{1}{2} F_{ij} T^j_{rt} g^{ir}$$

To start, let's decompose the tensors into their Minkowski and Lorentz coordinate components. We will use Latin indices for the Minkowski coordinates and

upper case Greek letters for the Lorentz coordinates. As t can be a coordinate of our choosing we denote it with a prime so as not to confuse it with the Minkowski coordinates. Thus the equation becomes:

$$\begin{aligned}\nabla^i F_{it'} + \nabla^\Sigma F_{\Sigma t'} &= \frac{1}{2}(F_{ij}T^j_{rt'}g^{ir} + F_{\Upsilon j}T^j_{rt'}g^{\Upsilon r} + F_{i\Omega}T^\Omega_{rt'}g^{ir} + F_{\Upsilon\Omega}T^\Omega_{rt'}g^{\Upsilon r} \\ &\quad + F_{\Upsilon\Omega}T^\Omega_{\Lambda t'}g^{\Upsilon\Lambda} + F_{\Upsilon j}T^j_{\Lambda t'}g^{\Upsilon\Lambda} + F_{i\Omega}T^\Omega_{\Lambda t'}g^{i\Lambda}) \qquad (9.34)\end{aligned}$$

The metric is symmetric and diagonalised, hence the off diagonal components are always zero and we may therefore drop any terms with a metric that combines Latin and Greek indices:

$$\nabla^i F_{it'} + \nabla^\Sigma F_{\Sigma t'} = \frac{1}{2}(F_{ij}T^j_{rt'}g^{ir} + F_{i\Omega}T^\Omega_{rt'}g^{ir} + F_{\Upsilon\Omega}T^\Omega_{\Lambda t'}g^{\Upsilon\Lambda} + F_{\Upsilon j}T^j_{\Lambda t'}g^{\Upsilon\Lambda})$$

Now, observe that as t' is free we essentially have a statement of 10 different equations, one for each value of t'. We note that classical electromagnetism only concerns itself with the Minkowski dimensions, hence we only need to seek agreement in these coordinates. For this reason we choose to only consider the equations where t' is a Minkowski coordinate. If we also note that the components of the torsion are the structure coefficients for commutation relations we can use this to remove a few more terms. The torsion restricted to Minkowski coordinates for the commutation of two space-time transformations is zero, thus the first term is zero, and the torsion restricted to to the Lorentz coordinates for the commutation of a boost or rotation and a space-time transformation is zero hence the third term is zero as well. Thus we are left with:

$$\nabla^i F_{it'} + \nabla^\Sigma F_{\Sigma t'} = \frac{1}{2}(F_{i\Omega}T^\Omega_{rt'}g^{ir} + F_{\Upsilon j}T^j_{\Lambda t'}g^{\Upsilon\Lambda})$$

Now if we take the limit of $r \to \infty$, then we also see from the commutation table that the term $T^\Omega_{rt'} = 0$ as the table collapses to that of the Poincare Lie

algebra. Thus we are left with:

$$\nabla^i F_{it'} + \nabla^\Sigma F_{\Sigma t'} = \frac{1}{2} F_{\Upsilon j} T^j_{\Lambda t'} g^{\Upsilon\Lambda}$$

To get rid of the second and third term we note that the for the metric g_{ij} in ordinary units as $r \to \infty$, the Minkowski components approach zero, but for the metric g^{ij} it's the Lorentz components that become insignificant, hence in the limit of large r the third term is zero. As to the second term we assume the Minkowski components of A_i are independent of the Lorentz coordinates and we recall from the beginning of chapter 8 that we assume the Lorentz components make contributions of the order $\frac{1}{r}$, and therefore may be discarded in the limit of large r. Thus we have (assuming $C_{t'}$ is zero):

$$\nabla^i F_{it'} = 0$$

There is actually another way to demonstrate this result, which is perhaps more elegant and satisfying from the perspective of the model as it relies on the operator identities that are particular to the model. Consider the definition of the field tensor, using tensor quantities:

$$\begin{aligned}
\frac{1}{2} g^{ir} F_{im} T^m_{rj} &= \frac{1}{2} g^{ir} (\nabla_i A_m - \nabla_m A_i - T^k_{im} A_k) T^m_{rj} \\
&= \frac{1}{2} g^{ir} (T^m_{rj} \nabla_i A_m - T^m_{rj} \nabla_m A_i - T^m_{rj} T^k_{im} A_k) \\
&= \frac{1}{2} (g^{ir} T^m_{rj} \nabla_i A_m + g^{mr} T^i_{rj} \nabla_m A_i + g^{ik} T^m_{rj} T^r_{im} A_k) \\
&= T^m_{rj} \nabla^r A_m - 3 A_j \qquad (9.35)
\end{aligned}$$

We note also that the invariant curl operator is defined as $\blacktriangledown\mathbf{x} = \nabla^i T_i \binom{*}{*}$. Looking at the first term of 9.34 it is tempting to identify it as the curl of A_j. If the action of the curl operator on a contravariant vector is

$$\nabla^i (T_i \binom{*}{*} v^k) = T^k_{ij} \nabla^i v^j$$

Its action on a covariant vector is

$$\nabla^i(T_i(^*_*)v_k) = -T^j_{ik}\nabla^i v_j$$

So we can see that the first term is in fact the negative of the curl of A_j. Consider, now, equation 6.29:

$$\blacktriangledown\mathbf{x}\,(\blacktriangledown f) = -3\blacktriangledown f$$

Thus if we let $\blacktriangledown f = v^k$ we see this is expressed explicitly as

$$T^k_{ij}\nabla^i v^j = -3v^k$$

We wish to rewrite this for a covariant vector field

$$\begin{aligned} T^k_{ij}\nabla^i v^j &= T^k_{ij}\nabla^i(g^{xj}v_x) \\ &= g^{xj}T^k_{ij}\nabla^i(v_x) \\ &= -g^{kj}T^x_{ij}\nabla^i(v_x) \\ &= g^{kj}(\blacktriangledown\mathbf{x}\,v)_j \end{aligned} \tag{9.36}$$

Thus for a covariant vector field the relationship is the same (provided v_k is the gradient of a scalar field)

$$\blacktriangledown\mathbf{x}\,v_k = -T^j_{ik}\nabla^i v_j = -3v_k$$

The relevance of this to 9.34 is that if we rearrange this equation we get

$$T^j_{ik}\nabla^i v_j - 3v_k = 0$$

Which upon replacing v_x with A_x we see is the right-hand side of 9.34.

Thus we have the condition that if the electromagnetic potential A_i is the

gradient of a scalar field then equation 9.34 is identically zero. It should be noted that we have encountered this possibility before, it turns out this is equivalent to the condition that the bilinear form $s_{\alpha\beta}$ is totally invariant. It was this condition that Crump proved implied the existence of only trivial EM phenomena, [2]. As a result we note that $A_i = \blacktriangledown f$ is the zero field condition. So, by identity the expression:

$$\nabla^i F_{it} = 0$$

is true.

Interestingly, the equivalence

$$\frac{1}{2} g^{ir} F_{im} T^m_{rj} = (\blacktriangledown \times A)_j - 3A_j$$

Allows us to make one more observation. If we now consider equation 6.30 recalling that the curvature vector $R_k = 0$, we see

$$\blacktriangledown \bullet \, (\frac{1}{2} g^{ir} F_{im} T^m_{rj}) = \blacktriangledown \bullet \, (\blacktriangledown \times A) - 3 \blacktriangledown \bullet \, A$$

For which the right hand side, by the identity, is zero. The implication of this being that

$$\begin{aligned} \blacktriangledown \bullet \, (\nabla^i F_{ij}) &= \blacktriangledown \bullet \, (\frac{1}{2} g^{ir} F_{im} T^m_{rj}) \\ \blacktriangledown \bullet \, (\nabla^i F_{ij}) &= 0 \end{aligned}$$

Thus this equation implies

$$\nabla^i F_{ij} = C'_j$$

Where C'_j is a new divergence-less quantity given by:

$$C'_j = \frac{1}{2} g^{ir} F_{im} T^m_{rj} + C_j$$

If we identify C_j the current density vector then we view this one as a modified charge density vector, and we note that using our approximation technique from before that the difference between the two drops to zero as $r \to \infty$ and we are left with

$$\nabla^i F_{ij} = C_j$$

Which in the event that $C_j = 0$, is the source free case.

Thus we have been able to show that the divergence of our field tensor is, itself, equal to a divergence-less vector quantity and that in the event of there being no real EM phenomena ($A_j = \nabla_j f$), is zero. In doing so we have shown that the modified Ampere-Gauss equation is consistent with the Ampere-Gauss equation for curved space [11].

To complete our examination we would like to show that in flat space this result gives us the usual form of the Ampere-Gauss equation. To do this consider the modified AG equation, and expand the covariant derivative:

$$\begin{aligned}
\nabla^i F_{ij} &= \frac{1}{2} g^{ir} F_{im} T^m_{rj} + C_j \\
\partial^i F_{ij} - g^{it}\Gamma^s_{ti} F_{sj} - g^{it}\Gamma^s_{tj} F_{is} &= \frac{1}{2} g^{ir} F_{im} T^m_{rj} + C_j \\
\partial^i F_{ij} - g^{it}\Gamma^s_{(ti)} F_{sj} - g^{it}\Gamma^s_{(tj)} F_{is} + \frac{1}{2} g^{it} F_{is} T^s_{tj} &= \frac{1}{2} g^{ir} F_{im} T^m_{rj} + C_j \\
\partial^i F_{ij} - g^{it}\Gamma^s_{(ti)} F_{sj} - g^{it}\Gamma^s_{(tj)} F_{is} &= C_j
\end{aligned}$$

Observing that our flat space condition is that the Christoffel symbols are zero ($\Gamma^k_{(ij)} = 0$), we get our result:

$$\partial^i F_{ij} = C_j$$

Which considering our interpretation of C_j as the current density vector, is the classical flat space form of the Ampere-Gauss equation.

9.3.2 Examining Gravity

Having determined that the identity proven in 9.2 gives us equations for electromagnetism consistent with the Ampere-Gauss equation, we would now like to switch our attention to the gravitational component, which will be referred to as the **Ampere-Gauss-Hawthorn** equation (or AGH for short) and see how it constrains the curvature of the manifold and hence the gravitational force.

Consider equation 9.31

$$\nabla^i R^x_{ij} - \frac{1}{2} g^{ir} R^x_{im} T^m_{rj} = D^x_j$$

As mentioned before, by analogy with equation 9.30 we would expect this equation to represent the source equation for gravity, where the tensor D^x_j is tentatively proposed to represent the energy-momentum tensor. However, if we make this identification then to successfully argue our case we must show that this equation reduces to Einstein's equation-if not identically then at least in the limit of large r. This immediately presents a problem in that equation 9.31 is a differential equation with covariant derivatives of the reduced curvature tensor and Einstein's equation is not. On top of this we make the observation that we already have equations consistent with Einstein's field equations, observe: we know from the Bianchi identities that the 10-dimensional Einstein tensor is divergence-less

$$\nabla^i (R_{ij} - \frac{1}{2} g_{ij} R) = 0$$

Which if we "undo" the covariant derivative leaves the right hand side arbitrary provided it's divergence-less, hence:

$$R_{ij} - \frac{1}{2} g_{ij} R = E_{ij} \tag{9.37}$$

Where $\nabla^i E_{ij} = 0$. We will call these the **Einstein-Hawthorn** field equations. Using equation 6.18 to rewrite this we get:

$$\hat{R}_{ij} - \frac{1}{2} g_{ij} \hat{R} - 6 g_{ij} = E_{ij} \tag{9.38}$$

Thus the Einstein-Hawthorn field equations are consistent with Einstein's field equations provided $E_{ij} = \kappa \tau_{ij}$, where τ_{ij} is the energy-momentum tensor of relativity and κ is a constant of proportionality. However, this is the degree of freedom that motivated Einstein to introduce the *cosmological constant* NOT the energy-momentum tensor. There are two reasons though, that prevent us from making that identification:

a) The accepted form of the cosmological term is: Λg_{ij}. We note that this is totally invariant, which is a condition we cannot guarantee for E_{ij}

b) We already have an appropriate candidate for the cosmological term arising naturally from the correspondence between the Einstein-Hawthorn tensor and the Einstein tensor.

Thus, in the event that we take E_{ij} to be the energy-momentum tensor, then it would appear that we have two source terms related to gravitational phenomena. Thus we either have equivalent but aesthetically different formulations of the same phenomena, or we have independent equations describing independent phenomena caused by independent sources. Which it is, it is not clear, we therefore hold off on any physical interpretations as of yet and look only for what may be distilled from the mathematics.

9.3.2.1 Corollaries and Limiting Cases

Here we will explore a few identities and approximations that we may make using the Einstein-Hawthorn field equations and the AGH equations. To start we will give an equivalent expressions for the AGH equation:

Corollary 9.2 *An equivalent form of the AGH equation is given by:*

$$\frac{1}{6}g^{xb}T^k_{bt}\nabla^i\hat{R}^t_{ijk} - \frac{1}{12}g^{ir}g^{xb}\hat{R}^t_{imk}T^k_{bt}T^m_{rj} - \frac{3}{4}1^x_j = D^x_j \tag{9.39}$$

Recalling that the hat denotes the curvature constructed from the symmetric part of the vector connection.

Proof. Substitute the expression given for the Riemannian curvature tensor from equation 6.18 into equation 9.31 and simplify.

□

We can produce another relationship regarding the curvature and the tensor D^x_j by contracting equation 9.31 with the torsion.

Corollary 9.3 *If R^s_{ijk} is the Riemannian curvature tensor and D^x_j is the divergence-less tensor from before, then together they satisfy the following equation:*

$$g^{ir}(\nabla_r R^s_{ijk} + R^s_{kmi}T^m_{jr}) = D^x_j T^s_{xk} \tag{9.40}$$

Proof. If we take our initial equation 9.31 and contract with the torsion tensor T^s_{xk}:

$$\begin{aligned}
\nabla^i R^x_{ij}T^s_{xk} &= \frac{1}{2}g^{ir}R^x_{im}T^m_{rj}T^s_{xk} + D^x_j T^s_{xk} \\
\nabla^i R^s_{ijk} &= \frac{1}{2}g^{ir}(R^x_{im}T^s_{xk})T^m_{rj} + D^x_j T^s_{xk} \\
&= \frac{1}{2}g^{ir}R^s_{imk}T^m_{rj} + D^x_j T^s_{xk} \\
&= -\frac{1}{2}g^{ir}(R^s_{mki} + R^s_{kim})T^m_{rj} + D^x_j T^s_{xk} \\
&= \frac{1}{2}g^{ir}(R^s_{kmi} - R^s_{kim})T^m_{rj} + D^x_j T^s_{xk} \\
&= \frac{1}{2}(g^{ir}R^s_{kmi}T^m_{rj} + g^{mr}R^s_{kim}T^i_{rj}) + D^x_j T^s_{xk} \\
&= g^{ir}R^s_{kmi}T^m_{rj} + D^x_j T^s_{xk}
\end{aligned} \tag{9.41}$$

Rearranging this gives

$$g^{ir}(\nabla_r R^s_{ijk} + R^s_{kmi}T^m_{jr}) = D^x_j T^s_{xk} \tag{9.42}$$

Thus completing the proof.

□

We next consider our first constraint on the tensor D^x_j.

Corollary 9.4 *The divergence-less tensor D^x_j must satisfy the differential equation:*

$$g_{is}\nabla^i D + D^x_j T^j_{sx} = 0 \tag{9.43}$$

where $D = trace(D^x_j)$.

Proof. Letting $x = j$ in equation 9.31 we see

$$\begin{aligned} \nabla^i R^j_{ij} - \frac{1}{2}g^{ir}R^j_{im}T^m_{rj} &= D^j_j \\ \nabla^i R_i - \frac{1}{2}g^{ir}R_{ir} &= D \\ -\frac{1}{2}R &= D \\ \therefore R &= -2D \end{aligned} \tag{9.44}$$

Again, considering equation 9.31, but now contracted through by T^j_{sx}

$$\begin{aligned} \nabla^i R^x_{ij}T^j_{sx} - \frac{1}{2}g^{ir}R^x_{im}T^m_{rj}T^j_{sx} = D^x_j T^j_{sx} \\ \nabla^i R_{is} - \frac{1}{2}g^{ir}R^x_{im}T^m_{rj}T^j_{sx} = D^x_j T^j_{sx} \end{aligned} \tag{9.45}$$

Examining the second term and considering the first Bianchi:

$$R^x_{im}T^j_{sx} = -R^x_{ms}T^j_{ix} - R^x_{si}T^j_{mx}$$

Substituting this back into the second term we see

$$\begin{aligned}
\frac{1}{2}g^{ir}(-R^x_{ms}T^j_{ix} - R^x_{si}T^j_{mx})T^m_{rj} &= -\frac{1}{2}g^{ir}R^x_{ms}T^j_{ix}T^m_{rj} - \frac{1}{2}g^{ir}R^x_{si}T^j_{mx}T^m_{rj} \\
&= -\frac{1}{2}g^{mr}R^x_{ms}(6g_{xr}) + \frac{1}{2}g^{ir}R^x_{si}(6g_{xr}) \\
&= 3(-1^m_x R^x_{ms} + 1^i_x R^x_{si}) \\
&= 3(R_s + R_s) \\
&= 0 \qquad (9.46)
\end{aligned}$$

Therefore equation 9.45 reduces to

$$\nabla^i R_{is} = D^x_j T^j_{sx} \qquad (9.47)$$

Thus we have equations relating the curvature scalar to the trace of D^x_j (equation 9.44) and the divergence of the Ricci tensor to $D^x_j T^j_{sx}$ (equation 9.47), we note also that as the 10-dimensional Einstein tensor is divergence-less then we also have an equation relating the divergence of the Ricci tensor to the divergence of the curvature scalar:

$$\nabla^i R_{is} = \frac{1}{2}g_{is}\nabla^i R$$

Substituting the right hand side of this equation in for the divergence of the Ricci tensor in equation 9.47 and using the expression for R in terms of D we get

$$\frac{1}{2}g_{is}\nabla^i(-2D) = D^x_j T^j_{sx}$$

Therefore, rearranging we see:

$$g_{is}\nabla^i D + D^x_j T^j_{sx} = 0$$

Thus concluding the proof of the proposition.

□

Now we turn our attention to the case where space is empty. We use the empty space condition put forward in [2]: $\Gamma^k_{ij} = -\frac{1}{2}T^k_{ij}$. We arrive at this condition simply by boiling off all but the necessary parts of the connection. As our minimum requirement is that at each point the Lie algebra structure is preserved, the torsion must be non-zero. As the antisymmetric component of the connection is $-\frac{1}{2}T^k_{ij}$, this is the only component we can't set to zero. Thus we have our "empty space" condition. This assumption is strengthened by noting that the symmetric component of the connection is the Christoffel symbol.

Proposition 9.5 *In the absence of matter (empty space), the divergence-less tensor D^x_j satisfies the equation:*

$$D^x_j = -\frac{3}{4}1^x_j \tag{9.48}$$

Proof. Consider the empty space condition $\Gamma^k_{ij} = -\frac{1}{2}T^k_{ij}$. Under this condition the curvature tensor takes on the form: $R^s_{ijk} = \frac{1}{4}T^x_{ij}T^s_{xk}$. With this result in mind consider now the third line of the proof of the previous proposition:

$$\begin{aligned}
\nabla^i R^s_{ijk} &= \frac{1}{2}g^{ir}R^s_{imk}T^m_{rj} + D^x_j T^s_{xk} \\
\nabla^i(\frac{1}{4}T^x_{ij}T^s_{xk}) &= \frac{1}{2}g^{ir}(\frac{1}{4}T^x_{im}T^s_{xk})T^m_{rj} + D^x_j T^s_{xk} \\
0 &= \frac{1}{8}g^{ir}(T^x_{im}T^m_{rj})T^s_{xk} + D^x_j T^s_{xk} \\
&= (\frac{1}{8}g^{ir}(T^x_{im}T^m_{rj}) + D^x_j)T^s_{xk}
\end{aligned} \tag{9.49}$$

As the $\{T^s_{xk}\}$ are linearly independent and using a Casimir identity this implies:

$$\begin{aligned}
D^x_j &= -\frac{1}{8}g^{ir}(T^x_{im}T^m_{rj}) \\
&= -\frac{1}{8}(6.1^x_j) \\
&= -\frac{3}{4}1^x_j
\end{aligned} \tag{9.50}$$

□

This result may also be achieved by setting $\hat{R}^t_{ijk} = 0$ in equation 9.38, which we observe from the definition of the curvature tensor is equivalent to the empty space condition. This same technique can be used to relate the reduced curvature tensor to torsion in empty space.

Proposition 9.6 *The empty space condition implies the relationship:*

$$R^s_{ij} = \frac{1}{4} T^s_{ij} \tag{9.51}$$

Proof. Consider again the empty space condition: $R^s_{ijk} = \frac{1}{4} T^x_{ij} T^s_{xk}$. Observe that the left hand side of this expression can factorised into the reduced curvature tensor and a torsion:

$$\begin{aligned} R^s_{ijk} &= \frac{1}{4} T^x_{ij} T^s_{xk} \\ R^t_{ij} T^s_{tk} &= \frac{1}{4} T^x_{ij} T^s_{xk} \end{aligned} \tag{9.52}$$

Rearranging this last expression we get

$$(R^x_{ij} - \frac{1}{4} T^x_{ij}) T^s_{xk} = 0 \tag{9.53}$$

As $\{T^s_{xk}\}$ form a linearly independent basis of the adjoint representation, the coefficient $R^x_{ij} - \frac{1}{4} T^x_{ij}$, must be identically zero. Therefore we can see that in empty space the reduced curvature tensor is simply: $R^x_{ij} = \frac{1}{4} T^x_{ij}$.

□

Having derived a few corollaries and investigated a few approximations we would now like to try and establish a relationship between the divergence-less constants D^x_j and E_{ij}.

Proposition 9.7 *If $D = trace(D^x_j)$ and $E = trace(E_{ij})$, then $D = \frac{1}{8} E$.*

Proof. Consider the Einstein-Hawthorn equation:

$$R_{ij} - \frac{1}{2} g_{ij} R = E_{ij}$$

Taking the trace of this we get

$$-4R = E \tag{9.54}$$

But $R = -2D$, therefore

$$8D = E \tag{9.55}$$

Dividing through by 8 completes the proof.

□

We also note that if we consider the identity given by equation 9.48, we can assign numerical values to R, D, and E.

Corollary 9.8 *Using the empty space condition and considering equations 9.44, 9.48, and 9.55, we can determine explicit values for D, R, and E:*

$$D = -\frac{15}{2}, \;\; R = 15, \;\; \textit{and} \;\; E = -60$$

Proof. Equation 9.48 tells us that in empty space the divergence-less tensor satisfies the relationship:

$$D^x_j = -\frac{3}{4}1^x_j$$

Letting $x = j$ we see this expression becomes

$$\begin{aligned} D &= -\frac{3}{4} \times 10 \\ &= -\frac{15}{2} \end{aligned} \tag{9.56}$$

Equation 9.44 tells us $R = -2D$, therefore

$$R = 15 \tag{9.57}$$

And, as D and E are related via equation 9.55, E satisfies:

$$E = -60 \tag{9.58}$$

Completing the the proof.

□

Proposition 9.9 *It may be shown that the tensors E_{rs} and D^x_j satisfy the equation:*

$$g^{ir}(E_{rs}R^s_{ij} + R^s_{jxr}R^x_{is}) = \nabla_x(D^x_j - 1^x_j D) \tag{9.59}$$

Proof. Consider the covariant derivative of the Einstein-Hawthorn equation contracted with the upper index:

$$\nabla_x \nabla^i R^x_{ij} + \nabla_x(\frac{1}{2}g^{ir}R^x_{im}T^m_{jr}) = \nabla_x D^x_j \tag{9.60}$$

As $\nabla_x R^x_{ij} = 0$ this reduces to:

$$g^{ir}\nabla_x\nabla_r R^x_{ij} = \nabla_x D^x_j \tag{9.61}$$

Again, as $\nabla_x R^x_{ij} = 0$, the term on the left is equal to the commutator of ∇_r and ∇_x acting on R^x_{ij} thus we see

$$\begin{aligned}
\nabla_x\nabla_r R^x_{ij} &= \nabla_x D^x_j \\
g^{ir}T^k_{xr}\nabla_k R^x_{ij} + g^{ir}R_{xr}(^*_*)R^x_{ij} &= \nabla_x D^x_j \\
-g^{rk}\nabla_k R^x_{ij}T^i_{xr} + g^{ir}R_{xr}(^*_*)R^x_{ij} &= \nabla_x D^x_j \\
-\nabla^r R_{rj} + g^{ir}R_{xr}(^*_*)R^x_{ij} &= \nabla_x D^x_j
\end{aligned} \tag{9.62}$$

We have from a previous identity that $-\nabla^i R_{ij} = D^x_s T^s_{xj}$ and similarly as D^x_j satisfies equation 9.43 we have that

$$\nabla^i R_{ij} = -g_{ij}\nabla^i D$$

Substituting this into equation 9.62 and rearranging we get:

$$g^{ir}R_{xr}\binom{*}{*}R^x_{ij} = \nabla_x(D^x_j - 1^x_j D) \tag{9.63}$$

Expanding out the left hand side we see

$$\begin{aligned} g^{ir}(R^x_{xrs}R^s_{ij} - R^s_{xri}R^x_{sj} - R^s_{xrj}R^x_{si}) &= g^{ir}(R_{rs}R^s_{ij} - R^s_{xri}R^x_{sj} - R^s_{xrj}R^x_{si}) \\ &= g^{ir}R_{rs}R^s_{ij} + g^{is}R^m_{xr}T^r_{mi}R^x_{sj} \\ &\quad -g^{ir}R^s_{xrj}R^x_{is} \\ &= g^{ir}(R^s_{rjx} + R^s_{jxr})R^x_{is} \end{aligned} \tag{9.64}$$

Observe

$$g^{ir}R^s_{rjx}R^x_{is} = g^{ir}R^m_{rj}T^s_{mx}R^x_{is} = g^{ir}R_{im}R^m_{rj}$$

Therefore putting it all together the equation becomes

$$g^{ir}R_{rs}R^s_{ij} + g^{ir}R^s_{jxr}R^x_{is} = \nabla_x(D^x_j - 1^x_j D) \tag{9.65}$$

If we observe that $R_{ij} = E_{ij} + \frac{1}{2}g_{ij}R$ then we get:

$$\begin{aligned} g^{ir}(E_{rs} + \frac{1}{2}g_{rs}R)R^s_{ij} + g^{ir}R^s_{jxr}R^x_{is} &= \nabla_x(D^x_j - 1^x_j D) \\ g^{ir}(E_{rs}R^s_{ij} + R^s_{jxr}R^x_{is}) &= \nabla_x(D^x_j - 1^x_j D) \end{aligned} \tag{9.66}$$

Proving the proposition.

□

This equation is decidedly ugly. However it does achieve something: it gives us an equation that relates E_{ij} with D^x_j. The importance of this is that we would like to identify E_{ij} as the energy-momentum tensor of relativity, however due to its relation to C_j, we expect D^x_j to be some sort of gravitational source term. Therefore if both of them represent sources in some form then there should be some sort of equivalence between them. This equation does provide

this relationship to some extent, it is far from clear what sort of relationship it implies, though. Most importantly, it fails to tell us whether the AGH equation and the Einstein-Hawthorn equation describe the same phenomena and are therefore equivalent expressions or if they are indeed independent.

In fact, this last equation is a fairly good metaphor for this section in that while the relationship is true and presumably (hopefully) highlights something meaningful, we have yet to draw any solid conclusion from it. So while we have expanded on our initial discoveries and derived several corollaries, we are still, largely, in the same position we were in at the beginning of the section. The most salient issue we'd like to clear up is what exactly the AGH equation is implying. While our Ampere-Gauss equation and Einstein-Hawthorn equation are essentially variations on the classical theme, the AGH equation is unique to this model. In order to understand the role the AGH equation plays we need an in-depth investigation into its implications.

9.3.2.2 Determining a Value for r

Having considered the some corollaries and approximations, we now wish to turn our attention to the Einstein-Hawthorn equation:

$$\hat{R}_{ij} - \frac{1}{2} g_{ij} \hat{R} - 6 g_{ij} = E_{ij}$$

We observe that if $E_{ij} = \kappa \tau_{ij}$, then this is equivalent to Einstein's field equations with a cosmological constant: $\Lambda g_{ij} = -6 g_{ij}$. The significance of this equation is it gives a very exact value for Λ, namely $\Lambda = -6$ and as $[\Lambda] = m^{-2}$ this is 6 natural inverse units of length squared. Recalling that a natural unit of length is r seconds times c metres per second, if we recall also that there is an experimental bound on the upper limit of the absolute value of the cosmological constant, $\Lambda \leq 10^{-46} m^{-2}$ in ordinary units, then we now have a means of determining a bound on the value of r. Consider the value of Λ in natural

units and convert back to ordinary units:

$$\begin{aligned} |6| &= |\Lambda| \\ \frac{6}{r^2c^2} &= 10^{-46} \end{aligned} \tag{9.67}$$

Rearranging for r we get

$$r = \sqrt{\frac{2}{3}10^{30}} \tag{9.68}$$

This gives an approximate value of $r = 8.2 \times 10^{15}$ seconds. If we convert this to years, we find this corresponds to an r of approximately 26 million years. As this was determined using an upper bound for Λ, this figure represents a lower bound for r. It should be noted that we should not confuse r with the actual global radius of the universe. As we see that r is associated with the Lie algebra we note therefore that it is a local quantity. Specifically, r is a measure of the extent to which spatial translations commute, it is *not* a prediction of the age or size of the universe. With that in mind we are therefore not obliged to try and reconcile our r value with the current measured value of the age of the universe, which is 13.772 ± 0.059 billion years (thus, several orders of magnitude larger than our r value)[38].

Thus we have finally been able to determine a potential lower bound for our parameter r. The importance of this figure is that it gives us a clue as to distance scales over which observed behaviour would start to deviate from the Poincare paradigm and thus whether it satisfies our previously mentioned Goldilocks condition. As galactic diameters range between 3,000-300,000 light-years, we see that these are approx. $.001r - .01r$, thus we would expect there to be a small, but potentially measurable effect in galactic dynamics corresponding to an r value of approx. 26 million years. Therefore we have the model's first experimental prediction:

Prediction. *There should be, corresponding to an r value of 26 million years,*

aberrations in the dynamics of galaxies, both gravitationally and electromagnetically, that remain unobserved or undiagnosed in the current cosmological model. We expect these discrepancies to be more pronounced in larger galaxies.

The obvious omission is what exactly these effects are thus the prediction is somewhat vague. Note the inclusion of the electromagnetic component, it is necessary to include this as we recall our classical electrodynamics approximation is only valid in regions where r can be assumed infinitely large, obviously we can see that this is not the case in galactic dynamics. Its vagueness, however should not detract from its significance as, by being the first prediction about observable behaviour, it represents the point at which the model passes from a mathematical framework into a genuine scientific theory.

Chapter 10

Conclusion

10.1 Discussion of Results

As was hoped we have been able to demonstrate a natural geometric origin to the equations governing gravity, with the derivation highlighting an intimate connection between electromagnetism and gravity. It was proven that within the identities:

$$R^{\beta}_{is\alpha}T^{s}_{jk} + \nabla_i(R^{\beta}_{ij\alpha}) \overset{ijk}{=} 0$$

$$\nabla^i R^{\beta}_{ij\alpha} - \frac{1}{2}g^{ir}R^{\beta}_{im\alpha}T^{m}_{rj} = K^{\beta}_{j\alpha}$$

is encoded the governing equations for electromagnetism and gravity, arising in pairs giving one EM equation and one gravity equation each. The first identity gives us the Faraday-Gauss equation and the Einstein field equation with a non-zero cosmological constant (Einstein-Hawthorn equation) and the second equation gives us the Ampere-Gauss equation and the Ampere-Gauss-Hawthorn equation.

Using the cosmological constant predicted in the Einstein-Hawthorn equation we were able to give the first estimate as to the value of r. Its value was determined to be 6 in natural units, which given the experimental bound on the cosmological constant allowed us to determine a lower bound for r: approxi-

mately 26 million years. Determining this lower bound allowed us to predict the distance scale over which the effects of this model translate to measurable deviations from the Poincare regime. Thus it is believed that the effects of this model should show themselves at a galactic level.

10.2 Future Research Avenues

10.2.1 Extension of the Thesis

As to future interests regarding the work done in this thesis, it is still necessary to analyse the implications of the Einstein-Hawthorn and the AGH equations in depth and hopefully produce a clearer picture of how they relate to each other. It is also hoped that future investigations will be helpful in producing constraints on the divergence-less tensors C_j, D^x_j and E_{ij}, providing more insight into their nature.

10.2.2 Long Standing Issues

There are also a few long standing issues that need still to be addressed.

Boost and Rotation Dimensions. In light of the similarities of our equations for gravity and electromagnetism to their classical analogues, the need for a definitive interpretation of the components in the boost and rotation dimensions is emphasized.

Versor Representation. Almost all of the working has been possible without considering the nature of the versor component of the connection. For a complete picture of what the model is saying we need to better understand the role that the versors play. While we would like to associate the versor components with the remaining strong and weak forces, it can be observed that one barrier preventing such an association is the problem of describing quarks and neutrinos with the model. If one refers to the weight diagrams in [1], we notice that there is no representation

that allows for the distribution of charges into thirds that is necessary for the quark formalism. This is a problem that must be addressed.

Bullet Scalars. In order to allow for non-trivial EM fields we must introduce scalar fields that parallel transport non-trivially. It is not known if fields like this exist or what sort of physical significance they have.

Quantization. The theory laid out here is a classical theory, ultimately it will require quantization.

Appendix A

Representations of the Lie Algebra $so(2,3)$

A.1 Representations of the Lie Algebra so(2,3)

In this section we look into the various representations of the Lie algebra $so(2,3)$. Before giving a general description it is worthwhile to go over a few examples that are of particular importance to the model.

A.1.1 The Adjoint Representation

Let $\mathfrak{g}$ denote the Lie algebra and define the adjoint mapping as

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{gl}(\mathfrak{g})$$

Which explicitly acts on a element of the algebra as

$$\mathrm{ad}_X(Y) = [X, Y]$$

It may be shown that this operation preserves the Lie bracket, and thus defines a Lie homomorphism (p.55, [3])(and in the case that the Lie algebra is semi-simple, an isomorphism). We see, then, that the adjoint mapping is the representation of the algebra when acting on the 10D space spanned by the

natural representation.

Matrices of this representation may be generated by considering a coordinate vector in $\mathfrak{g}\,(t, x, y, z, a, b, c, i, j, k)^T$ and the commutation relations between elements of $\mathfrak{g}$:

$$\mathrm{ad}_T : X \to A$$

therefore if X and A are represented by vectors

$$X = (0, 1, 0, 0, 0, 0, 0, 0, 0, 0)^T \text{ and } A = (0, 0, 0, 0, 1, 0, 0, 0, 0, 0)^T$$

respectively, then $(\mathrm{ad}_T)_{52} = 1$. Thus the matrix for ad_T is

$$\mathrm{ad}_T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Repeating this process for matrices X through K produces the rest of the adjoint representation.

A.1.2 The Lie Algebra $sp(4, \mathbb{R})$

Consider a skew-symmetric bilinear form B acting on $u, v \in \mathbb{R}^4$

$$B[u, v] = \sum_{k=1}^{2} u_k v_{n+k} - u_{n+k} v_k$$

Let

$$\Omega = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

We then pick a basis such that $B[u, v]$ may be rewritten as:

$$B[u, v] = u^T \Omega v$$

We define the matrix group $Sp(4, \mathbb{R})$ as the set of 4×4 matrices that preserve this bilinear form. Thus if $M \in Sp(4, \mathbb{R})$, then M satisfies

$$B[Mu, Mv] = B[u, v]$$

As $u, v \in \mathbb{R}^4$ are arbitrary, this implies: $M^T \Omega M = \Omega$. This in turn implies the relationship $Y^T \Omega = -\Omega Y$ for an element Y of the Lie algebra. We may therefore proceed as in the case of the canonical representation and determine a basis for the associated Lie algebra, $sp(4, \mathbb{R})$. This basis is given in Figure A.1.

$$T = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$X = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad Y = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad Z = \frac{1}{2}\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$A = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad B = \frac{-1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad C = \frac{1}{2}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$I = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad J = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad K = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

Figure A.1 Representation of $sp(4,\mathbb{R}) \simeq so(2,3)$

If we consider the commutators of these matrices we find that they are identical to that of $so(2,3)$, hence $so(2,3)$ and $sp(4,\mathbb{R})$ are isomorphic.

A.1.3 Ω-Symmetric Representation

We may also construct a representation of $sp(4,\mathbb{R})$ by considering the matrices P that satisfy the relationship $P^T\Omega = \Omega P$. Matrices with this property will be called Ω-symmetric and a quick proof can show that if P is Ω-symmetric and $M \in sp(4,\mathbb{R})$ then $[M,P]$ is Ω-symmetric as well.

Proof.

$$\begin{aligned}
[M,P]\Omega &= PM\Omega - PM\Omega \\
&= M\Omega P^T + P\Omega M^T \\
&= -\Omega M^P T^T + \Omega P^T M^T \\
&= \Omega[M,P]^T \ \square
\end{aligned}$$

It is easy to show that this action preserves the Lie bracket. Therefore if we define a map ϕ, mapping $M \in sp(4,\mathbb{R})$ to $\phi(M)$ such that its action on an element $P \in \Omega$-symmetric matrices is equivalent to the adjoin action, i.e.

$$\phi(M)P = [M,P]$$

It may be seen that the matrices $\phi(M)$ form a representation of $sp(4,\mathbb{R})$ acting on the vector space of Ω-symmetric matrices.

We may determine the basis of the Ω-symmetric matrices as follows

$$\Omega = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \text{ and } P = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Ω-symmetry of P therefore implies

$$\begin{pmatrix} -C^T & A^T \\ -D^T & B^T \end{pmatrix} = \begin{pmatrix} C & D \\ -A & -B \end{pmatrix}$$

This implies that

$$C = \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix},\ A = D^T = \begin{pmatrix} a & d \\ \pm d & e \end{pmatrix} \text{ and} B = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$$

Hence, a general Ω-symmetric matrix has the form

$$P = \begin{pmatrix} a & d & 0 & b \\ \pm d & e & -b & 0 \\ 0 & c & a & d \\ -c & 0 & \pm d & e \end{pmatrix}$$

There are 6 independent components thus we may find 6 linearly independent basis elements, these are given in Figure A.2.

$$1 = \tfrac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad P_\lambda = \tfrac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$P_t = \tfrac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad P_x = \tfrac{1}{2}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P_y = \tfrac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad P_z = \tfrac{1}{2}\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Figure A.2 Basis for the 6D representation

Thus this representation of $sp(4,\mathbb{R})$ is 6-dimensional. However it is not an irreducible representation as it can be shown that the adjoint action of $sp(4,\mathbb{R})$ on this basis leaves the subspace spanned by 1 invariant. Therefore the 6D representation may be decomposed into an irreducible 1D representation and an irreducible 5D representation, which under the action of $sp(4,\mathbb{R})$ give the trivial representation and the natural representation, respectively.

A.2 Representation Theory of $so(2,3)$

Having presented a few examples of representations of $so(2,3)$, we will now give a brief overview of representation theory and its application to $so(2,3)$. What follows may be found in more depth in [1] and general theory regarding representations may be found in [3] and [4].

Our goal is to state the Theorem of the Highest Weight and discuss its application to $so(2,3)$, but before we do that we must clear up some terminology. We start off with a few definitions.

Definition A.1 *If $\mathfrak{g}$ is a complex semi-simple Lie algebra, then a Cartan sub-algebra of $\mathfrak{g}$ is a maximally commutative complex sub-algebra $\mathfrak{h}$ of $\mathfrak{g}$ such that for all $H \in \mathfrak{h}$, ad_H is diagonalisable.*

As all elements of a Cartan sub-algebra (CSA) commute they preserve each others eigenspaces, this motivates the definition of weights and weight vectors:

Definition A.2 *If π is a representation of $\mathfrak{g}$ on a vector space V with CSA $\mathfrak{h}$, and if $v \in V$ satisfies:*

$$\pi(H_i)v = \lambda_i v, \ \ \forall H \in \mathfrak{h} \, and \, \lambda_i \in \mathbb{C}$$

then v is called a weight vector. The ordered set $(\lambda_1, ..., \lambda_n)$ such that:

$$\begin{aligned} \pi(H_1)v &= \lambda_1 v \\ &\vdots \\ \pi(H_n)v &= \lambda_n v \end{aligned}$$

is called a weight.

Thus we can see that weight vectors and weights are simply eigenvectors and eigenvalues of the CSA. In the event that the representation is the adjoint representation then we have a special name for the weights and weight vectors:

in this representation weights are called **roots** and weight vectors are called **root vectors**.

For $so(2,3)$ specifically, it is simple (this is easily checked by examining the table of commutation relations) therefore we may look for a CSA.

To find the CSA we need to look for the largest commutative sub-algebra. This is easily achieved by considering the table of commutators. We observe that time translations commute with rotations, as do those of spatial translations and Lorentz boosts bar {X,A}, {Y,B} and {Z,C}. For ease of interpretation of eigenvalues, we choose to work with the time translations and rotations. Thus we pick {T,I} to work with. If we go about solving their characteristic polynomials we find they are diagonalisable over the complex numbers (and are therefore compact operators). According to the corollary on page 30 of [4](preservation of Jordan decomposition, specific case: $x_n = 0$), this relationship holds in any finite dimensional complex representation, i.e. they are commutative and diagonalisable in every representation.

Let $so(2,3)$ be represented on V, and let $v \in V$ be some non-zero vector such that

$$T(v) = qv \tag{A.1}$$

$$I(v) = sv \tag{A.2}$$

Where $q, s \in \mathbb{C}$. According to our previous definitions the pair (q, s) is a weight and v is a weight vector. We may classify weights in terms of being higher or lower such that the highest weight is that for which q_0 is the maximum eigenvalue of T and s_0 is the maximum eigenvalue of I that can be coupled with q_0. Now we may state the Theorem of the Highest Weight as found in [3].

Theorem A.1 (Theorem of the Highest Weight) *If L is a complex semi-simple Lie algebra, then:*

1. *Every irreducible representation has a highest weight.*
2. *Two irreducible representations with the same highest weight are equivalent.*

We can see therefore, that every irreducible representation of $so(2,3)$ can be classified according to its highest weight, with each weight furnishing us with a distinct irreducible representation. Weights may be represented pictorially in weight diagrams and each irreducible representation has unique set of weights. This is done for $so(2,3)$ in [1] on page 18. It can be shown that $so(2,3)$ has IRs of dimension 1, 4, 5, and 10, these correspond to the trivial, canonical $sp(4,R)$, canonical $so(2,3)$, and the adjoint representations respectively. Further representations may be constructed by taking direct sums or tensor products of these representations, though they will not necessarily be irreducible. All finite dimensional representations will be decomposable into direct sums of irreducible representations.

Appendix B

Proofs of Tensor Properties

In this appendix we will give the proofs of the propositions mentioned in section 4.1.

B.1 Tensor Derivations

Proposition B.1 *The following holds for all tensor derivations:*

i) If D and E are tensor derivations then so is [D,E].

ii) Every tensor derivation has a rank $\binom{i}{j}$ *and maps tensors of rank* $\binom{k}{l}$ *to tensors of rank* $\binom{k+i}{l+j}$.

iii) If D is a tensor derivation and S any tensor, then $S \otimes D$ *is a tensor derivation where* $(S \otimes D)(T) = S \otimes D(T)$

Proof.

For *i).* we must show that the commutator bracket of two tensor derivations satisfies linearity, the Leibnitz condition on tensor products and commutes with contraction:

Let a be some scalar, X and Y tensors and D and E are tensor derivations

Linearity.

$$\begin{aligned}
[D,E]aX &= DE(aX) - ED(aX) \\
&= D(aE(X)) - E(aD(X)) \\
&= aD(E(X)) - aE(D(X)) \\
&= a[D,E]X
\end{aligned}$$

$$\begin{aligned}
[D,E](X+Y) &= DE(X+Y) - ED(X+Y) \\
&= D(E(X)+E(Y)) - E(D(X)+D(Y)) \\
&= DE(X) + DE(Y) - ED(X) - ED(Y) \\
&= [D,E](X) + [D,E](Y)
\end{aligned}$$

Leibnitz.

$$\begin{aligned}
[D,E](XY) &= DE(XY) - ED(XY) \\
&= D(E(X)Y + XE(Y)) - E(D(X)Y + XD(Y)) \\
&= D(E(X))Y + E(X)D(Y) + D(X)E(Y) + XD(E(Y)) \\
&\quad -E(D(X))Y - D(X)E(Y) - E(X)D(Y) - XE(D(Y)) \\
&= (DE(X) - ED(X))Y + X(DE(Y) - ED(Y)) \\
&= [D,E](X)Y + X[D,E](Y)
\end{aligned}$$

Commutation with contraction.

As D and E are tensor derivations, they individually conserve contraction. Therefore any combination of the two will preserve the contraction, hence the commutator bracket will also preserve it. Thus [D, E] commutes with contraction.

Thus, *i)* is proven.

For *ii)*, let f be a function on the manifold, and consider it a tensor of rank zero. Consider the tensor derivation D, which maps tensors of rank zero to tensors D(f) of rank $\binom{i}{j}$. Let T be a tensor of rank $\binom{k}{l}$. Consider the action of D on the product fT:

$$D(fT) \;=\; D(f)T + fD(T)$$

We note that D(f)T has rank $\binom{i+k}{j+l}$ thus so does the second term. But f is rank zero therefore D(T) must have rank $\binom{i+k}{j+l}$. We conclude that a tensor derivation maps tensors of rank $\binom{k}{l}$ to tensors of rank $\binom{i+k}{j+l}$, thus we associate $\binom{i}{j}$ with the tensor derivation and call it the rank of the tensor derivation.

Thus *ii)* is proven.

iii). follows from the definition. Note

$$(S \otimes D)(T) = S \otimes D(T)$$

Therefore if T can be written as a tensor product XY, we see the above equivalent to

$$\begin{aligned} S \otimes D(T) &= S \otimes D(XY) \\ &= (S \otimes D(X))Y + X(S \otimes D(Y)) \end{aligned}$$

Thus *iii)* is proven.

□

Proposition B.2 *If E is a tensor derivation of rank $\binom{0}{0}$ with $E(f) = 0$ for*

all functions f on $\mathcal{M}$, then there exists a tensor Γ^i_j of rank $\binom{1}{1}$ so that

$$E(X^{\alpha_1\alpha_2\ldots\alpha_m}_{\beta_1\beta_2\ldots\beta_n}) = \sum_s \Gamma^{\alpha_s}_{\hat{\alpha}_s} X^{\alpha_1\ldots\hat{\alpha}_s\ldots\alpha_m}_{\beta_1\beta_2\ldots\beta_n} - \sum_t \Gamma^{\hat{\beta}_t}_{\beta_t} X^{\alpha_1\alpha_2\ldots\alpha_m}_{\beta_1\ldots\hat{\beta}_t\ldots\beta_n}$$

Proof. Let v be a vector field and f be a scalar field. The action of E on fv is given by $E(fv) = E(f)v + fE(v)$. Hence it acts linearly on a vector field and thus is equivalent to contraction with a tensor of rank $\{1, 1\}$.

Now, if we consider the vector fields $\{e_i\}$, that form a basis of the tangent space at each point, then v may be written $v = v^i e_i$. Considering the action of E on v, we obtain $E(v) = E(v^i)e_i + v^i E(e_i) = v^i \Gamma^j_i e_j$. Letting coordinates describe tensors we write: $E(v^i) = \Gamma^i_t v^t$.

Considering this action on the product $u_i v^i$ we find

$$\begin{aligned} E(u_i v^i) &= E(u_i)v^i + u_i E(v^i) \\ 0 &= E(u_i)v^i + u_i \Gamma^i_t v^t \end{aligned}$$

As v is arbitrary we can conclude that $E(u_i) = -\Gamma^t_i u_t$.

This argument may be extended inductively to tensors of arbitrary rank, thus proving the original proposition.

□

Proposition B.3 *Every tensor derivation of rank $\binom{m}{n}$ takes the form:*

$$D^{\lambda_1\ldots\lambda_m}_{\mu_1\ldots\mu_n} = (a^{\lambda_1\ldots\lambda_m}_{\mu_1\ldots\mu_n})^i \frac{\partial}{\partial x^i} + \Gamma^{\lambda_1\ldots\lambda_m}_{\mu_1\ldots\mu_n} \binom{*}{*}$$

where

$$\Gamma^{\lambda_1\ldots\lambda_m}_{\mu_1\ldots\mu_n} \binom{*}{*} (T^{\alpha_1\alpha_2\ldots\alpha_m}_{\beta_1\beta_2\ldots\beta_n}) = \sum_s (\Gamma^{\lambda_1\ldots\lambda_m}_{\mu_1\ldots\mu_n})^{\alpha_s}_{\hat{\alpha}_s} X^{\alpha_1\ldots\hat{\alpha}_s\ldots\alpha_m}_{\beta_1\beta_2\ldots\beta_n} - \sum_t (\Gamma^{\lambda_1\ldots\lambda_m}_{\mu_1\ldots\mu_n})^{\hat{\beta}_t}_{\beta_t} T^{\alpha_1\alpha_2\ldots\alpha_m}_{\beta_1\ldots\hat{\beta}_t\ldots\beta_n}$$

This is proven in by observing that the components of $D^{\lambda_1 \dots \lambda_m}_{\mu_1 \dots \mu_n}$ are all derivations of rank $\binom{0}{0}$.

□

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