Finding p-adic zeroes of the Kubota-Leopoldt zeta-function numerically

A thesis

submitted in partial fulfilment

of the requirements for the Degree

of

Master of Science

at the

University of Waikato

by

Nof Turki S Alharbi

THE UNIVERSITY OF

WAIKATO

Te Whare Wānanga o Waikato

University of Waikato

2014

Abstract

We first establish why the p-adic zeta function has a Dirichlet series expansion. We then compute an improved expansion, which allows us to express it as a power-series modulo p^n. Using this expansion, we compute all the zeros of $L_p(s, \chi\omega^j)$ for those quadratic characters χ of conductor < 200.

Acknowledgement

I take this opportunity to express my gratitude to the people who have been instrumental in the successful completion of this thesis. I would like to express the deepest appreciation to my supervisor Dr Daniel Delbourgo for the useful comments, remarks and engagement through the learning process of this master thesis. Furthermore I would like to acknowledge with much appreciation the king Abdullah scholarships program, who gave me this chance for foreign study. Last but not least, I would like to thank my husband, Abdulrahman, for his love, kindness and support he has shown during the past year it has taken me to finalize this thesis.

Contents

Introduction

The Riemann zeta function is one of the most studied objects in mathematics. If $s = \sigma + it$ with $\sigma > 1$, it is given by the Dirichlet expansion

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

and can be analytically continued to the punctured plane $\mathbb{C} - \{1\}$ with a simple pole at $s = 1$. This function was introduced by Leonhard Euler. He studied this function in the first half of the 18^{th} century without reference to the complex numbers. He also found the connection between the Riemann zeta function and prime numbers. This connection has its genesis in the Euler product formula. In addition, Euler computed the values of the Riemann zeta function at even positive integers and at negative integers, which yields a direct relationship to the Bernoulli numbers.

Later in 1859, Bernhard Riemann extended the domain of the zeta function to the complex plane. He studied zeros of the zeta function, which gave us a clear picture expressing the connection between the zeta function's zeros and the distribution of prime numbers.

The Riemann zeta function vanishes at the negative even integers as a consequence of the fact that $B_k = 0$ for odd $k > 1$. The negative even integers are called the trivial zeros. However the non-trivial zeros have taken far more attention, since their distribution is far less developed. Studying the distribution of non-trivial zeros yields impressive results related to the distribution of prime numbers. The non-trivial zeros lies in the strip

$$\{s \in \mathbb{C}, 0 < Re(s) < 1\},$$

in other words the non-trivial zeros lie to the left of $\sigma = 1$, and to the right of $\sigma = 0$.

A century later in 1964, Kubota and Leopoldt extended the domain of definition for the Riemann zeta function to the p-adic numbers. They gave a construction of a p-adic zeta function interpolating the classical L-function. The values of their p-adic L-function are the essentially same as the Riemann zeta function at negative odd integers. Beside this there is another expansion for the L-function established by Iwasawa and Coleman. They interpreted the p-adic L-function in terms of Iwasawa Theory.

The purpose of this thesis is to compute the zeros of the p-adic L-function twisted by a quadratic character. Firstly, we understand the relationship between the zeros of $L_p(s, \chi_d\omega^{1+\beta})$ and the zeros of a certain power series $F_\chi(T)$. The number of zeros of $F_\chi(T)$ is exactly the λ zeros of the distinguished polynomial in its Weierstrass factorization. We then compute the number of zeros associated to $L_p(s, \chi_d\omega^{1+\beta})$, and the coefficients of the p-adic power series $F_\chi(T)$. Finally, we extract the Iwasawa polynomial from the computed coefficients.

Here is a detailed plan. In the first chapter, I gather the basic information about p-adic numbers and p-adic analysis. Also the values of $\zeta(s)$ are expressed in term of the Bernoulli numbers for all negative integers and even positive integers. Moreover, we give a quick glimpse on some classical results on Dirichlet's L-function.

In the following chapter, Kubota-Leopoldt's p-adic L-function is defined, which interpolates the complex L-function associated to Dirichlet characters. Also in this chapter, I present an overview of the relevant distribution theory, which is an important tool in extending the definition of the p-adic zeta function. In particular, I introduce the k-th Bernoulli distribution and associated p-adic measure.

In the third chapter, we prove expansions for the p-adic zeta function. Quite surprisingly the Kubota and Leopoldt zeta function also has a Dirichlet series expansion. More precisely, we show that

$$\zeta(-s, \omega^{1+\beta}) = \frac{1}{2(1-\omega^{1+\beta}(2) < 2 >^{1+s})} \sum_{n=1}^{\infty} \Big(\sum_{\substack{m=p^{n-1} \\ p \nmid m}}^{p^n} (-1)^{m+1} \omega^{\beta}(m) < m >^{s} \Big).$$

The proof of Delbourgo [2] is related to p-adic fractional derivatives.

In Chapter 4, I express the p-adic L-function in the form of a power series. We will define the λ-invariant of $F_\chi(T)$, and similarly the λ-invariant of the distinguished polynomial $P_\chi(T)$. Chapter 5 gives our main steps for the computation in a PARI program; moreover we write our implementation for the approximations in a GP/script.

Finally in Chapter 6, we firstly tabulate λ-invariants associated to $L_p(s, \chi_d\omega^{1+\beta})$. We then tabulate the coefficients of the power series, and the Iwasawa polynomial for primes 3 and 5. This yields some new zeros for the p-adic zeta-function, additional to those computed in [4].

1 p-adic numbers

The purpose of this chapter is to develop some basic ideas of p-adic analysis. Here, we present the concept of distance (metric) between two rational numbers with some examples. In addition, some basic information about p-adic numbers is given. One can express any natural number as a sequence $(a_i)_{i\in\mathbb{N}}$ of p-adic digits. Also, I will define the Riemann zeta function with the Dirichlet character and properties of the classical Dirichlet L-function. Moreover, some details about generalised Bernoulli numbers is given, which are needed for some values of complex zeta function and Dirichlet L-function.

Definition 1.1.
Let X be a non-empty set, and $\ell : X \times X \to [0, \infty)$ a function. Then (X, ℓ) is called a metric space if for all x, y and $z \in X$:

1. $\ell(x, y) = 0$ *if and only if* $x = y$;
2. $\ell(x, y) = \ell(y, x)$;
3. $\ell(x, y) \leqslant \ell(x, z) + \ell(z, y)$.

Definition 1.2.
A norm $\|\|$ on a field F is a map from F to the non-negative real numbers, such that

1. $||x|| = 0$ *if and only if* $x = 0$;
2. $\|xy\| = \|x\|\|y\|$;
3. $\|x + y\| \leq \|x\| + \|y\|$;

for all $x, y \in F$.

It is clear that any metric is induced by a norm by the value $\ell(x, y) = ||x - y||$. For example, the absolute value on the rational number field $\mathbb{Q}$ is a norm, while the distance metric on $\mathbb{Q}$ is induced by $\ell(x, y) = |x - y|$. Furthermore, another metric on $\mathbb{Q}$ which is related to our topic is the p-adic metric $||_p$ where

$$|x|_p = \begin{cases} p^{-ord_p x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

and $ord_p x$ is called the p-adic valuation; it is defined as the highest power of p dividing x, where x is taken from $\mathbb{Q}$.

Definition 1.3.
A metric (respectively a norm) is called non-Archimedean if it satisfies the following stronger inequality respectively:

$$\ell(x, y) \leqslant max(\ell(x, z), \ell(z, y)),$$

$$||x+y|| \leq max(||x||, ||y||),$$

for all x, y and $z \in F$.

In other words, a metric is non-Archimedean if it comes from non-Archimedean norm as follows:

$$\begin{aligned}
\ell(x,y) &= ||x-y|| \\
&= ||(x-z)+(z-y)|| \\
&\leq max(||x-z||, ||z-y||) = max(\ell(x,z), \ell(z,y)).
\end{aligned}$$

Therefore, a non-Archimedean norm $|\quad|_p$ on $\mathbb{Q}$ induces a non-Archimedean metric on $\mathbb{Q}$. Likewise the ordinary absolute value $|\quad|$ which is Archimedean induced from Archimedean norm.

Definition 1.4.
A Cauchy sequence in a normed field F is a sequence (a_n) such that for all $\varepsilon > 0$, there exists $N > 0$ depending on ε, such that

$$|a_n - a_m| < \varepsilon$$

for all $n, m \geq N$.

If every Cauchy sequence has a limit in F, then F is complete. For example, most Cauchy sequences in $\mathbb{Q}$ do not converge to an element in $\mathbb{Q}$, hence the rational numbers $\mathbb{Q}$ are not complete with respect to the ordinary absolute value $|\quad|$.

Definition 1.5.
The p-adic rational numbers $\mathbb{Q}_p$ denotes the completion of $\mathbb{Q}$ with respect to the p-adic norm, where p is a fixed prime.

All elements of $\mathbb{Q}_p$ can be expressed in the form:

$$a_m p^m + a_{m+1}p^{m+1} + a_{m+2}p^{m+2} + \ldots$$

where $a_i \in \{0, 1, 2, \ldots, p-1\}$ and m is any integer. In addition, all $a \in \mathbb{Q}_p$ with $|a|_p \leq 1$ are called p-adic integers $\mathbb{Z}_p$ (which is a subring of $\mathbb{Q}_p$), so that

$$\mathbb{Z}_p = \{a \in \mathbb{Q}_p : |a|_p \leq 1\} = \{a \in \mathbb{Q}_p : ord_p(a) \geq 0\}.$$

Moreover, every element a in $\mathbb{Z}_p$ can be express p-adically with no negative power of p,

$$a = a_0 + a_1 p + a_2 p^2 + \cdots = \sum_{i=0}^{\infty} a_i p^i.$$

Since a has a unique representative Cauchy sequence a_i, where a is the limit of this sequence $0 \leq a_i \leq p^i$, for all $i = 1, 2, 3$ and $a_i \equiv 0 \bmod p^n$.

In particular, elements a of $\mathbb{Z}_p$ with $|a|_p = 1$ form a multiplicative subgroup of $\mathbb{Z}_p$ called "p-adic units". These p-adic units can be identified via:

$$\mathbb{Z}_p^* = \Big\{ \sum_{i=0}^{\infty} a_i p^i : a_i \in \{0, 1, 2, \dots, p-1\}, a_0 \neq 0 \Big\},$$

while

$$\mathbb{Z}_p = \Big\{ \sum_{i=0}^{\infty} a_i p^i : a_i \in \{0, 1, 2, \dots, p-1\} \Big\}.$$

Example 1.1.
We now describe an expression for $\sqrt{-1}$ *in* $\mathbb{Z}_5$*. Firstly, if*

$$\sqrt{-1} = a_0 + a_1 5 + a_2 5^2 + a_3 5^3 + \dots$$

then

$$-1 = (a_0^2 + 2a_0 a_1 5 + 5^2 a_2 a_0 + \dots)^2$$

i.e.

$$-1 = a_0^2 + 2a_0 a_1 5 + 5^2 a_2 a_0 + \dots.$$

Reducing modulo 5*, one obtains*

$$-1 \equiv a_0^2 \bmod 5 \quad \Longrightarrow a_0 = 3$$

$$-1 \equiv a_0^2 + 2a_0 a_1 5 \bmod 5^2$$

then

$$-1 \equiv 9 + 30a_1 \bmod 5^2 \quad \Longrightarrow a_1 = 4$$

$$-1 \equiv a_0^2 + 2a_0 a_1 5 + 5^2 a_2 a_0 \bmod 5^3$$

in which case

$$-1 \equiv 9 + 120 + 75a_2 \bmod 5^3 \quad \Longrightarrow a_2 = 0$$

so $\sqrt{-1} \in \mathbb{Z}_5$.

However, $\sqrt{-1} \notin \mathbb{Z}_7$ *because*

$$a_0^2 \equiv 6 \bmod 7$$

is impossible to solve in integers, where a_i *is taken from the set* $\{0, 1, 2, \dots, 6\}$.

In $\mathbb{Q}_p$ we define $D_a(c) = \{x : |x - a|_p \leq c\}$ to be the closed disc of radius c centered at a, and $D_a(\bar{c}) = \{x : |x - a|_p < c\}$ to be the open disc of radius centered at a. If $b \in D_a(c)$ then

$$|b - a|_p \leq c$$

and since $| \quad |_p$ is non-Archimedean norm on $\mathbb{Q}_p$, the latter implies that

$$|b-a|_p \leq max(|b|_p, |a|_p).$$

Then any interior point in a disc is its center.

In particular, the rational integers $\mathbb{Z}_p$ is a disc in $\mathbb{Q}_p$ centered at zero, so that

$$\mathbb{Z}_p = D_0(1) = \{a : |a|_p \leq 1\},$$

and it is viewed as the closure of the ordinary integers $\mathbb{Z}$ in $\mathbb{Q}_p$.

Now consider the set $p\mathbb{Z}_p$ is disjoint from $\mathbb{Z}_p^*$, and moreover $\mathbb{Z}_p = p\mathbb{Z}_p \cup \mathbb{Z}_p^*$. In fact $p\mathbb{Z}_p$ is a unique maximal ideal. Any element $a \in p\mathbb{Z}_p$ with valuation $ord_p a = k$ implies $a \notin p^{k+1}\mathbb{Z}_p$, so that

$$\mathbb{Z}_p \supset p\mathbb{Z}_p \cdots \supset p^k\mathbb{Z}_p \supset \cdots \supset \cap_{k\geq 0} p^k\mathbb{Z}_p = \{0\}.$$

Therefore $p^k\mathbb{Z}_p, k \in \mathbb{N}$, gives a sequence of neighbourhoods of zero.

1.1 Teichmüller representative

The Teichmüller character is a homomorphism of multiplicative groups $\omega : \mathbb{Z}_p^* \to \mathbb{Z}_p^*$ where for each $a \in \mathbb{Z}_p^*$, $\omega(a)$ is the unique $(p-1)^{st}$ root of unity such that $\omega(a) \equiv a \bmod p$. Here $\omega(a)$ is called the Teichmüller representative of a. Each $a \in \mathbb{Z}_p^*$ can be uniquely decomposed in the form $a = \omega(a) < a >$ where $< a >$ lies in the principal units

$$1 + p\mathbb{Z}_p = \{1 + pa : a \in \mathbb{Z}_p\}.$$

In addition, any $x \in \mathbb{Q}_p$ can be written as

$$x = p^{ord_p x} a, \quad a \in \mathbb{Z}_p^*$$

and decomposed further into

$$x = p^{ord_p x}\omega(a) < a > .$$

Example 1.2.
In the group $\mathbb{Z}_5^*$, we shall now compute $\omega(2)$ and $< 2 >= 2/\omega(2)$. Firstly

$$\omega(2) = a_0 + 5a_1 + 5^2 a_2 + \dots$$

and

$$\omega(a) \equiv a \bmod p,$$

in which case

$$a_0 \equiv 2 \bmod 5 \quad \implies a_0 = 2.$$

Also $\omega(a)^{p-1} \equiv 1$ *in* $\mathbb{Z}_p$*, hence*

$$1 \equiv \omega(2)^4 = (2 + 5a_1)^4 \bmod 25$$
$$= 2^4 + 4 \times 2^3 \times 5a_1 + \ldots \bmod 25$$

implies that

$$a_1 = 1$$

and

$$\omega(2) = 7 + 5^2 a_2.$$

Once again $\omega(a)^{p-1} \equiv 1$ *in* $\mathbb{Z}_p$*, so that*

$$1 = \omega(2)^4 = (7 + 5^2 a_2 + \ldots)4 \bmod 125$$

implies $a_2 = 2$*, and* $\omega(2) = 57$.

Lastly $< 2 >\equiv 2/\omega(2) \bmod 5^3$*, which means*

$$\omega(2)^{-1} \equiv 57^{-1} \equiv -57 \equiv 68 \bmod 5^3$$

and

$$< 2 >\equiv 2 \times 68 \equiv 11 \bmod 5^3.$$

It can be concluded that, in $\mathbb{Z}_5^*$*, one can express* 2 *as*

$$2 \equiv \omega(2) \times < 2 >\equiv 57 \times 11 \bmod 5^3.$$

In example 1.1, we found $\sqrt{-1} \notin \mathbb{Z}_7$, because the congruence $a_0^2 \equiv 6 \bmod 7$ can not be solved. Solving this congruence is the same solving the equation $x^2 - 6 = 0$ in $\mathbb{Z}_7$. In particular, let α be the solution for the above equation in $\mathbb{Z}_7$, then

$$\alpha = a_0 + a_1 7 + a_2 7^2 + \ldots.$$

Therefore, solving an equation in the p-adic integers means solving each coefficient of $a_0, a_1, a_2, \ldots$ at modulo $p, p^2, p^3, \ldots$ respectively. If one coefficient has no solution, then there is no solution for the equation. This can be easily shown by "Hensel's lemma".

Theorem 1.1. *(Hensel's lemma in [8])*
Let $f(x) \in \mathbb{Z}_p[x]$ *polynomial with p-adic integers coefficients. Let* $a_0 \in \mathbb{Z}_p$ *such that* $f(a_0) \equiv 0 \bmod p$*,* $f'(a_0) \not\equiv 0 \bmod p$*. Then there is a unique* $a \in \mathbb{Z}_p$ *such that* $f(a) = 0$ *and* $a \equiv a_0 \bmod p$*.*

Proof.
We must establish the existence of a sequence of rational integers satisfying:

1. $f(a_n) \equiv 0 \bmod p^{n+1}$;

2. $a_n \equiv a_{n-1} \bmod p^n$;

3. $0 \leq a_n < p^{n+1}, n \geq 1$.

The existence and uniqueness of a_n can be proved by induction on n. If $n = 1$, then choose $\check{a_0}$ to be the unique integer in $\{0, 1, \dots, p-1\}$ satisfying

$$\check{a_0} \equiv a_0 \bmod p.$$

Now putting $a_1 = \check{a_0} + b_1 p$, where $0 \leq b_1 \leq p-1$ as a_1 satisfies (2) and (3), then

$$\begin{aligned} f(a_1) = f(\check{a_0} + b_1 p) &= \sum_{i=0} c_i(\check{a_0} + b_1 p)^i \\ &= \sum c_i \check{a_0}^i + (\sum i c_i \check{a_0}^{i+1}) b_1 p \bmod p^2 \\ &= f(\check{a_0}) + f'(\check{a_0}) b_1 p. \end{aligned}$$

Since $f(a_0) \equiv 0 \bmod p$ implies that

$$f(\check{a_0}) \equiv \alpha p \bmod p^2 \quad , \alpha \in \{0, 1, \dots, p-1\}, \tag{1.1}$$

we need $f(a_0) \equiv 0 \bmod p^2$. We must therefore have

$$\alpha p + f'(\check{a_0}) b_1 p \equiv 0 \bmod p^2$$

implies

$$\alpha + f'(\check{a_0}) b_1 \equiv 0 \bmod p. \tag{1.2}$$

By the assumption $f'(\check{a_0}) \not\equiv 0 \bmod p$ in this theorem, and (1.2) can be solved by choosing $b_1 \in \{0, 1, \dots, p-1\}$, so b_1 is determined uniquely.

Suppose we have obtained $a_1, a_2, \dots, a_{n-1}$. Then as in the first case,

$$f(a_n) = f(\check{a_{n-1}} + b_n p^n) = f(\check{a_{n-1}}) + f'(\check{a_{n-1}}) b_n p^n \quad \bmod p^{n+1}.$$

By the assumption in the theorem, $f'(a_{n-1}) \not\equiv 0 \bmod p^n$ hence

$$f(\check{a_{n-1}}) \equiv \alpha p^n \bmod p^{n+1}, \quad \alpha \in \{0, 1, \dots, p^n - 1\}$$

which implies that

$$\alpha p^n + f'(\check{a_{n-1}}) b_n p^n \equiv 0 \bmod p^{n+1} \tag{1.3}$$

and

$$\alpha + f'(\check{a_{n-1}}) b_n \equiv 0 \bmod p. \tag{1.4}$$

As we did before, (1.4) can be solved by choosing $b_1 \in \{0, 1, \dots, p-1\}$, so b_1 is determined uniquely. Moreover this satisfies $f(a_n) \equiv 0 \bmod p^{n+1}$, therefore

$$a = \check{a_0} + b_1 p + b_2 p^2 + \dots$$

such that

$$f(a) \equiv f(a_n) \equiv 0 \bmod p^{n+1}$$

and then $f(a) = 0$ as required.

□

1.2 Zeta functions and Bernoulli numbers

Here the Riemann zeta function is introduced, which is a key player in of our thesis.

Definition 1.6.
The classical Riemann ζ-function is defined by the formula

$$\zeta(s) = \sum_{n\geq 1} 1/n^s, \qquad s = \sigma + it \in \mathbb{C}$$

with σ and t real numbers and $\sigma > 1$.

Proposition 1.1.
The sum of the infinite series $\sum_{n\geq 1} 1/n^s$ converges when $Re(s) > 1$.

Proof.
If $s = \sigma + it$, then

$$\begin{aligned}\zeta(\sigma + it) &= \sum_{n\geq 1} \frac{1}{n^s} \\ &= \sum_{n\geq 1} \frac{1}{n^{\sigma+it}} \\ &= \sum_{n\geq 1} \frac{1}{n^{\sigma} n^{it}} \\ &= \sum_{n\geq 1} \frac{e^{-it}\log n}{n^{\sigma}}.\end{aligned}$$

We can see $\sum_{n\geq 1} \frac{1}{n^\sigma}$ is convergent for $\sigma > 1$, so that

$$\begin{aligned}\left|\zeta(s)\right| &\leq \sum_{n\geq 1} \frac{1}{n^{\sigma}} \\ &\leq 1 + \int_1^{\infty} \frac{1}{x^{\sigma}} dx \\ &= 1 + \frac{1}{\sigma - 1}.\end{aligned}$$

□

There is a famous expansion for the Riemann zeta function

$$\zeta(s) = \prod_p \frac{1}{1 - (1/p^s)}$$

which is known by Euler product formula.

Now, we present the definition of Bernoulli numbers. The purpose of that is to express ζ in terms of the Bernoulli numbers for all the negative integers, and all the even positive integers. Consider Taylor series expansion of

$$F(t) = \frac{te^t}{e^t - 1}$$

about $t = 0$.

Definition 1.7.
Expanding $F(t)$ into a power series of t

$$F(t) = \sum_{n \geq 0} B_n \frac{t^n}{n!}$$

the coefficients B_n are called Bernoulli numbers.

For example, the first few B_n are $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 = -\frac{1}{30}$, where $B_n = 0$ for odd $n > 1$, since $F(-t) = F(t) - t$.

Definition 1.8.
Consider the function F(t) in two variables t and x :

$$F(t, x) = \frac{te^{xt}}{e^t - 1} = (\sum_{n=0}^{\infty} B_n \frac{t^n}{n!})(\sum_{n=0}^{\infty} \frac{(xt)^n}{n!}).$$

Then

$$B_n(x) = \sum_{i=0}^{n} \left(\begin{array}{c} n \\ i \end{array} \right) B_i x^{n-i}, n \geq 0$$

where the $B_n(x)$ are "Bernoulli polynomials" with rational coefficients.

The first few $B_n(x)$ are $B_0(x) = 1$, $B_1(x) = x - \frac{1}{2}$, $B_2(x) = x^2 - x + \frac{1}{6}$, $B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$. It is also clear that $B_n(0) = B_n$.

1.2.1 Dirichlet characters

Definition 1.9.
Let n be a positive integer. A map $\chi : \mathbb{Z} \to \mathbb{C}$ is called a Dirichlet character to the modulus n if it satisfies:

1. *$\chi(a)$ depends only on the residue class of $a \bmod n$;*
2. *χ is multiplicative, i.e. for any $a, b \in \mathbb{Z}$ $\chi(ab) = \chi(a)\chi(b)$;*
3. *$\chi(a) \neq 0$ if and only if $(a, n) = 1$.*

Definition 1.10.
Let χ' be a Dirichlet character to a modulus m, where $m|n$ and $m < n$. The character χ is said to be induced from χ', if for all $a \in \mathbb{Z}$, $\chi(a)$ is given by

$$\chi(a) = \begin{cases} \chi' & \text{if } (a, n) = 1 \\ 0 & \text{if } (a, n) > 1. \end{cases}$$

Then χ is a Dirichlet character to the modulus n. Otherwise, if χ is not induced from any character to a modulus m, then χ to modulus n is primitive.

If the character χ to a modulus n is primitive, then n is the conductor of χ and it is denoted by f_χ. Also, if χ_1, χ_2 are two primitive Dirichlet characters and f_1 and f_2 be conductors respectively, then there is a unique primitive Dirichlet character χ with conductor f dividing $f_1 f_2$, such that

$$\chi(a) = \chi_1(a)\chi_2(a), \qquad (a, f_1 f_2) = 1.$$

The set of all primitive Dirichlet characters form abelian group. The identity of the group is the principal character defined as $\chi^0(a) = 1$ of conductor 1.

1.2.2 Generalization of B_n and $B_n(x)$

Let χ be a Dirichlet character with conductor f. Now consider

$$F_\chi(t) = \sum_{a=1}^{f} \frac{\chi(a) t e^{at}}{e^{ft} - 1}$$

so that

$$F_\chi(t, x) = F_\chi(t) e^{xt} = \sum_{a=1}^{f} \frac{\chi(a) t e^{(a+x)t}}{e^{ft} - 1}.$$

By expanding this as a power series in t,

$$F_\chi(t) = \sum_{n=0}^{\infty} B_{n,\chi} \frac{t^n}{n!}$$

and

$$F_\chi(t, x) = \sum_{n=0}^{\infty} B_{n,\chi}(x) \frac{t^n}{n!}.$$

In particular, one can write

$$B_{n,\chi}(x) = \sum_{i=0}^{n} \binom{n}{i} B_{i,\chi} x^{n-i}.$$

Let $\mathbb{Q}(\chi)$ denotes the field generated over $\mathbb{Q}$ by the values of $\chi(a)$. Thus $B_{n,\chi}$ lies in $\mathbb{Q}(\chi)$, and it is called generalised Bernoulli numbers. Also $B_{n,\chi}(x)$ is generalised Bernoulli polynomials in $\mathbb{Q}(\chi)[x]$.

In the case $\chi = \chi^0$ and $f = 1$,

$$F_\chi(t) = F(t) \quad and \quad F_\chi(t,x) = F(t,x)$$

therefore

$$B_{n,\chi^0} = B_n, \quad B_{n,\chi^0}(x) = B_n(x).$$

1.3 The values of $\zeta(s)$ at negative and positive integers

The interesting feature of the Riemann zeta function is that it can be expressed at negative integers in terms of Bernoulli numbers. This means zeta functions have rational values, and therefore, we can interpolate zeta functions p-adically. Here we present two important special values of zeta function as follows:

1. The value at negative integers:

$$\zeta(1-n) = \frac{-B_n}{n}, \quad n \in \mathbb{N}.$$

   One can see $\zeta(s)$ is zero when n takes even negative integers values, because Bernoulli numbers equal zero when n is odd. They are called trivial zeros of Riemann zeta functions.

2. The value at even positive integers:

$$\zeta(2n) = (-1)^n \pi^{2n} \frac{2^{2n-1}}{(2n-1)!} \frac{-B_{2n}}{2n}, \quad n \in \mathbb{N}.$$

   Moreover, the functional equation of the complex zeta function relates $\zeta(1-2n)$ with value of $\zeta(2n)$, i.e.

$$\begin{aligned}
\zeta(2n) &= (-1)^n \pi^{2n} \frac{2^{2n}}{2(2n)!} - B_{2n} \\
&= (-1)^n (2\pi)^{2n} \frac{-B_{2n}}{2n(2n-1)!} \\
&= (-1)^n \frac{(2\pi)^{2n}}{(2n-1)!} \frac{-B_{2n}}{2n} \\
&= (-1)^n \frac{(2\pi)^{2n}}{(2n-1)!} \zeta(1-2n).
\end{aligned}$$

1.4 The Dirichlet L-function

Definition 1.11.
Let χ be a Dirichelet character. The Dirichelet L-function associated to χ is defined by

$$L(s,\chi) = \sum_{n\geq 1} \chi(n) n^{-s},$$

for Re$(s) > 1$.

If $\chi = \chi^0$ then $L(s,\chi^0) = L(s,1) = \zeta(s)$.

Remark 1.1.

1. *Dirichlet's L-functions can similarly be written as an Euler product*

$$L(s,\chi) = \prod_p \big(1 - \frac{\chi(p)}{p^s}\big)^{-1},$$

where Re$(s) > 1$ and the product is taken over all primes p. In particular, we can view $\zeta(s)$ as a Dirichlet L-function for the principal character $\chi^0 \bmod 1$.

2. *The values*

$$L(1-k,\chi) = -\frac{B_{k,\chi}}{k}$$

are rational numbers where k is positive integer, and $B_{k,\chi}$ is generalized Bernoulli numbers.

2 The method of p-adic interpolation

This chapter contains all the necessary background material on p-adic L-functions attached to abelian number fields, through their twists by Dirichlet characters. We employ the language of measure theory.

2.1 Kubota-Leopldt p-adic L-function

The main purpose of this section is to construct a p-adic analogue of the classical Dirichlet L-function $L(s, \chi)$. Kubota-Leopoldt solved this problem by constructing a p-adic L-function which takes the same values as $L(s, \chi)$ when $s \in \{0, -1, -2, \dots\}$. For the classic construction, I follow Iwasawa's book [6].

From now let p be an odd prime number, $\mathbb{Z}_p$ be the p-adic integers, and let $\mathbb{C}_p$ denote the completion of the algebraic closure of $\mathbb{Q}_p$. If we use $|p| = p^{-1}$ to denote the normalization of the p-adic absolute value, then $\mathbb{C}_p$ is a topological field in the metric defined by the p-adic valuation, where the topology is the p-adic topology of $\mathbb{Q}_p$.

Define $\omega(a)$ to be the unique $(p-1)^{st}$ root of unity in $\mathbb{Z}_p$ satisfying $\omega(a) \equiv a \bmod p$, for each $a \in \mathbb{Z}_p^*$, where ω is the p-adic Teichmüllier character. In fact, ω is a Dirichlet character on $\mathbb{Z}$ of order $p-1$ and conductor p. Let K be a finite extension of $\mathbb{Q}_p$ in $\mathbb{C}_p$, and define

$$K[[x]] = \left\{A = A(x) = \sum_{i=0}^{\infty} a_i x^i : a_i \in K\right\}$$

to be the ring of all power series in x, then $A(x)$ converges at $x = s$ in $\mathbb{C}_p$, if and only if $|a_i s^i|_p \to 0$ as $i \to \infty$.

Definition 2.1.
Let P_K denote the set of all power series A in $K[[x]]$ with $||A|| < \infty$.

According to the definition of a norm, $||A||$ be a norm on P_K if

1. $||A|| \geq 0$ and $||A|| = 0$ if and only if $A = 0$;
2. $||A + B|| \leq \max(||A||, ||B||)$ and $A, B \in P_K$;

3. $||cA|| = |c|||A||$ and $c \in K$;
4. $||AB|| \leq ||A|| \times ||B||$ and $A, B \in P_K$.

Therefore, P_K is a subalgebra of $K[[x]]$ containing the polynomial ring $K[x]$, i.e.

$$K[x] \subseteq P_K \subseteq K[[x]]$$

Lemma 2.1.
P_K is complete in the norm $||A||$ so that it has the structure of a Banach algebra over the local field K.

Proof.
Let A_n be a Cauchy sequence in P_K with respect to $|| \quad ||$, say that

$$A_n = \sum_{k=0}^{\infty} a_{n,k} T^k, \quad a_{n,k} \in K$$

The lemma can be proved by the following three steps:

1. For each $k \geq 0, lim_{n\to\infty} a_{n,k} = a_k$ exists in K. This means the sequence$(a_{n,k})$ is convergent
2. $A = A(x) = \sum_{n=0}^{\infty} a_k T^k \in P_K$. if $a_k = \lim_{n\to\infty} a_{n,k}$.
3. $\lim_{n\to\infty} A_n = A$ in the norm topology of P_K.

$Proof\ of$ 1.
For any $\varepsilon > 0, \exists N \in \mathbb{N}$, such that $n, m \geq N$

$$\implies ||A_n - A_m|| < \varepsilon,$$

so

$$||a_{n,k} - a_{m,k}|| < \varepsilon,$$

so for fixed $k, (a_{n,k}) \subseteq K$ is Cauchy so it is convergent.

$Proof\ of$ 2.
Here (A_n) is bounded, as it is Cauchy. Let (A_n) is bounded by $C > 0$, then $\forall n, k$

$$|a_{n,k}| \leq ||A_n|| \leq C,$$

so $|a_k| \leq C$

$$\implies A \in P_K$$

.

$Proof\ of$ 3.
For any $\varepsilon > 0, \exists N$ such that $n, m \geq N$, $\implies ||A_n - A_m|| < \varepsilon$, then for any $k \geq 0$

$$|a_{n,k} - a_{m,k}| \geq ||A_n - A_m|| < \varepsilon$$

so for fixed k

$$|a_k - a_{m,k}| \leq \varepsilon \quad , m \geq N$$

Hence

$$|A - A_m| < \varepsilon$$

so A_n converges to A. $\square$

For any $n \in \mathbb{N}$, we define a polynomial $\binom{x}{n}$ of degree n in $K[x]$ by:

$$\binom{x}{n} = \frac{x(x-1)\dots(x-n+1)}{n!} = \frac{1}{n!}\prod_{k=1}^{n-1}(x-n).$$

It is clear that $||\binom{x}{n}|| \leq |\frac{1}{n!}|$ when $x \in \mathbb{Z}_p$. Now let b_n be a sequence of elements of K, then we define

$$c_n = \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}b_i,$$

so that

$$\sum_{n=0}^{\infty} c_n \frac{t^n}{n!} = \sum_{n=0}^{\infty}(\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}b_i)\frac{t^n}{n!}$$
$$= (\sum_{n=0}^{\infty} b_n \frac{t^n}{n!})(\sum_{n=0}^{\infty}(-1)^n \frac{t^n}{n!}).$$

Then multiplying both side by e^t,

$$\sum_{n=0}^{\infty} b_n \frac{t^n}{n!} = (\sum_{n=0}^{\infty} c_n \frac{t^n}{n!})(\sum_{n=0}^{\infty} \frac{t^n}{n!})$$
$$= \sum_{n=0}^{\infty}(\sum_{i=0}^{n}\binom{n}{i}c_i)\frac{t^n}{n!},$$

so that,

$$b_n = \sum_{i=0}^{n}\binom{n}{i}c_i, \quad n \geq 0.$$

Theorem 2.1.
Let $r \in \mathbb{R}$, such that $0 < r < |p|^{\frac{1}{p-1}}$, and assume $|c_n| \leq Cr^n$, for all $n \in \mathbb{N}$, with $C > 0$. Then there exists a unique power series $A(x)$ in P_K satisfying the following properties:

1. *$A(x)$ converges for $|s| < |p|^{\frac{1}{p-1}} r^{-1}$;*
2. *For all $n \in \mathbb{N}$,*

$$A(n) = b_n.$$

Proof.

1. Let

$$A_k(x) = \sum_{i=0}^{k} c_i \binom{x}{i} = \sum_{n=0}^{\infty} a_n^{(k)} x^n, \quad a_n^{(k)} \in K$$

clearly $A_k(n) = b_n$, $A_k(x)$ is a polynomial of degree $\leq k$, so we have

$$a_n^{(k)} = 0, \quad if k < n.$$

By the given assumption on c_n

$$\left|\left|c_n \binom{x}{n}\right|\right| \leq \left|c_n\right| \times \left|\frac{1}{n!}\right| \leq \left|c_n\right| p^{-\frac{n}{p-1}} \leq C(\left|p\right|^{-\frac{1}{p-1}})^n = C\delta^n \tag{2.1}$$

where $\delta = |p|^{\frac{-1}{p-1}} r < 1$.

For $l > k$,

$$||A_l - A_k|| \leq \max(\left||c_i \binom{x}{i}\right|| \leq C\delta^{k+1}, \quad k < l) \tag{2.2}$$

since $\delta < 1$, so (A_k) is Cauchy sequence, then $A = lim_{k\to\infty} A_k \quad \in P_K$ with respect to $|| \quad ||$. Now let

$$A = \sum_{l=0}^{\infty} a_l x^l \quad and \quad A_k = \sum_{l=0}^{\infty} a_{l,k} x^l$$

then $a_{l,k} \to a_k$ as l increases, and $deg(A_{k-1}) \leq k-1$. It follows that $a_{k,k-1} = 0$ and for $l \geq k$, we use the bound in (2.2),

$$|a_{l,k}| = |a_{l,k} - a_{k,k-1}| \leq ||A_l - A_{k-1}|| \leq C\delta^{k+1}.$$

Consequently

$$|a_{l,k}| \leq C\delta^{k+1},$$

and so $A(x)$ converges at $s \in \mathbb{C}_p$ with

$$|s| < \delta^{-1} = |p|^{\frac{1}{p-1}} r^{-1}.$$

2. Now fix $s \in \mathbb{C}_p$ such that $|s| < \delta^{-1}$. Then $A_k(s) \to A(s)$. Let $b_{l,k} = a_l - a_{l,k}$, so that

$$A(s) - A_k(s) = \sum_{l=0}^{\infty} b_{l,k} s^l \quad as \quad k \to \infty.$$

It is enough to show that $\max(|b_{l,k} s^l|) \to 0$. If $l > k$

$$|a_l s^l - a_{l,k} s^l| = |(a_l - a_{l,k}) s^l| \leq ||A - A_k|| |s|^l \leq C\delta^{k+1} |s|^l$$

so $|b_{l,k}s^l| \leq C\delta^{k+1}|s|^l$ and for $l \leq k$

$$|b_{l,k}s^l| \leq ||A - A_k|||s|^l \leq C\delta^k|s|^l \leq \begin{cases} C\delta^k & \text{if } |s| \leq 1 \\ C(\delta|s|)^k & \text{if } |s| > 1 \end{cases}.$$

If we call $m = \max(\delta, \delta|s|) < 1$, then

$$|b_{l,k}s^l| = |A(s) - A_k(s)| \leq Cm^k, \quad k \geq 0$$

which implies

$$A(s) = \lim_{k\to\infty} A_k(s).$$

Now for each $n \leq k, n \in \mathbb{Z}$ one has

$$A_k(n) = \sum_{i=0}^{k} c_i \binom{n}{i} = b_n$$

hence $A(n) = b_n$.

□

2.2 The classical p-adic L-function

For each integer n, define the character

$$\chi_n = \chi\omega^{-n}$$

where χ is primitive Dirichlet character of conductor f with values in $\mathbb{C}_p$, and ω be Teichmüller character with conductor q. If the conductor f_n of χ_n is coprime to p, then f_n divides fq. Therefore, f_n and f differ only by a factor which is a power of p. So if $a \in \mathbb{Z}$, with $(a,p) = 1$, then $(a, f_n) = (a, f)$ and $\chi_n(a) = \chi(a)\omega(a)^{-1}$.

Let $K = \mathbb{Q}_p(\chi)$ be the generated field over $\mathbb{Q}_p$ by the values of $\chi(a), a \in \mathbb{Z}$. Then K is a finite extension of $\mathbb{Q}_p$ in $\mathbb{C}_p$, as $\chi(a) \in \mathbb{C} \subseteq \mathbb{C}_p$. Let (b_n) be the sequence of elements of K given by

$$b_n = (1 - \chi_n(p)p^{n-1})B_{n,\chi_n}$$

where $n \geq 0$, with B_{n,χ_n} is the generalised Bernoulli number. We now define

$$c_n = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} b_i, \quad n \leq 0.$$

Lemma 2.2.
One has the bound

$$|c_n| \leq |\frac{1}{q^2 f}||q|^n,$$

for all $n \geq 0$.

Proof. For any integer $k \geq 0$ define $S_{n,\chi}(k) = \sum_{a=1}^{k} \chi(a)a^n$, where $n \geq 0$. Now let

$$F_\chi(t,x) - F_\chi(t,x-f) = \sum_{a=1}^{f} \chi(a)t \quad e^{(a+x-f)t},$$

and

$$B_{n,\chi}(x) - B_{n,\chi}(x-f) = n\sum_{a=1}^{f} \chi(a)(a+x-f)^{n-1},$$

for $n \geq 0$. If we replace n by $n+1$ and add the equalities $x = f, 2f, \ldots, kf$, one obtains

$$S_{n,\chi}(kf) = \frac{1}{n+1}(B_{n+1,\chi}(kf) - B_{n+1,\chi}(0)),$$

for $n, k \geq 0$. Now if we let $k = p^h$, one has

$$S_{n,\chi}(p^h f) = \frac{1}{n+1}(B_{n+1,\chi}(p^h f) - B_{n+1,\chi}(0))$$

then

$$B_{n+1,\chi}(x) - B_{n+1,\chi}(0) = (n+1)B_{n,\chi}x + (\text{terms of degree} \geq 2 \text{ in } x).$$

If χ equals the principal character so that $f = 1$, then we obtain

$$B_n = \lim_{h\to\infty} \frac{1}{p^h} S_n(p^h).$$

Therefore, in $\mathbb{Q}_p(\chi)$

$$B_{n,\chi} = \lim_{h\to\infty} \frac{1}{p^h f} S_{n,\chi}(p^h f)$$

since $f_n = p^\alpha f$, in which case

$$\begin{aligned} B_{n,\chi_n} &= \lim_{h\to\infty} \frac{1}{p^h f_n} S_{n,\chi_n}(p^h f_n) \\ &= \lim_{h\to\infty} \frac{1}{p^h f} S_{n,\chi_n}(p^h f) \\ &= \lim_{h\to\infty} \frac{1}{p^h f} \sum_{a=1}^{p^h f} \chi_n(a)a^n. \end{aligned}$$

If we now return to the definition of b_n, one has

$$\begin{aligned} b_n &= (1 - \chi_n(p)p^{n-1})B_{n,\chi_n} \\ &= B_{n,\chi_n} - \chi_n(p)p^{n-1}B_{n,\chi_n} \\ &= \lim_{h\to\infty} \frac{1}{p^h f} \sum_{c=1}^{p^h f} \chi(c)c^n - \lim_{h\to\infty} \frac{\chi_n(p)p^{n-1}}{p^{h-1} f} \sum_{a=1}^{f} \chi_n(a)a^n \end{aligned}$$

$$= \lim_{h\to\infty} \frac{1}{p^h f} \sum_{c=1}^{p^h f} \chi(c) c^n - \lim_{h\to\infty} \frac{1}{p^h f} \sum_{a=1}^{p^{h-1} f} \chi_n(ap)(ap)^n$$

and removing the repeated terms,

$$b_n = \lim_{h\to\infty} \frac{1}{p^h f} \sum_{a=1,(a,p)=1}^{p^h f} \chi_n(a) a^n.$$

However

$$\chi_n(a) a^n = \chi(a)\omega(a)^{-n}\omega(a)^n < a >^n$$
$$= \chi(a) < a >^n$$

so that

$$b_n = \lim_{h\to\infty} \frac{1}{q^h f} \sum_{a=1}^{g^h f} \chi(a) < a >^n \ . \tag{2.3}$$

Now plugging (2.3) into the definition of c_n,

$$c_n = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \lim_{h\to\infty} \frac{1}{q^n f} \sum_{a=1,(a,p)=1}^{g^h f} \chi(a) < a >^i$$

$$= \lim_{h\to\infty} \frac{1}{q^h f} \sum_{a=1,(a,p)=1}^{g^h f} \chi(a) \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} < a >^i$$

$$= \lim_{h\to\infty} \frac{1}{q^h f} \sum_{a=1,(a,p)=1} \chi(a)(< a > -1)^n$$

$$= \lim_{h\to\infty} \frac{1}{q^h f} c_n(h)$$

where one sets

$$c_n(h) = \sum_{a=1,(a,p)=1}^{g^h f} \chi(a)(< a > -1)^n.$$

Now, we will prove that for all $n \geq 0$,

$$\frac{1}{q^h f} c_n(h) \equiv 0 \bmod \frac{q^n}{q^2 f}$$

and

$$c_n(h) \equiv 0 \bmod q^{n+h-2}.$$

We prove it by induction on h. In the base case $h = 1$,

$$< a >\equiv 1 \bmod q$$

so that

$$(< a > -1)^n \equiv 0 \bmod q^n$$

and

$$c_n(1) \equiv 0 \bmod q^n.$$

Now assume $h \geq 1$, and suppose that

$$c_n(h) \equiv 0 \bmod q^{n+h-2}.$$

Then "a" in the range $1 \leq a \leq q^{h+1}f$ can be written uniquely as

$$a = u + q^h fv$$

where $1 \leq u \leq q^h f, 0 \leq v \leq q - 1$. In particular

$$u \equiv a \bmod q^n f$$

which means

$$\omega(u) = \omega(a).$$

Consequently

$$< a >=< u > +q^h f\omega(u)^{-1}v$$

and

$$(< a > -1)^n = \sum_{i=0}^{n} \binom{n}{i} (< u > -1)^i (q^n f\omega(u)^{-1}v)^{n-i}$$

since $< u >\equiv 1 \bmod q$. Note that the i-th term of last sum is divisible by $q^{i+h(n-i)}$. For $n - i \leq 1$, one has

$$i + (n - i)h = n + (n - i)(h - 1) \geq n + h - 1$$

therefore

$$(< a > -1)^n \equiv (< u > -1)^n \bmod q^{n+h-1}.$$

Moreover since $a \equiv u \bmod f$ and $\chi(a) = \chi(u)$, then

$$\chi(a)(< a > -1)^n \equiv \chi(u)(< u > -1)^n \bmod q^{n+h-1}.$$

By taking the sum over "a" in the range $1 \leq a \leq q^{h+1}f$ and $(a, p) = 1$,

$$\sum_{a=1}^{q^{h+1}f} \chi(a)(< a > -1)^n \equiv \sum_{v=0}^{p-1} \sum_{u=1,(u,p)=1}^{p^h f} \chi(u)(< u > -1)^n \bmod q^{n+h-1}$$

hence

$$c_n(h + 1) \equiv qc_n(h) \equiv 0 \bmod q^{n+h-1}.$$

then we obtain a congruence

$$\frac{1}{q^{h+1}f}c_n(h+1) \equiv \frac{1}{q^h f}c_n(h) \bmod \frac{q^n}{q^2 f}$$

for all $h \geq 1$, since c_n is the limit of $q^{-h}f^{-1}c_n(h)$ and

$$|c_n| = lim_{h\to\infty} \frac{1}{|q^{n+h}f|}|c_n(h)| \leq \frac{1}{|q^2 f|}|q^n|.$$

□

Now we present Kubota-Leopoldt and Iwasawa's construction of the p-adic zeta-function.

Theorem 2.2.
There exists a unique p-adic continuous function $L_p(s,\chi)$, with the following properties:

(i) There is a Taylor series expansion

$$L_p(s,\chi) = \begin{cases} \frac{1-\frac{1}{p}}{s-1} + \sum_{n=0}^{\infty} a_n(s-1)^n & \text{if } \chi = \chi^0 = 1 \\ \sum_{n=0}^{\infty} a_n(s-1)^n & \text{if } \chi \neq \chi^0 = 1 \end{cases}$$

with $a_n \in K = \mathbb{Q}_p(\chi)$, and $L_p(s,\chi)$ converges on the disk

$$D_1(r) = \{s : s \in \mathbb{C}_p, |s-1|_p < r\}$$

where $r = \left|p\right|^{\frac{1}{p-1}} \left|q\right|^{-1}$.

(ii) It satisfies the interpolation rule

$$\begin{aligned} L_p(1-n,\chi) &= -(1-\chi_n(p)p^{n-1})\frac{B_{n,\chi_n}}{n} \\ &= (1-\chi_n(p)p^{n-1})L(1-n,\chi_n), \end{aligned}$$

for $n = 1, 2, 3, \ldots$.

Proof.
By Theorem 2.1, there exists a power series $A_\chi(x)$ in $K[[x]]$ converging at every s in $\mathbb{C}_p$ bounded by

$$|s| < |p|^{\frac{1}{p-1}}|q|^{-1}.$$

Applying Theorem 2 to the sequences b_n and c_n, one has for each $n \geq 1$ that

$$A_\chi(n) = (1-\chi_n(p)p^{n-1})B_{n,\chi_n}.$$

Similarly if $n = 0$ then

$$A_\chi(0) = (1-\chi(p)p^{n-1})B_{0,\chi} = \begin{cases} 1-\frac{1}{p} & \text{if } \chi = 1 \\ 0 & \text{if } \chi \neq 1 \end{cases}.$$

□

In the p-adic L-function's interpolation property,

$$L_p(1-n,\chi) = -(1-\chi_n(p)p^{n-1})\frac{B_{n,\chi_n}}{n}$$

for every positive integer n. If $p-1|n$, then $L_p(1-n,\chi)$ will be equivalent to the classical Dirichlet L-function, except for the Euler factor at p.

If χ is an odd Dirichlet character $\chi(-1) = -1$, and $n \equiv 0 \bmod p-1$, then n is even which implies $B_{n,\chi} = 0$, hence $L_p(s,\chi)$ is identically zero. Otherwise, if $\chi(-1) = 1$, then $B_{n,\chi} \neq 0$ as $n \equiv 0 \bmod p$, so $L_p(s,\chi)$ is not identically zero.

Also $L_p(0,\chi)$ vanishes at $s = 0$, when $\chi_1(p) = 1$ as

$$L_p(0,\chi) = (1-\chi_1(p))L(0,\chi_1).$$

The p-adic L-function as defined above gives the values of $L_p(s,\chi)$ for $s = 0,-1,-2,\ldots$, but what about the values of the same function when $s = 1,2,3,\ldots$? Leopoldt found a remarkable formula for $L_p(1,\chi)$, described as follows. If χ is a non-principal Dirichlet character, then

$$L_p(1,\chi) = -(1-\frac{\chi(p)}{p})\frac{\tau(\chi)}{f}\sum_{a=1}^{f}\widetilde{\chi}(a)log_p(1-\zeta^{-a})$$

where the root of unity

$$\zeta = e^{\frac{-2\pi i}{f}} \in \mathbb{C} \subseteq \mathbb{C}_p,$$

the Gauss sum $\tau(\chi) = \sum_{a=1}^{f}\chi(a)\zeta^a$, and $\widetilde{\chi}$ = the conjugate character of χ so that

$$\widetilde{\chi}(a) = \begin{cases} \chi(a)^{-1} & \text{if } (a,f) = 1 \\ 0 & \text{if } (a,f) \neq 1 \end{cases}.$$

2.3 p-adic distribution

This section starts with some definitions and propositions, which describe how to interpolate the Riemann zeta function p-adically. Firstly, fixing $a \in \mathbb{Q}_p$ and $N \in \mathbb{Z}$, we introduce the set

$$a + p^N\mathbb{Z}_p = \left\{x \in \mathbb{Q}_p : |x-a|_p \leq p^{-N}\right\},$$

which plays the role of an interval(disc). All intervals of this form are viewed as a basis of open sets on $\mathbb{Q}_p$, and are donated by $a + (p^N)$.

Definition 2.2.
Let X be a compact open subset of $\mathbb{Q}_p$. Then μ is called a p-adic distribution on X, if μ is additive map from the set of compact-open sets in X to $\mathbb{Q}_p$.

That means, if $U \subset X$ is a finite disjoint union of compact-open subsets, $\{U_i\}_{i=1}^{n}$, say, then

$$\mu(U) = \sum_{i=1}^{n} \mu(U_i).$$

Definition 2.3.
A p-adic distribution μ on X is a bounded measure, if there exists a positive real number B such that

$$|\mu(U)|_p \leq B$$

for all compact-open sets $U \subseteq X$.

Proposition 2.1.
Every map μ from the set of intervals contained in X to $\mathbb{Q}_p$ with the additivity property

$$\mu(a + (p^N)) = \sum_{b=0}^{p-1} \mu(a + bp^N + (p^{N+1}))$$

whenever $a + (p^N) \subset X$, extends uniquely to a p-adic distribution on X.

Proof.
Let suppose U be a compact-open subset of X, so U is a finite disjoint union of interval

$$U = \bigcup_{i=1}^{n} I_i,$$

where $I_i = a_i + (p^N)$ for some a_i and N. One then defines

$$\mu(U) = \sum \mu(I_i).$$

We shall prove that $\mu(U)$ is well-defined; suppose that we have two decompositions

$$U = \bigcup I_i = \bigcup I_i^{'},$$

where $\{I_i\}$ and $\{I_i^{'}\}$ are different partitions. Suppose that

$$I_i \bigcap I_j^{'} = I_{ij} \neq \varnothing,$$

so we can write

$$I_i = \bigcup I_{ij}.$$

By Heine-Borel property there exists $N^{'} > N$ such that

$$I_{ij} = a_i + \sum_{k=N}^{N'-1} a_{jk} p^k + (p^{N'})$$

for all j. Then

$$\mu(I_i) = \sum_j \mu(I_{ij}),$$

so that

$$\mu(U) = \sum_i \mu(I_i) = \sum_{i,j} \mu(I_{ij}).$$

Similarly

$$\mu(U) = \sum_j \mu(I_j^{'}) = \sum_{i,j} \mu(I_{ij}),$$

so μ is independent of the choice of partitions. Now we would like to show μ is a finitely additive map. Let U be compact-open in X, and decompose

$$U = \bigcup_{i=1}^{n} U_i,$$

where U_i are disjoint compact-open subsets. Since each U_i can be written as a finite disjoint union of intervals, say I_{ij}, therefore

$$\begin{aligned} \mu(U) &= \mu(\bigcup I_{ij}) \\ &= \sum_{i,j} \mu(I_{ij}) \\ &= \sum_j \sum_i \mu(I_{ij}) \\ &= \sum_i \mu(U_i) \end{aligned}$$

as required. □

Let us give some examples of p-adic distributions. The first of these is kth Bernoulli distribution ($\mu_{B,k}$), which is defined as

$$\mu_{B,k}(a + (p^N)) = p^{N(k-1)} B_k(\frac{a}{p^N}).$$

Proposition 2.2.
For each fixed $k \geq 1$, $\mu_{B,k}$ extends to a distribution on $\mathbb{Z}_p$.

Proof.
By Proposition 2.1, it is enough to show that,

$$\mu_{B,k}(a + (p^N)) = \sum_{b=0}^{p-1} \mu_{B,k}(a + bp^N + (p^{N+1})).$$

Now from the definition of $\mu_{B,k}$,

$$\sum_{b=0}^{p-1} \mu_{B,k}(a + bp^N + (p^{N+1}) = p^{(N+1)(k-1)} \sum_{b=0}^{p-1} B_k(\frac{a + bp^N}{p^{N+1}}),$$

so that

$$B_k(\frac{a}{p^N}) = p^{k-1}\sum_{b=0}^{p-1} B_k(\frac{a}{p^{N+1}} + \frac{b}{p}).$$

Suppose $\alpha = \frac{a}{p^N}$, then

$$\begin{aligned}
\sum_{b=0}^{p-1}\sum_{k=0}^{\infty} B_k(\alpha + \frac{b}{p})\frac{t^k}{k!} &= \sum_{b=0}^{p-1} \frac{t\epsilon^{(\frac{\alpha+b}{p})t}}{\epsilon^t - 1} \\
&= \frac{te^{\alpha t}}{e^t - 1}\sum_{b=0}^{p-1} e^{\frac{t}{p}b} \\
&= \frac{te^{\alpha t}}{e^t - 1} \times \frac{e^{(\frac{t}{p})p} - 1}{e^{\frac{t}{p}} - 1} \\
&= p\frac{(\frac{t}{p})e^{\alpha t}}{e^{\frac{t}{p}} - 1} \\
&= p\sum_{k=0}^{\infty} \frac{B_k(\alpha)}{p^k}\frac{t^k}{k!}.
\end{aligned}$$

□

Definition 2.4.
Let μ be a p-adic bounded measure on X, where X is a compact-open set in $\mathbb{Q}_p$, and let $f : X \to \mathbb{Q}_p$ be a continuous function. The Nth Riemann sum is given by

$$S_{N,\{x_{a_i},N\}} = \sum_{i=1}^{m} f(x_{a_i,N})\mu(a_i + (p^N)),$$

where X is a disjoint union of $\{a_i+(p^N)\}_{i=1}^m$, and $x_{a_i,N}$ denotes a given arbitrary point in $a_i + (p^N)$.

Theorem 2.3.
The limit of $S_{N,\{a_i,N\}}$ when $N \to \infty$ exists in $\mathbb{Q}_p$, and is indepentent of the choice of $\{x_{a_i,N}\}$.

Proof.
Let $B > 0$ such that $|\mu(U)|_p \leq B$ for all compact -open subset U of X. Suppose $\epsilon > 0$ with N large enough satisfying that:

1. X is a finite disjoint union of $\{a_i + (p^N)\}_{i=1}^n$ and $0 < a_i < p^N - 1$.
2. For any x and y with $x \equiv y \bmod p^N$, one has the bound

$$|f(x) - f(y)|_p < \frac{\epsilon}{B}.$$

For any $N' > N$, then

$$X = \bigcup_{a=1}^{n} (a_i + (p^N))$$

can be sub-partitioned into

$$X = \bigcup_{i,j} (a_{i,j} + (p^{N'})).$$

Therefore,

$$\begin{aligned} S_{N,\{x_{a_i},N\}} &= \sum_i f(x_{a_i,N})\mu(a_i(p^N)) \\ &= \sum_i f(x_{a_i,N}) \sum_j \mu(a_{ij} + (p^{N'})) \\ &= \sum_{i,j} f(x_{a_i,N})\mu(a_{ij} + (p^{N'})), \end{aligned}$$

and moreover

$$\begin{aligned} \Big| S_{N,\{x_{a_i},N\}} - S_{N',\{x_{a_{ij}},N'\}} \Big|_p &= \Big| \sum_{i,j} (f(x_{a_i,N}) - f(x_{a_{ij},N'}))\mu(a_{ij} + (p^{N'})) \Big|_p \\ &\leq \max_{i,j} \Big| f(x_{a_i,N}) - f(x_{a_{ij},N'}) \Big|_p \Big| \mu(a_{ij} + (p^{N'})) \Big|_p \\ &\leq \epsilon B. \end{aligned}$$

Therefore, the Riemann sums form a Cauchy sequence, hence there is a limit in $\mathbb{Q}_p$.
Now we need to show the limit is independent of the choice of points. Let $y_{a_i,N}$ be any point from each disc $a_i + (p^N)$, so that

$$\begin{aligned} \Big| S_{N,\{x_{a_i},N\}} - S_{N,\{y_{a_i},N\}} \Big|_p &= \Big| \sum_{i,j} (f(x_{a_i,N}) - f(y_{a_i,N}))\mu(a + (p^N)) \Big|_p \\ &\leq \max_{i,j} \Big| f(x_{a_i,N}) - f(y_{a_i,N}) \Big|_p \Big| \mu(a + (p^N)) \Big|_p \\ &\leq \epsilon B. \end{aligned}$$

The result follows. □

Definition 2.5.
Let $f : X \to \mathbb{Q}_p$ be a continuous function, and μ a p-adic bounded measure on X. One defines $\int f\mu$ to be the limit of Riemann sums in the above theorem.

Corollary 2.1.
If $f, g : X \to \mathbb{Q}_p$ are two continuous functions such that

$$|f(x) - g(x)|_p \leq \varepsilon,$$

for all $x \in X$ *and if*

$$\mu(U) \leq B$$

for all compact-open $U \subset X$, *then*

$$\left| \int f\mu - \int g\mu \right|_p \leq \varepsilon B.$$

2.4 p-adic zeta function

Let $f(s) = n^s$, where n is fixed positive real number. This function is a continuous function of a real variable.

We can consider n as an element of $\mathbb{Q}_p$, so that $n^s \in \mathbb{Z}_p$, for every non-negative integer s. We might naively try to extend $f(s) = n^s$ to a continuous function for all $s \in \mathbb{Z}_p$.

Suppose we have the continuity of $f(s) = n^s$ with $s \in \mathbb{Z}_p$, and assume s and s' very close p-adically with $s < s'$. If $n \in p\mathbb{Z}_p$ then

$$\left|n^s - n^{s'}\right|_p = \left|n^s\right|_p \left|1 - n^{s'-s}\right|_p = \left|n^s\right|_p$$

so n^s can not be continuous!

Now restrict n to be a p-adic unit; we need to show that $\left|n^s - n^{s'}\right|_p$ converges to zero if s and s' are congruent modulo $p-1$. Fix an integer

$$s_0 \in \{1, 2, \ldots, p-2\}$$

and introduce the set

$$A_{s_0} = \{s \in \mathbb{Z}, s > 0 : s \equiv s_0 (\mathrm{mod} p - 1)\}.$$

Then for all $s \in A_{s_0}$,

$$\begin{aligned} \left|n^{s_0} - n^s\right|_p &= \left|n^{s_0}\right|_p \left|1 - n^{s-s_0}\right|_p \\ &= \left|n^{s_0}\right|_p \left|1 - n^{(p-1)t}\right|_p \end{aligned}$$

for some integer t. But

$$n^{p-1} \equiv 1 \bmod p,$$

so we can write

$$n^{p-1} = 1 + mp$$

for some m. If s and s_0 are very close, let

$$s - s_0 = (p-1)p^N s'$$

for some rational integer s'. Then

$$\begin{aligned}\left|n^{s_0} - n^s\right|_p &= \left|n^{s_0}\right|_p \left|1 - n^{(p-1)p^N s'}\right|_p \\ &= \Big| - \sum_{k=1}^{p^N s'} \binom{p^N s'}{k} (mp)^k \Big|_p \\ &\leq \left|p^{N+1}\right|_p \\ &= \frac{1}{p^{N+1}}\end{aligned}$$

which establishes the continuity of $f(s) = n^s$.

Proposition 2.3.
A_{s_0} is dense in $\mathbb{Z}_p$.

Proof.
We need to show that, for any $s \in \mathbb{Z}_p$, there exists a sequence $\{s_i\}$ in A_{s_0} that converges to s. Let us expand

$$s = \sum_{i=0}^{\infty} a_i p^i$$

and write

$$s_i = \sum_{j=0}^{i} a_j p^j + (s_0 - a_0 - a_1 - \cdots - a - i)p^i.$$

Then s_i lies in A_{s_0} for all i since

$$p^j \equiv 1 \bmod (p-1)$$

for all j. This follows from the progression

$$p^j - 1 = (p-1)(p^{j-1} + \cdots + p + 1) \equiv 0 \bmod (p-1)$$

hence

$$s_i \equiv a_0 + a_1 + \cdots + a_i + s_0 - a_0 - \cdots - a_i = s_0 \bmod (p-1)$$

so s_i converges to s, provided

$$\left|s_i - s\right|_p = \Big|(s_0 - a_0 - \cdots - a_i)p^i - \sum_{j=i+1}^{\infty} a_j p^j\Big|_p \leq \frac{1}{p^i} \to 0$$

as $i \to \infty$.

Applying this proposition, we can extend a continuous function $f(s) = n^s$ to $\mathbb{Z}_p$. Furthermore, we have shown that, if x is a p-adic unit and

$$k \equiv k' \bmod ((p-1)p^N),$$

then

$$\left|x^{k-1} - x^{k'-1}\right|_p \leq \frac{1}{p^{N+1}},$$

and so

$$\left|\int_{\mathbb{Z}_p^*} x^{k-1}\mu_{1,\alpha} - \int_{\mathbb{Z}_p^*} x^{k'-1}\mu_{1,\alpha}\right|_p \leq \frac{1}{p^{N+1}}.$$

□

Unfortunately, the *kth* Bernoulli distributions ($\mu_{B,k}$) are not bounded measures for any positive integer k. A bounded measure can be obtained by regularizing the Bernoulli distribution as we now describe.

Definition 2.6.
Let $\alpha \in \mathbb{Z}_p$ with $\alpha \neq 1$ and $p \nmid \alpha$. The kth regularized Bernoulli distribution on $\mathbb{Z}_p$ is defined by

$$\mu_{k,\alpha}(U) = \mu_{B,k}(U) - \alpha^{-k}\mu_{B,k}(\alpha U),$$

where $\alpha U = \left\{x \in \mathbb{Q}_p : \frac{x}{\alpha} \in U\right\}$.

Proposition 2.4.
The kth regularized Bernoulli distribution ($\mu_{k,\alpha}$) is a p-adic distribution.

Proof.
Let U be a finite disjoint union of intervals

$$U = \left\{U_i\right\}_{i=1}^{n}.$$

Then αU is a disjoint union of $\left\{\alpha U_i\right\}_{i=1}^{n}$, since

$$\begin{aligned} \alpha U &= \left\{x \in \mathbb{Q}_p : \frac{x}{\alpha} \in U\right\} \\ &= \left\{x \in \mathbb{Q}_p : \frac{x}{\alpha} \in \bigcup U_i\right\} \\ &= \bigcup \left\{x \in \mathbb{Q}_p : x \in \alpha U_i\right\}. \end{aligned}$$

Therefore,

$$\begin{aligned} \mu_{k,\alpha}(U) &= \mu_{B,k}(U) - \alpha^{-k}\mu_{B,k}(\alpha U) \\ &= \sum_i \mu_{B,k}(U_i) - \alpha^{-k}\sum_i \mu_{B,k}(\alpha U_i) \\ &= \sum_i \mu_{B,k}(U_i) - \alpha^{-k}\mu_{B,k}(\alpha U_i) \\ &= \sum_i \mu_{k,\alpha}(U_i), \end{aligned}$$

therefore $\mu_{k,\alpha}$ is a distribution too. □

Now, for $\alpha \in \mathbb{Z}_p$, we write $\{\alpha\}_N$ for the unique rational integer such that

$$0 \leq \{\alpha\}_N \leq p^N - 1$$

and

$$\{\alpha\}_N \equiv \alpha \bmod p^N.$$

Let $U = a + (p^N)$, where $a \in \{0, 1, \ldots, p^N - 1\}$, then

$$\begin{aligned}
\alpha U &= \big\{x \in \mathbb{Z}_p : \frac{x}{\alpha} \in U\big\} \\
&= \big\{x \in \mathbb{Z}_p : \big|\frac{x}{\alpha} - a\big|_p \leq p^{-N}\big\} \\
&= \big\{x \in \mathbb{Z}_p : \big|\frac{1}{\alpha}\big|_p \big|x - \alpha a\big|_p \leq p^{-N}\big\} \\
&= \big\{\alpha a\big\}_N + (p^N).
\end{aligned}$$

Now we compute $\mu_{k,\alpha}$, when $k = 0$,

$$\begin{aligned}
\mu_{0,\alpha}(U) &= \mu_{0,\alpha}(a + (p^N)) \\
&= \mu_0(a + (p^N)) - \mu_0(\{\alpha a\}_N + (p^N)) \\
&= p^{-N} - p^{-N} = 0.
\end{aligned}$$

Lemma 2.3.
For $k = 1$, $\mu_{k,\alpha}$ is a bounded measure.

Proof.

$$\begin{aligned}
\mu_{1,\alpha}(a + (p^N)) &= \mu_1(a + (p^N)) - \alpha^{-1}\mu_1(\{\alpha a\}_N + (p^N)) \\
&= B_1\big(\frac{a}{p^N}\big) - \alpha^{-1} B_1\big(\frac{\{\alpha a\}_N}{p^N}\big) \\
&= \frac{a}{p^N} - \frac{1}{2} - \frac{1}{\alpha}\big(\frac{\{\alpha a\}_N}{p^N} - \frac{1}{2}\big) \\
&= \frac{1}{\alpha}\big[\frac{\alpha a}{p^N}\big] + \frac{1}{2}\big(\frac{1}{\alpha} - 1\big).
\end{aligned}$$

Now,we need to show $\big|\mu_{1,\alpha}(U)\big|_p$ is bounded for $a \in \big\{0, 1, \ldots, p^N - 1\big\}$.
Firstly, $\frac{1}{\alpha} \in \mathbb{Z}_p$, since $p \nmid \alpha$ and $\alpha \in \mathbb{Z}_p$. Therefore, if $p \neq 2$, then $\frac{1}{2}\big(\alpha^{-1} - 1\big) \in \mathbb{Z}_p$. Also if $p = 2$, then

$$\alpha^{-1} - 1 \equiv 0 \bmod 2,$$

so we can write

$$\frac{1}{\alpha} = 1 + \sum_{i=0}^{\infty} a_i 2^i,$$

where $a_i \in \{0, 1\}$. Thus

$$\frac{1}{2}(\alpha^{-1} - 1) \in \mathbb{Z}_p.$$

In addition, for any interval $a+(p^N)$,

$$\mu_{1,\alpha}(a+(p^N)) \in \mathbb{Z}_p,$$

so $\frac{1}{\alpha}\left[\frac{\alpha a}{p^N}\right] \in \mathbb{Z}_p$. We conclude that

$$\left|\mu_{1,\alpha}(U)\right|_p \leq 1,$$

hence $\mu_{1,\alpha}$ is a bounded measure.

□

Theorem 2.4.
Let d_k be the least common denominators of the coefficients of $B_k(x)$. Then

$$d_k\mu_{k,\alpha}(a+(p^N)) \equiv d_k k a^{k-1}\mu_{1,\alpha}(a+(p^N)) \quad (mod p^N).$$

Proof.
From the definition of $\mu_{k,\alpha}$,

$$\begin{aligned} d_k\mu_{k,\alpha}(a+(p^N)) &= d_k(\mu_k(a+(p^N)) - \alpha^{-k}\mu_k(\{a\alpha\}_N+(p^N))) && (2.4)\\ &= d_k p^{N(k-1)}B_k(\frac{a}{p^N}) - d_k\alpha^{-k}p^{N(k-1)}B_k(\frac{\{a\alpha\}_N}{p^N}). && (2.5)\end{aligned}$$

The polynomial $B_k(x)$ has degree k, therefore

$$\begin{aligned} B_k(x) &= \sum_{i=0}^{k}\binom{k}{i}B_i x^{k-i}\\ &= B_0x^k + kB_1x^{k-1} + \dots\\ &= x^k - \binom{k}{2}x^{k-1} + \dots + B_k.\end{aligned}$$

Now, the first part of the right hand side of (2.5) is

$$d_k p^{N(k-1)}B_k(\frac{a}{p^N}) \equiv d_k p^{N(k-1)}(\frac{a^k}{p^{Nk}} - \frac{k}{2}\frac{a^{k-1}}{2p^{N(k-1)}}) \bmod p^N$$

$$= d_k a^{k-1}(\frac{a}{p^N} - \frac{k}{2}). \qquad (2.6)$$

Also, the second part of the right hand side of (2.5) yields

$$\begin{aligned} d_k\alpha^{-k}p^{N(k-1)}B_k(\frac{\{a\alpha\}_N}{p^N}) &\equiv d_k\alpha^{-k}p^{N(k-1)}(\frac{\{a\alpha\}_N^k}{p^{Nk}} - \frac{k}{2}\frac{\{a\alpha\}_N^{k-1}}{p^{N(k-1)}}) \bmod p^N\\ &= d_k\alpha^{-k}p^{N(k-1)}((\frac{a\alpha}{p^N} - [\frac{a\alpha}{p^N}])^k - \frac{k}{2}(\frac{a\alpha}{p^N} - [\frac{a\alpha}{p^N}])^{k-1})\\ &\equiv d_k\alpha^{-k}p^{N(k-1)}(\frac{a^k\alpha^k}{p^{Nk}} - k\frac{a^{k-1}\alpha^{k-1}}{p^{N(k-1)}}[\frac{a\alpha}{p^N}] - \frac{k}{2}(\frac{a^{k-1}\alpha^{k-1}}{p^{N(k-1)}})) \bmod p^N\end{aligned}$$

$$= d_k a^{k-1}\big(\frac{a}{p^N} - \frac{k}{\alpha}\big[\frac{a\alpha}{p^N} - \frac{k}{2\alpha}\big). \tag{2.7}$$

By inserting (2.6) and (2.7) in (2.5)

$$\begin{aligned} d_k\mu_{k,\alpha}(a+(p^N)) &\equiv d_k k a^{k-1}\big(\frac{1}{\alpha}\big[\frac{a\alpha}{p^N}\big] + \frac{k}{2}\big(\frac{1}{\alpha} - 1\big)\big) \bmod p^N \\ &= d_k k a^{k-1}\mu_{1,\alpha}(a+(p^N)), \end{aligned}$$

as required. □

Now, let us take a closer look at this above congruence

$$d_k\mu_{k,\alpha}(a+(p^N)) \equiv d_k k a^{k-1}\mu_{1,\alpha}(a+(p^N)) \quad \bmod p^N.$$

If we divide both sides by d_k, we must replace p^N by $p^{N-ord_p d_k}$. Here $ord_p d_k$ is constant, so it does not play a role for large N.
Therefore, we can write the above congruence as

$$\mu_{k,\alpha}(a+(p^n)) \equiv k a^{k-1}\mu_{1,\alpha}(a+(p^N)) \bmod p^{N-ord_p d_k}.$$

By the above theorem, we have

$$\begin{aligned} \big|\mu_{k,\alpha}(a+(p^N))\big|_p &\le \max\big(\big|\frac{p^N}{d_k}\big|_p, \big|k a^{k-1}\mu_{1,\alpha}(a+(p^N))\big|_p\big) \\ &\le \max\big(\big|\frac{1}{d_k}\big|_p, \big|\mu_{1,\alpha}(a+(p^N))\big|_p\big). \end{aligned}$$

Since $\mu_{1,\alpha}$ is a bounded measure, it follow that $\mu_{k,\alpha}$ is a bounded measure. The regularized Bernoulli distributions are related as follows.

Proposition 2.5.
Let $f : \mathbb{Z}_p \to \mathbb{Z}_p$ be the function $f(x) = x^{k-1}$ where k is positive integer. Let X be a compact-open subset of $\mathbb{Z}_p$, then

$$\int_X 1\mu_{k,\alpha} = k\int_X x^{k-1}\mu_{1,\alpha}.$$

Proof.
Let X be a finite union of intervals, say $X = \bigcup(a+(p^N))$, where N is large enough. Then

$$\begin{aligned} \int_X 1\mu_{k,\alpha} &= \sum_a \int_{a+(p^N)} 1\mu_{k,\alpha} \\ &= \sum_a \mu_{k,\alpha}(a+(p^N)). \end{aligned}$$

From the above theorem

$$\mu_{k,\alpha}(a+(p^N)) \equiv ka^{k-1}\mu_{1,\alpha}(a+(p^N)) \mod p^{N-ord_p d_k},$$

so that

$$\begin{aligned}\int_X 1\mu_{k,\alpha} &\equiv k\sum_a a^{k-1}\mu_{1,\alpha}(a+(p^N)) \mod p^{N-ord_p d_k} \\ &\equiv k\sum_a f(a)\mu_{1,\alpha}(a+(p^N)).\end{aligned}$$

By allowing $N \to \infty$,

$$\sum_{0\le a<p^N} f(a)\mu_{1,\alpha}(a+(p^N)) = \int_X x^{k-1}\mu_{1,\alpha},$$

from which we obtain the desired result. □

We can now write

$$\int_{\mathbb{Z}_p^*} x^{k-1}\mu_{1,\alpha} = \frac{1}{k}\int_{\mathbb{Z}_p^*} 1\mu_{k,\alpha},$$

so that

$$\frac{1}{k}\int_{\mathbb{Z}_p^*} 1\mu_{k,\alpha} = \frac{1}{k}\mu_{k,\alpha}(\mathbb{Z}_p^*).$$

From the definition of $\mu_{k,\alpha}$ one has

$$\begin{aligned}\mu_{k,\alpha}(\mathbb{Z}_p^*) &= \mu_{B,k}(\mathbb{Z}_p^*) - \alpha^{-k}\mu_{B,k}(\alpha\mathbb{Z}_p^*) \\ &= (1-\alpha^{-k})\mu_{B,k}(\mathbb{Z}_p^*) \\ &= (1-\alpha^{-k})(1-p^{k-1})B_k.\end{aligned}$$

However

$$\mu_{B,k}(\mathbb{Z}_p) = p^0 B_k(0) = B_k$$

and

$$\mu_{B,k}(p\mathbb{Z}_p) = p^{k-1}B_k(0) = p^{k-1}B_k,$$

in which case

$$\begin{aligned}\mu_{B,k}(\mathbb{Z}_p^*) &= \mu_{B,k}(\mathbb{Z}_p) - \mu_{B,k}(p\mathbb{Z}_p) \\ &= B_k - p^{k-1}B_k \\ &= B_k(1-p^{k-1}).\end{aligned}$$

Therefore, one concludes that

$$(1-p^{k-1})(-\frac{B_k}{k}) = \frac{1}{\alpha^{-k}-1}\int_{\mathbb{Z}_p} x^{k-1}\mu_{1,\alpha}.$$

Definition 2.7.
Let α be a rational integer with $\alpha \neq 1$ and $p \nmid \alpha$. Then for any $k \geq 0$, define

$$\zeta_p(1-k) = \frac{1}{\alpha^{-k}-1}\int_{\mathbb{Z}_p{}^*} x^{k-1}\mu_{1,\alpha}.$$

Likewise if $B \in \mathbb{Z}$, $p \nmid B$, $B \neq 1$ then

$$(B^{-k}-1)^{-1}\int_{\mathbb{Z}_p^*} x^{k-1}\mu_{1,B} = (\alpha^{-k}-1)^{-1}\int_{\mathbb{Z}_p^*} x^{k-1}\mu_{1,\alpha}.$$

Since both sides equal

$$(1-p^{k-1})(-\frac{B_k}{k}),$$

the right expression is independent of α. Moreover

$$\zeta_p(1-k) = ((1-p^{k-1})\frac{-B_k}{k},$$

shows that $\zeta_p(1-k)$ can be obtained by removing the p-factor from the Euler identity for $\zeta(1-k)$ as $\frac{B_k}{k} = -\zeta(1-k)$.

The following result is often called the Kummer congruences.

Theorem 2.5. *(Kummer)*

1. *If $p-1 \nmid k$, then $\left|\frac{B_k}{k}\right|_p \leq 1$ (this means that $\frac{B_k}{k}$ is p-adic integer).*
2. *If $p-1 \nmid k$ and $k \equiv k^{'} (\mathrm{mod}(p-1)p^N)$, then*

$$(1-p^{k-1})\frac{B_k}{k} \equiv (1-p^{k^{'}-1})\frac{B_k^{'}}{k^{'}} \mod p^{N+1}.$$

Proof.

1. If $k=1$, then for any $p > 2$,

$$\left|B_1\right|_p = 1.$$

In the case $k > 1$, let α be such that $2 \leq \alpha \leq p-1$ with $p-1$ is the smallest positive integer satisfying

$$\alpha^{p-1} - 1 \equiv 0 \mod p.$$

Since α is a p-adic unit, it can be identified with a $(p-1)^{st}$ root of unity. Let us assume

$$(\alpha^k - 1) \not\equiv 0 \mod p,$$

which implies $\alpha^{-k}-1$ is a p-adic unit. Then

$$\begin{aligned}\left|\frac{B_k}{k}\right|_p &= \left|1-p^{k-1}\right|_p^{-1}\left|\alpha^{-k}-1\right|_p^{-1}\left|\int_{\mathbb{Z}_p^*} x^{k-1}\mu_{1,\alpha}\right|_p \\ &\leq \left|\mu_{1,\alpha}\mathbb{Z}_p^*\right|_p \\ &\leq 1.\end{aligned}$$

2. Again suppose $2 \leq \alpha \leq p-1$, and

$$k \equiv k' \text{ mod } (p-1)p^N.$$

In particular

$$\alpha^k \equiv \alpha^{k'} \mod p^{N+1}$$

and therefore,

$$(1-p^{k-1})\frac{B_k}{k} \equiv (1-p^{k'-1})\frac{B_{k'}}{k'} \text{ mod } p^{N+1}.$$

□

3 A Dirichlet series expansion for the p-adic zeta function

Fractional derivatives allow the exotic possibility of taking the differentiation operator $D^n = (\frac{d}{dx})^n$, where n is a real number power or complex number power. The term orders relate to the function composition, in the same way that $f^2(x) = f(f(x))$. In this chapter, I follow Delbourgo's paper [2] to expand the p-adic zeta function in such a way.

3.1 Fractional derivations

We begin by stating the full expansion.

Theorem 3.1. *(Delbourgo [3]) Let $s \in \mathbb{Z}_p$ and $\beta \equiv 0 \bmod p-1$; then*

$$L_p(s, \chi\omega^{1+\beta}) = (2\omega^{\beta}(2) < 2 >^{-s} -1)^{-1} \times \lim_{t\to\infty} \Big(\sum_{\substack{m=1 \\ p\nmid m}}^{p^{t\phi(2f_\chi)}} a_m(\chi)\omega^{\beta}(m) < m >^{-s} \Big),$$

where $a_m = L(0,\chi) + \sum_{j=1}^{m-1} \chi(j) - 2\sum_{j=1}^{\lfloor \frac{m-1}{2} \rfloor} \chi(j)$

Here, we shall only prove this theorem in the very special case of $\chi = 1$. The proof relates $\zeta_p(-s, \omega^{1+\beta})$ to a p-adic fractional derivative.

Let K be a finite Galois extension of $\mathbb{Q}_p$, and $\mathcal{O}$ be the ring of integral elements over $\mathbb{Z}_p$. Also, let $\Lambda = \mathcal{O}[[T]]$ denote the ring of power series. Define the polynomial $P_n(T) = (1+T)^{p^n} - 1$, where $n \leq 1$. We name $\Lambda = \mathcal{O}[[T]]$ the Iwasawa ring. In addition, the Iwasawa ring has a division algorithm property. More precisely, if $F(T)$ and $Q_n \in \Lambda$, then there are $P_n(T), R_n(T) \in \mathcal{O}[T]$ such that

$$F(T) = P_n(T)Q_n(T) + R_n(T),$$

where $deg(R_n(T)) < deg(P_n(T)) = p^n$.

Lemma 3.1.
Let us define a polynomial

$$\Theta_n(T) = p^{-n}\Big(\frac{T^{p^n} - 1}{T - 1}\Big);$$

then

$$R_n(T) = \sum_{\alpha\in\mu_{p^n}} F(\alpha^{-1} - 1)\Theta_n(\alpha(1+T)),$$

where μ_{p^n} denotes the group of p^n-th roots of unity.

Proof.
These two polynomials have degree $< p^n$, so all we need to prove is that

$$R_n(\xi - 1) = \Theta_n(\xi - 1),$$

at every $\xi \in \mu_{p^n}$. Now evaluating at $T = \xi - 1$, we have

$$\begin{aligned}\sum_{\alpha\in\mu_{p^n}} F(\alpha^{-1} - 1)\Theta_n(\alpha\xi) &= F(\xi - 1)\Theta_n(\xi^{-1}\xi) + \sum_{\alpha\neq\xi^{-1}} F(\alpha^{-1} - 1) \times 0 \\ &= F(\xi - 1)\Theta_n(1) + \sum_{\alpha\neq\xi^{-1}} F(\alpha^{-1} - 1) \times 0.\end{aligned}$$

Since $\Theta_n(1) = 1$, we obtain $F(\xi - 1) = R_n(\xi - 1)$, Clearly, both polynomials agree on the set $\{\xi - 1 : \xi \in \mu_{p^n}\}$, and therefore in general too. □

For all $F(T) \in \Lambda$ define the idempotent ψ by

$$\psi F(T) = F(T) - \frac{1}{p}\sum_{\xi\in\mu_p} F(\xi(1 + T) - 1).$$

Let $T = exp(Z)-1$, so that $F(exp(Z)-1) \in K[[Z]]$. Then differentiating k-times

$$\big(\frac{d}{dZ}\big)^k F(exp(Z) - 1) = ((1 + T)\frac{d}{dT})^k \circ F(T).$$

Now if we consider an arbitrary element

$$G(T) = \sum_{m=0}^{deg(G)} g_m(1 + T) \quad \in \mathcal{O}[T],$$

then

$$\psi G(T) = \sum_{\substack{m=0 \\ p\nmid m}}^{deg(G)} g_m(1 + T)^m,$$

so that

$$\big((1 + T)\frac{d}{dT}\big) \circ \psi G(T) = \sum_{m=0}^{deg(G)} m \times g_m(1 + T)^m.$$

Therefore, the operator $(1 + T)\frac{d}{dT} : \Lambda^{\psi=1} \to \Lambda^{\psi=1}$ is one to one and onto.

Lemma 3.2.
For all $k \in \mathbb{N}$,

$$((1 + T)\frac{d}{dT})^k \circ \psi F(T) \equiv ((1 + T)\frac{d}{dT})^k \circ \psi R_n(T) \pmod{(p, T)^n}$$

Proof.
Firstly, if $k = 1$ then

$$\big((1+T)\frac{d}{dT}\big) \circ \psi\big(P_n(T)Q_n(T)\big) = (1+T)\big(\frac{d}{dT}(P_n(T)\psi Q_n(T))\big).$$

One can manipulate the right hand side as follows:

$$\begin{aligned} R.H.S. &= (1+T)\big(\frac{d}{dT}(((1+T)^{p^n}-1)\psi Q_n(T))\big) \\ &= (1+T)\big(p^n(1+T)^{p^n-1}\psi Q_n(T) + ((1+T)^{p^n}-1)\frac{d}{dT}\psi Q_n(T)\big) \\ &= p^n(1+T)^{p^n}\psi Q_n(T) + P_n(T)(1+T)\frac{d}{dT}\psi Q_n(T). \end{aligned}$$

Now we have $(1+T)^{p^n}\psi Q_n(T) \in \Lambda$, and $(1+T)\frac{d}{dT}\psi Q_n(T) \in \Lambda$, so that $p^n(1+T)^{p^n}\psi Q_n(T) + P_n(T)(1+T)\frac{d}{dT}\psi Q_n(T)$ is in $p^n\Lambda + P_n\Lambda \subset (p,T)^n$. Then by induction on k, we have the desired congruence. □

Now let $\mathfrak{F}_\beta$ be the set of positive integers congruent to β modulo $p-1$. Consider the decomposition $m = \omega(m) < m >$, where $m \in \mathbb{Z}_p^*$; then if $k \in \mathfrak{F}_\beta$, one has

$$\begin{aligned} m^k &= \omega^k(m) < m >^k \\ &= \omega^\beta(m) \times exp(k \log m). \end{aligned}$$

When $p \nmid m$ the function m^k is continuous, otherwise m^k tends to zero as $k \to \infty$ if $p|m$.

Definition 3.1.
For any $s \in \mathbb{Z}_p$, define the operator

$$D_\beta^s : \mathcal{O}[T]^{\psi=1} \to \mathcal{O}[T]^{\psi=1}$$

to be the limit of $\big((1+T)\frac{d}{dT}\big)^k$ as $k \to s$ inside $\mathbb{Z}_p \bigcap \mathfrak{F}_\beta$.

This mapping is well-defined, since $\mathfrak{F}_\beta$ is clearly dense in $\mathbb{Z}_p$. Also it acts on the polynomial $\psi G(T) \in \mathcal{O}[T]^{\psi=1}$ by

$$D_\beta^s \circ \psi G(T) = \sum_{\substack{m=0 \\ p\nmid m}} \omega^\beta(m) < m >^s (1+T)^m,$$

for all $s \in \mathbb{Z}_p$.

For any $F(T) \in \Lambda$ with convergents $\{R_n(T)\}_{n\in\mathbb{N}}$, then from Lemma 3.2

$$((1+T)\frac{d}{dT})^k \circ \psi F(T) \equiv ((1+T)\frac{d}{dT})^k \circ \psi R_n(T) \pmod{(p,T)^n},$$

so that the action of the operator D^s_β on $\psi F(T)$ extends by continuity, i.e.

$$D^s_\beta \circ \psi F(T) = lim_{n\to\infty}(D^s_\beta \circ \psi R_n(T))$$

and is well-defined. Also for each $\mathfrak{F}_\beta$ the operator $D^s_\beta : \Lambda^{\psi=1} \to \Lambda^{\psi=1}$ is the unique extension from $G[T]^{\psi=1}$ to Λ. One can see the operator D^s_β has $p-1$ branches, the same as the p-adic L-function.

Lemma 3.3.

1. *For $\beta_1, \beta_2 \in \mathbb{Z}/(p-1)\mathbb{Z}$, and $s_1, s_2 \in \mathbb{Z}_p$, we have* $D^{s_1}_{\beta_1} \circ D^{s_2}_{\beta_2} = D^{s_1+s_2}_{\beta_1+\beta_2}$.
2. *If $k \in \mathfrak{F}_\beta$, then* $D^k_\beta = ((1+T)\frac{d}{dT})^k$.
3. *For all $s \in \mathbb{Z}_p$, the p-fold composition* $D^s_\beta \circ \cdots \circ D^s_\beta = D^{ps}_\beta$.
4. *If $a, b \in \mathbb{N}$ such that $gcd(b, p(p-1)) = 1$ and $\beta \equiv ab^{-1} \bmod (p-1)$, then the b-fold composition* $D^{\frac{a}{b}}_\beta \circ \cdots \circ D^{\frac{a}{b}}_\beta = \big((1+T)\frac{d}{dT}\big)^a$.

Proof.

1.

$$\begin{aligned} D^{s_1}_{\beta_1} \circ D^{s_2}_{\beta_2} &= \lim_{k\to s_1} \big((1+T)\frac{d}{dT}\big)^k \circ \lim_{k\to s_2} \big((1+T)\frac{d}{dT}\big)^k \\ &= \lim_{k\to s_1+s_2} \big((1+T)\frac{d}{dT}\big)^k \\ &= D^{s_1+s_2}_{\beta_1+\beta_2}. \end{aligned}$$

2. It is clear that, if $k \in \mathfrak{F}_\beta$

$$D^k_\beta = \lim_{k\to k} \big((1+T)\frac{d}{dT}\big)^k = \big((1+T)\frac{d}{dT}\big)^k.$$

3. From the density of $\mathbb{N}$ in $\mathbb{Z}_p$, without loss of generality assume that $s \in \mathbb{N}$; then

$$D^s_\beta \circ \cdots \circ D^s_\beta = D_\beta^{\overbrace{s+s+\cdots+s}^{p \text{ times}}} = D^{ps}_\beta.$$

4. Exploiting 1, the b-fold composition is as follows:

$$\begin{aligned} D^{a/b}_\beta \circ \cdots \circ D^{a/b}_\beta = D^{ba/b}_\beta &= D^a_\beta \\ &= \lim_{k\to a} \big((1+T)\frac{d}{dT}\big)^k \\ &= \big((1+T)\frac{d}{dT}\big)^a. \end{aligned}$$

□

3.2 Approximating the zeta-function

Let $F(T) \in \Lambda$ with convergent polynomials $\{R_n(T)\}_{n\in\mathbb{N}}$. In the Λ-adic topology, if n is very large then the ideal $(p,T)^n$ is very small.

Lemma 3.4.
For all $n \in \mathbb{N}$,

$$D_\beta^s \circ \psi F(T) = \sum_{\alpha\in\mu_{p^n}} F(\alpha^{-1}-1)p^{-n} \sum_{\substack{m=0 \\ p\nmid m}} \omega^\beta(m) < m >^s \alpha^m (1+T)^m \mod (p,T)^n.$$

Proof.
From Lemma 3.2

$$\big((1+T)\frac{d}{dT}\big)^s \circ \psi F(T) \equiv \big((1+T)\frac{d}{dT}\big)^s \circ \psi R_n(T) \mod (p,T)^n$$

and if $s \in \mathfrak{F}_\beta$ then from Lemma 3.3(2), $D_\beta^s = \big((1+T)\frac{d}{dT}\big)^s$. We conclude that

$$D_\beta^s \circ \psi F(T) \equiv D_\beta^s \circ \psi R_n(T) \mod (p,T)^n$$

when $s \in \mathbb{Z}_p \bigcap \mathfrak{F}_\beta$.

Now, it is enough to show $D_\beta^s \circ \psi R_n(T)$ equals the right hand side in the above lemma. From Lemma 3.1

$$\begin{aligned} R_n(T) &= \sum_{\alpha\in\mu_{p^n}} F(\alpha^{-1}-1)\Theta_n(\alpha(1+T)) \\ &= \sum_{\alpha\in\mu_{p^n}} F(\alpha^{-1}-1)p^{-n} \sum_{m=0}^{p^n-1} \alpha^m(1+T)^m. \end{aligned}$$

Then from the action of D_β^s on $\psi R_n(T)$, we have

$$D_\beta^s \circ \psi R_n(T) = \sum_{\alpha\in\mu_{p^n}} F(\alpha^{-1}-1)p^{-n} \sum_{\substack{m=0 \\ p\nmid m}}^{p^n-1} \omega^\beta(m) < m >^s \alpha^m (1+T)^m \mod (T,p)^n.$$

□

This lemma gives a nice numerical approximation to the fractional derivative. This sequence also assists us to find a connection between fractional derivatives and L-functions.

Proposition 3.1.
Let $\mathcal{L}_2 = \frac{1}{T+2}$ *and fix a class* β *modulo* $p-1$*. Then for all* $s \in \mathbb{Z}_p$,

$$-\zeta_p(-s, \omega^{1+\beta}) = \frac{D^s_\beta \circ \psi\mathcal{L}_2(0)}{(1-\omega^{1+\beta}(2) < 2 >^{1+s})}.$$

Proof.
The power series $\mathcal{L}_2(T)$ has no poles in $\mathbb{Z}_p$ except at $T = -2$, so $\mathcal{L}_2(T) \in \Lambda$. Let us now rewrite $\mathcal{L}_2(T)$ in a different form, namely

$$\mathcal{L}_2(T) = \frac{1}{T} - \frac{2}{(1+T)^2 - 1}.$$

Substituting $T = \exp(Z) - 1$, then

$$\mathcal{L}_2(\exp(Z) - 1) = \frac{1}{Z}\left(\frac{Z}{\exp(Z) - 1} - \frac{2Z}{\exp(2Z) - 1}\right).$$

Note that

$$\frac{Z}{\exp(Z) - 1} = \sum_{n=0}^{\infty} B_n \frac{Z^n}{n!}.$$

In the article [2] it is reworded that for all $k \in \mathbb{N}$,

$$\begin{aligned}\big((1+T)\frac{d}{dT}\big)^k \psi\mathcal{L}_2(T)\big|_{T=0} &= \big(\exp(Z)\frac{d}{dZ}\big)^k \psi\mathcal{L}_2(\exp(Z)-1)\big|_{Z=0} \\ &= (1-2^{1+k})(1-p^k)\frac{B_{1+k}}{1+k}.\end{aligned}$$

One can see for $k \in \mathfrak{F}_\beta$ that

$$\big((1+T)\frac{d}{dT}\big)^k \psi\mathcal{L}_2(T)\big|_{T=0} = D^k_\beta \circ \psi\mathcal{L}_2(0).$$

Now let $\chi_n = 1$ (the principal character) which implies $\chi = \omega^n$. The Kubota and Leopoldt p-adic L-function interpolates

$$\begin{aligned}L_p(1-n, \omega^n) &= (1-p^{n-1})\frac{-B_n}{n} \\ &= (1-p^{n-1})L(1-n, \omega^n).\end{aligned}$$

Now fix β to be a congruence class modulo $p-1$. Then the each branche $\zeta_p(s, \omega^\beta)$ is a continuous function except for a pole at $s = 1$, if $\beta = -1$, so that

$$(1-p^k)\frac{B_{1+k}}{1+k} = -\zeta_p(-s, \omega^{1+\beta}).$$

Also, the Euler factor $(1-2^{1+k})$ is interpolated p-adically by

$$(1-\omega^{1+\beta}(2) < 2 >^{1+s}),$$

so we are done. ∎

Here, we give a proof for Theorem 3.1 in the special case when $\chi = 1$.

Proof.
Recall that

$$L_p(s, \omega^{1+\beta}) = \zeta_p(s, \omega^{1+\beta}).$$

Now we will prove the expansion

$$\zeta(-s, \omega^{1+\beta}) = \frac{1}{2(1 - \omega^{1+\beta}(2) < 2 >^{1+s})} \sum_{n=1}^{\infty} \Big(\sum_{\substack{m=p^{n-1} \\ p \nmid m}}^{p^n} (-1)^{m+1} \omega^{\beta}(m) < m >^s \Big),$$

(Dellbourgo [2]) where $s \in \mathbb{Z}_p$ and $\beta \equiv 0 \bmod p-1$.

From Proposition 3.1 we have

$$(1 - \omega^{1+\beta}(2) < 2 >^{1+s})\zeta_p(-s, \omega^{1+\beta}) = -D^s_\beta \circ \psi \mathcal{L}_2(0),$$

and by Lemma 3.3,

$$D^s_\beta \circ \psi \mathcal{L}_2(0) \equiv \sum_{\substack{m=1 \\ p \nmid m}}^{p^n} \omega^{\beta}(m) < m >^s p^{-n} \sum_{\alpha \in \mu_{p^n}} \alpha^m \mathcal{L}_2(\alpha^{-1} - 1) \quad (\bmod p^n).$$

Let us recall $\Omega_m(\mathcal{L}_2) = p^{-n} \sum_{\alpha \in \mu_{p^n}} \alpha^m \mathcal{L}_2(\alpha^{-1} - 1)$. Hence we need to show that $\Omega_m(\mathcal{L}_2) = \frac{(-1)^m}{2}$, where $0 \leq m \leq p^n - 1$.

It can be proved by induction on m; we do the calculation inside the complex numbers, since $\Omega_m(\mathcal{L}_2) \in \mathbb{Q}(\mu_{p^n})$. If $m = 0$, then

$$\begin{aligned} \Omega_0(\mathcal{L}_2) &= p^{-n} \sum_{\alpha \in \mu_{p^n}} \frac{1}{\alpha^{-1} + 1} \\ &= p^{-n} \sum_{\alpha \in \mu_{p^n}} \frac{1 + \alpha}{2 + 2Re(\alpha)} \\ &= p^{-n} \sum_{\alpha \in \mu_{p^n}} \frac{(1 + Re(\alpha)) + iIm(\alpha)}{2 + 2Re(\alpha)} \\ &= \frac{1}{2p^n} \sum_{\alpha \in \mu_{p^n}} 1 + i \frac{Im(\alpha)}{1 + Re(\alpha)}. \end{aligned}$$

For any $\alpha \in \mu_{p^n}$,

$$\frac{Im(\alpha^{-1})}{1 + Re(\alpha^{-1})} = -\frac{Im(\alpha)}{1 + Re(\alpha)},$$

so that $\Omega_0(\mathcal{L}_2) = \frac{1}{2}$. If $1 \leq m \leq p^n - 1$ clearly $m \not\equiv 0 \bmod p^n$, hence

$$\Omega_m(\mathcal{L}_2) = p^{-n} \sum_{\alpha \in \mu_{p^n}} \frac{\alpha^m}{\alpha^{-1} + 1}$$

$$= p^{-n} \sum_{\alpha \in \mu_{p^n}} \frac{\alpha^m((\alpha^{-1}+1) - \alpha^{-1})}{\alpha^{-1}+1}$$
$$= p^{-n} \sum_{\alpha \in \mu_{p^n}} \alpha^m - \frac{\alpha^{m-1}}{\alpha^{-1}+1}$$
$$= 0 - \Omega_{m-1}(\mathcal{L}_2).$$

Therefore, we conclude that

$$\Omega_{m-1}(\mathcal{L}_2) = \frac{(-1)^{m-1}}{2}.$$

□

Let us now take a closer look at the formula

$$\zeta(-s, \omega^{1+\beta}) = \frac{1}{2(1-\omega^{1+\beta}(2) < 2 >^{1+s})} \sum_{n=1}^{\infty} \Big(\sum_{\substack{m=p^{n-1} \\ p \nmid m}}^{p^n} (-1)^{m+1} \omega^{\beta}(m) < m >^s \Big).$$

We shall define

$$\Delta_\beta(n,s) = \sum_{\substack{m=p^{n-1} \\ p \nmid m}} (-1)^{m+1} \omega^{\beta}(m) < m >^s .$$

If $s = -k \in \mathbb{Z}_p$ with $\beta \equiv -k \bmod p-1$, then

$$\omega^\beta < m >^s = \omega^\beta(m) < m >^{-k} = m^{-k}.$$

Now one can partition each summation by

$$\sum_{\substack{m=p^{n-1} \\ p \nmid m}}^{p^n} (-1)^{m+1} < m >^s = \sum_{\substack{m=p^{n-1} \\ p \nmid m}}^{p^n} (-1)^{m+1} m^{-k}$$
$$= \sum_{\substack{m=p^{n-1} \\ m=odd \\ p \nmid m}} \frac{1}{m^k} - \sum_{\substack{m=p^{n-1} \\ m=even \\ p \nmid m}} \frac{1}{m^k}.$$

Furthermore

$$\omega^{1+\beta}(2) < 2 >^{1+s} = \omega^{1+\beta}(2) < 2 >^{1-k}$$
$$= \omega(2) < 2 > \omega^{\beta}(2) < 2 >^{-k}$$
$$= 2(2^{-k}),$$

so that

$$\zeta_p(k,\omega^{1-k}) = \frac{1}{2(1-2^{1-k})}\sum_{n=1}^{\infty}\Big(\sum_{\substack{m=p^{n-1}\\ m=odd\\ p\nmid m}}\frac{1}{m^k} - \sum_{\substack{m=p^{n-1}\\ m=even\\ p\nmid m}}\frac{1}{m^k}\Big).$$

If $k=1$ then $\beta\equiv -1$ and $(1-2^{1-k})=0$. Therefore the ω^0-branch of the p-adic zeta function has a pole at $k=1$. If $k\leq 0$, and $k\in\mathfrak{F}_\beta$, then

$$\begin{aligned}\zeta_p(k,\omega^{1-k}) &= (1-p^{-k})\zeta(k)\\ &= (1-p^{-k})\frac{B_{1-k}}{1-k}.\end{aligned}$$

Therefore, we obtain a congruence

$$(1-p^{-k})\frac{B_{1-k}}{1-k} \equiv \frac{-1}{2(1-2^{1-k}}\Big(\sum_{\substack{m=p^{n-1}\\ m=odd\\ p\nmid m}}\frac{1}{m^k} - \sum_{\substack{m=p^{n-1}\\ m=even\\ p\nmid m}}\frac{1}{m^k}\Big) \bmod p^n,$$

which implies

$$(1-2^{1-k})(1-p^{-k})\frac{B_{1-k}}{1-k} \equiv \frac{-1}{2}\sum_{\substack{m=1\\ p\nmid m}}^{p^n}\frac{(-1)^{m+1}}{m^k} \bmod p^n.$$

unfortunately $k\geq 2$ we cannot interpolate the classical Riemann zeta function from the values $\zeta_p(k,\omega^{1-k})$.

4 How to obtain a distinguished polynomial from a power series

Recall that Kubota and Leopoldt introduced the p-adic L-function $L_p(s, \chi)$ attached to a Dirichlet character χ, which is always supposed to be primitive. The p-adic L-function $L_p(s, \chi)$ is a continuous function converging on the open disc $D_s = \left\{s : s \in \bar{\mathbb{Q}}_p, |s|_p < p^{\frac{p-2}{p-1}}\right\}$.

Here, we will express the p-adic L-function in the form of a power series. Let K be a finite unamified extensions of $\mathbb{Q}_p$. Also, let $\mathcal{O}$ be the ring of integer of K. For $F \in \mathcal{O}[[T]]$ let $F = \sum_{j=0}^{\infty} a_j T^j$. We need to introduce some notation as follows:

$$\mu := \min \left\{v(a_j) : j \geq 0\right\}$$

and

$$\lambda := \min \left\{j \geq 0 : v(a_j) = \mu\right\}.$$

Definition 4.1.
A polynomial $P(T) \in \mathcal{O}[T]$ is called distinguished if it has the form

$$P(T) = T^n + b_{n-1}T^{n-1} + \cdots + b_0,$$

where each b_j lies in the maximal ideal of $\mathcal{O}$.

4.1 Weierstrass Preparation Theorem

Theorem 4.1. *(p-adic Weierstrass Preparation Theorem)*
Let $F \in \mathcal{O}[[T]]$ be a power series, with μ and λ as defined above. Then the power series F can be factored into

$$F(T) = p^{\mu} U(T) P(T),$$

where $U(T)$ is a unit in $\mathcal{O}[[T]]$, and P is a distinguished polynomial.

This theorem show that F has exactly the same number of zeros as P, namely $deg(P)$. Also, the region of these zeros is $\left\{T \in \bar{\mathbb{Q}}_p : |T| < 1\right\}$.

Proposition 4.1. *(Iwasawa's Theorem)*
There exists a unique $F_\chi \in \mathbb{K}[[T]]$ such that

$$F_\chi((1+p)^{-s} - 1) = (1 - \chi(2) < 2 >^{1-s}) L_p(s, \chi),$$

for all $s \in \bar{\mathbb{Q}}_p$ with $|s| \leq 1$.

It has been proved by Ferrero and Washington that the μ-invariant of $F(T)$ is zero [4]. By the Weierstrass Preparation theorem

$$F(T) = F_\chi(T) = U(T)P(T),$$

where U is a unit; therefore one defines the λ-invariant equivalently by

$$\lambda = \min \{ j \geq 0 : v_p(a_j) = 0 \},$$

since $\mu = 0$. Now, we can write

$$\begin{aligned} F_\chi(T) &= U_\chi(T)P_\chi(T) \\ &= U_\chi(T)(T^\lambda + b_{\lambda-1}T^{\lambda-1} + \cdots + b_1 + b_0). \end{aligned}$$

where $U_\chi(T)$ is a unit power series in $\mathcal{O}_\chi[[T]]$ and the coefficients b_j are in $p\mathbb{Z}_p$. We call $P_\chi(T)$ the Iwasawa polynomial. Here, $\lambda_\chi = deg(P_\chi)$, which is the same as the λ-invariant of $F_\chi(T)$.

Now, let us suppose $\xi \in \bar{\mathbb{Q}}_p$ is a zero of P_χ. Since P_χ is a distinguished polynomial, we can write P_χ in the form

$$P_\chi(T) = T^\lambda + \sum_{i<\lambda} c_i T^i.$$

with $\left|c_i\right|_p \leq \frac{1}{p}$ for $i < \lambda$. Then

$$P_\chi(\xi) = \xi^\lambda + \sum_{i<\lambda} c_i \xi^i = 0,$$

which implies

$$\left|\xi^\lambda\right|_p = \Big|\sum_{i<\lambda} c_i \xi^i\Big|_p \leq \frac{1}{p} \quad .$$

We conclude that

$$\left|\xi\right|_p \leq p^{-\frac{1}{\lambda}} \quad .$$

Note that $F_\chi(T)$ has λ zeros, and they are the same as the zeros of $P_\chi(T)$. These zeros exist in the disc

$$D_T = \{ T \in \bar{\mathbb{Q}}_p : v_p(T) > 0 \},$$

which corresponds to

$$D_s = \{ s : s \in \bar{\mathbb{Q}}_p, |s|_p < p^{\frac{p-2}{p-1}} \}$$

(the region of $L_p(s,\chi)$'s zeros).

4.2 The coefficients of the power series

Now let us fix integers $N, t \geq 1$ such that $p^N < p^{t\phi(2f_\chi)}$, and $\alpha \in \{1, \ldots, 2f_\chi - 1\}$ satisfies

$$\alpha \equiv p^{-1}(\text{mod} 2f_\chi).$$

For each $x, m \in \mathbb{N}$ with $p \nmid m$, define

$$\theta(x, m) \in \{1, \ldots, 2f_x p^N - 1\}$$

to be the unique representative for which

$$\theta(x, m) \equiv m + (p\alpha)^N (x - m) \bmod 2f_\chi p^N,$$

and set

$$\gamma_{N,t} = \Big\lfloor \frac{p^{t\phi(2f_\chi)}}{2f_\chi p^N} \Big\rfloor.$$

Theorem 4.2. $\big(2\omega^\beta(2) < 2 >^{-s} -1\big) L_p(s, \chi\omega^{1+\beta}) \equiv \sum_{\substack{m=1 \\ p\nmid m}}^{p^N} \omega^\beta(m) < m >^{-s}$ $\times \sum_{x=1}^{2f_\chi} a_\chi(x) \times \big(\gamma_{N,t} + \delta_{\theta(x,m) < p^{t\phi(2f_\chi)} - 2f_\chi p^N \gamma_{N,t}}\big) \quad \bmod p^N,$

where $\delta_{a<b} = \begin{cases} 1 & \textit{if } a < b \\ 0 & \textit{if } a \geq b \end{cases}$.

Proof.
Theorem 7 implies that the

$$L.H.S. \equiv \sum_{\substack{m=1 \\ p\nmid m}}^{p^{t\phi(2f_\chi)}} a_m(\chi)\omega^\beta(m) < m >^{-s} \bmod p^{t\phi(2f_\chi)}.$$

In particular, the coefficients $a_m(x)$ are periodic of modulo $2f_\chi$, hence

$$L.H.S. \equiv \sum_{x=1}^{2f_\chi} a_x(\chi) \times \sum_{\substack{m=1 \\ p\nmid \\ m\equiv x \bmod 2f_\chi}} \omega^\beta < m >^{-s} \bmod p^{t\phi(2f_\chi)}.$$

Let us now suppose that

$$\alpha \equiv p^{-1}(\text{mod} 2f_\chi)$$

with $\alpha \in \{1, \ldots, 2f_\chi - 1\}$. Then

$$\sum_{\substack{m=1 \\ p\nmid m \\ m\equiv x \bmod 2f_\chi}}^{p^{t\phi(2f_\chi)}} \omega^\beta(m) < m >^{-s} \equiv \sum_{\substack{m'=1 \\ p\nmid m'}}^{p^N} \sum_{\substack{m=1 \\ m\equiv x (\bmod 2f_\chi) \\ m\equiv m' \bmod p^N}}^{p^{t\phi(2f_\chi)}} \omega^\beta(m') < m' >^{-s},$$

since
$$\omega^{\beta}(m) < m >^{-s} \equiv \omega^{\beta}(m')< m' >^{-s} \bmod p^N$$
where $m \equiv m' \bmod p^N$.

The two congruences
$$m \equiv x (\mathrm{mod} 2f_\chi)$$
and
$$m \equiv m' (\mathrm{mod} p^N)$$
can be combined into the single congruence
$$m \equiv m' + (p\alpha)^N (x - m')(\mathrm{mod} 2f_\chi p^N).$$
It follow that
$$\sum_{\substack{m=1 \\ p\nmid m \\ m\equiv x \bmod 2f_\chi}}^{p^{t\phi(2f_\chi)}} \omega^{\beta}(m) < m >^{-s} \equiv \sum_{\substack{m'=1 \\ p\nmid m'}}^{p^N} \omega^{\beta}(m')< m' >^{-s} \times \#\Upsilon^{(x,m')},$$
where $\Upsilon^{(x,m')} = \{m| \quad 1 \leq m < p^{t\phi(2f_\chi)}, m \equiv m' + (p\alpha)^N (x - m') \bmod 2f_\chi p^N\}$. By inserting this in our formula, one obtains
$$L.H.S. \equiv \sum_{\substack{m'=1 \\ p\nmid m'}}^{p^N} \omega^{\beta}(m') < m' >^{-s} \times \sum_{x=1}^{2f_\chi} a_x(\chi) \times \#\Upsilon^{(x,m')} \bmod p^N.$$

To compute the number of m in $\Upsilon^{(x,m')}$, let $\theta \in \Big\{1, \dots, 2f_\chi p^N - 1\Big\}$ with $p \nmid \theta$, and define
$$\Upsilon_\theta = \Big\{m|1 \leq m < p^{t\phi(2f_\chi)}, m \equiv \theta \bmod 2f_\chi p^N\Big\}.$$
The interval $[1, p^{t\phi(2f_\chi)}) \bigcap \mathbb{N}$ can be divided into equal subintervals, and the number of subintervals is $\lfloor \frac{p^{t\phi(2f_\chi)}}{2f_\chi p^N} \rfloor$. The remaining interval on the right is $\big[2f_\chi p^N \times \lfloor \frac{p^{t\phi(2f_\chi)}}{2f_\chi p^N} \rfloor, p^{t\phi(2f_\chi)}\big]$.

There is exactly one solution in each subinterval, so that
$$\#\Upsilon_\theta = \lfloor \frac{p^{t\phi(2f_\chi)}}{2f_\chi p^N} \rfloor + \#\Upsilon'_\theta,$$
where
$$\Upsilon'_\theta = \Big\{m| \quad 2f_\chi p^N \times \lfloor \frac{p^{t\phi(2f_\chi)}}{2f_\chi p^N} \rfloor < m < p^{t\phi(2f_\chi)}, m \equiv \theta (\mathrm{mod} 2f_\chi p^N)\Big\}.$$

Now the number $\Upsilon_\theta^{'}$ will be zero or one, depending on whether

$$\theta + 2f_\chi p^N \times \lfloor \frac{p^{t\phi(2f_\chi)}}{2f_\chi p^N} \rfloor$$

is greater than $p^{t\phi(2f_\chi)}$, or not. Consequently

$$\Upsilon_\theta^{'} = \begin{cases} 1 & \text{if } \theta < p^{t\phi(2f_\chi)} - 2f_\chi p^N \times \lfloor \frac{p^{t\phi(2f_\chi)}}{2f_\chi p^N} \rfloor \\ 0 & \text{otherwise} \end{cases} .$$

We can conclude that, if $\gamma_{N,t} = \lfloor \frac{p^{t\phi(2f_\chi)}}{2f_\chi p^N} \rfloor$, then

$$\#\Upsilon_\theta = \gamma_{N,t} + \delta_{\theta < p^{t\phi(2f_\chi)} - 2f_\chi p^N \gamma_{N,t}}.$$

As a special case,

$$\#\Upsilon^{(x,m')} = \gamma_{N,t} + \delta_{\theta(x,m') < p^{t\phi(2f_\chi)} - 2f_\chi p^N \gamma_{N,t}},$$

where $\theta(x, m') \in \{1, \ldots, 2f_\chi p^N - 1\}$ satisfies

$$\theta(x, m') \equiv m' + (p\alpha)^N (x - m') \bmod 2f_\chi p^N.$$

In particular, we obtain

$$L.H.S. = \sum_{\substack{m'=1 \\ p \nmid m'}}^{p^N} \omega^\beta(m') < m' >^{-s} \times \sum_{x=1}^{2f_\chi} a_x(\chi) \times \#\Upsilon^{(x,m')} \bmod p^N$$

where $\#\Upsilon^{(x,m')} = \gamma_{N,t} + \delta_{\theta(x,m') < p^{t\phi(2f_\chi)} - 2f_\chi p^N \gamma_{N,t}}$. Upon replacing m' with m, the theorem statement follows. □

Recall from Theorem 3.1, we have

$$L_p(s, \chi\omega^{1+\beta}) = (2\omega^\beta(2) < 2 >^{-s} -1)^{-1} \times \lim_{n\to\infty} \Big(\sum_{\substack{m=1 \\ p \nmid m}}^{p^{n\phi(2f_\chi)}} a_m(\chi)\omega^\beta(m) < m >^{-s} \Big),$$

or more precisely,

$$(2\omega^\beta(2) < 2 >^{-s} -1) L_p(s, \chi\omega^{1+\beta}) \equiv \sum_{\substack{m=1 \\ p \nmid m}}^{p^{n\phi(2f_\chi)}} a_m(\chi)\omega^\beta(m) < m >^{-s} \quad \bmod p^{n\phi(2f_\chi)}.$$

In addition, we also know

$$F_\chi((1+p)^{-s}-1) = U_\chi((1+p)^{-s}-1) P_\chi((1+p)^{-s}-1) = (2\omega^\beta(2) < 2 >^{-s} -1) L_p(s, \chi\omega^{1+\beta}),$$

which implies that

$$F_\chi((1+p)^{-s}-1) \equiv \sum_{\substack{m=1\\ p\nmid m}}^{p^{n\phi(2f_\chi)}} a_m(\chi)\omega^\beta(m) < m >^{-s} \quad \mathrm{mod} p^{n\phi(2f_\chi)}.$$

Hence we have a polynomial approximation in $\mathcal{O}[T]$, namely

$$F_\chi(T) \equiv \sum_{\substack{m=1\\ p\nmid m}}^{p^{n\phi(2f_\chi)}} a_m(\chi)\omega^\beta(m) \times (1+T)^{\lambda_n(m)} \quad \mathrm{mod}\ p^{n\phi(2f_\chi)}$$

where $\lambda_n(m)$ is an integer from the set $\{0,1,\ldots,p^n-1\}$ congruent to $\frac{log_p(m)}{log_p(1+p)}$ mod $p^{n\phi(2f_\chi)}$. Now, by the Binomial Theorem the

$$\begin{aligned} R.H.S. &= \sum_{\substack{m=1\\ p\nmid m}}^{p^{n\phi(2f_\chi)}} a_m(\chi)\omega^\beta(m) \sum_{t=0}^{\lambda_n(m)} \binom{\lambda_n(m)}{t} T^t \\ &= \sum_{\substack{m=1\\ p\nmid m}}^{p^{n\phi(2f_\chi)}} a_m(\chi)\omega^\beta(m) \sum_{t=0}^{p^{n\phi(2f_\chi)}} \binom{\lambda_n(m)}{t} \delta_{t\le\lambda_n(m)} T^t \end{aligned}$$

where $\delta_{t\le\lambda_n(m)} = \begin{cases} 1 & \text{if } t \le \lambda_n(m) \\ 0 & \text{otherwise} \end{cases}$. Thus

$$R.H.S. = \sum_{t=0}^{p^{n\phi(2f_\chi)}} T^t \times \sum_{\substack{m=1\\ p\nmid m}}^{p^{n\phi(2f_\chi)}} \delta_{t\le\lambda_n(m)} \binom{\lambda_n(m)}{t} a_m(\chi)\omega^\beta(m).$$

If we define $C_{\chi,t} = \sum_{\substack{m=1\\ p\nmid m}}^{p^{n\phi(2f_\chi)}} \delta_{t\le\lambda_n(m)} \binom{\lambda_n(m)}{t} a_m(\chi)\omega^\beta(m)$, then

$$F_\chi(T) \equiv \sum_{t=0}^{p^{n\phi(2f_\chi)}} T^t \times C_{\chi,t} \quad \mathrm{mod}\ p^{n\phi(2f_\chi)}.$$

Proposition 4.2.
If $F_\chi(T) \in \mathcal{O}_\chi[[T]]$ is the power series corresponding to

$$(2\omega^\beta(2) < 2 >^{-s} -1)L_p(s, \chi\omega^{1+\beta}),$$

then for all N such that $p^N < p^{n\phi(2f_\chi)}$:

$$F_\chi(T) \equiv \sum_{j=0}^{p^N} C_j^{(p^N)} T^j \quad \mathrm{mod}\ (1+T)^{p^N} - 1,$$

where

$$C_j^{(p^N)} := \sum_{\substack{m=1 \\ p \nmid m}}^{p^N} \delta_{j \le \lambda_N(m)} \binom{\lambda_N(m)}{j} \omega^\beta(m) \times \sum_{x=1}^{2f_\chi} a_x(\chi) \delta_{\theta(x,m) < p^{t\phi(2f_\chi)} - 2f_\chi p^N \gamma_{N,t}}.$$

Proof.
From the discussion after Theorem 4.2, we can write

$$F_\chi(T) \equiv \sum_{\substack{m=1 \\ p \nmid m}}^{p^N} \omega^\beta(m)(1+T)^{\lambda_N(m)} \sum_{x=1}^{2f_\chi} a_x(\chi)(\gamma_{N,t} + \delta_\theta).$$

Since $\sum_{x=1}^{2f_\chi} a_x(\chi)$ is always equal to zero, we can rewrite the power series congruence as follows:

$$F_\chi(T) \equiv \sum_{\substack{m=1 \\ p \nmid m}}^{p^N} \omega^\beta(m) \sum_{x=1}^{2f_\chi} a_x(\chi)\delta_\theta \times (1+T)^{\lambda_N(m)}.$$

By again using the Binomial Theorem,

$$\begin{aligned} (1+T)^{\lambda_N(m)} &= \sum_{j=0}^{\lambda_N(m)} \binom{\lambda_N(m)}{j} T^j \\ &= \sum_{j=0}^{p^N} \delta_{j \le \lambda_N(m)} \binom{\lambda_N(m)}{j} T^j. \end{aligned}$$

Lastly inserting this in the formula of $F_\chi(T)$, one obtains

$$\begin{aligned} F_\chi(T) &\equiv \sum_{\substack{m=1 \\ p \nmid m}}^{p^N} \omega^\beta(m) \sum_{x=1}^{2f_\chi} a_x(\chi)\delta_\theta \times \sum_{j=0}^{p^N} \delta_{j \le \lambda_N(m)} \binom{\lambda_N(m)}{j} T^j \\ &= \sum_{j=0}^{p^N} T^j \times \sum_{\substack{m=1 \\ p \nmid m}}^{p^N} \delta_{j \le \lambda_N(m)} \binom{\lambda_N(m)}{j} \omega^\beta(m) \times \sum_{x=1}^{2f_\chi} a_x(\chi)\delta_\theta \\ &= \sum_{j=0}^{p^N} T^j C_j^{(p^N)}, \end{aligned}$$

as required. □

Corollary 4.1.
If the λ-invariant of $F_\chi(T)$ is less than p^2, then it must equal

$$\min\left\{j : C_j^{(p^2)} \not\equiv 0 \bmod p\right\}.$$

Proof. Assume that $\lambda(F_\chi) < p^2$. Since $\mu(F_\chi) = 0$, clearly we must have

$$ord_p(C_j(F_\chi)) > 0$$

for all $j \in \{0, \ldots, \lambda(F_\chi) - 1\}$, whilst $ord_p(C_\lambda(F_\chi)) = 0$. Moreover, we know that

$$F_\chi(T) \;=\; \sum_{j=0}^{\infty} C_j(F_\chi)T^j \;\equiv\; \sum_{j=0}^{p^2} C_j^{(p^2)} T^j \quad \bmod (1+T)^{p^2} - 1$$

where

$$C_j^{(p^2)} := \sum_{\substack{m=1 \\ p \nmid m}}^{p^2} \delta_{j \le \lambda_2(m)} \binom{\lambda_2(m)}{j} \omega^\beta(m) \times \sum_{x=1}^{2f_\chi} a_x(\chi) \delta_{\theta(x,m) < p^{t\phi(2f_\chi)} - 2f_\chi p^2 \gamma_{2,t}}.$$

Furthermore $(1+T)^{p^2} - 1 \equiv T^{p^2} \bmod p$, and T^{p^2} has strictly higher degree than T^λ, hence the result follows. $\square$

5 Finding the zeros of $L_p(s, \chi_d \omega^{1+\beta})$

Here, we introduce the main steps to compute the zeros of the p-adic zeta function. We use a PARI program to compute our zeros, and we wrote a GP/script to insert data into the PARI program. We first compute the λ-invariant of $L_p(s, \chi_d \omega^{1+\beta})$ for $d \in \mathbb{N}$, d square-free with $p \nmid d$, and $\beta \in \{1, 3, \ldots, p-2\}$. Also we compute the λ-invariant of $L_p(s, \chi_d \omega^{1+\beta})$ for $-d \in \mathbb{N}$ d square-free with $p \nmid d$, and $\beta \in \{0, 2, \ldots, p-3\}$. We choose d to range from -200 to 200. Secondly, we explain how to compute the coefficients of the p-adic power series arising from the p-adic zeta function to desired precision. Finally, we extract the Iwasawa polynomial from this computed power series.

5.1 Computing the λ-invariant of $L_p(s, \chi_d \omega^{1+\beta})$

Let us first compute the λ-invariant of the power series associated to

$$F_\chi((1+p)^{-s} - 1) = (2\omega^\beta(2) < 2 >^{-s} - 1) L_p(s, \chi\omega^{1+\beta}),$$

which is the index of the first coefficient not divisible by p.

Let us examine the factor $(2\omega^\beta(2) < 2 >^{-s} - 1)$. Then

$$2\omega^\beta(2) < 2 >^{-s} - 1 = 0 \quad \text{at} \quad s = 1$$

$$\iff 2\omega^\beta < 2 >^{-s} = 1 \quad \text{at} \quad s = 1$$

$$\iff \omega^{\beta+1}(2) < 2 >^{1-s} = 1 \quad \text{at} \quad s = 1$$

$$\iff \omega^{\beta+1}(2) = 1$$

$$\iff 2^{\beta+1} \equiv 1 \bmod p$$

$$\iff \beta \equiv -1 \bmod p - 1.$$

Thus there is a zero at $s = 1$ for the factor $(2\omega^\beta(2) < 2 >^{-s} - 1)$ if and only if $\beta \equiv -1$, so that λ-invariants of the power series measure the number of zeros of the p-adic L-functions, including the contribution from the factor $(2\omega^\beta(2) < 2 >^{-s} - 1)$. Since we are looking for zeros of $L_p(s, \chi_d \omega^{1+\beta})$, we have the following relation:

$$\#\text{of zeros for} L_p(s, \chi_d \omega^{1+\beta}) = \begin{cases} \lambda(F_\chi) & \text{if } \beta \not\equiv -1 \bmod p - 1 \\ \lambda(F_\chi) - 1 & \text{if } \beta \equiv -1 \bmod p - 1 \end{cases}.$$

We implement this as "laminv(beta)".

5.2 Computing the coefficients of power series

Let χ be a quadratic character. For each $m \in \{1, \ldots, 2f_\chi\}$, one has

$$a_m(\chi) := -\sum_{a=1}^{f_\chi} \chi(a) \times \frac{a}{f_\chi} + \sum_{j=1}^{m-1} \chi(j) - 2 \sum_{j=1}^{\lfloor \frac{m-1}{2} \rfloor} \chi(j).$$

Our program computes each coefficient $a_m(\chi)$ for $\chi = \chi_d$ with

$$d \in \{-200, -199, \ldots, 199, 200\}.$$

Note that the discriminant of $\mathbb{Q}(\sqrt{d})$ is

$$D = \begin{cases} d & \text{if } d \equiv 1 \bmod 4 \\ 4d & \text{if } d \equiv 2 \quad \text{or} \quad 3 \bmod 4 \end{cases}.$$

In fact $|D|$ equals the conductor f_χ, by the conductor-discriminant formula. Assume K is an abelian extension of $\mathbb{Q}$. Let r_1, r_2 be the number of real/complex embeddings of the number field K. Then

$$\prod_{\chi : Gal(K/\mathbb{Q}) \to \mathbb{C}^\times} f_\chi = (-1)^{r_2} \times disc(K/\mathbb{Q}).$$

For example, If $K = \mathbb{Q}(\sqrt{d})$ then

$$L.H.S. = 1 \times f_{\chi_d} = R.H.S. = \big|D\big|.$$

Furthermore by Proposition 4.2, we have the approximation

$$F_\chi(T) \equiv \sum_{j=0}^{p^N} C_j^{(p^N)} T^j \mod (1+T)^{p^N} - 1,$$

where the coefficients of the power series are given by

$$C_j^{(p^N)} = \sum_{\substack{m=1 \\ p \nmid m}}^{p^N} \delta_{j \leq \lambda_N(m)} \binom{\lambda_N(m)}{j} \omega^\beta(m) \times \sum_{x=1}^{2f_\chi} a_x(\chi) \delta_{\theta(x,m) < p^{t\phi(2f_\chi)} - 2f_\chi p^N \gamma_{N,t}}.$$

We implement this approximation as "coeffappr(j,beta)".

5.3 Computing the Iwasawa polynomial

Suppose $F(T) = \sum_{i=0}^{\infty} a_i T^i$ where $F(T) \in \mathcal{O}[[T]]$ is a power series with trivial μ-invariant. Its Weierstrass factorization is as follows

$$F(T) = U(T)P(T).$$

Here $U(T)$ is unit power series in $\mathcal{O}[[T]]$, and $P(T)$ is a distinguished polynomial of degree λ.

Proposition 5.1 (Ellenberg, J.S., Jain,S., Venkatesh, A.,[4]).
Assume one has the coefficient a_i of $F_\chi(T)$ to precision $0(p^{K+1-i})$. Then for each $k \leq \lfloor \frac{K}{\lambda} \rfloor$, one can compute the coefficient $c_0, c_1, \ldots, c_{K-\lambda k}$ of $P(T)$ to precision $0(p^k)$.

Proof.
Step 1: suppose we have

$$F(T) = U(T)(T^\lambda + c_{\lambda-1}T^{\lambda-1} + c_{\lambda-2}T^{\lambda-2} + c_{\lambda-3}T^{\lambda-3} + \cdots + c_0).$$

so we can write

$$T^\lambda + c_{\lambda-1}T^{\lambda-1} + c_{\lambda-2}T^{\lambda-2} + c_{\lambda-3}T^{\lambda-3} + \cdots + c_0 = \frac{1}{U(T)}F(T).$$

If $\frac{1}{U(T)} = \sum_{j=0}^{\infty} b_j T^j$ and $F(T) = \sum_{i=0}^{\infty} a_i T^i$, then

$$T^\lambda + c_{\lambda-1}T^{\lambda-1} + c_{\lambda-2}T^{\lambda-2} + c_{\lambda-3}T^{\lambda-3} + \cdots + c_0 = \sum_{j=0}^{\infty} b_j T^j \times \sum_{i=0}^{\infty} a_i T^i, \tag{5.1}$$

and the right hand side is

$$a_0b_0 + (a_0b_1 + a_1b_0)T + (a_0b_2 + a_1b_1 + a_2b_0)T^2 + (a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0)T^3 + \ldots.$$

Now we have $\sum_{i+j=\lambda} a_i b_j = 1$, so that

$$a_0b_\lambda + a_1b_{\lambda-1} + a_2b_{\lambda-2} + a_3b_{\lambda-3} + \cdots + a_{\lambda-3}b_3 + a_{\lambda-2}b_2 + a_{\lambda-1}b_1 + a_\lambda b_0 = 1.$$

Since $a_i \equiv 0 \bmod p$ for $i < \lambda$, we obtain the congruence

$$\begin{aligned} T^\lambda + c_{\lambda-1}T^{\lambda-1} + c_{\lambda-2}T^{\lambda-2} + \cdots + c_0 &\equiv T^\lambda + 0T^{\lambda-1} + 0T^{\lambda-2} + \cdots + c_0 \\ &\equiv T^\lambda \\ &\equiv (a_0b_\lambda + a_1b_{\lambda-1} + \cdots + a_{\lambda-1}b_1 + a_\lambda b_0)T^\lambda \\ &\equiv (0 + 0 + \cdots + a_\lambda b_0)T^\lambda \mod p. \end{aligned}$$

The latter implies that

$$a_\lambda b_0 \equiv 1 \mod p,$$

or equivalently,

$$b_0 \equiv a_\lambda^{-1} \mod p. \tag{5.2}$$

Because the L.H.S. of (5.1) has a trivial coefficient in $T^{\lambda+1}$,

$$a_0b_{\lambda+1} + a_1b_\lambda + a_2b_{\lambda-1} + \cdots + a_{\lambda-1}b_2 + a_\lambda b_1 + a_{\lambda+1}b_0 \equiv 0 \mod p.$$

Again $a_i \equiv 0 \bmod p$ for $i < \lambda$, so that

$$a_\lambda b_1 + a_{\lambda+1} b_0 \equiv 0 \bmod p.$$

As a consequence

$$b_1 \equiv -a_\lambda^{-1}(a_{\lambda+1} b_0) \bmod p. \tag{5.3}$$

Repeating the same argument with the coefficient of $T^{\lambda+2}$, we find

$$a_0 b_{\lambda+2} + a_1 b_{\lambda+1} + a_2 b_\lambda + \cdots + a_\lambda b_2 + a_{\lambda+1} b_1 + a_{\lambda+2} b_0 \equiv 0 \bmod p.$$

hence

$$a_\lambda b_2 + a_{\lambda+1} b_1 + a_{\lambda+2} b_0 \equiv 0 \bmod p,$$

or equivalently,

$$b_2 = -a_\lambda^{-1}(a_{\lambda+1} b_1 + a_{\lambda+2} b_0). \tag{5.4}$$

Proceeding inductively, one obtains

$$b_3 \equiv -a_\lambda^{-1}(a_{\lambda+1} b_2 + a_{\lambda+2} b_1 + a_{\lambda+3} b_0) \quad \bmod p \tag{5.5}$$

$$b_4 \equiv -a_\lambda^{-1}(a_{\lambda+1} b_3 + a_{\lambda+2} b_2 + a_{\lambda+3} b_1 + a_{\lambda+4} b_0) \quad \bmod p \tag{5.6}$$

$$b_5 \equiv -a_\lambda^{-1}(a_{\lambda+1} b_4 + a_{\lambda+2} b_3 + a_{\lambda+4} b_1 + a_{\lambda+5} b_0) \quad \bmod p, \tag{5.7}$$

and in general,

$$b_i \equiv -a_\lambda^{-1} \sum_{j=1}^{i} a_{\lambda+j} b_{i-j} \quad \bmod p.$$

Step 2: we now explain how to obtain the coefficients b_s to greater and greater accuracy. Assume we know the b_i's to a fixed accuracy modulo $p^{k'}$.

Firstly, since the coefficient of T^λ is equal to one in Equation(5.1), we obtain

$$\sum_{j=0}^{\lambda} a_{\lambda-j} b_j = 1$$

or equivalently,

$$b_0 = \frac{1}{a_\lambda} \times (1 - \sum_{i=1}^{\lambda} a_{\lambda-j} b_j).$$

This formula increases the accuracy of b_0 by a power of p, i.e. to modulus $p^{k'+1}$.

Secondly, assume that $s > 0$. Then as the coefficient of $T^{\lambda+s}$ is equal to zero in Equation(5.1), we now obtain

$$\sum_{i=0}^{\lambda+s} a_i b_{\lambda+s-i} = 0$$

or equivalently,

$$b_s = -\frac{1}{a_\lambda} \times \sum_{\substack{i=0 \\ i\neq\lambda}}^{\lambda+s} a_i b_{\lambda+s-i}.$$

Once more the accuracy of b_s will be increased by a power of p, i.e. to modulus $p^{k'+1}$. We now increase s by 1, and repeat the process. Finally, performing this algorithm $\lfloor \frac{K}{\lambda} \rfloor$-times, the proposition follows.

□

We have implemented the approximation of $P(T)$ coefficients as "bvecp[1]" and the $P(T)$ as "cvecp".

We should also remark that once one knows the Iwasawa polynomial to a given accuracy, it is straightforward to determine its zeros. If the degree of the polynomial is ≤ 4 then one simply applies either the quadratic, the cubic, or the quartic formula. If the degree is ≥ 5 then there is a p-adic version of Newton's method, that can be implemented at relatively low computational cost.

6 Tables

Note that λ-invariant associated to $L_p(s, \chi_d \omega^{1+\beta})$ is given by

$$\lambda = \lambda(p, d, \beta) := \begin{cases} \lambda(F_\chi) & \text{if } \beta \not\equiv -1 \bmod p-1 \\ \lambda(F_\chi) - 1 & \text{if } \beta \equiv -1 \bmod p-1 \end{cases}.$$

Tabulated below are exclusively the values of p, d, β where $\lambda(p, d, \beta) > 0$, in the range we are searching through.

Table 6.1: $\lambda = 1$

p	d	β
3	29	1
3	43	1
3	58	1
3	67	1
3	74	1
3	79	1
3	82	1
3	85	1
3	106	1
3	113	1
3	122	1
3	131	1
3	137	1
3	142	1
3	173	1
3	182	1
5	14	1
5	26	1
5	31	1
5	38	3
5	39	3
5	42	3
5	51	3
5	53	3
5	59	1
5	62	3
5	69	3
5	73	3
5	82	3
5	86	1
5	89	3
5	107	3
5	123	1
5	129	1
5	134	3
5	139	3
5	143	3
5	161	3
5	183	3
5	186	3
5	187	1
5	191	1
5	191	3
5	199	1
5	-127	2
7	-73	2
7	-188	2
7	-188	4
7	-145	4
11	-14	4
11	-19	2
11	-56	4

Table 6.2: $\lambda = 2$

p	d	β
3	62	1
3	77	1
3	83	1
3	103	1
3	139	1
3	151	1
3	179	1
3	181	1
3	199	1
3	-14	0
3	-35	0
3	-47	0
3	-65	0
3	-74	0
3	-101	0
3	-107	0
3	-113	0
3	-149	0
3	-158	0
3	-173	0
5	23	1
5	37	1
5	109	3
5	127	1
5	127	3
5	149	1
5	-11	0
5	-26	0
5	-34	0
5	-41	0
5	-46	0
5	-51	0
5	-114	0
5	-166	2

p	d	β
7	6	5
7	173	3
7	-34	0
7	-34	2
7	-65	4
7	-73	0
7	-89	0
7	-111	0
7	-118	0
7	-195	0
11	21	3
11	86	3
11	-14	2
11	-15	6
11	-19	0
11	-56	2
11	-61	0
11	-94	0
11	-107	0
11	-127	0
11	-182	0

Table 6.3: $\lambda = 3$

p	d	β
3	-41	0
3	-86	0
5	114	3
5	-123	2
5	-166	0
7	-143	0
7	-145	0
7	-151	0
7	-194	0

Table 6.4: $\lambda = 4$

p	d	β
7	123	5

When $\lambda(p,d,\beta) \geq 1$, we compute the coefficients of the power series from Proposition 4.2, as follows:

$$C_j^{(p^N)} = \sum_{\substack{m=1 \\ p\nmid m}}^{p^N} \delta_{j\leq\lambda_N(m)} \binom{\lambda_N(m)}{j} \omega^\beta(m) \times \sum_{x=1}^{2f_\chi} a_x(\chi)\delta_{\theta(x,m)<p^{t\phi(2f_\chi)}-2f_\chi p^N \gamma_{N,t}}$$

for $j = 0, 1, 2, 3, \ldots, 10$. In addition, we tabulate the Iwasawa polynomial associated to the power series. Note that $\deg(P_\chi(T)) = \lambda(F_\chi)$. As an illustration of our notation, if we write

$3^2 + 2*3^3 + 2*3^4 + 2*3^5 + 2*3^6 + 2*3^7 + 2*3^8 + 2*3^9 + 2*3^{10} + 2*3^{11}$

then we mean that the coefficient a_0 has been computed up to accuracy $0(3^{12})$.

Table 6.5: The compution of coefficients and polynomials

$p = 3,\ d = 29,\ \beta = 1,\ \lambda(F_\chi) = 2,\ \lambda(p,d,\beta) = 1$	
a_0	$3^2 + 2*3^3 + 2*3^4 + 2*3^5 + 2*3^6 + 2*3^7 + 2*3^8 + 2*3^9 + 2*3^{10} + 2*3^{11}$
a_1	$2*3 + 3^2 + 3^3 + 2*3^6 + 2*3^7 + 2*3^9 + 2*3^{10}$
a_2	$1 + 2*3 + 2*3^3 + 2*3^4 + 2*3^7 + 2*3^8$
a_3	$3^2 + 2*3^3 + 2*3^6 + 3^7 + 2*3^8$
a_4	$2*3 + 2*3^2 + 3^6$
a_5	$3^2 + 3^3 + 3^5$
a_6	$1 + 2*3 + 3^2 + 2*3^3 + 2*3^4 + 3^5$
a_7	$1 + 2*3 + 2*3^2$
a_8	$2*3^3$
a_9	$1 + 3 + 2*3^2 + 2*3^3$
a_{10}	$2 + 2*3 + 3^3$
$P_\chi(T) = T^2 + (3^2 + 3^3 + 3^4)T + 3^4$	

$p = 3,\ d = 43,\ \beta = 1,\ \lambda(F_\chi) = 2,\ \lambda(p,d,\beta) = 1$	
a_0	$2*3^2 + 3^3 + 2*3^4 + 2*3^5 + 2*3^6 + 2*3^7 + 2*3^8 + 2*3^9 + 2*3^{10} + 2*3^{11}$
a_1	$3^3 + 2*3^4 + 2*3^9 + 2*3^{10}$
a_2	$1 + 3 + 3^2 + 3^3 + 3^6 + 2*3^7 + 2*3^8$
a_3	$1 + 2*3 + 2*3^3 + 2*3^4 + 3^5 + 3^6 + 2*3^7 + 3^8$
a_4	$2 + 3 + 3^2 + 3^3 + 3^4 + 3^6 + 3^7$
a_5	$3^2 + 3^6$
a_6	$2 + 3 + 3^2 + 2*3^3 + 3^4 + 2*3^5$
a_7	$2 + 2*3 + 2*3^2 + 2*3^3 + 2*3^4$
a_8	$3 + 3^2 + 2*3^3$
a_9	2
a_{10}	0
$P_\chi(T) = T^2 + (3^2 + 2*3^3)T + (2*3^2 + 2*3^3 + 3^4)$	

$p=3,\ d=58,\ \beta=1,\ \lambda(F_\chi)=2,\ \lambda(p,d,\beta)=1$	
a_0	$2*3^2+3^3+2*3^4+2*3^5+2*3^6+2*3^7+2*3^8+2*3^9+2*3^{10}+2*3^{11}$
a_1	$2*3^2+3^3+3^4+3^6+2*3^7+3^9$
a_2	$1+2*3^2+3^3+3^4+2*3^5+2*3^6+3^8$
a_3	$1+2*3^3+3^5+3^7+3^8$
a_4	$2+3^2+2*3^3+2*3^4$
a_5	$1+3+2*3^4+3^5$
a_6	$1+2*3+3^3+2*3^4$
a_7	$2*3+3^4$
a_8	3^2
a_9	$2+2*3+2*3^2$
a_{10}	0
$P_\chi(T)=T^2+(2+3^4)T+(2*3^2+3^3)$	

$p=3,\ d=67,\ \beta=1,\ \lambda(F_\chi)=2,\ \lambda(p,d,\beta)=1$	
a_0	$2*3^2+3^3+2*3^4+2*3^5+2*3^6+2*3^7+2*3^8+2*3^9+2*3^10+2*3^{11}$
a_1	$3+2*3^3+3^4+3^5+3^6+3^7+3^8$
a_2	$2+3+2*3^2+3^3+3^4+2*3^7+3^8+2*3^9$
a_3	$2*3+2*3^3+2*3^4+2*3^6+2*3^7+2*3^8$
a_4	$3+2*3^3+2*3^4+2*3^5+2*3^7$
a_5	$1+2*3^2+2*3^4+3^5+3^6$
a_6	$2+3^2+3^3$
a_7	$2*3+3^3+2*3^4$
a_8	3^2+3^3
a_9	$3+2*3^2$
a_{10}	0
$P_\chi(T)=T^2+(2*3+2*3^4)T+(3^2+2*3^4)$	

$p=3,\ d=74,\ \beta=1,\ \lambda(F_\chi)=2,\ \lambda(p,d,\beta)=1$	
a_0	$2*3^2+3^3+2*3^4+2*3^5+2*3^6+2*3^7+2*3^8+2*3^9+2*3^{10}+2*3^{11}$
a_1	$3+2*3^2+3^3+3^4+2*3^5+2*3^7+3^8+3^9+2*3^{10}$
a_2	$2+3^3+2*3^4+3^5+3^6+2*3^9$
a_3	$3+3^3+2*3^5+3^7$
a_4	$2*3+3^2+2*3^3+2*3^4+2*3^5+2*3^7$
a_5	$1+3+2*3^2+3^3+2*3^4+3^5$
a_6	$2+2*3^2+2*3^3$
a_7	$2+3^2+2*3^3+3^4$
a_8	$1+2*3^2$
a_9	$2+2*3$
a_{10}	0
$P_\chi(T)=T^2+(2*3+2*3^2)T+(3^2+2*3^3)$	

$p=3,\ d=79,\ \beta=1,\ \lambda(F_\chi)=2,\ \lambda(p,d,\beta)=1$	
a_0	$2*3^2+3^3+2*3^4+2*3^5+2*3^6+2*3^7+2*3^8+2*3^9+2*3^{10}+2*3^{11}$
a_1	$3+2*3^3+3^6+3^8+3^9$
a_2	$2+2*3+3^2+3^3+3^4+3^5+2*3^6$
a_3	$1+3+2*3^2+2*3^4+2*3^5+3^6+2*3^8$
a_4	$2+3^3+3^5+2*3^7$
a_5	$3^2+3^4+2*3^5+3^6$
a_6	$2*3+3^2+2*3^3+2*3^4$
a_7	$1+2*3+2*3^2+2*3^3$
a_8	$2+3+3^2+3^3$
a_9	$2*3+3^2$
a_{10}	0
$P_\chi(T)=T^2+(2*3+2*3^2+2*3^3+3^4)T+(3^2+2*3^3+2*3^4)$	

$p=3,\ d=82,\ \beta=1,\ \lambda(F_\chi)=2,\ \lambda(p,d,\beta)=1$	
a_0	$2*3^2+3^3+2*3^4+2*3^5+2*3^6+2*3^7+2*3^8+2*3^9+2*3^{10}+2*3^{11}$
a_1	$2*3^2+3^4+3^6+2*3^7+2*3^8+2*3^9+2*3^{10}$
a_2	$1+2*3+2*3^2+3^3+2*3^5+3^6$
a_3	$3+3^2+3^3+3^4+3^5+2*3^6+3^7+2*3^8$
a_4	$2+3^2+3^3+2*3^5+2*3^6+2*3^7$
a_5	$3+2*3^4+3^5+3^6$
a_6	$1+3+3^2+3^3+3^4$
a_7	$2+3+2*3^2+2*3^3$
a_8	$3+3^2+3^3$
a_9	$1+3^2$
a_{10}	0
$P_\chi(T)=T^2+(2*3^2)T+(2*3^2+2*3^4)$	

$p=3,\ d=85,\ \beta=1,\ \lambda(F_\chi)=2,\ \lambda(p,d,\beta)=1$	
a_0	$2*3^2+3^3+2*3^4+2*3^5+2*3^6+2*3^7+2*3^8+2*3^9+2*3^{10}+2*3^{11}$
a_1	$3^5+3^6+2*3^8+2*3^9+3^{10}$
a_2	$1+2*3^2+2*3^3+3^4+2*3^5+2*3^7+3^8$
a_3	$3+3^2+2*3^3+3^4+2*3^5+2*3^7+2*3^8$
a_4	$2*3^2+3^4+2*3^5+2*3^6+2*3^7$
a_5	$3+2*3^5+3^6$
a_6	$1+3^2+3^3+2*3^4+3^5$
a_7	$2+2*3^3$
a_8	$2+3+3^2+3^3$
a_9	$1+2*3^2$
a_{10}	0
$P_\chi(T)=T^2+(3^3+2*3^4)T+(2*3^2+3^3+3^4)$	

$p = 3,\ d = 106,\ \beta = 1,\ \lambda(F_\chi) = 2,\ \lambda(p, d, \beta) = 1$	
a_0	$2*3^2+3^3+2*3^4+2*3^5+2*3^6+2*3^7+2*3^8+2*3^9+2*3^{10}+2*3^{11}$
a_1	$2*3^2+3^3+3^5+3^8+2*3^9+3^{10}$
a_2	$1+2*3^2+3^6+3^8$
a_3	$1+2*3^2+3^3+2*3^4+2*3^5+2*3^6+2*3^7+3^8$
a_4	$2+2*3+3^2+3^3+2*3^4+3^5+3^6+3^7$
a_5	$1+2*3^2+3^3+2*3^5+2*3^6$
a_6	$2+2*3+3^2+2+3^3+2*3^3+2*3^4+3^5$
a_7	$1+3^3+3^4$
a_8	$2+3+2*3^3$
a_9	$3+3^2$
a_{10}	0
$P_\chi(T) = T^2 + (2*3^2+3^3)$	

$p = 3,\ d = 113,\ \beta = 1,\ \lambda(F_\chi) = 2,\ \lambda(p, d, \beta) = 1$	
a_0	$3^2+2*3^3+2*3^4+2*3^5+2*3^6+2*3^7+2*3^8+2*3^9+2*3^{10}+2*3^{11}$
a_1	$2*3^3+3^4+3^6+2*3^7+2*3^8+2*3^9+2*3^{10}$
a_2	$2+3+3^2+2*3^4+2*3^5+3^6+2*3^7+3^8$
a_3	$2*3^2+2*3^4+2*3^6+3^7$
a_4	$1+2*3^2+3^3+3^5+2*3^7$
a_5	$1+2*3+2*3^2+2*3^3$
a_6	$3+3^2+2*3^4+3^5$
a_7	$2*3^2+2*3^3+3^4$
a_8	$2+3$
a_9	$3+3^2$
a_{10}	0
$P_\chi(T) = T^2 + 3^3T + (2*3^2+3^3+3^4)$	

$p = 3,\ d = 122,\ \beta = 1,\ \lambda(F_\chi) = 2,\ \lambda(p, d, \beta) = 1$	
a_0	$2*3^2+3^3+2*3^4+2*3^5+2*3^6+2*3^7+2*3^8+2*3^9+2*3^{10}+2*3^{11}$
a_1	$3+2*3^4+2*3^5+2*3^7+3^9+3^{10}$
a_2	$2+2*3^2+3^3+3^4+2*3^6+3^8$
a_3	$2+2*3^3+2*3^4+3^5+2*3^7+2*3^8$
a_4	$1+2*3^2+2*3^6+2*3^7$
a_5	$3+3^2+3^3+2*3^4+3^5$
a_6	$1+3+2*3^2+2*3^3+2*3^4+2*3^5$
a_7	$2+2*3^2+2*3^4$
a_8	$2*3+2*3^2+3^3$
a_9	$1+3+3^2$
a_{10}	0
$P_\chi(T) = T^2 + (2*3+3^2+3^3)T + (3^2+3^3+2*3^4)$	

$p = 3,\ d = 131,\ \beta = 1,\ \lambda(F_\chi) = 2,\ \lambda(p, d, \beta) = 1$	
a_0	$2 * 3^2 + 3^3 + 2 * 3^4 + 2 * 3^5 + 2 * 3^6 + 2 * 3^7 + 2 * 3^8 + 2 * 3^9 + 2 * 3^{10} + 2 * 3^{11}$
a_1	$2 * 3^2 + 2 * 3^3 + 3^4 + 2 * 3^9 + 3^{10}$
a_2	$1 + 3 + 2 * 3^2 + 2 * 3^5 + 2 * 3^6 + 3^9$
a_3	$2 + 2 * 3^2 + 2 * 3^3 + 3^4 + 3^5 + 3^5 + 3^6 + 2 * 3^7$
a_4	$2 + 2 * 3 + 2 * 3^4 + 3^5 + 3^7$
a_5	$1 + 2 * 3^2 + 2 * 3^3 + 2 * 3^4 + 2 * 3^5 + 2 * 3^6$
a_6	$1 + 2 * 3 + 2 * 3^2 + 3^3 + 3^4 + 3^5$
a_7	$2 * 3^2 + 2 * 3^3 + 2 * 3^4$
a_8	$2 + 3 + 2 * 3^3$
a_9	$1 + 3^2$
a_{10}	0
$P_\chi(T) = T^2 + (3^2 + 2 * 3^3)T + (2 * 3^2 + 2 * 3^3 + 3^4)$	

$p = 3,\ d = 137,\ \beta = 1,\ \lambda(F_\chi) = 2,\ \lambda(p, d, \beta) = 1$	
a_0	$3^2 + 2 * 3^3 + 2 * 3^4 + 2 * 3^5 + 2 * 3^6 + 2 * 3^7 + 2 * 3^8 + 2 * 3^9 + 2 * 3^{10} + 2 * 3^{11}$
a_1	$2 * 3 + 2 * 3^3 + 2 * 3^4 + 2 * 3^5 + 2 * 3^6 + 3^7 + 2 * 3^8 + 2 * 3^9$
a_2	$1 + 3^2 + 3^4 + 3^5 + 3^7 + 3^9$
a_3	$2 + 3 + 2 * 3^2 + 3^5 + 2 * 3^7 + 2 * 3^8$
a_4	$2 + 2 * 3 + 2 * 3^3 + 3^4 + 3^5 + 3^6 + 2 * 3^7$
a_5	$2 + 3 + 2 * 3^2 + 3^4 + 2 * 3^5$
a_6	$3^2 + 2 * 3^3 + 3^4$
a_7	$2 + 2 * 3^2 + 3^3$
a_8	$1 + 2 * 3^2 + 2 * 3^3$
a_9	$2 + 3 + 3^2$
a_{10}	0
$P_\chi(T) = T^2 + (2 * 3 + 2 * 3^4)T + (3^2 + 2 * 3^4)$	

$p = 3,\ d = 142,\ \beta = 1,\ \lambda(F_\chi) = 2,\ \lambda(p, d, \beta) = 1$	
a_0	$3^2 + 2 * 3^4 + 2 * 3^5 + 2 * 3^6 + 2 * 3^7 + 2 * 3^8 + 2 * 3^9 + 2 * 3^{10} + 2 * 3^{11}$
a_1	$2 * 3^2 + 3^6 + 3^7 + 3^8 + 3^{10}$
a_2	$2 + 3 + 2 * 3^2 + 3^3 + 2 * 3^4 + 3^5 + 2 * 3^6 + 3^7$
a_3	$2 + 2 * 3 + 3^3 + 3^4 + 3^5 + 2 * 3^6$
a_4	$3 + 3^2 + 2 * 3^3 + 3^4 + 2 * 3^5 + 3^7$
a_5	$2 * 3^2 + 3^3 + 2 * 3^4 + 2 * 3^5 + 2 * 3^6$
a_6	$1 + 3 + 3^2 + 2 * 3^4 + 2 * 3^5$
a_7	$1 + 2 * 3 + 3^3 + 2 * 3^4$
a_8	$2 * 3 + 3^2$
a_9	$2 + 3^2$
a_{10}	0
$P_\chi(T) = T^2 + (2 * 3^2 + 2 * 3^3 + 3^4)T + (2 * 3^2 + 3^4)$	

$p=3,\ d=173,\ \beta=1,\ \lambda(F_\chi)=2,\ \lambda(p,d,\beta)=1$	
a_0	$3^3+2*3^4+2*3^5+2*3^6+2*3^7+2*3^8+2*3^9+2*3^{10}+2*3^{11}$
a_1	$2*3+2*3^3+2*3^4+3^5+2*3^6+3^7+2*3^8+2*3^9+3^{10}$
a_2	$2+2*3^3+2*3^5+3^6+2*3^7$
a_3	$2+2*3+3^4+2*3^6+2*3^8$
a_4	$2+2*3^2+3^4+2*3^6+2*3^7$
a_5	$2+2*3^2+3^3+2*3^4+3^5+2*3^6$
a_6	$2+3+3^2+3^5$
a_7	$1+2*3+2*3^2+3^4$
a_8	$1+3+3^2+3^3$
a_9	$2+3+3^2$
a_{10}	0
$P_\chi(T)=T^2+(3+2*3^2)T+(2*3^2+2*3^4)$	

$p=3,\ d=182,\ \beta=1,\ \lambda(F_\chi)=2,\ \lambda(p,d,\beta)=1$	
a_0	$3^2+2*3^4+2*3^5+2*3^6+2*3^7+2*3^8+2*3^9+2*3^{10}+2*3^{11}$
a_1	$2*3+3^4+3^8+2*3^9$
a_2	$1+2*3+2*3^2+3^3+3^4+3^6+3^7+3^8$
a_3	$2+2*3+3^2+2*3^3+2*3^4+2*3^5+3^6+2*3^7+2*3^8$
a_4	$1+3+3^2+3^3+2*3^6+2*3^6+2*3^7$
a_5	$3^2+2*3^3+3^4+2*3^5+3^6$
a_6	$1+2*3^2+2*3^4+3^5$
a_7	$1+2*3^2+2*3^3+2*3^4$
a_8	$1+3+3^3$
a_9	3
a_10	0
$P_\chi(T)=T^2+(2*3+2*3^2+3^4)T+(3^2+2*3^3)$	

$p=3,\ d=-2,\ \beta=0,\ \lambda(F_\chi)=1,\ \lambda(p,d,\beta)=1$	
a_0	0
a_1	$1+2*3+3^2+2*3^4+2*3^7+3^8+3^9+2*3^{10}$
a_2	$1+3^2+3^3+3^5+2*3^6+3^9$
a_3	$2+3+2*3^3+2*3^4+2*3^5+2*3^7+3^8$
a_4	$3+2*3^3+2*3^4+3^5+3^7$
a_5	$1+2*3+2*3^3+3^4+2*3^5+2*3^6$
a_6	$2*3^3$
a_7	$2+3^2+3^3+2*3^4$
a_8	$2+2*3+2*3^2$
a_9	$3+3^2$
a_{10}	0
$P_\chi(T)=T$	

$p = 3,\ d = -5,\ \beta = 0,\ \lambda(F_\chi) = 1,\ \lambda(p, d, \beta) = 1$	
a_0	0
a_1	$1 + 3 + 3^2 + 2 * 3^3 + 3^4 + 2 * 3^6 + 2 * 3^7 + 2 * 3^8 + 2 * 3^9 + 2 * 3^{10}$
a_2	$2 * 3 + 3^2 + 2 * 3^4 + 3^7 + 2 * 3^8 + 2 * 3^9$
a_3	$1 + 3^2 + 3^5$
a_4	$2 * 3 + 3^3 + 3^4 + 2 * 3^6 + 3^7$
a_5	$1 + 3 + 2 * 3^2$
a_6	$1 + 3^3$
a_7	$2 + 2 * 3 + 3^2 + 3^4$
a_8	2*3
a_9	$1 + 3$
a_{10}	0
$P_\chi(T) = T$	

$p = 3,\ d = -11,\ \beta = 0,\ \lambda(F_\chi) = 1,\ \lambda(p, d, \beta) = 1$	
a_0	0
a_1	$2 + 3 + 2 * 3^2 + 2 * 3^3 + 3^4 + 2 * 3^5 + 2 * 3^6 + 2 * 3^7$
a_2	$2 + 2 * 3^5 + 2 * 3^7$
a_3	$2 + 3^4 + 2 * 3^7 + 2 * 3^8$
a_4	$3 + 2 * 3^2 + 3^3 + 3^5 + 3^6 + 2 * 3^7$
a_5	$3^2 + 2 * 3^5$
a_6	$1 + 3 + 3^2 + 3^3 + 2 * 3^4$
a_7	$2 + 2 * 3 + 3^2 + 2 * 3^3 + 3^4$
a_8	$2 * 3 + 2 * 3^2 + 3^3$
a_9	$2 + 2 * 3 + 2 * 3^2$
a_{10}	0
$P_\chi(T) = T$	

$p = 3,\ d = -17,\ \beta = 0,\ \lambda(F_\chi) = 1,\ \lambda(p, d, \beta) = 1$	
a_0	0
a_1	$2 + 3 + 2 * 3^2 + 3^3 + 2 * 3^5 + 2 * 3^6 + 3^7 + 3^8 + 3^9 + 2 * 3^{10}$
a_2	$1 + 2 * 3 + 3^2 + 2 * 3^3 + 2 * 3^5 + 3^8 + 3^9$
a_3	$2 * 3 + 3^3 + 3^4 + 2 * 3^5 + 3^6$
a_4	$1 + 3 + 3^4 + 3^6 + 2 * 3^7$
a_5	$1 + 3^2 + 2 * 3^3 + 2 * 3^4 + 2 * 3^5 + 3^6$
a_6	$2 + 3^2 + 3^3 + 3^5$
a_7	$1 + 2 * 3 + 3^2 + 3^4$
a_8	$3 + 2 * 3^2$
a_9	2
a_{10}	0
$P_\chi(T) = T$	

$p = 3,\ d = -23,\ \beta = 0,\ \lambda(F_\chi) = 1,\ \lambda(p, d, \beta) = 1$	
a_0	0
a_1	$1 + 2 * 3 + 3^2 + 3^3 + 2 * 3^4 + 3^5 + 3^6 + 2 * 3^9$
a_2	$2 + 2 * 3 + 2 * 3^2 + 3^3 + 2 * 3^4 + 3^6 + 3^7 + 2 * 3^9$
a_3	$3 + 3^5 + 3^6 + 2 * 3^8$
a_4	$2 + 3 + 2 * 3^2 + 3^3 + 2 * 3^5 + 2 * 3^6 + 3^7$
a_5	$3 + 2 * 3^2 + 3^3 + 3^5$
a_6	1
a_7	$2 + 3^2 + 3^3 + 2 * 3^4$
a_8	$2 * 3 + 2 * 3^2 + 2 * 3^3$
a_9	$1 + 2 * 3 + 3^2$
a_{10}	0
$P_\chi(T) = T$	

$p = 3,\ d = -26,\ \beta = 0,\ \lambda(F_\chi) = 1,\ \lambda(p, d, \beta) = 1$	
a_0	0
a_1	$1 + 2 * 3 + 3^3 + 2 * 3^4 + 2 * 3^5 + 3^{10}$
a_2	$1 + 3 + 2 * 3^2 + 3^3 + 2 * 3^4 + 2 * 3^7$
a_3	$1 + 3 + 3^3 + 2 * 3^6 + 3^8$
a_4	$2 * 3^3 + 2 * 3^5 + 2 * 3^7$
a_5	$1 + 2 * 3^2 + 3^3 + 3^5 + 3^6$
a_6	$1 + 2 * 3 + 3^2 + 2 * 3^3 + 3^4$
a_7	$1 + 3 + 3^3 + 3^4$
a_8	$3^2 + 3^3$
a_9	$1 + 2 * 3^2$
a_{10}	0
$P_\chi(T) = T$	

$p = 3,\ d = -29,\ \beta = 0,\ \lambda(F_\chi) = 1,\ \lambda(p, d, \beta) = 1$	
a_0	0
a_1	$1 + 2 * 3 + 2 * 3^2 + 2 * 3^3 + 2 * 3^4 + 3^5 + 3^6 + 2 * 3^7 + 3^8 + 3^{10}$
a_2	$2 * 3^2 + 2 * 3^4 + 2 * 3^5 + 3^9$
a_3	$1 + 2 * 3 + 3^2 + 2 * 3^3$
a_4	$2 * 3 + 3^3 + 3^4 + 2 * 3^6 + 3^7$
a_5	$2 + 3 + 2 * 3^2 + 2 * 3^4 + 3^5 + 2 * 3^6$
a_6	$2 + 2 * 3 + 2 * 3^2 + 3^3 + 3^4 + 2 * 3^5$
a_7	$1 + 2 * 3 + 3^3 + 2 * 3^4$
a_8	$1 + 3^2$
a_9	$2 * 3$
a_{10}	0
$P_\chi(T) = T$	

$p=3,\ d=-31,\ \beta=0,\ \lambda(F_\chi)=1,\ \lambda(p,d,\beta)=1$	
a_0	$2*3$
a_1	$1+3^2+3^3+3^4+3^5+2*3^6+3^9+2*3^{10}$
a_2	$1+2*3^2+2*3^4+3^5+3^6+2*3^7+2*3^8+2*3^9$
a_3	$2+3+2*3^2+2*3^3+3^5+2*3^7$
a_4	$3+3^2+3^4+3^7$
a_5	$1+2*3+3^4$
a_6	$2+2*3+3^2+2*3^4+2*3^5$
a_7	3
a_8	1
a_9	$1+2*3+2*3^2$
a_{10}	0
$P_\chi(T)=T+(2*3+3^2+2*3^3+3^4+3^5+3^6+2*3^7+3^9)$	

$p=3,\ d=-38,\ \beta=0,\ \lambda(F_\chi)=1,\ \lambda(p,d,\beta)=1$	
a_0	0
a_1	$1+3+3^2+3^3+2*3^4+3^5+2*3^6+3^7+2*3^9+3^{10}$
a_2	$1+2*3^5+3^6+2*3^7+3^8$
a_3	$3^2+2*3^3+2*3^4+3^5+3^7$
a_4	$1+2*3+3^2+2*3^3+3^5+3^6+3^7$
a_5	$2+3+3^4+2*3^5+3^6$
a_6	$2+2*3+2*3^2+3^3+2*3^4$
a_7	$2*3^2$
a_8	$3+3^2$
a_9	$2+3+3^2$
a_{10}	0
$P_\chi(T)=T$	

$p=3,\ d=-53,\ \beta=0,\ \lambda(F_\chi)=1,\ \lambda(p,d,\beta)=1$	
a_0	0
a_1	$2+2*3+3^2+2*3^3+3^4+2*3^5+2*3^6+3^8+2*3^9$
a_2	$2+3^2+2*3^3+3^4+3^5+2*3^8+3^9$
a_3	$2*3^2+2*3^3+3^5+2*3^6+2*3^7$
a_4	$3^5+2*3^6+3^7$
a_5	$2+3+2*3^3+3^6$
a_6	$2*3+3^2$
a_7	$2*3^2+2*3^4$
a_8	$3+2*3^2$
a_9	$2*3+2*3^2$
a_{10}	0
$P_\chi(T)=T$	

$p = 3,\ d = -59,\ \beta = 0,\ \lambda(F_\chi) = 1,\ \lambda(p,d,\beta) = 1$	
a_0	0
a_1	$2 + 3^3 + 3^4 + 2 * 3^6 + 2 * 3^7 + 3^8 + 2 * 3^9$
a_2	$2 + 2 * 3 + 3^2 + 2 * 3^4 + 3^6 + 2 * 3^7 + 3^8 + 3^9$
a_3	$2 + 3 + 3^4 + 3^6 + 2 * 3^7 + 2 * 3^8$
a_4	$2 + 2 * 3 + 2 * 3^4 + 3^7$
a_5	$1 + 2 * 3^2 + 2 * 3^3 + 3^6$
a_6	3
a_7	$2 + 2 * 3^2 + 2 * 3^4$
a_8	$2 + 3^3$
a_9	$1 + 3$
a_{10}	0
$P_\chi(T) = T$	

$p = 3,\ d = -61,\ \beta = 0,\ \lambda(F_\chi) = 1,\ \lambda(p,d,\beta) = 1$	
a_0	$3 + 3^2$
a_1	$1 + 3 + 2 * 3^2 + 2 * 3^3 + 2 * 3^4 + 3^6 + 3^7 + 3^9 + 3^{10}$
a_2	$1 + 2 * 3^2 + 3^3 + 2 * 3^4 + 2 * 3^5 + 3^6 + 3^8 + 2 * 3^9$
a_3	$2 + 2 * 3^2 + 3^3 + 2 * 3^6 + 3^8$
a_4	$1 + 2 * 3^3 + 3^7$
a_5	$1 + 3 + 2 * 3^5 + 2 * 3^6$
a_6	$3 + 3^2 + 3^3 + 3^4 + 2 * 3^5$
a_7	$2 * 3 + 2 * 3^2 + 3^3 + 3^4$
a_8	$2 + 3^3$
a_9	$2 + 2 * 3 + 3^2$
a_{10}	0
$P_\chi(T) = T + (3 + 3^2 + 2 * 3^4 + 3^5 + 3^7 + 2 * 3^8 + 3^9)$	

$p = 3,\ d = -62,\ \beta = 0,\ \lambda(F_\chi) = 1,\ \lambda(p,d,\beta) = 1$	
a_0	0
a_1	$2 + 3 + 2 * 3^4 + 3^5 + 2 * 3^6 + 2 * 3^8 + 2 * 3^9 + 3^{10}$
a_2	$2 + 2 * 3 + 2 * 3^3 + 2 * 3^4 + 3^5 + 2 * 3^6 + 2 * 3^7 + 3^8 + 3^9$
a_3	$1 + 2 * 3 + 3^3 + 3^4 + 2 * 3^5 + 3^6 + 3^7 + 3^8$
a_4	$1 + 2 * 3^3 + 2 * 3^4 + 3^5 + 3^7$
a_5	$2 + 3 + 3^2 + 3^3 + 3^4 + 2 * 3^5$
a_6	$2 * 3 + 3^2 + 3^3 + 3^4 + 3^5$
a_7	$2 * 3 + 3^2 + 3^3 + 2 * 3^4$
a_8	$2 + 2 * 3 + 3^2 + 2 * 3^3$
a_9	$1 + 2 * 3^2$
a_{10}	0
$P_\chi(T) = T$	

$p=3,\ d=-71,\ \beta=0,\ \lambda(F_\chi)=1,\ \lambda(p,d,\beta)=1$	
a_0	0
a_1	$2+3+3^2+3^4+2*3^5+3^7+3^9+3^{10}$
a_2	$2+3+2*3^3+3^4+3^5+2*3^6+2*3^7$
a_3	$1+2*3+2*3^2+2*3^6+2*3^7+3^8$
a_4	$1+3^4+2*3^5+2*3^7$
a_5	$2+3^2+3^3+3^4+3^5$
a_6	$1+2*3+3^5$
a_7	$3+2*3^2+3^3+3^4$
a_8	$1+2*3+3^2+3^3$
a_9	$2+3$
a_{10}	0
$P_\chi(T)=T$	

$p=3,\ d=-77,\ \beta=0,\ \lambda(F_\chi)=1,\ \lambda(p,d,\beta)=1$	
a_0	0
a_1	$2+3+2*3^2+2*3^5+3^7+3^8+2*3^9+2*3^{10}$
a_2	$2*3^2+3^3+2*3^4+2*3^5+2*3^6$
a_3	$3+3^3+3^4+3^5+2*3^6+3^8$
a_4	$1+2*3+3^2+2*3^3+2*3^4+2*3^5+3^6+3^7$
a_5	$2+3^2+3^4+2*3^5+2*3^6$
a_6	$2+3+2*3^3+3^4+2*3^5$
a_7	$2+2*3+2*3^2+2*3^3$
a_8	$1+2*3+3^3$
a_9	1
a_{10}	0
$P_\chi(T)=T$	

$p=3,\ d=-83,\ \beta=0,\ \lambda(F_\chi)=1,\ \lambda(p,d,\beta)=1$	
a_0	0
a_1	$1+3+2*3^2+3^6+2*3^7$
a_2	$2+2*3+3^3+3^4+3^6+3^7+3^9$
a_3	$1+2*3+2*3^2+3^3+3^4+2*3^5+3^8$
a_4	$1+2*3+3^4+3^6+2*3^7$
a_5	$2+3^2+2*3^3+3^4+3^5+2*3^6$
a_6	$2*3+2*3^3+2*3^4+3^5$
a_7	$3^2+2*3^3+2*3^4$
a_8	$2+2*3+3^3$
a_9	1
a_{10}	0
$P_\chi(T)=T$	

$p=3,\ d=-89,\ \beta=0,\ \lambda(F_\chi)=1,\ \lambda(p,d,\beta)=1$	
a_0	0
a_1	$2+3+2*3^2+3^3+2*3^4+3^5+2*3^6+3^7+2*3^8+3^9$
a_2	$2+2*3+3^2+3^3+3^4+2*3^5+3^6+2*3^7+2*3^8$
a_3	$2+3+3^3+2*3^4+3^5+3^6+2*3^7+2*3^8$
a_4	$3+2*3^2+3^3+3^4+2*3^5+2*3^6$
a_5	$2*3+3^3+3^4+2*3^5+2*3^6$
a_6	$2*3+3^3+2*3^5$
a_7	$1+2*3+3^2+3^3+2*3^4$
a_8	$2+2*3^2$
a_9	$2*3^2$
a_{10}	0
$P_\chi(T)=T$	

$p=3,\ d=-95,\ \beta=0,\ \lambda(F_\chi)=1,\ \lambda(p,d,\beta)=1$	
a_0	0
a_1	$2+3^3+2*3^4+3^5+2*3^6+2*3^7+2*3^8+2*3^{10}$
a_2	$1+3^4+2*3^6+3^7+2*3^8+2*3^9$
a_3	$2+3+2*3^3+3^4+2*3^5+2*3^7$
a_4	$2*3^3+2*3^4+3^5+2*3^6+2*3^7$
a_5	$1+2*3+2*3^4+2*3^5$
a_6	$3^2+3^3+3^4$
a_7	$2+3^2+3^3+3^4$
a_8	$3+3^2+2*3^3$
a_9	$2+3+2*3^2$
a_{10}	0
$P_\chi(T)=T$	

$p=3,\ d=-106,\ \beta=0,\ \lambda(F_\chi)=1,\ \lambda(p,d,\beta)=1$	
a_0	$3+3^2$
a_1	$2+3+2*3^2+3^3+3^4+2*3^5+2*3^10$
a_2	$1+2*3+3^3+3^5+3^6+3^8+2*3^9$
a_3	$1+3^2+3^3+2*3^4+2*3^5+2*3^8$
a_4	$2+3+2*3^2+3^3+3^5+2*3^6+2*3^7$
a_5	$2+3+3^2+3^3+3^5+2*3^6$
a_6	$3^2+2*3^4+2*3^5$
a_7	3^2+3^4
a_8	$3+3^3$
a_9	$2+3$
a_{10}	0
$P_\chi(T)=T+(2*3+3^2+3^3+2*3^5+2*3^6+3^7+2*3^8+2*3^9+2*3^{10})$	

$p = 3$, $d = -110$, $\beta = 0$, $\lambda(F_\chi) = 1$, $\lambda(p, d, \beta) = 1$	
a_0	0
a_1	$1 + 3 + 3^2 + 2 * 3^4 + 2 * 3^6 + 2 * 3^7 + 2 * 3^{10}$
a_2	$1 + 3 + 3^2 + 3^4 + 2 * 3^5 + 3^7 + 3^8 + 2 * 3^9$
a_3	$2 + 3 + 3^2 + 2 * 3^3 + 2 * 3^4 + 3^6 + 3^7$
a_4	$2 + 3 + 3^2 + 2 * 3^4 + 2 * 3^6 + 3^7$
a_5	$2 * 3 + 2 * 3^2 + 3^4 + 2 * 3^5$
a_6	$3 + 3^2 + 2 * 3^4 + 2 * 3^5$
a_7	$2 * 3 + 2 * 3^2 + 3^3$
a_8	$3 + 3^3$
a_9	$2 * 3$
a_{10}	0
$P_\chi(T) = T$	

$p = 3$, $d = -118$, $\beta = 0$, $\lambda(F_\chi) = 1$, $\lambda(p, d, \beta) = 1$	
a_0	0
a_1	$1 + 2 * 3^3 + 2 * 3^5 + 2 * 3^6 + 3^8 + 3^9 + 3^{10}$
a_2	$1 + 2 * 3^2 + 2 * 3^4 + 2 * 3^6 + 2 * 3^7 + 2 * 3^8 + 2 * 3^9$
a_3	$3 + 3^2 + 2 * 3^3 + 2 * 3^4 + 2 * 3^5 + 3^7 + 3^8$
a_4	$2 * 3 + 3^3 + 3^4 + 2 * 3^6 + 2 * 3^7$
a_5	$3 + 2 * 3^2 + 2 * 3^4$
a_6	$2 + 2 * 3 + 3^3 + 3^4 + 2 * 3^5$
a_7	$2 * 3^4$
a_8	$2 + 3^2 + 2 * 3^3$
a_9	$1 + 2 * 3$
a_{10}	0
$P_\chi(T) = T$	

$p = 3$, $d = -119$, $\beta = 0$, $\lambda(F_\chi) = 1$, $\lambda(p, d, \beta) = 1$	
a_0	$3 + 3^2$
a_1	$2 + 2 * 3 + 3^2 + 3^3 + 3^4 + 2 * 3^5 + 3^7 + 3^9 + 3^{10}$
a_2	$3^2 + 3^3 + 2 * 3^4 + 3^5 + 3^6 + 2 * 3^7 + 3^9$
a_3	$2 + 2 * 3 + 2 * 3^2 + 3^3 + 2 * 3^4 + 2 * 3^5 + 2 * 3^6 + 2 * 3^7 + 3^8$
a_4	$2 + 2 * 3 + 3^3 + 2 * 3^4 + 3^5$
a_5	$2 * 3 + 3^2 + 3^5$
a_6	$1 + 2 * 3 + 2 * 3^2$
a_7	$2 + 2 * 3 + 3^2 + 2 * 3^3 + 3^4$
a_8	$1 + 3 + 3^2 + 3^3$
a_9	2
a_{10}	0
$P_\chi(T) = T + (2 * 3 + 3^2 + 3^3 + 2 + 3^4 + 2 * 3^5 + 2 * 3^7 + 3^9 + 3^{10})$	

$p = 3$, $d = -122$, $\beta = 0$, $\lambda(F_\chi) = 1$, $\lambda(p, d, \beta) = 1$	
a_0	0
a_1	$2 + 2*3 + 2*3^2 + 2*3^3 + 2*3^4 + 2*3^5 + 2*3^6 + 2*3^7 + 2*3^8 + 3^9 + 2*3^{10}$
a_2	$1 + 3 + 2*3^2 + 2*3^3 + 3^4 + 3^6 + 2*3^7 + 2*3^9$
a_3	$3^2 + 2*3^3 + 2*3^4 + 2*3^5 + 2*3^6 + 2*3^8$
a_4	$2 + 2*3 + 3^2 + 2*3^3 + 2*3^4 + 2*3^5 + 2*3^6 + 2*3^7$
a_5	$2 + 3^2 + 2*3^4 + 3^5$
a_6	$2 + 3 + 2*3^3 + 2*3^4$
a_7	$1 + 3 + 2*3^2 + 3^3$
a_8	$2 + 3 + 2*3^2$
a_9	$2 + 2*3 + 3^2$
a_{10}	0
$P_\chi(T) = T$	

$p = 3$, $d = -131$, $\beta = 0$, $\lambda(F_\chi) = 1$, $\lambda(p, d, \beta) = 1$	
a_0	0
a_1	$2 + 2*3 + 3^2 + 3^3 + 2*3^4 + 3^5 + 2*3^6 + 2*3^7 + 2*3^9 + 3^{10}$
a_2	$2 + 2*3 + 2*3^2 + 2*3^3 + 3^5 + 2*3^6 + 2*3^7 + 3^9$
a_3	$1 + 2*3^5 + 3^7 + 2*3^8$
a_4	$2*3 + 2*3^2 + 3^3 + 3^4 + 2*3^7$
a_5	$3^3 + 2*3^5$
a_6	$1 + 3^2 + 3^4 + 2*3^5$
a_7	$1 + 2*3 + 2*3^2 + 2*3^4$
a_8	$2 + 3^2 + 3^3$
a_9	$2 + 3$
a_{10}	0
$P_\chi(T) = T$	

$p = 3$, $d = -134$, $\beta = 0$, $\lambda(F_\chi) = 1$, $\lambda(p, d, \beta) = 1$	
a_0	0
a_1	$1 + 2*3 + 3^2 + 2*3^3 + 2*3^4 + 2*3^5 + 3^6 + 3^7 + 2*3^9$
a_2	$2 + 3 + 3^2 + 2*3^3 + 3^4 + 3^6 + 2*3^7$
a_3	$2*3 + 3^4 + 3^5 + 2*3^6 + 3^7$
a_4	$2 + 2*3 + 3^5 + 3^7$
a_5	$2*3 + 3^2 + 2*3^4 + 3^5 + 3^6$
a_6	$1 + 2*3^2 + 2*3^3 + 3^4$
a_7	$1 + 3^2 + 2*3^4$
a_8	$1 + 3^3$
a_9	$2*3$
a_{10}	0
$P_\chi(T) = T$	

$p = 3,\ d = -137,\ \beta = 0,\ \lambda(F_\chi) = 1,\ \lambda(p, d, \beta) = 1$	
a_0	0
a_1	$1 + 2*3 + 3^4 + 3^5 + 3^7 + 2*3^8 + 3^9 + 3^{10}$
a_2	$2*3 + 2*3^2 + 3^3 + 2*3^4 + 2*3^5 + 2*3^6 + 3^7 + 3^8 + 3^9$
a_3	$2 + 2*3^2 + 2*3^4 + 2*3^5 + 3^6 + 2*3^8$
a_4	$2 + 3 + 2*3^2 + 2*3^3 + 3^4 + 3^5 + 2*3^7$
a_5	$1 + 2*3 + 2*3^2 + 3^3 + 2*3^4 + 2*3^5 + 3^6$
a_6	$2 + 2*3 + 3^4$
a_7	$2*3 + 2*3^2$
a_8	$1 + 2*3 + 2*3^3$
a_9	$2 + 2*3$
a_{10}	0
$P_\chi(T) = T$	

$p = 3,\ d = -139,\ \beta = 0,\ \lambda(F_\chi) = 1,\ \lambda(p, d, \beta) = 1$	
a_0	$2*3$
a_1	$2 + 2*3 + 3^2 + 2*3^5 + 2*3^6 + 3^7 + 2*3^8 + 3^9$
a_2	$1 + 3 + 3^2 + 2*3^3 + 2*3^4 + 2*3^5 + 2*3^6 + 3^7 + 3^8$
a_3	$2 + 2*3 + 3^2 + 3^3 + 2*3^4 + 3^5 + 3^6 + 2*3^7$
a_4	$3 + 2*3^3 + 3^4 + 2*3^5 + 2*3^6 + 3^7$
a_5	$2*3 + 3^2 + 3^3 + 3^4 + 2*3^5 + 3^6$
a_6	$1 + 2*3 + 2*3^3 + 2*3^4$
a_7	$2 + 2*3^2 + 3^3 + 2*3^4$
a_8	$1 + 3^2 + 2*3^3$
a_9	$1 + 3$
a_{10}	0
$P_\chi(T) = T + (3 + 3^2 + 3^6 + 3^8 + 2*3^{10})$	

$p = 3,\ d = -143,\ \beta = 0,\ \lambda(F_\chi) = 1,\ \lambda(p, d, \beta) = 1$	
a_0	0
a_1	$1 + 3 + 3^3 + 2*3^5 + 2*3^6 + 2*3^8 + 2*3^9 + 2*3^{10}$
a_2	$2 + 3^5 + 3^6 + 3^8 + 2*3^9$
a_3	$1 + 3^2 + 3^3 + 2*3^4 + 3^5 + 2*3^6 + 2*3^7$
a_4	$1 + 3 + 3^4 + 3^5 + 3^6 + 2*3^7$
a_5	$2 + 3^4 + 3^5 + 2*3^6$
a_6	$1 + 2*3^2 + 3^5$
a_7	$2*3^2 + 2*3^3$
a_8	2
a_9	$1 + 2*3^2$
a_{10}	0
$P_\chi(T) = T$	

$p=3,\ d=-146,\ \beta=0,\ \lambda(F_\chi)=1,\ \lambda(p,d,\beta)=1$	
a_0	0
a_1	$2+2*3+3^2+2*3^5+3^7+2*3^9+3^{10}$
a_2	$2*3^2+2*3^3+2*3^5+3^6+3^8+2*3^9$
a_3	$3^3+3^4+2*3^6+3^8$
a_4	$2+2*3^2+2*3^4+2*3^5+2*3^7$
a_5	$1+2*3+3^2+3^3+2*3^4$
a_6	$2+3+2*3^2+3^4+3^5$
a_7	$2*3+3^2+2*3^3+2*3^4$
a_8	$2*3+2*3^3$
a_9	$1+3^2$
a_{10}	0
$P_\chi(T)=T$	

$p=3,\ d=-155,\ \beta=0,\ \lambda(F_\chi)=1,\ \lambda(p,d,\beta)=1$	
a_0	0
a_1	$1+2*3^2+3^3+2*3^5+3^6+2*3^8+2*3^9+3^{10}$
a_2	$1+2*3^2+2*3^3+2*3^4+2*3^5+2*3^7$
a_3	$2+2*3+2*3^3+3^4+3^7+2*3^8$
a_4	$2+3+3^2+2*3^3+3^5+3^7$
a_5	$2+3+2*3^2+3^5+2*3^6$
a_6	$3+2*3^2+3^3+2*3^4+2*3^5$
a_7	$1+3+3^3+3^4$
a_8	$2+3+3^2$
a_9	$1+2*3^2$
a_{10}	0
$P_\chi(T)=T$	

$p=3,\ d=-157,\ \beta=0,\ \lambda(F_\chi)=1,\ \lambda(p,d,\beta)=1$	
a_0	$3+3^2$
a_1	$1+3+2*3^3+3^4+3^5+3^6+3^7+3^9+2*3^{10}$
a_2	$1+3+3^2+2*3^4+2*3^5+3^6+3^7$
a_3	$1+2*3+3^3+3^4+3^6+3^8$
a_4	$1+2*3+2*3^2+3^7$
a_5	$2+3+3^2+3^3+2*3^4+3^5$
a_6	$3^2+2*3^3+3^4+2*3^5$
a_7	$2+3^2+3^3+3^4$
a_8	$2+3^2+3^3$
a_9	$1+3$
a_{10}	0
$P_\chi(T)=T+(3+3^2+3^3+2*3^4+2*3^7+3^9+2*3^{10})$	

$p=3,\ d=-161,\ \beta=0,\ \lambda(F_\chi)=1,\ \lambda(p,d,\beta)=1$	
a_0	0
a_1	$1+2*3+2*3^2+2*3^4+2*3^6+3^7+3^{10}$
a_2	$1+3+3^3+3^5+3^9$
a_3	$3+2*3^2+2*3^3+2*3^5+2*3^6+2*3^7+3^8$
a_4	$2+2*3^3+2*3^4+3^5+2*3^6+3^7$
a_5	$2+2*3+2*3^2+3^3+2*3^4$
a_6	$3^2+3^4+2*3^5$
a_7	3^3
a_8	$2+3+3^3$
a_9	$2+3^2$
a_{10}	0
$P_\chi(T)=T$	

$p=3,\ d=-167,\ \beta=0,\ \lambda(F_\chi)=1,\ \lambda(p,d,\beta)=1$	
a_0	0
a_1	$2+2*3+3^2+2*3^5+3^6+3^7+3^8+2*3^{10}$
a_2	$2*3+3^2+2*3^3+2*3^4+2*3^6+3^8+3^9$
a_3	$1+3^3+2*3^5+3^7+3^8$
a_4	$2*3+2*3^2+2*3^5+3^6$
a_5	$2+3+2*3^3+3^5+3^6$
a_6	$2+2*3^2+3^3+3^4+2*3^5$
a_7	$2+2*3^3$
a_8	$1+2*3+3^3$
a_9	$2*3+2*3^2$
a_{10}	0
$P_\chi(T)=T$	

$p=3,\ d=-170,\ \beta=0,\ \lambda(F_\chi)=1,\ \lambda(p,d,\beta)=1$	
a_0	0
a_1	$2+2*3^2+2*3^4+3^6+3^7+2*3^8+3^9+3^{10}$
a_2	$2*3+2*3^2+2*3^3+2*3^5+3^9$
a_3	$3^2+3^6+3^7$
a_4	$2+3+3^2+2*3^3+3^5+2*3^6+3^7$
a_5	$1+2*3+2*3^2+3^4+3^5+2*3^6$
a_6	$2+2*3^2+2*3^5$
a_7	$1+3+3^2$
a_8	$2+3+3^3$
a_9	2
a_{10}	0
$P_\chi(T)=T$	

$p = 3,\ d = -179,\ \beta = 0,\ \lambda(F_\chi) = 1,\ \lambda(p, d, \beta) = 1$	
a_0	0
a_1	$2 + 3 + 2 * 3^2 + 2 * 3^3 + 3^4 + 2 * 3^6 + 2 * 3^7 + 3^8$
a_2	$1 + 2 * 3^2 + 2 * 3^4 + 3^7 + 2 * 3^8 + 2 * 3^9$
a_3	$2 + 3 + 3^3 + 2 * 3^4 + 2 * 3^6 + 2 * 3^7 + 2 * 3^8$
a_4	$1 + 2 * 3 + 2 * 3^2 + 2 * 3^7$
a_5	$2 + 2 * 3 + 2 * 3^2 + 3^4 + 2 * 3^6$
a_6	$2 + 3^2 + 2 * 3^4 + 2 * 3^5$
a_7	$1 + 2 * 3 + 2 * 3^3$
a_8	1
a_9	0
a_{10}	0
$P_\chi(T) = T$	

$p = 3,\ d = -182,\ \beta = 0,\ \lambda(F_\chi) = 1,\ \lambda(p, d, \beta) = 1$	
a_0	0
a_1	$1 + 2 * 3 + 2 * 3^2 + 2 * 3^3 + 2 * 3^4 + 3^6 + 2 * 3^7 + 2 * 3^8 + 3^9 + 3^{10}$
a_2	$1 + 3 + 3^2 + 3^3 + 2 * 3^4 + 3^6 + 2 * 3^7 + 2 * 3^9$
a_3	$3 + 3^3 + 2 * 3^7 + 2 * 3^8$
a_4	$1 + 3 + 2 * 3^2 + 2 * 3^3 + 2 * 3^4 + 3^5 + 3^7$
a_5	$2 + 3^2 + 2 * 3^4 + 3^6$
a_6	$2 + 3^2 + 2 * 3^3 + 2 * 3^4$
a_7	$2 + 2 * 3 + 2 * 3^3 + 2 * 3^4$
a_8	$2 + 2 * 3 + 2 * 3^2 + 3^3$
a_9	$2 * 3 + 3^2$
a_{10}	0
$P_\chi(T) = T$	

$p = 3,\ d = -185,\ \beta = 0,\ \lambda(F_\chi) = 1,\ \lambda(p, d, \beta) = 1$	
a_0	0
a_1	$1 + 3 + 2 * 3^2 + 2 * 3^3 + 3^4 + 2 * 3^5 + 3^6 + 2 * 3^8 + 2 * 3^9$
a_2	$1 + 3 + 3^2 + 3^3 + 2 * 3^4 + 3^5 + 2 * 3^6 + 3^7 + 2 * 3^8 + 2 * 3^9$
a_3	$3 + 2 * 3^3 + 3^4 + 3^5 + 2 * 3^6 + 3^7 + 3^8$
a_4	$1 + 2 * 3 + 2 * 3^2 + 3^4 + 2 * 3^5 + 2 * 3^7$
a_5	$2 + 2 * 3 + 3^2 + 3^4 + 3^5 + 3^6$
a_6	$1 + 3 + 3^2 + 3^3 + 3^5$
a_7	$3 + 3^2 + 2 * 3^3 + 3^4$
a_8	$1 + 3^2$
a_9	$1 + 2 * 3$
a_{10}	0
$P_\chi(T) = T$	

$p = 3,\ d = -191,\ \beta = 0,\ \lambda(F_\chi) = 1,\ \lambda(p, d, \beta) = 1$	
a_0	0
a_1	$1 + 3^2 + 2 * 3^5 + 3^6 + 3^7 + 2 * 3^{10}$
a_2	$2 + 3^2 + 3^4 + 2 * 3^5 + 2 * 3^6 + 3^7 + 2 * 3^9$
a_3	$3 + 3^3 + 3^4 + 3^5 + 3^7$
a_4	$3 + 2 * 3^7$
a_5	$2 * 3^4 + 3^5 + 3^6$
a_6	$2 + 3 + 3^3 + 2 * 3^4 + 3^5$
a_7	$1 + 2 * 3 + 3^2$
a_8	$3 + 2 * 3^3$
a_9	3
a_{10}	0
$P_\chi(T) = T$	

$p = 3,\ d = -197,\ \beta = 0,\ \lambda(F_\chi) = 1,\ \lambda(p, d, \beta) = 1$	
a_0	0
a_1	$2 + 3 + 3^2 + 3^4 + 2 * 3^5 + 3^6 + 3^8 + 2 * 3^9 + 2 * 3^{10}$
a_2	$1 + 3^2 + 2 * 3^3 + 3^4 + 2 * 3^8 + 2 * 3^9$
a_3	$1 + 3 + 2 * 3^2 + 2 * 3^3 + 3^5 + 3^6$
a_4	$3^2 + 3^3 + 2 * 3^4 + 3^5 + 2 * 3^7$
a_5	$1 + 2 * 3 + 2 * 3^2$
a_6	$3^2 + 2 * 3^4 + 3^5$
a_7	$2 * 3 + 3^3 + 2 * 3^4$
a_8	$2 + 2 * 3 + 2 * 3^2 + 3^3$
a_9	$2 + 3 + 2 * 3^2$
a_{10}	0
$P_\chi(T) = T$	

$p = 3,\ d = -199,\ \beta = 0,\ \lambda(F_\chi) = 1,\ \lambda(p, d, \beta) = 1$	
a_0	0
a_1	$2 + 3 + 2 * 3^2 + 2 * 3^3 + 2 * 3^4 + 2 * 3^6 + 2 * 3^7 + 2 * 3^8 + 3^9 + 2 * 3^{10}$
a_2	$2 + 2 * 3 + 3^2 + 3^4 + 3^6 + 2 * 3^7 + 2 * 3^8 + 3^9$
a_3	$1 + 3 + 3^2 + 2 * 3^4$
a_4	$3 + 3^2 + 3^3 + 3^4$
a_5	$3 + 2 * 3^3 + 3^4 + 3^5$
a_6	$2 + 2 * 3 + 2 * 3^2 + 2 * 3^3 + 3^5$
a_7	$1 + 3 + 2 * 3^2 + 3^3$
a_8	$3^2 + 3^3$
a_9	$1 + 2 * 3 + 2 * 3^2$
a_{10}	0
$P_\chi(T) = T + (3^2 + 3^3 + 2 * 3^4 + 2 * 3^5 + 3^6 + 2 * 3^7 + 3^9 + 2 * 3^{10})$	

$p=5,\ d=-127,\ \beta=2,\ \lambda(F_\chi)=1,\ \lambda(p,d,\beta)=1$	
a_0	$4*5+3*5^2+4*5^3+4*5^4+4*5^5+4*5^6+4*5^7+4*5^8+4*5^9+4*5^{10}+4*5^{11}$
a_1	$3+3*5+2*5^2+4*5^3+2*5^4+2*5^6+2*5^7+5^8+4*5^{10}$
a_2	$5+2*5^2+3*5^3+2*5^4+3*5^5+3*5^7+3*5^8+5^9$
a_3	$1+2*5^2+4*5^3+3*5^4+2*5^5+2*5^6+3*5^7+5^8$
a_4	$4*5+3*5^2+5^3+2*5^4+3*5^5+4*5^6+2*5^7$
a_5	$3+2*5+3*5^2+2*5^4+5^5+2*5^6$
a_6	$2+2*5+5^2+4*5^3+4*5^5$
a_7	$3+5^2+4*5^3$
a_8	$4+5+2*5^2+5^3$
a_9	$3+3*5+4*5^2$
a_{10}	$2+4*5$
$P_\chi(T)=T+(3*5+5^2+4*5^4+5^5+5^6+3*5^7+5^8+3*5^9)$	

$p=3,\ d=62,\ \beta=1,\ \lambda(F_\chi)=3,\ \lambda(p,d,\beta)=2$	
a_0	$2*3^2+3^3+2*3^4+2*3^5+2*3^6+2*3^7+2*3^8+2*3^9+2*3^{10}+2*3^{11}$
a_1	$2*3+3^2+3^3+3^5+3^6+3^7+2*3^8+3^9+3^{10}$
a_2	$2*3+3^2+3^5+3^6+2*3^7+3^8$
a_3	$1+3^2+3^3+2*3^4+3^8$
a_4	$2+2*3+2*3^3+2*3^4+3^5$
a_5	$2+2*3+3^2+2*3^3+3^4+3^6$
a_6	$1+2*3^2+2*3^3+2*3^4+3^5$
a_7	$1+2*3+2*3^3$
a_8	$1+2*3+3^2+2*3^3$
a_9	$1+3+2*3^2$
a_{10}	0
$P_\chi(T)=T^3+(3)T^2+(2*3)T+(2*3^2)$	

$p=3,\ d=77,\ \beta=1,\ \lambda(F_\chi)=3,\ \lambda(p,d,\beta)=2$	
a_0	$2*3^2+3^3+2*3^4+2*3^5+2*3^6+2*3^7+2*3^8+2*3^9+2*3^{10}+2*3^{11}$
a_1	$2*3+2*3^4+2*3^5+3^6+2*3^7+3^8$
a_2	$3+3^4+3^8$
a_3	$1+2*3+2*3^2+3^5+3^6+2*3^7+2*3^8$
a_4	$3^2+2*3^3+2*3^4+3^5+3^6+2*3^7$
a_5	$2*3^2+3^3+2*3^4+3^6$
a_6	$3^2+2*3^3+2*3^4+3^5$
a_7	$2+2*3+2*3^4$
a_8	3^2
a_9	$1+2*3^2$
a_{10}	0
$P_\chi(T)=T^3+(3+3^2)T^2+(2*3+2*3^2)T+(2*3^2)$	

$p=3,\ d=83,\ \beta=1,\ \lambda(F_\chi)=3,\ \lambda(p,d,\beta)=2$	
a_0	$2*3^2+3^3+2*3^4+2*3^5+2*3^6+2*3^7+2*3^8+2*3^9+2*3^{10}+2*3^{11}$
a_1	$2*3+2*3^2+2*3^3+2*3^4+3^5+3^6+3^8+3^9$
a_2	$3^2+2*3^5+3^7+2*3^9$
a_3	$1+2*3+2*3^2+2*3^4+2*3^6+3^7$
a_4	$1+2*3+2*3^4$
a_5	$2*3+3^4+2*3^5+3^6$
a_6	$2+3+2*3^2+2*3^3+2*3^5$
a_7	$2+2*3+3^2+3^3$
a_8	$2+3+2*3^2+3^3$
a_9	1
a_{10}	0
$P_\chi(T)=T^3+(3+2*3^2)T^2+(2*3+3^2)T+(2*3^2)$	

$p=3,\ d=103,\ \beta=1,\ \lambda(F_\chi)=3,\ \lambda(p,d,\beta)=2$	
a_0	$2*3^2+3^3+2*3^4+2*3^5+2*3^6+2*3^7+2*3^8+2*3^9+2*3^{10}+2*3^{11}$
a_1	$2*3+2*3^2+2*3^4+3^5+2*3^6+2*3^7+2*3^8+2*3^{1}0$
a_2	$3^4+3^7+3^8+3^9$
a_3	$1+3+2*3^2+2*3^3+3^4+2*3^5+2*3^6+3^8$
a_4	$2+3+3^4+2*3^5+3^7$
a_5	$2+2*3+3^2+2*3^3+3^4+2*3^5+3^6$
a_6	$1+2*3+2*3^2+2*3^4$
a_7	$1+3+3^2+2*3^4$
a_8	$2+2*3+3^2+3^3$
a_9	$3+2*3^2$
a_{10}	0
$P_\chi(T)=T^3+(2*3)T^2+(2*3)T+2*3^2$	

$p=3,\ d=139,\ \beta=1,\ \lambda(F_\chi)=3,\ \lambda(p,d,\beta)=2$	
a_0	$2*3^2+3^3+2*3^4+2*3^5+2*3^6+2*3^7+2*3^8+2*3^9+2*3^{10}+2*3^{11}$
a_1	$2*3+3^4+2*3^8+3^9$
a_2	$3+3^2+2*3^3+2*3^4+3^5+3^6+2*3^8+3^9$
a_3	$1+2*3+3^2+3^3+2*3^4+3^7+3^8$
a_4	$2+3^2+2*3^5$
a_5	$1+3+2*3^2+3^3+3^4+2*3^5+3^6$
a_6	$2+3+3^2+3^3+3^4+3^5$
a_7	$2+3+3^3$
a_8	$2+3^2$
a_9	3^2
a_{10}	0
$P_\chi(T)=T^3+(2*3^2)T^2+(2*3+2*3^2)T+(2*3^2)$	

$p=3,\ d=151,\ \beta=1,\ \lambda(F_\chi)=3,\ \lambda(p,d,\beta)=2$	
a_0	$2*3^2+3^3+2*3^4+2*3^5+2*3^6+2*3^7+2*3^8+2*3^9+2*3^{10}+2*3^{11}$
a_1	$2*3+2*3^2+2*3^3+3^5+2*3^6+3^7+3^8+2*3^9$
a_2	$2*3^2+3^3+2*3^5+2*3^8+3^9$
a_3	$1+3^2+2*3^4+3^6+2*3^7$
a_4	$1+3+3^2+3^3+2*3^4+3^6$
a_5	$2+3^5$
a_6	$3+2*3^4$
a_7	$2+2*3+2*3^2+3^3+2*3^4$
a_8	$1+2*3+3^3$
a_9	$2+2*3$
a_{10}	0
$P_\chi T=T^3+(3)T^2+(2*3+3^2)T+(2*3^2)$	

$p=3,\ d=179,\ \beta=1,\ \lambda(F_\chi)=3,\ \lambda(p,d,\beta)=2$	
a_0	$2*3^2+3^3+2*3^4+2*3^5+2*3^6+2*3^7+2*3^8+2*3^9+2*3^{10}+2*3^{11}$
a_1	$2*3+3^4+2*3^6+3^{10}$
a_2	$3+3^2+2*3^5$
a_3	$1+2*3^2+2*3^3+2*3^4+2*3^5+3^6+3^7$
a_4	$2*3+2*3^5+3^6+2*3^7$
a_5	$2+2*3+2*3^2+3^4$
a_6	$2+2*3^3+3^4+2*3^5$
a_7	$2+2*3+3^3+3^4$
a_8	$2+3^2+3^3$
a_9	$2+3+2*3^2$
a_{10}	0
$P_\chi(T)=T^3+(3)T^2+(2*3+2*3^2)T+(2*3^2)$	

$p=3,\ d=181,\ \beta=1,\ \lambda(F_\chi)=3\ \lambda(p,d,\beta)=2$	
a_0	$2*3^2+3^3+2*3^4+2*3^5+2*3^6+2*3^7+2*3^8+2*3^9+2*3^{10}+2*3^{11}$
a_1	$2*3+2*3^3+3^4+3^5+2*3^7+2*3^8+3^{10}$
a_2	$3+2*3^2+3^6+2*3^9$
a_3	$1+3^3+2*3^4+2*3^7$
a_4	$1+3+3^6+2*3^7$
a_5	$2+3+2*3^3+2*3^4+3^5$
a_6	$2+2*3+3^3+2*3^5$
a_7	$2*3+2*3^2+2*3^3+3^4$
a_8	$2*3^3$
a_9	0
a_{10}	0
$P_\chi(T)=T^3+(2*3+3^2)T^2+(2*3+3^2)T+(2*3^2)$	

$p=3,\ d=199,\ \beta=1,\ \lambda(F_\chi)=3,\ \lambda(p,d,\beta)=2$	
a_0	$2*3^2+3^3+2*3^4+2*3^5+2*3^6+2*3^7+2*3^8+2*3^9+2*3^{10}+2*3^{11}$
a_1	$2*3+2*3^2+2*3^3+3^6+3^8+2*3^9$
a_2	$3^2+3^3+2*3^4+2*3^6+3^7+2*3^8$
a_3	$1+3+2*3^2+2*3^3+3^6+3^7$
a_4	$3+3^2+3^4+2*3^5+2*3^6$
a_5	$2*3^2+3^3+3^6$
a_6	$1+2*3+3^2+3^3$
a_7	$2+2*3+3^2$
a_8	3^2
a_9	$1+3+3^2$
a_{10}	0
$P_\chi(T)=T^3+(2*3)T+(2*3^2)$	

$p=3,\ d=-14,\ \beta=0,\ \lambda(F_\chi)=2,\ \lambda(p,d,\beta)=2$	
a_0	0
a_1	$3+3^3+2*3^4+2*3^5+2*3^6+3^7+3^8+3^9+2*3^{10}$
a_2	$2+2*3^2+2*3^4+3^6$
a_3	$1+2*3^2+2*3^5+2*3^7$
a_4	$2*3+2*3^2+2*3^3+2*3^5+3^6$
a_5	$1+3+3^2+2*3^3+2*3^4+2*3^5+2*3^6$
a_6	$2*3+2*3^2$
a_7	$2+3+3^3+3^4$
a_8	$2+3$
a_9	$2+3+3^2$
a_{10}	0
$P_\chi(T)=T^2+(2*3+2*3^3+2*3^4)T$	

$p=3,\ d=-35,\ \beta=0,\ \lambda(F_\chi)=2,\ \lambda(p,d,\beta)=2$	
a_0	0
a_1	$2*3+2*3^2+3^3+2*3^4+3^5+3^6+2*3^7+3^8+2*3^{10}$
a_2	$2+2*3^2+2*3^3+2*3^4+3^5+2*3^6+3^7+3^8+3^9$
a_3	$1+3+3^3+2*3^4+2*3^5+2*3^7+2*3^8$
a_4	$2+3^2+2*3^3+2*3^4+3^5+3^7$
a_5	$1+3^3+3^5+3^6$
a_6	$2+3^2+2*3^3+2*3^5$
a_7	$2*3^2+3^4$
a_8	$2+3^2+3^3$
a_9	3
a_{10}	0
$P_\chi(T)=T^2+(3+3^3+3^4)T$	

$p=3,\ d=-47,\ \beta=0,\ \lambda(F_\chi)=2,\ \lambda(p,d,\beta)=2$	
a_0	0
a_1	$3^2+3^5+2*3^6+2*3^8+2*3^9+2*3^{10}$
a_2	$1+3+3^3+3^6+2*3^8+3^9$
a_3	$2+3+3^2+2*3^3+3^5+3^7+2*3^8$
a_4	$2+3^2+3^4+2*3^5+3^6+3^7$
a_5	$1+3+2*3^3+2*3^4+2*3^6$
a_6	$2*3+3^2+3^3+2*3^4$
a_7	$3+2*3^2+2*3^3+3^4$
a_8	$2*3+3^2+3^3$
a_9	$2*3+3^2$
a_{10}	0
$P_\chi(T)=T^2+(3^2+2*3^3+2*3^4)T$	

$p=3,\ d=-65,\ \beta=0,\ \lambda(F_\chi)=2,\ \lambda(p,d,\beta)=2$	
a_0	0
a_1	$3+2*3^2+2*3^4+2*3^5+3^8+2*3^9+3^{10}$
a_2	$2+3^2+2*3^3+2*3^4+3^5+3^6+3^8$
a_3	$2+3+3^2+2*3^3+3^7+2*3^8$
a_4	$2*3^3+2*3^6$
a_5	$1+2*3+2*3^2+3^3+3^4+2*3^5+2*3^6$
a_6	$3+3^2+3^4+2*3^5$
a_7	$2+3+2*3^2+3^4$
a_8	$2+3+2*3^2+3^3$
a_9	3
a_{10}	0
$P_\chi(T)=T^2+(2*3+3^3)T$	

$p=3,\ d=-74,\ \beta=0,\ \lambda(F_\chi)=2,\ \lambda(p,d,\beta)=2$	
a_0	0
a_1	$3^2+2*3^3+3^4+2*3^6+3^8+3^9+2*3^{10}$
a_2	$1+3^2+3^3+3^4+2*3^7+2*3^8$
a_3	$1+3+2*3^2+2*3^3+3^4+2*3^6+2*3^7$
a_4	$1+2*3^3+2*3^4+2*3^5+2*3^6+2*3^7$
a_5	$2+3+3^2+2*3^3+2*3^5$
a_6	$2+2*3+3^2+3^3+3^5$
a_7	$1+3+3^2+2*3^3+2*3^4$
a_8	3^2+3^3
a_9	$2*3^2$
a_{10}	0
$P_\chi(T)=T^2+(3^2+2*3^3+3^4)T$	

$p=3,\ d=-101,\ \beta=0,\ \lambda(F_\chi)=2,\ \lambda(p,d,\beta)=2$	
a_0	0
a_1	$3^2+2*3^3+3^5+3^6+2*3^7+2*3^{10}$
a_2	$2+2*3^2+3^4+2*3^5+3^7+3^9$
a_3	$1+3^3+3^5+3^6+2*3^7+3^8$
a_4	$1+2*3+3^2+3^3+2*3^4$
a_5	$2*3+2*3^3+3^4+3^6$
a_6	$1+3+2*3^2+2*3^3+2*3^4$
a_7	$2*3+2*3^2+2*3^3+2*3^4$
a_8	$2*3^2+2*3^3$
a_9	$2*3$
a_{10}	0
$P_\chi(T)=T^2+(2*3^2+2*3^3+3^4)T$	

$p=3,\ d=-107,\ \beta=0,\ \lambda(F_\chi)=2,\ \lambda(p,d,\beta)=2$	
a_0	0
a_1	$2*3^2+2*3^3+3^4+2*3^5+3^6+3^7+3^8+3^9+2*3^{10}$
a_2	$1+3+2*3^3+3^4+2*3^5+3^6+2*3^7+2*3^8+2*3^9$
a_3	$1+3+2*3^2+3^3+3^4+3^7+3^8$
a_4	$2+3+3^2+2*3^3+2*3^5+2*3^7$
a_5	$2*3^2+2*3^3+3^4+3^5$
a_6	$1+2*3^2$
a_7	$2+2*3+2*3^2+3^3+3^4$
a_8	$3+3^3$
a_9	$2+3+3^2$
a_{10}	0
$P_\chi(T)=T^2+(2*3^2+2*3^4)T$	

$p=3,\ d=-113,\ \beta=0,\ \lambda(F_\chi)=2,\ \lambda(p,d,\beta)=2$	
a_0	0
a_1	$3+2*3^2+2*3^3+2*3^4+2*3^5+2*3^6+3^7+2*3^8+2*3^{10}$
a_2	$1+3+2*3^2+2*3^3+2*3^4+3^5+3^7+3^9$
a_3	$2*3+2*3^2+3^3+3^4+2*3^8$
a_4	$2+3^2+3^4+3^5+3^7$
a_5	$2*3^3$
a_6	$2*3+2*3^2+3^4$
a_7	$2*3+2*3^3+2*3^4$
a_8	$2*3+3^3$
a_9	$1+2*3$
a_{10}	0
$P_\chi(T)=T^2+(3+3^2+2*3^3+3^4)T$	

$p=3,\ d=-149,\ \beta=0,\ \lambda(F_\chi)=2,\ \lambda(p,d,\beta)=2$	
a_0	0
a_1	$2*3^2+2*3^5+2*3^6+2*3^8$
a_2	$1+3+2*3^2+3^3+3^4+2*3^5+3^6+2*3^8+2*3^9$
a_3	$2+3+3^2+3^3+2*3^4+2*3^5+3^6+3^7+2*3^8$
a_4	$3+3^2+2*3^3+3^5+2*3^6+2*3^7$
a_5	$2+3+3^2+2*3^3+2*3^4+3^6$
a_6	$1+2*3+3^2+3^3+2*3^5$
a_7	$2*3+3^2+2*3^3+3^4$
a_8	$2+3+3^3$
a_9	3^2
a_{10}	0
$P_\chi(T)=T^2+(2*3^2+3^3+2*3^4)T$	

$p=3,\ d=-158,\ \beta=0,\ \lambda(F_\chi)=2,\ \lambda(p,d,\beta)=2$	
a_0	0
a_1	$2*3+2*3^2+3^3+3^5+3^6+2*3^7+2*3^{10}$
a_2	$2+3+2*3^2+3^3+3^4+3^6+3^7+2*3^8+3^9$
a_3	$3^3+3^4+2*3^5+2*3^6+3^8$
a_4	$2+3+3^3+3^4+3^7$
a_5	3^2+3^3
a_6	$1+3^2+3^3+3^4+2*3^5$
a_7	$1+2*3^4$
a_8	$2+3+2*3^2$
a_9	2
a_{10}	0
$P_\chi(T)=T^2+(3+2*3^2+2*3^4)T$	

$p=3,\ d=-173,\ \beta=0,\ \lambda(F_\chi)=2,\ \lambda(p,d,\beta)=2$	
a_0	0
a_1	$3+2*3^2+2*3^4+2*3^5+2*3^6+3^7+3^8+2*3^{10}$
a_2	$1+3^2+3^4+2*3^5+2*3^6$
a_3	$2+2*3+2*3^2+2*3^3+2*3^4+2*3^6+3^8$
a_4	$2+2*3+3^2+3^3+2*3^5+2*3^6+2*3^7$
a_5	$1+3+2*3^2+3^3+2*3^4+2*3^5+3^6$
a_6	$1+2*3^2+3^5$
a_7	$2+2*3^2+2*3^3$
a_8	$2+3+3^2+2*3^3$
a_9	$1+3^2$
a_{10}	0
$P_\chi(T)=T^2+(3+3^2+3^3+3^4)T$	

$p=5,\ d=23,\ \beta=1,\ \lambda(F_\chi)=2,\ \lambda(p,d,\beta)=2$	
a_0	$5+3*5^2+4*5^3+2*5^4+5^5+4*5^6+2*5^8+5^9+5^{10}$
a_1	$4*5+4*5^2+2*5^3+5^4+2*5^5+2*5^6+5^8+5^9+5^{10}$
a_2	$4+2*5+2*5^3+4*5^4+3*5^5+2*5^6+3*5^7+4*5^8+3*5^9$
a_3	$2+2*5+3*5^2+4*5^3+4*5^5+2*5^6+5^7$
a_4	$2*5+2*5^2+5^3+3*5^4+5^6+5^7$
a_5	$4+2*5+2*5^2+4*5^3+3*5^4+4*5^5+2*5^6$
a_6	$4+2*5^2+3*5^3+3*5^4+5^5$
a_7	$2+5^4$
a_8	$2*5+4*5^2+4*5^3$
a_9	$1+2*5$
a_{10}	$3+5$
$P_\chi(T)=T^2+(4*5+2*5^2+2*5^3+4*5^4)T+(4*5+5^2+4*5^3+5^4)$	

$p=5,\ d=37,\ \beta=1,\ \lambda(F_\chi)=2,\ \lambda(p,d,\beta)=2$	
a_0	$4*5+2*5^2+4*5^3+2*5^4+5^5+4*5^6+2*5^8+5^9+5^{10}$
a_1	$2*5+5^2+3*5^4+2*5^5+5^6+5^{10}$
a_2	$3+2*5+4*5^2+3*5^3+4*5^4+4*5^5+3*5^6+3*5^7+5^8+5^9$
a_3	$3*5+3*5^2+3*5^6+4*5^7+5^8$
a_4	$4+4*5+3*5^2+3*5^3+4*5^4+5^5+5^7$
a_5	$3+5+3*5^2+3*5^3+4*5^4+5^5+5^6$
a_6	$2+3*5^3+3*5^4+4*5^5$
a_7	$3*5+3*5^2+4*5^3+4*5^4$
a_8	$2+5+4*5^2$
a_9	$2+5+3*5^2$
a_{10}	$1+3*5$
$P_\chi(T)=T^2+(4*5+3*5^3+3*5^4)T+(3*5+2*5^2+2*5^3+5^4)$	

$p=5,\ d=109,\ \beta=3,\ \lambda(F_\chi)=3,\ \lambda(p,d,\beta)=2$	
a_0	$5^2+2*5^3+3*5^4+4*5^6+2*5^7+3*5^8+3*5^9+4*5^{10}+4*5^{11}$
a_1	$5+4*5^4+3*5^5+4*5^6+5^7+2*5^8+5^9+2*5^{10}$
a_2	$5^3+2*5^5+2*5^7+2*5^8+4*5^9$
a_3	$3+4*5+5^2+4*5^3+5^4+2*5^5+5^7$
a_4	$3+5+2*5^2+4*5^3+3*5^4$
a_5	$2+3*5+3*5^2+2*5^3+5^4+5^5+2*5^6$
a_6	$2+4*5+2*5^2+2*5^3+4*5^4$
a_7	$1+4*5+2*5^3$
a_8	$3+3*5+2*5^3$
a_9	$2+3*5$
a_{10}	5
$P_\chi(T)=T^3+(3*5+4*5^2)T^2+(2*5+2*5^2)T+(2*5^2)$	

$p = 5,\ d = 127,\ \beta = 1,\ \lambda(F_\chi) = 2,\ \lambda(p, d, \beta) = 2$	
a_0	$5 + 5^2 + 5^3 + 5^4 + 2 * 5^6 + 2 * 5^7 + 3 * 5^8 + 5^{10} + 4 * 5^{11}$
a_1	$3 * 5^2 + 2 * 5^3 + 3 * 5^4 + 3 * 5^6 + 5^7 + 4 * 5^8 + 4 * 5^9 + 5^{10}$
a_2	$3 + 3 * 5^2 + 5^3 + 4 * 5^4 + 2 * 5^6 + 4 * 5^7 + 2 * 5^8 + 4 * 5^9$
a_3	$3 + 5 + 2 * 5^3 + 3 * 5^4 + 3 * 5^5 + 4 * 5^6 + 3 * 5^7 + 2 * 5^8$
a_4	$3 + 5 + 4 * 5^3 + 5^4 + 4 * 5^5 + 2 * 5^6 + 5^7$
a_5	$4 + 2 * 5 + 4 * 5^2 + 2 * 5^3 + 2 * 5^4 + 5^5 + 3 * 5^6$
a_6	$2 + 4 * 5 + 2 * 5^2 + 5^4 + 2 * 5^5$
a_7	$2 + 2 * 5 + 5^2 + 2 * 5^4$
a_8	$4 + 3 * 5 + 2 * 5^2 + 3 * 5^3$
a_9	$3 + 4 * 5 + 5^2$
a_{10}	$2 + 2 * 5$
$P_\chi(T) = T^2 + (3 * 5 + 4 * 5^2 + 2 * 5^3 + 5^4)T + (2 * 5 + 3 * 5^3 + 3 * 5^4)$	

$p = 5,\ d = 127,\ \beta = 3,\ \lambda(F_\chi) = 3,\ \lambda(p, d, \beta) = 2$	
a_0	$3 * 5^2 + 3 * 5^3 + 3 * 5^4 + 4 * 5^5 + 2 * 5^6 + 2 * 5^7 + 5^8 + 4 * 5^9 + 3 * 5^{10}$
a_1	$3 * 5 + 5^2 + 3 * 5^3 + 3 * 5^5 + 5^6 + 4 * 5^8 + 2 * 5^9$
a_2	$4 * 5 + 4 * 5^2 + 3 * 5^3 + 4 * 5^4 + 3 * 5^5 + 2 * 5^6 + 3 * 5^7 + 3 * 5^8 + 2 * 5^9$
a_3	$4 + 2 * 5 + 2 * 5^2 + 5^3 + 2 * 5^4 + 2 * 5^8$
a_4	$2 * 5^2 + 2 * 5^3 + 5^4 + 4 * 5^5 + 4 * 5^6$
a_5	$2 + 2 * 5 + 2 * 5^3 + 2 * 5^4 + 3 * 5^5 + 3 * 5^6$
a_6	$3 * 5 + 2 * 5^3 + 2 * 5^4 + 2 * 5^5$
a_7	$4 + 3 * 5^4$
a_8	$3 * 5^2 + 4 * 5^3$
a_9	$4 + 3 * 5 + 4 * 5^2$
a_{10}	0
$P_\chi(T) = T^3 + (5 + 4 * 5^2)T^2 + (2 * 5 + 5^2)T + (2 * 5^2)$	

$p = 5,\ d = 149,\ \beta = 1,\ \lambda(F_\chi) = 2,\ \lambda(p, d, \beta) = 2$	
a_0	$3 * 5 + 2 * 5^2 + 2 * 5^4 + 3 * 5^5 + 4 * 5^7 + 2 * 5^8 + 3 * 5^9 + 4 * 5^{11}$
a_1	$3 * 5 + 5^3 + 4 * 5^6 + 3 * 5^8 + 5^9 + 3 * 5^{10}$
a_2	$3 + 5 + 5^2 + 4 * 5^3 + 3 * 5^4 + 2 * 5^5 + 3 * 5^6 + 3 * 5^7 + 5^9$
a_3	$4 + 4 * 5 + 3 * 5^2 + 4 * 5^3 + 2 * 5^4 + 5^5 + 4 * 5^7 + 4 * 5^8$
a_4	$2 + 5 + 5^2 + 4 * 5^3 + 3 * 5^4 + 3 * 5^5 + 2 * 5^7$
a_5	$2 + 2 * 5^2 + 2 * 5^4 + 5^5 + 5^6$
a_6	$4 + 2 * 5 + 5^2 + 3 * 5^4 + 2 * 5^5$
a_7	$2 + 5 + 3 * 5^2 + 4 * 5^4$
a_8	$4 + 3 * 5 + 3 * 5^3$
a_9	$1 + 5$
a_{10}	$2 + 5$
$P_\chi(T) = T^2 + (3 * 5 + 2 * 5^2 + 2 * 5^4)T + (5 + 3 * 5^3 + 5^4)$	

$p = 5,\ d = -11,\ \beta = 0,\ \lambda(F_\chi) = 2,\ \lambda(p, d, \beta) = 2$	
a_0	0
a_1	$5 + 3 * 5^2 + 3 * 5^3 + 5^8 + 4 * 5^9 + 4 * 5^{10}$
a_2	$3 + 3 * 5^2 + 4 * 5^3 + 2 * 5^4 + 2 * 5^7 + 2 * 5^8 + 4 * 5^9$
a_3	$4 + 3 * 5 + 3 * 5^2 + 2 * 5^3 + 4 * 5^4 + 2 * 5^5 + 2 * 5^6 + 3 * 5^7$
a_4	$4 + 3 * 5 + 5^2 + 3 * 5^3 + 4 * 5^5 + 4 * 5^6 + 3 * 5^7$
a_5	$3 + 2 * 5^2 + 5^4 + 3 * 5^5 + 3 * 5^6$
a_6	$3 + 4 * 5^2 + 3 * 5^3 + 2 * 5^4 + 2 * 5^5$
a_7	$4 + 2 * 5 + 4 * 5^2 + 2 * 5^3 + 5^4$
a_8	$1 + 5 + 5^2 + 5^3$
a_9	$4 + 5 + 2 * 5^2$
a_{10}	$1 + 4 * 5$
$P_\chi(T) = T^2 + (2 * 5 + 5^2 + 2 * 5^3 + 5^4)T$	

$p = 5,\ d = -26,\ \beta = 0,\ \lambda(F_\chi) = 2,\ \lambda(p, d, \beta) = 2$	
a_0	0
a_1	$5 + 3 * 5^2 + 3 * 5^3 + 4 * 5^5 + 5^6 + 4 * 5^7 + 2 * 5^9 + 4 * 5^{10}$
a_2	$2 + 4 * 5 + 3 * 5^2 + 5^4 + 5^5 + 2 * 5^7 + 4 * 5^8$
a_3	$1 + 4 * 5 + 4 * 5^2 + 5^3 + 4 * 5^4 + 4 * 5^5 + 2 * 5^6 + 4 * 5^7$
a_4	$4 + 5 + 2 * 5^3 + 2 * 5^4 + 5^5 + 4 * 5^6 + 3 * 5^7$
a_5	$3 * 5^2 + 3 * 5^3 + 3 * 5^5 + 4 * 5^6$
a_6	$5^2 + 4 * 5^3 + 4 * 5^5$
a_7	$2 * 5 + 3 * 5^3 + 5^4$
a_8	$4 + 4 * 5^2 + 4 * 5^3$
a_9	$4 + 4 * 5$
a_{10}	3
$P_\chi(T) = T^2 + (3 * 5 + 2 * 5^2 + 2 * 5^4)T$	

$p = 5,\ d = -34,\ \beta = 0,\ \lambda(F_\chi) = 2,\ \lambda(p, d, \beta) = 2$	
a_0	0
a_1	$3 * 5 + 5^2 + 5^4 + 2 * 5^5 + 5^7 + 3 * 5^8 + 2 * 5^9$
a_2	$3 + 2 * 5 + 4 * 5^2 + 2 * 5^3 + 3 * 5^4 + 3 * 5^5 + 2 * 5^7 + 2 * 5^8$
a_3	$3 * 5 + 2 * 5^2 + 3 * 5^3 + 2 * 5^4 + 2 * 5^5 + 5^6 + 4 * 5^8$
a_4	$1 + 5 + 5^3 + 3 * 5^4 + 2 * 5^5 + 3 * 5^6 + 4 * 5^7$
a_5	$1 + 5 + 3 * 5^2 + 3 * 5^3 + 5^4 + 2 * 5^5 + 4 * 5^6$
a_6	$2 + 2 * 5^3 + 4 * 5^4$
a_7	$3 * 5 + 3 * 5^3 + 2 * 5^4$
a_8	$2 * 5 + 4 * 5^2 + 4 * 5^3$
a_9	$5 + 2 * 5^2$
a_{10}	$1 + 2 * 5$
$P_\chi(T) = T^2 + (5 + 3 * 5^2 + 2 * 5^4)T$	

$p=5,\ d=-41,\ \beta=0,\ \lambda(F_\chi)=2,\ \lambda(p,d,\beta)=2$	
a_0	0
a_1	$2*5+2*5^2+2*5^3+2*5^4+2*5^5+5^6+2*5^7+5^9$
a_2	$4+2*5^2+5^3+2*5^4+3*5^5+3*5^6+3*5^8+3*5^9$
a_3	$4+5+2*5^2+5^4+2*5^6+3*5^7+3*5^8$
a_4	$1+5+5^3+3*5^4+2*5^5+3*5^6+4*5^7$
a_5	$4+2*5+2*5^2+4*5^3+3*5^5+2*5^6$
a_6	$3+3*5+2*5^2+5^3+5^4$
a_7	$4+4*5+4*5^2+2*5^3+2*5^4$
a_8	$4+4*5+4*5^2$
a_9	$3+3*5+4*5^2$
a_{10}	$1+4*5$
$P_\chi(T)=T^2+(3*5+4*5^2+2*5^3+2*5^4)T$	

$p=5,\ d=-46,\ \beta=0,\ \lambda(F_\chi)=2,\ \lambda(p,d,\beta)=2$	
a_0	0
a_1	$3*5+5^2+4*5^3+2*5^6+2*5^7+4*5^8+3*5^9+3*5^{10}$
a_2	$1+3*5+4*5^2+4*5^3+3*5^5+3*5^6+5^7+2*5^8$
a_3	$4*5+2*5^2+3*5^3+2*5^5+5^7$
a_4	$4+2*5+5^2+2*5^3+4*5^4+2*5^5+4*5^6$
a_5	$4+2*5+4*5^2+3*5^4+5^5+4*5^6$
a_6	$2+5+3*5^2+3*5^3+5^4$
a_7	$2*5+2*5^2$
a_8	$3+4*5+3*5^2$
a_9	$4*5+2*5^2$
a_{10}	$2+5$
$P_\chi(T)=T^2+(3*5+4*5^2+2*5^3+2*5^4)T$	

$p=5,\ d=-51,\ \beta=0,\ \lambda(F_\chi)=2,\ \lambda(p,d,\beta)=2$	
a_0	0
a_1	$5^3+3*5^4+4*5^5+2*5^7+4*5^8+4*5^{10}$
a_2	$3+2*5+5^2+5^3+4*5^5+2*5^6+5^7+2*5^8+2*5^9$
a_3	$5+3*5^2+2*5^3+2*5^4+5^5+4*5^7$
a_4	$2+3*5+3*5^2+4*5^3+2*5^4+5^6+2*5^7$
a_5	$4+2*5+2*5^2+3*5^3+2*5^4+4*5^5$
a_6	$3*5+2*5^3+3*5^4$
a_7	$2+3*5^2+3*5^4$
a_8	$2+4*5+5^2+4*5^3$
a_9	$1+5^2$
a_{10}	$3+2*5$
$P_\chi(T)=T^2+(2*5^3+5^4)T$	

$p = 5,\ d = -114,\ \beta = 0,\ \lambda(F_\chi) = 2,\ \lambda(p, d, \beta) = 2$	
a_0	0
a_1	$5 + 2*5^2 + 2*5^3 + 4*5^5 + 3*5^6 + 3*5^8 + 3*5^9$
a_2	$2 + 5 + 4*5^2 + 5^3 + 4*5^4 + 4*5^5 + 5^7 + 4*5^8 + 4*5^9$
a_3	$2 + 4*5 + 4*5^2 + 4*5^3 + 3*5^4 + 4*5^5$
a_4	$3 + 5 + 3*5^2 + 5^4 + 5^5 + 3*5^6 + 5^7$
a_5	$2 + 2*5 + 3*5^2 + 4*5^3 + 3*5^4 + 5^5$
a_6	$3 + 2*5^2 + 3*5^4 + 3*5^5$
a_7	$3 + 4*5 + 5^3$
a_8	$4 + 3*5 + 4*5^2 + 5^3$
a_9	$3 + 3*5 + 4*5^2$
a_{10}	$2 + 2*5$
$P_\chi(T) = T^2 + (3*5 + 3*5^2 + 2*5^4)T$	

$p = 5,\ d = -166,\ \beta = 2,\ \lambda(F_\chi) = 2,\ \lambda(p, d, \beta) = 2$	
a_0	$3*5 + 2*5^2 + 4*5^3 + 4*5^4 + 4*5^5 + 4*5^6 + 4*5^7 + 4*5^8 + 4*5^9 + 4*5^{10} + 4*5^{11}$
a_1	$2*5^2 + 4*5^3 + 2*5^4 + 3*5^5 + 3*5^6 + 4*5^7 + 5^8 + 2*5^9 + 2*5^{10}$
a_2	$4 + 4*5 + 2*5^2 + 4*5^3 + 2*5^4 + 2*5^5 + 4*5^6 + 3*5^7 + 2*5^8 + 4*5^9$
a_3	$2*5 + 3*5^2 + 4*5^3 + 2*5^5 + 5^6 + 4*5^7$
a_4	$4 + 2*5 + 4*5^3 + 3*5^4 + 5^5 + 5^6$
a_5	$2 + 3*5 + 3*5^2 + 2*5^3 + 3*5^4 + 3*5^5 + 3*5^6$
a_6	$1 + 2*5 + 2*5^2 + 2*5^3 + 3*5^4 + 3*5^5$
a_7	$5 + 5^3 + 2*5^4$
a_8	$4 + 5 + 5^2 + 4*5^3$
a_9	$1 + 5^2$
a_{10}	$2*5$
$P_\chi(T) = T^2 + (4*5^2 + 3*5^3 + 4*5^4)T + (2*5 + 5^2 + 2*5^3 + 5^4)$	

$p = 7,\ d = 6,\ \beta = 5,\ \lambda(F_\chi) = 3,\ \lambda(p, d, \beta) = 2$	
a_0	$6*7^2 + 6*7^3 + 4*7^4 + 7^5 + 4*7^6 + 2*7^7 + 6*7^8 + 4*7^9 + 4*7^{10} + 2*7^{11}$
a_1	$6*7 + 7^2 + 7^3 + 4*7^5 + 7^6 + 2*7^7 + 5*7^8 + 4*7^9 + 6*7^{10}$
a_2	$7^2 + 7^4 + 5*7^5 + 5*7^6 + 5*7^9$
a_3	$4 + 6*7 + 5*7^2 + 3*7^3 + 5*7^4 + 2*7^5 + 2*7^6 + 4*7^7$
a_4	$1 + 5*7 + 2*7^2 + 7^3 + 4*7^4 + 5*7^5 + 6*7^6 + 7^7$
a_5	$3 + 2*7^2 + 6*7^3 + 5*7^4 + 5*7^5 + 4*7^6$
a_6	$1 + 6*7 + 6*7^2 + 3*7^3 + 5*7^4 + 3*7^5$
a_7	$5 + 2*7 + 3*7^2 + 6*7^3 + 7^4$
a_8	$4*7 + 3*7^2 + 2*7^3$
a_9	$6 + 2*7 + 6*7^2$
a_{10}	$2 + 7$
$P_\chi(T) = T^3 + (4*7 + 4*7^2)T^2 + (5*7 + 4*7^2)T + (5*7^2)$	

$p=3,\ d=-41,\ \beta=0,\ \lambda(F_\chi)=3,\ \lambda(p,d,\beta)=3$	
a_0	0
a_1	$2*3^3+2*3^4+3^5+2*3^6+2*3^8+3^{10}$
a_2	$2*3+2*3^2+2*3^4+3^5+2*3^6$
a_3	$2+3+3^4+2*3^5+3^6+2*3^7+3^8$
a_4	$2*3+3^2+3^4+2*3^6+2*3^7$
a_5	$2+2*3+2*3^2+3^3+2*3^4$
a_6	$1+3+3^3+2*3^5$
a_7	0
a_8	$2*3^3$
a_9	$2+2*3$
a_{10}	0
$P_\chi(T)=T^3+(3+2*3^2)T^2$	

$p=3,\ d=-86,\ \beta=0,\ \lambda(F_\chi)=3,\ \lambda(p,d,\beta)=3$	
a_0	0
a_1	$3+3^3+3^4+3^5+2*3^6+2*3^8+3^9+3^{10}$
a_2	$3+3^2+2*3^3+3^4+2*3^5+2*3^8+3^9$
a_3	$1+3+3^3+3^4+3^5+2*3^6+3^8$
a_4	$1+3+2*3^2+2*3^3+3^5+3^7$
a_5	$3+3^2+2*3^5+2*3^6$
a_6	$2*3^4+3^5$
a_7	$2+3+2*3^3$
a_8	$1+3+2*3^2+2*3^3$
a_9	$2+3+3^2$
a_{10}	0
$P_\chi(T)=T^3+(3^2)T^2+(3+2*3^2)T$	

$p=5,\ d=114,\ \beta=3,\ \lambda(F_\chi)=4,\ \lambda(p,d,\beta)=3$	
a_0	$4*5^2+4*5^3+3*5^5+4*5^6+3*5^7+2*5^8+4*5^9+5^{10}+4*5^{11}$
a_1	$4*5+4*5^3+3*5^4+2*5^5+2*5^6+2*5^7+4*5^8+3*5^9+5^{10}$
a_2	$2*5+2*5^3+5^4+4*5^6+3*5^7+2*5^8+4*5^9$
a_3	$4*5+4*5^2+2*5^3+5^4+5^5+5^6+3*5^7$
a_4	$4+4*5+2*5^2+5^3+2*5^4+5^5+4*5^7$
a_5	$5+5^2+3*5^4+4*5^5+3*5^6$
a_6	$1+2*5^2+3*5^3+2*5^4+2*5^5$
a_7	$4+4*5+4*5^2+4*5^4$
a_8	5^3
a_9	$3+5+2*5^2$
a_{10}	$4+3*5$
$P_\chi(T)=T^4+(4+3*5)T^3+(4*5)T^2$	

Conclusion

This project afforded me a good opportunity to study algebraic methods in number theory, and to compute several new p-adic L-function zeroes corresponding to the arithmetic of quadratic extensions of the rationals. However, due to limitations of time and resources, many avenues remain unexplored.

Given more time, I would have liked to understand more the connection between the p-adic zeta-function, and the arithmetic of towers of class groups (over the cyclotomic $\mathbb{Z}_p$-extension of the quadratic field). This connection is neatly encapsulated in the Main Conjecture of Iwasawa theory, proved by Mazur and Wiles.

In particular, each zero that I computed has a precise interpretation in terms of the structure of the associated Iwasawa module; more precisely, it should be an eigenvalue of the γ-operator, where $\Gamma =< \gamma >$ denotes the Galois group of the $\mathbb{Z}_p$-extension.

Another interesting project would be to use the p-adic Dirichlet expansions to tabulate λ-invariant data over a large range of cubic number fields (see [4] for the quadratic case). This would be completely new, and it would be interesting to compare these distributions with those arising from the Cohen-Lenstra heuristics.

Nevertheless, I am delighted to have achieved the aims of my project, and to have devised a new methods of calculating these p-adic invariants by computer algebra means.

PARI-GP Programs

Program A

```
\\allocatemem(50000000);
\\ Quadratic field parameters
d = 29;
if(gcd(d-1,4)==4 , D=abs(d), D=abs(4*d) );

quadjacobi(j) = {
if(gcd(j,D)==1, kronecker(sign(d) * D, j), 0);
}
A(m)= {
   return(-sum(j=1, D, j*quadjacobi(j) , 0)/D + sum(j=1, m-1, quadjacobi(j),
0) -(2*sum(j=1, floor((m-1)/2), quadjacobi(j) , 0)))
}

a = vector(2*D);
for(i=1, 2*D, a[i]=A(i));
if(d==1, a=[-1/2,1/2], );

\\Default parameters
pbase = 3;       \\ the prime p in p-adic
acc = 4;         \\ number of p-adic places to use
N = acc;         \\ where our approx. cuts out at
sumtrunc = 1+floor(N/eulerphi(2*D)); \\ the power of p to truncate

\\ Basic functions
ord_p(n)= valuation (n,pbase)
padic(x) = x + O(pbase^ (acc));    \\ convert a rational to a p-adic number
pu_pow(x,s) = exp(s * log(x))    \\    <x>^ s
pu_pow_i(x,s) = x^ s/teichmuller(x)^ (s%(pbase-1))

\\ Nof's super-dooper improved algorithm which works out
\\ p-adic zeta function with twist omega ^ (1+beta), and with an
\\ extra Euler factor at 2,
q = floor(pbase ^ (sumtrunc*eulerphi(2*D)) / (2*D*pbase^ N));
u = lift(Mod(1,2*D)/Mod(pbase,2*D));
th(x, m) = lift( Mod(m,2*D*(pbase^ N))+Mod((x-m)*((pbase*u)^ N)-1 , 2*D*(pbase^
N)) )+1;
mom(m) = sum(x=1, 2*D, if(th(x,m)< (pbase ^ (sumtrunc*eulerphi(2*D))-
2*D*(pbase^ N)*q), a[x] , 0) );
zetanofp(s, beta) = {
   return((sum(m=1, pbase^ (N), padic(if(gcd(pbase,m)==1, padic(mom(m))*(teichmuller(padic(m))^
beta) * pu_pow(padic(m),-s), padic(0))) , 0)
));
}
```

```
\\ Truncation of log(m)/log(1+p) modulo p^ N
lambdaN(m) = lift(Mod((log(m + O(pbase^ (N+1)))/log(pbase+1 + O(pbase^
(N+1)))), pbase^ N))

\\ Work out the j-th coefficient of the poly approx. mod p^ N
coeffappr(j, beta) = sum(m=1, pbase^ N, if( gcd(pbase,m)==1 && j<=lambdaN(m)
, padic(binomial(lambdaN(m),j)) * teichmuller(padic(m))^ (beta) * padic(mom(m)),
padic(0) ) );

\\ The interpolation polynomials F_ N^ (beta)
polyzeta(beta) = sum(m=1, pbase^ N, if(gcd(pbase,m)==1, padic(mom(m)) *
teichmuller(padic(m))^ (beta) * (1+X)^ (lambdaN(m)), 0));

\\ Returns the lambda-invariant of the beta-branch
laminv(beta) = {
for(i=0, 100, if(valuation(coeffappr(i, beta),pbase)==0, return(i-valuation(gcd(pbase,2^
(beta+1)-1),pbase) ); break));
}
```

Program B

```
\\ Default parameters
pbase = ;        \\ the prime p in p-adic
laminv = ;
CapK = 11;
littleK = floor(CapK/laminv);
acc = CapK+1;        \\ number of p-adic places to use

\\ Basic functions
ord_p(n) = valuation(n,pbase);
padic(x) = x + O(pbase^ (acc));

\\ Coefficients of f(T), to be inputted by hand, unfortunately
a = [padic(0), padic(0), padic(0), padic(0), padic(0), padic(0), padic(0), padic(0),
padic(0), padic(0), padic(0), padic(0), padic(0)];

\\ Work out the coefficients b_n modulo p
bmodp = vector(CapK-laminv+1);
bmodp[1] = Mod(1/a[1+laminv],pbase);
for(s=1, CapK-laminv, bmodp[s+1] = Mod(-1/a[1+laminv],pbase) * Mod(sum(i=1,
s, a[1+laminv+i] * bmodp[1+s-i], 0), pbase));

\\ Work out the characteristic zero coefficients b_n
bvecp = vector(CapK-laminv+1);
for(i=0, CapK-laminv, bvecp[i+1] = padic(lift(bmodp[i+1])) );

\\ Iteratively compute better and better bvecp's
for(X=1, CapK, bvecp[1] = (1/a[1+laminv]) * (1-sum(j=1, laminv, a[1+laminv-
j]*bvecp[1+j], 0) ); for(s=1, CapK-2*laminv, bvecp[1+s] = (-1/a[1+laminv]) *
(sum(i=0, laminv+s, a[1+i]*bvecp[1+laminv+s-i], 0) - a[1+laminv]*bvecp[1+s])
); );

\\ Compute c's from the a's and b's
cvecp = vector(1+laminv);
cvecp[1+laminv] = 1 +O(pbase^ (littleK));
for(n=0, laminv-1, cvecp[n+1] = sum(i=0, n, a[i+1]*bvecp[n-i+1], 0)+O(pbase^
(littleK)) );
```

Bibliography

[1] Childress, N., Gold, R.(1985). Zeros of p-adic L-functions. Zaklad Metod Numerycznych, 63-71.

[2] Delbourgo, D. (2006). A Dirichlet series expansion for the p-adic zeta-function. J. Aust.Math.Soc, 81, 215-224.

[3] Delbourgo, D. (2008). The convergence of Euler products over p-adic number fields. Proceedings of the Edinburgh Mathematical Society, 52, 583-606.

[4] Ellenberg, J.S., Jain,S., Venkatesh, A. Modelling λ-invariant by p-adic random matrices. 1-19.

[5] Ernvall,R., Metsankyla,T. (1992). Computation of the zeros of p-adic L-functions. Mathematics of computation, 58,815-830.

[6] Iwasawa, K. (1972). Lectures on p-adic L-functions. New Jersey: Princeton.

[7] Koblitz, N. (1980). p-adic analysis: a short course on recent work. Cambridge: the Press Syndicate of the University of Cambridge.

[8] Koblitz, N.(1977). p-adic numbers, p-adic analysis, and zeta-functions. New York: Springer-Verlag.