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ANALYTICAL DESCRIPTION OF STEADY MAGNETIC RECONNECTION IN HALL MAGNETOHYDRODYNAMICS

Yuri E. Litvinenko[1]
Institute for the Study of Earth, Oceans, and Space, University of New Hampshire, Durham, NH 03824-3525, USA
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ABSTRACT

Steady magnetic reconnection in the framework of incompressible Hall magnetohydrodynamics is considered. The principal role of the Hall effect in the formation of the structure of the reconnecting current sheet is emphasized. Analytical expressions for the velocity and the magnetic field in the sheet are derived, based on the approximation of a weak two dimensionality of the planar components of the solution. The analytical solution illustrates key features of Hall magnetic reconnection, including the reconnection rate enhancement and the sheet thinning due to the Hall effect, the presence of a quadrupolar axial (out-of-the-plane) magnetic field that controls the geometry of the reconnecting planar magnetic field, and the dynamical coupling of the axial and planar components of the solution, with the coupling strength that is proportional to the ion skin depth. Scalings for the sheet thickness, width, and the reconnection inflow and outflow speeds in terms of the electric resistivity and the axial magnetic field are determined. Implications of the results for fast magnetic reconnection in a weakly collisional plasma of the solar corona are discussed.

Key words: MHD – plasmas – Sun: flares – Sun: magnetic fields

1. INTRODUCTION

In many astrophysical situations where fast magnetic reconnection is believed to occur, resistive reconnection timescales are too slow to explain the observed energy release times. In addition, the formal thickness of the resistive current layer, predicted in the traditional Sweet–Parker model (Parker 1957), is comparable to kinetic scales such as the ion skin depth. These facts strongly motivate the inclusion of collisionless effects in resistive magnetohydrodynamic (MHD) models for reconnection (e.g., Sonnerup 1979).

Numerical studies show that the Hall term in the generalized Ohm's law may play a key role in rapid magnetic reconnection. As long as the Hall effect is taken into consideration, simulations predict enhanced reconnection rates that appear to be virtually independent of other model assumptions and the codes employed (Birn et al. 2001, 2005). Hence, Hall MHD can provide the simplest physical model for describing fast magnetic reconnection in weakly collisional plasmas. Applications of Hall MHD reconnection are believed to include magnetotail substorms and solar flares (see, e.g., Bhattacharjee 2004, for a review).

Recent numerical studies focus on the dynamics and energetics of Hall MHD reconnection in various magnetic geometries (e.g., Bhattacharjee et al. 2005; Cassak et al. 2006; Craig & Litvinenko 2008). More detailed kinetic studies extend and test predictions of Hall MHD models by exploring the structure of the electron dissipation region and the associated electron outflow jets (e.g., Daughton et al. 2006; Drake et al. 2008).

Analytical models can help in developing some insight into how fast reconnection occurs in weakly collisional plasmas. Of particular interest would be interpretation of space and laboratory observations that have already revealed distinct features of Hall magnetic reconnection (Mozer et al. 2002; Ren et al. 2005). Although exact steady solutions for Hall MHD reconnection are available, they are limited to one-dimensional current sheets (Dorelli 2003; Craig & Watson 2003, 2005). The collapse to a one-dimensional current sheet in Hall MHD can be described by a time-dependent self-similar solution (Litvinenko 2007).

The goal of this paper is to develop analytical solutions for steady magnetic reconnection in Hall MHD, which do not rely on the assumption of one dimensionality of a current sheet. The problem is made amenable to analytical treatment by making the assumption of weak two dimensionality of the planar velocity and magnetic field components within the sheet. Previously the method had been successfully used to describe the structure of a reconnecting current sheet in standard resistive MHD (Biskamp 1986; Jamitzky & Scholer 1995).

2. MHD EQUATIONS AND THE ROLE OF THE HALL TERM

An incompressible, nonviscous, magnetized plasma is considered. Throughout the paper, magnetic fields, lengths, and plasma densities are normalized to reference values B_0, L, and n. Velocities are scaled by the Alfvén speed v_A, and times are scaled by the Alfvén time L/v_A. For example, $B_0 = 10^2$ G, $L = 10^{9.5}$ cm, $n = 10^9$ cm^{-3}, and $v_A = 10^9$ cm s^{-1} can be adopted for a typical solar active region.

The governing equations are those of Hall MHD in dimensionless form. Specifically, the plasma velocity **v** and the magnetic field **B** are found by solving the momentum equation

$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p + \mathbf{J}\times\mathbf{B}, \quad (1)$$

Ohm's law

$$\mathbf{E} + \mathbf{v}\times\mathbf{B} = \eta\mathbf{J} + d_i(\mathbf{J}\times\mathbf{B} - \nabla p_e), \quad (2)$$

the continuity equation

$$\nabla\cdot\mathbf{v} = 0, \quad (3)$$

and Maxwell's equations

$$\nabla\times\mathbf{E} = -\partial_t\mathbf{B}, \quad \nabla\cdot\mathbf{B} = 0. \quad (4)$$

[1] Current address: Department of Mathematics, University of Waikato, P.O. Box 3105, Hamilton, New Zealand.

Here, $\mathbf{J} = \nabla \times \mathbf{B}$ is the electric current density, and p and p_e are the total and the electron plasma pressures, respectively. It is assumed for simplicity that the electron pressure tensor can be approximated with the diagonal term, so that its curl vanishes.

Resistive effects are controlled by the dimensionless resistivity $\eta = c^2/(4\pi v_A L\sigma)$, based on the Spitzer conductivity σ. The relative effect of the Hall term is quantified by the dimensionless ion skin depth, $d_i = c/(L\omega_{pi})$, based on the proton plasma frequency $\omega_{pi} = (4\pi ne^2/m_i)^{1/2}$. The adopted reference parameter values in a million-degree plasma of the solar corona lead to $\eta \simeq 10^{-14.5}$ and $d_i \simeq 10^{-6.5}$. The Hall term effects become significant when $\tau {d_i}^2/\eta \geqslant 1$, where τ is a typical dynamical timescale (e.g., Craig & Litvinenko 2008). This condition is easily satisfied in the solar corona where ${d_i}^2/\eta \simeq 10^{1.5} \gg 1$, even if the plasma evolves very rapidly, say on the Alfvénic timescale $\tau \simeq 1$.

In what follows, z is assumed to be an ignorable coordinate. This allows us to identically satisfy the constraints $\nabla \cdot \mathbf{v} = 0$ and $\nabla \cdot \mathbf{B} = 0$ by introducing the stream function representation for the plasma velocity,

$$\mathbf{v}(x, y, t) = \nabla\phi \times \hat{\mathbf{z}} + W\hat{\mathbf{z}}, \tag{5}$$

and the flux function representation for the magnetic field,

$$\mathbf{B}(x, y, t) = \nabla\psi \times \hat{\mathbf{z}} + Z\hat{\mathbf{z}}. \tag{6}$$

In a steady case, the resulting $2\frac{1}{2}$-dimensional Hall MHD equations are as follows:

$$[\nabla^2\phi, \phi] = [\nabla^2\psi, \psi], \tag{7}$$

$$-E + [\psi, \phi] = \eta\nabla^2\psi + d_i[\psi, Z], \tag{8}$$

$$[W, \phi] = [Z, \psi], \tag{9}$$

$$[Z, \phi] = \eta\nabla^2 Z + [W, \psi] + d_i[\nabla^2\psi, \psi] \tag{10}$$

(e.g., Craig & Watson 2005). Here, $\mathbf{E} = E\hat{\mathbf{z}}$ is a steady uniform electric field and the Poisson brackets are typified by $[\psi, \phi] = \partial_x\psi\partial_y\phi - \partial_y\psi\partial_x\phi$.

Dorelli (2003) realized that magnetic merging in Hall MHD, driven by the stagnation-point flow $\phi = -u_0xy$, can be analytically described by setting $\psi = \psi(x)$, $Z = h_0xy$, and $W = W(x)$ (see also Craig & Watson 2005). Here, u_0 is a dimensionless inflow speed at the boundary, and the axial magnetic field Z reflects the quadrupolar pattern of the axial field, which is believed to be an important signature of reconnection in Hall MHD (e.g., Sonnerup 1979; Uzdensky & Kulsrud 2006). Now the momentum and induction equations reduce to ordinary differential equations that are solved to give $W(x) = (h_0/u_0)\psi(x)$ and the merging magnetic field

$$B_y(x) = -\frac{d\psi}{dx} = \frac{E}{\eta\mu}\,\mathrm{daw}(\mu x). \tag{11}$$

Here, $\mathrm{daw}(x) = \int_0^x \exp(s^2 - x^2)ds$ is the Dawson function, and $\mu^2 = (u_0 + d_ih_0)/2\eta$. Dorelli's solution is a Hall MHD generalization of the well-known flux pile-up merging in resistive MHD (Parker 1973; Sonnerup & Priest 1975; Craig & Henton 1995).

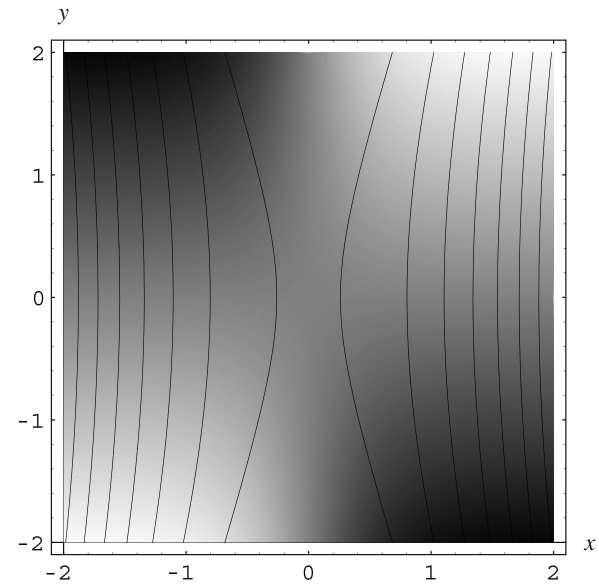

Figure 1. Magnetic field geometry in Hall reconnection. The reconnection inflow is along the x-axis. The reconnecting current sheet is in the $x = 0$ plane. The electric current is along the z-axis. The shading indicates the magnitude of the quadrupolar axial (out-of-the-plane) magnetic field.

The solution identifies the current sheet half-thickness

$$l = \mu^{-1} = \left(\frac{2\eta}{u_0 + d_ih_0}\right)^{1/2}. \tag{12}$$

Therefore, as emphasized by Dorelli (2003), the Hall effect leads to thinner current sheets when the orientation of the axial magnetic field is such that $d_ih_0 > 0$. For a fixed magnetic field at the entrance to the sheet, $B_y(l)$, thinner sheets correspond to faster plasma inflows and larger merging rates. Thus, the model can describe the speed-up of reconnection due to the Hall effect.

Dorelli's solution shows that the Hall effect should have a strong impact on magnetic reconnection. The solution, however, is limited in two important respects. First, generalizations that describe two-dimensional reconnecting planar magnetic fields do not appear possible, limiting the method to the description of merging of straight magnetic field lines in an infinitely long current sheet. As a result, the analytical description does not allow proper coupling of the planar flow and the axial magnetic field. Second, as pointed out by Craig & Watson (2005), a significant modification of the resistive solution due to the Hall effect occurs only if

$$h_0 \geqslant \frac{u_0}{d_i}. \tag{13}$$

Because $d_i \ll 1$, an unrealistically large axial magnetic field $B_a = h_0$ is required in order to model fast reconnection, characterized by the inflow speed that is a significant fraction of the Alfvén speed, so that u_0 approaches unity. Therefore, only minor Hall-effect modifications of the resistive reconnection regime can be described by the solution.

The coupling of the reconnection flow and the axial magnetic field, controlled by the Hall term, as well as the resulting qualitative change in the geometry of the reconnection site, can be illustrated by an exact self-similar solution that describes a hyperbolic planar magnetic field (Figure 1), driven by a stagnation-point plasma flow in the vicinity of a magnetic neutral line:

$$\psi = \alpha x^2 - \beta y^2, \tag{14}$$

$$\phi = -\gamma xy. \tag{15}$$

The corresponding axial velocity and magnetic field are given by

$$W = fx^2 + gy^2, \tag{16}$$

$$Z = hxy. \tag{17}$$

These forms can be used to describe the time-dependent process of current sheet formation. For certain initial conditions, the solution exhibits a finite-time singularity that describes the collapse to a current sheet (Litvinenko 2007).

In a steady case, the forms above do not result in a nontrivial solution in the limit $d_i = 0$. This corresponds to a well-known result for current sheet solutions in resistive MHD: magnetic field lines osculate rather than form a finite angle at the magnetic null, as long as the solution can be represented as a Taylor series (Cowley 1975; Yeh 1976). By contrast, exact steady X-point solutions ($\alpha(t) = \alpha_0$, etc.) are possible if $d_i \neq 0$, provided the following relationships among the parameters are satisfied:

$$-\alpha_0/f_0 = \beta_0/g_0 = -\gamma_0/h_0 = d_i, \tag{18}$$

$$E = 2\eta(\beta_0 - \alpha_0). \tag{19}$$

It is also straightforward to write down a formal infinite series solution that generalizes that in resistive MHD.

Although the exact quadratic solution strongly suggests that the Hall effect is of central importance in determining the reconnection geometry, it is only valid in the immediate vicinity of a neutral line. The solution has another significant weakness. It might seem that the form of the out-of-the-plane magnetic field $Z = h_0 xy$ near the neutral line has the observationally required quadrupolar pattern of the axial magnetic field. Yet Equation (18) demands that $h_0 < 0$. The sign of h_0 is inconsistent with the theoretically expected orientation of the axial magnetic field in Hall magnetic reconnection (see Figure 24 in Sonnerup 1979), which has also been detected in space (Mozer et al. 2002) and laboratory experiments (Ren et al. 2005; Brown et al. 2006).

Further progress in the analytical description of Hall MHD reconnection could be achieved with a solution that incorporates both a two-dimensional planar magnetic field (perhaps reducing to the field profile in Dorelli's solution in some limit) and a more general spatial dependence of the axial (out-of-the-plane) magnetic field. The solution should also describe the structure of a current sheet rather than just the immediate vicinity of a neutral line. A possible approach is presented in the remainder of this paper.

3. ANALYTICAL SOLUTION FOR THE RECONNECTING CURRENT SHEET STRUCTURE IN HALL MHD

Extending the analysis of magnetic reconnection in resistive MHD (Biskamp 1986; Jamitzky & Scholer 1995), consider the structure of a reconnecting sheet, located in the $x = 0$ plane (Figure 1). In what follows, the dimension of the sheet along the x-axis is referred to as its thickness $2l$, and the dimension of the sheet along the y-axis as its width $2w$. The key assumption in the analytical approach is that the structure of the sheet is quasi-one-dimensional. In other words, variations across the sheet are much stronger than those along it: $\frac{\partial}{\partial x} \gg \frac{\partial}{\partial y}$. The assumption makes it possible to search for a solution in the following form:

$$\psi = \psi_0(x) + \psi_2(x)y^2/2! + \cdots, \tag{20}$$

$$\phi = \phi_1(x)y + \phi_3(x)y^3/3! + \cdots, \tag{21}$$

$$W = W_0(x) + W_2(x)y^2/2! + \cdots, \tag{22}$$

$$Z = Z_1(x)y + Z_3(x)y^3/3! + \cdots. \tag{23}$$

A formal small parameter $\epsilon \sim \eta^{1/2} \sim d_i \sim E$ identifies a rescaled coordinate across the sheet, x/ϵ. In principle, the approach can be used to describe separator reconnection if $Z \neq 0$ at the origin, although this possibility is not pursued in what follows.

As discussed previously for the resistive case (Biskamp 1986; Jamitzky & Scholer 1995), the assumed weak two dimensionality of the solution formally means that only the leading-order terms in ϵ should be kept when the above forms are substituted into the steady Hall MHD equations of the previous section. For instance, $\nabla^2\psi \approx \psi_0'' + \psi_2'' y^2/2! + \cdots$ since $\psi_0'' \sim \epsilon^{-2}$ and thus $\psi_0'' \gg \psi_2$. The assumption of a weakly two-dimensional geometry is not valid, however, if the solution develops strong gradients along the sheet, which is the case near the magnetic X-point (see the previous section). Hence, the approximation breaks down in the vicinity of $x = y = 0$.

Now balancing the terms at equal powers of y leads to an infinite system of ordinary differential equations, the first few of which are as follows:

$$-E + \psi_0'\phi_1 = \eta\psi_0'' + d_i\psi_0' Z_1, \tag{24}$$

$$W_0'\phi_1 + \psi_0' Z_1 = 0, \tag{25}$$

$$\phi_1\phi_1''' - \phi_1'\phi_1'' = \psi_0'''\psi_2 - \psi_0'\psi_2'', \tag{26}$$

$$Z_1'\phi_1 - Z_1\phi_1' = \eta Z_1'' + W_0'\psi_2 - W_2\psi_0' + d_i(\psi_0'''\psi_2 - \psi_0'\psi_2''), \tag{27}$$

$$W_2'\phi_1 + W_0'\phi_3 - 2W_2\phi_1' + \psi_2' Z_1 + \psi_0' Z_3 - 2\psi_2 Z_1' = 0, \tag{28}$$

$$\psi_2'\phi_1 + \psi_0'\phi_3 - 2\psi_2\phi_1' = \eta\psi_2'' + d_i(\psi_2' Z_1 + \psi_0' Z_3 - 2\psi_2 Z_1'). \tag{29}$$

Once two functions, say $\psi_0(x)$ and $Z_1(x)$, are specified, terms up to a desired order in the series expansions can be successively calculated. It is worth noting that the exact Dorelli's solution (Dorelli 2003) can be recovered from the approximate equations above. When the leading-order terms, given by Dorelli's solution, are substituted into the infinite system, all other terms vanish. This happens because the current sheet is strictly one dimensional in the exact solution, which is consistent with the assumed quasi-one-dimensionality in the expansion scheme.

In order to determine a more general solution that reveals the influence of the Hall effect on magnetic reconnection, consider the following profile of the zero-order electric current density:

$$J_0 = \frac{E}{\eta}\operatorname{sech}^2\tilde{x}. \tag{30}$$

Here, a rescaled variable

$$\tilde{x} = \frac{x}{a_0\eta^{1/2}} \tag{31}$$

is used, where a parameter a_0 specifies the sheet half-thickness $l = a_0\eta^{1/2}$. This profile, originally introduced to describe a current sheet in a collisionless plasma (Harris 1962), later had been extensively used to model the current sheet structure in resistive MHD (Biskamp 1986). Hall MHD effects will be formally manifested as a dependence of a_0 on d_i. The zero-order flux function, specified by J_0, is as follows:

$$\psi_0 = -a_0^2 E \ln(\cosh \tilde{x}). \qquad (32)$$

Notably, the assumed magnetic field profile has no flux pile-up. Hence, the problem of specifying the reconnecting magnetic field magnitude at the entrance to the current sheet, which is a recurring difficulty in all analytical models of flux pile-up merging (e.g., Litvinenko & Craig 2000; Dorelli 2003), simply does not occur. Hall MHD simulations show that flux pile-up is indeed reduced as the electric resistivity is decreased (Dorelli & Birn 2003).

The quadrupolar axial magnetic field, observed to be associated with Hall magnetic reconnection, motivates the following form for the leading-order axial field:

$$Z_1 = b_0 \tanh \tilde{x}. \qquad (33)$$

The selection of ψ_0 and Z_1 leads to a nontrivial solution for the current sheet structure in Hall MHD, which is characterized by the parameters a_0 and b_0. Note for clarity that, although in principle b_0 is a free parameter, $b_0 > 0$ corresponds to the physically correct orientation of the quadrupolar axial magnetic field, associated with Hall magnetic reconnection (Sonnerup 1979; Mandt et al. 1994). It should also be noted that the selected axial field profile Z_1 is not localized at $x = 0$. Hence, the analysis can lead to profiles of other quantities, which are not localized within the sheet either. The issue is further discussed in the Appendix. The focus of this paper though is on the properties of the current sheet rather than a global geometry.

Equation (24) gives the dimensionless inflow velocity profile:

$$v_x = \phi_1 = \left(d_i b_0 - \frac{\eta^{1/2}}{a_0}\right) \tanh \tilde{x}. \qquad (34)$$

The formula clearly demonstrates the coupling of the planar plasma flow and the axial magnetic field. The requirement that the inflow velocity profile matches an ideal solution outside the sheet, $\phi_1 = -E \tanh \tilde{x}$, defines the parameter

$$a_0 = \frac{\eta^{1/2}}{E + d_i b_0} \qquad (35)$$

and hence the current sheet half-thickness

$$l = \frac{\eta}{E + d_i b_0}. \qquad (36)$$

The speed v_o of the reconnection outflow (along the y-axis) is determined by evaluating $-\partial_x \phi(x, y)$ at the point $x = 0, y = w$. The leading-order result is as follows:

$$v_o = \frac{w}{l} E. \qquad (37)$$

Here, w is the sheet half-width.

An advantage of the two-dimensional solution of this paper is that, in contrast to the one-dimensional Dorelli's solution, the present analysis imposes a much weaker requirement on the magnetic field magnitude B_a outside the sheet in order for the Hall effect to be significant. If the axial field gradient is estimated as $b_0 \simeq B_a/w$, then the Hall effect significantly modifies the resistive solution if

$$B_a \geqslant \frac{w}{d_i} E_0, \qquad (38)$$

where E_0 is a field that would be predicted by a purely resistive solution. Typically, E_0 is on the order of $\eta^{1/2}$ as long as the resistive model incorporates an extended Sweet–Parker reconnecting current sheet (e.g., Litvinenko & Craig 2000 and references therein). It is important that the sheet half-width w in Hall MHD reconnection is not defined by the global length scale (e.g., Birn et al. 2001; Cassak et al. 2006). As a result, $b_0 \gg 1$ is possible when $w \ll 1$, notwithstanding $B_a \simeq 1$. By contrast, $B_a \simeq 1$ leads to $h_0 \simeq 1$ in Dorelli's solution. Hence, Equation (38) is much easier to satisfy than Equation (13).

Qualitatively, Equation (36) is similar to Dorelli's result, given by Equation (12). Both describe the thinning of a current sheet due to the Hall effect, which sustains the fast reconnection rate. The key point, however, is that the present result for the thickness of a resistive reconnection layer agrees quantitatively with scaling arguments and Hall MHD numerical simulations, presented by Wang et al. (2001). Assuming that the reconnection electric field E is small enough, Equation (36) reduces to the scaling

$$l \simeq \frac{\eta w}{d_i B_a}. \qquad (39)$$

This is essentially Equation (14) in Wang et al. (2001) in a slightly different notation. In particular, the key scaling $l \sim \eta$ was confirmed by Hall MHD simulations (Wang et al. 2001). Thus, an enhanced reconnection rate in Hall MHD is associated with a narrow profile of the electric current density J_z, defined by resistivity. It is worth emphasizing that Equation (36) results from an explicit analytical solution for the structure of the sheet rather than from less rigorous scaling arguments.

Now the leading-order term for the axial flow speed follows from Equation (25):

$$W_0 - W_0(0) = \frac{a_0^3 b_0 E}{(d_i a_0 b_0 - \eta^{1/2})} \ln(\cosh \tilde{x}). \qquad (40)$$

This expression is written for the general case of arbitrary a_0 and b_0. In the physically interesting case of an externally imposed reconnection electric field E, specified by Equation (35), the expression for the axial flow profile simplifies somewhat

$$W_0 - W_0(0) = -\frac{\eta b_0}{(E + d_i b_0)^2} \ln(\cosh \tilde{x}). \qquad (41)$$

Note that the speed increases linearly with x outside the current sheet. The result that the profile W_0 is not localized at $x = 0$ can be traced to the assumed form of the axial magnetic field Z_1. The issue is further discussed in the Appendix.

As an interesting aside, note that the solution with the assumed Z_1 and the corresponding W_0 does not vanish in the limit $d_i = 0$. Thus, the quadrupolar axial magnetic field can be realized already in traditional resistive MHD, and the quadrupolar structure of the out-of-the-plane field by itself is not a unique feature of Hall MHD reconnection. This result is in agreement with previous arguments (Bian & Vekstein 2007).

The next step is to determine the y-dependent correction to the flux function ψ. As in the corresponding resistive MHD case (Biskamp 1986; Jamitzky & Scholer 1995), Equation (26) is integrated to give an expression for $\psi_2(x)$. The expression contains two constants. One of them follows from the symmetry constraint $B_y = 0$ at $x = 0$. The other constant is determined from the requirement that ϕ_3 be nonsingular. The procedure, while straightforward, turns out to be more time-consuming than that for the resistive current sheet. First, the axial flow component W_2 is determined from Equation (27) in terms of ψ_2:

$$W_2 = \frac{2\eta^{1/2}}{a_0^3 E}\mathrm{sech}^2\tilde{x}\left[b_0 - \frac{2d_i}{a_0^3\eta^{1/2}}\left(1 - \frac{d_i a_0 b_0}{\eta^{1/2}}\right)^2 \tanh^2\tilde{x}\right] + \frac{a_0 b_0}{(\eta^{1/2} - d_i a_0 b_0)}\psi_2(x). \quad (42)$$

Next, Equations (28) and (29) are used to eliminate Z_3 and obtain an equation for ϕ_3 in terms of ψ_2 and W_2. Finally, ψ_2 and W_2 are substituted into the resulting equation and the condition that ϕ_3 remain nonsingular at $x = 0$ is employed to determine the sought-after constant. The resulting expression for ψ_2 is as follows:

$$\psi_2 = \frac{\eta}{a_0^4 E}\left(1 - \frac{d_i a_0 b_0}{\eta^{1/2}}\right) \times \left[\tilde{x}\tanh\tilde{x} - \left(1 - \frac{d_i a_0 b_0}{\eta^{1/2}}\right)\tanh^2\tilde{x} - \frac{d_i a_0 b_0}{\eta^{1/2}}\right]. \quad (43)$$

This completes the calculation of the leading y-dependent terms (up to and including $\sim y^2$) in the steady Hall MHD solution for the reconnecting current sheet structure.

Recall that Equation (43) in the limit $d_i = 0$ should reduce to the previously obtained resistive MHD solution. It appears that there is a factor of 2 difference with Equation (23) in Biskamp (1986) and complete agreement with Equation (55) in Jamitzky & Scholer (1995).

For small x, the lowest-order terms in Equations (32) and (43) give the following expression for the flux function $\psi(x, y)$ in the current sheet:

$$\psi \approx -\frac{E}{2\eta}x^2 - \frac{d_i b_0}{2a_0^3 E}(\eta^{1/2} - d_i a_0 b_0)y^2. \quad (44)$$

Using Equation (35) leads to the following expression for the flux function in the sheet:

$$\psi \approx -\frac{E}{2\eta}\left[x^2 + \frac{d_i b_0}{E}(E + d_i b_0)^2 y^2\right]. \quad (45)$$

The flux function corresponds to a magnetic O-point at the origin in the case $b_0 > 0$. Consistent with the quadratic-form solution of the previous section, X-point solutions are also formally possible if the axial field parameter $b_0 < 0$.

The O-point structure of the solution is an unexpected result. One might argue that the solution of this paper describes the local structure of a current sheet rather than a global reconnection geometry, and therefore the O-point is not inconsistent with a global magnetic geometry of Hall MHD reconnection, characterized by a background X-point and a current sheet at the origin (Wang et al. 2001; Bhattacharjee 2004). Yet, as argued by the referee, the X-point topology is a local signature, which is why one cannot help suspecting that the result is a mathematical artifact. The key point, however, is that the solution in the limit $d_i = 0$ predicts the magnetic field lines to osculate. This is a well-known result in standard resistive MHD: Taylor series expansions at a magnetic null show that the magnetic field lines osculate rather than cross at a finite angle in two-dimensional reconnection (Cowley 1975; Yeh 1976). Several attempts had been made to either confirm or disprove this counterintuitive result in resistive MHD (e.g., Biskamp 1994; Heerikhuisen et al. 2000 and references therein), yet the resolution of the problem still lies deeper than the plummet of our intelligence can sound. The Hall MHD solution of this paper is a straightforward extension of the resistive MHD solution, and it leads to an even more unexpected result–an O-point at the origin. More detailed analytical or numerical studies of the reconnecting current sheet structure are needed to determine whether the O-point structure can be realized physically.

The electric current density in the sheet can be calculated using the analytical expressions for ψ_0 and ψ_2. The resulting expression for the reconnection-related axial component J_z of the current density is as follows:

$$J_z(0, y) = \frac{E}{\eta}\left(1 - \frac{y^2}{w^2}\right), \quad (46)$$

where

$$\frac{w}{l} = \frac{1}{(E + d_i b_0)}\left(\frac{E}{d_i b_0}\right)^{1/2}. \quad (47)$$

This equation illustrates the tendency for current localization in Hall MHD, well established in numerical studies (see Bhattacharjee 2004 and references therein). Clearly, w can be identified with the half-width of the sheet. Thus, in contrast to resistive reconnection solutions (Biskamp 1986; Jamitzky & Scholer 1995), a localization mechanism is present in Hall MHD ($d_i > 0$), which leads to $w < 1$ (or $w < L$ in dimensional units).

Finally, Equations (37) and (47) can be combined to relate the outflow speed and the reconnection electric field:

$$v_o = \frac{E}{(E + d_i b_0)}\left(\frac{E}{d_i b_0}\right)^{1/2}. \quad (48)$$

Although the reconnection rate remains undetermined, it is worth pointing out that Alfvénic outflows ($v_o \simeq 1$) would imply dimensionless electric fields on the order of $d_i b_0$. It was argued that sheet widths can be as small as $10d_i$ in the Hall MHD reconnection regime (e.g., Birn et al. 2005; Cassak et al. 2006). If the axial field gradient is estimated as $b_0 \simeq B_a/w$ and the field outside the sheet $B_a \simeq 1$, then $w \simeq 10d_i$ implies fast reconnection, characterized by $E \simeq d_i/w \simeq 0.1$.

4. DISCUSSION

The relatively simple analytical approach of this paper appears to capture some key features of Hall MHD reconnection. As in previous analytical studies (Dorelli 2003; Craig & Watson 2005), the solution displays a resistive small scale, incorporates a quadrupolar axial magnetic field, and describes the reconnection rate enhancement due to the sheet thinning by the Hall effect. In contrast to the previous studies, based on the assumption of a one-dimensional current sheet, the present model allows us to make significant further progress by explicitly describing a two-dimensional reconnecting current sheet of a finite width. As a result, the model reveals dynamical coupling of the axial and planar components of the solution with the coupling strength that is proportional to the ion skin depth. This allows us to quantify the dependence of the reconnection inflow speed and the sheet thickness on the resistivity and the axial magnetic field, which agrees with numerical results (Wang et al. 2001).

Two parameters appear in the model of this paper. As Equations (35) and (36) show, the parameter a_0 controls the sheet thickness $2l$. The essential point is that the model remains formally valid for any given reconnection electric filed E. In particular, a resistivity-independent E can be specified. This is in sharp contrast to traditional resistive reconnection models of Sweet–Parker type, in which the electric field is limited by resistivity. Physically, the Hall effect provides a mechanism for adjusting the parameters of the current sheet to match the electric field that can be independently specified by the boundary conditions in the ideal inflow region. Thus, the sheet thinning is responsible for the resistivity-independent reconnection rate in Hall MHD (see also Wang et al. 2001).

The sheet thickness $2l$ is the smallest length scale in the Hall MHD reconnection model, which determines the validity range of the model. For the parameters adopted in Section 2 and the dimensionless half-width w of the sheet in the range between $10d_i$ and 1, Equation (39) gives l in the range between $10^{-13.5}/B_a$ and $10^{-8}/B_a$. The key point is that, unless the axial field $B_a \leqslant 1$ is significantly weakened at the entrance to the sheet, the formal thickness is likely to fall below the electron skin depth $d_e = c/(L\omega_{pe}) \simeq 10^{-8}$ for typical parameters of a solar active region. Physically, when $l \simeq d_e$, Hall MHD can no longer adequately describe the structure of the sheet, and electron-inertia effects must be taken into consideration (e.g., Craig & Watson 2003). Alternatively, if the threshold for a current-driven instability is exceeded, turbulent, anomalous resistivity in the sheet should be incorporated into the model (e.g., Litvinenko & Craig 2000). Thus, the analysis of this paper shows that the length scale, associated with the Hall effect and classical electric resistivity, is likely to be so small that Hall MHD cannot give a compete description of magnetic reconnection in the flaring solar corona. In spite of this limitation, the present model appears to provide some valuable insights into the mechanism of collisionless reconnection. The corrections that should result from the required modifications is an interesting topic for further study.

The parameter b_0 defines the local magnetic geometry near the neutral line, described by Equation (44). The case $b_0 > 0$ agrees with the pattern of the out-of-the-plane magnetic field, identified in both laboratory and space observations of collisionless reconnection (Mozer et al. 2002; Ren et al. 2005; Brown et al. 2006). In this case, the solution for the planar magnetic field in the sheet corresponds to electric current localization at the intersection of two separatrices, which has been well established in numerical simulations of reconnection in Hall MHD (Bhattacharjee 2004). The current localization, quantified by Equation (47) for the aspect ratio of the sheet, shows that the sheet width is not determined by a global length scale, in agreement with previous arguments (e.g., Birn et al. 2005; Cassak et al. 2006).

It is less clear whether a solution with $b_0 < 0$ can be physically meaningful. One interesting possibility is that a formal solution for the current sheet structure with $b_0 < 0$ describes a global reconnection solution characterized by an octupolar planar magnetic field, as suggested by Bulanov et al. (1992).

Finally, although the analysis of this paper resulted in formulas for the reconnection inflow and outflow speeds and the current sheet dimensions (see Equations (34), (36), (47), and (48)), it should be remembered that the new analytical results rely on the assumed forms of the leading-order profiles ψ_0 and Z_1. As discussed in the Appendix, a more realistic Z_1 can lead to modified reconnection scalings. While more work is clearly necessary, the analytical approach shows promise for interpreting observations of magnetic reconnection and guiding new experimental and numerical studies. Accurate estimates of the sheet thickness would also be of great interest in a more general context of self-regulated coronal heating (Cassak et al. 2008).

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APPENDIX

SOLUTION WITH A LOCALIZED AXIAL MAGNETIC FIELD

The technique used in this paper allows, at least in principle, to derive analytical perturbative Hall MHD reconnection solutions, once two lowest-order terms in the expansion are specified. The particular choice employed in the paper is motivated both by physical considerations and analytical tractability. Whereas the Harris (1962) profile ψ_0 for the zero-order current sheet should be a reasonable choice under quite general conditions, the assumed form of the axial magnetic field is at variance with numerical simulations and laboratory and magnetospheric observations. The data suggest that the axial magnetic field is localized at the reconnection site. Obviously, the assumed profile, $Z_1 = b_0 \tanh \tilde{x}$, does not have this property.

As pointed out by the referee, a more realistic profile would be

$$Z_1 = b_0 \tanh \tilde{x}\, \mathrm{sech}^2 \tilde{x} \qquad \text{(A1)}$$

that goes to zero at large distances from the current sheet. Unfortunately, the mathematical complexity of the technique would increase significantly. This is especially true for the calculation of ψ_2, which is required to determine the width of the sheet. If the complete calculation turns out to be possible, it will be presented elsewhere. Nevertheless, the sheet thickness and the reconnection inflow profile follow already from the first-order calculation, and it is worthwhile to present these results here.

As in the paper, Equation (24) is solved to give the inflow velocity profile for Z_1 above. The resulting velocity profile is as follows:

$$v_x = \phi_1 = (-E + d_i b_0\, \mathrm{sech}^2 \tilde{x}) \tanh \tilde{x}. \qquad \text{(A2)}$$

This result should be compared with the inflow profile $\phi_1 = -E \tanh \tilde{x}$, based on the nonlocalized Z_1. Interestingly, Equation (A2) shows that the Hall MHD reconnection electric field should exceed $d_i b_0$ in order to ensure the reconnection inflow without a velocity reversal and the reconnection outflow. The solution in the paper led to a similar scaling only when an additional assumption of the Alfvénic outflow was made.

The expression for the current sheet half-thickness

$$l = \frac{\eta}{E} \qquad \text{(A3)}$$

replaces Equation (36). Both formulas describe the thinning of a current sheet, which is necessary to sustain rapid reconnection inflow. As long as E scales as $d_i b_0$, the new expression for l agrees with the numerical result (39), obtained by Wang et al. (2001).

As in the paper, the axial flow velocity is determined from Equation (25). The assumed form of a localized Z_1 leads to the following axial flow profile:

$$W_0 - W_0(0) = -\frac{\eta}{2d_i E} \ln\left(\frac{1 - d_i b_0 \operatorname{sech}^2\tilde{x}/E}{1 - d_i b_0/E}\right). \qquad \text{(A4)}$$

In contrast to the velocity profile described by Equation (41), based on the nonlocalized Z_1, the modified solution does not lead to unlimited axial velocities far from the current sheet.

Thus, the first-order analysis, based on Equation (24) and a localized profile of the axial magnetic field, confirms some of the conclusions reached in the paper. At the same time, the more realistic magnetic geometry leads to a solution, characterized by new interesting features, justifying further investigation.

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