

Maximal partial Latin cubes

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Abstract

We prove that each maximal partial Latin cube must have more than 29.289% of its cells filled and show by construction that this is a nearly tight bound. We also prove upper and lower bounds on the number of cells containing a fixed symbol in maximal partial Latin cubes and hypercubes, and we use these bounds to determine for small orders n the numbers k for which there exists a maximal partial Latin cube of order n with exactly k entries. Finally, we prove that maximal partial Latin cubes of order n exist of each size from approximately half-full ($n^3/2$ for even $n \geq 10$ and $(n^3 + n)/2$ for odd $n \geq 21$) to completely full, except for when either precisely 1 or 2 cells are empty.

1 Introduction

The famous theorem known as the Evans Conjecture, posed by Evans [7] and proved independently by Anderson and Hilton [1], Häggkvist [9], and Smetaniuk [15], states that it is possible to complete each partial Latin square of order n with at most $n - 1$ elements. In contrast, simple examples such as

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show that it is not always possible to complete such squares if they contain n or more elements. A partial Latin square like the one above is *maximal* if none of its empty cells

can be filled. A natural question arises: how many elements may there be in a maximal partial Latin square? This question was answered almost completely in [11].

In the present paper, we investigate an analogous question, namely that of determining the possible numbers of elements in maximal partial Latin *cubes*.

We prove that each maximal partial Latin cube of order n must have more than $(1 - 1/\sqrt{2})n^3 > 0.29289n^3$ of its n^3 cells filled (cf. Theorem 4) and we show by construction that this is close to being a tight bound (cf. Theorem 15). We further prove upper and lower bounds on the number of entries of a fixed symbol, or in a fixed layer, in maximal partial Latin cubes, and we generalise these bounds with respect to maximal partial Latin *hypercubes* (cf. Theorem 10). Using these bounds, we determine for orders 2, 3 (and almost for order 4), which numbers k there exists a maximal partial Latin cube with exactly k entries (cf. Theorem 11). We also prove that maximal partial Latin cubes of order n with each size from approximately half-full ($n^3/2$ for even n and $(n^3 + n)/2$ for odd n) to completely full exist (for $n \geq 21$), except for when either precisely 1 or 2 cells are empty (cf. Theorems 28 and 30).

2 Preliminaries

Let $[n] = \{1, 2, \dots, n\}$. A *partial Latin square* of order n is a set L of ordered triples from $[n]^3$ such that if (r, c, s) and (r', c', s') are distinct elements of L , at most one of the following is true: $r = r'$; $c = c'$; $s' = s$. We typically represent a Latin square as an $n \times n$ array, with *symbol* s in *row* r and *column* c if and only if $(r, c, s) \in L$. Each symbol then occurs at most once per row and at most once per column. A partial Latin square with n^2 elements is a *Latin square*.

We may extend Latin squares by an extra dimension in the following manner. A *partial Latin cube* L of order n is a subset of $[n]^4$ such that if (ℓ, r, c, s) and (ℓ', r', c', s') are distinct elements of L , then at most two of the following are true: $\ell = \ell'$; $r = r'$; $c = c'$; $s = s'$. Here, ℓ is the *layer*; r is the *row*; c is the *column*; and s is the *symbol*.

We typically represent a partial Latin cube as a set of partial Latin squares (layers) such that no symbol occurs in the same row and column of two distinct layers. Thus the set $L_\ell := \{(r, c, s) \mid (\ell, r, c, s) \in L\}$ is a partial Latin square for each $\ell \in [n]$; these partial Latin squares form the layers of L . If $(\ell, r, c, s) \in L$, then we may say that symbol s is in *cell* (ℓ, r, c) of L or that symbol s is in *cell* (r, c) of layer ℓ of L . In either case, we say that this cell is *filled*; conversely, if $(\ell, r, c, s) \notin L$ for all symbols s , then the cell (ℓ, r, c) is *empty*.

The *lines* of a partial Latin cube are obtained by fixing any two of the first three coordinates and allowing the remaining coordinate to vary. The lines obtained by fixing the second and third coordinates will be known as *stacks*. If the layers of the partial Latin cube are stacked directly on top of each other, then the stacks are the vertical columns formed.

Thus, a partial Latin cube can be seen as a 3-dimensional array with symbols occurring at most once in each line. A partial Latin cube of order n with size n^3 is a *Latin cube*.

A partial Latin square (cube) L of order n is said to be *maximal* if $L = L'$ whenever

L' is a partial Latin square (cube) of the same order n and $L \subseteq L'$. In other words, L is maximal if it is impossible to add any new elements to L . Here is a maximal partial Latin cube of order 2, represented by its two layers:

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| | 2 | 1 | |

In [11], the spectrum of sizes of maximal partial Latin squares is studied and determined with only a small range of possible exceptions. In this paper, we investigate the spectrum of sizes of maximal partial Latin cubes. However, we begin by summarizing results on maximal partial Latin squares, some of which we will use in later sections. First, we emphasize the values which are not in the spectrum.

Let $ML(n)$ be the set of integers t for which there exists a maximal partial Latin square of order n with size t (i.e. exactly t filled cells).

Theorem 1. [11] *Let t be an integer such that either:*

1. $t < \frac{1}{2}n^2$; or
2. $t = \frac{1}{2}n^2 + k$, $1 \leq k \leq \frac{1}{2}n$ where k is odd and n is even; or
3. $t = \lceil \frac{1}{2}n^2 \rceil + k$, $1 \leq k < \frac{1}{2}(n - 1)$ where k is odd and n is odd; or
4. $t = n^2 - 1$.

Then $t \notin ML(n)$.

Next we describe values which are known to be in the spectrum:

Theorem 2. [11] *Let t be an integer such that $\frac{1}{2}n^2 \leq t < n^2 - 1$ and $t \neq \lceil \frac{1}{2}n^2 \rceil + k$, where k is odd and $1 \leq k \leq n - 1$. Then $t \in ML(n)$.*

This leaves the following open problem.

Open problem [11] Determine whether $t \in ML(n)$, where:

1. $t = \frac{1}{2}n^2 + k$, n even, k odd, $\frac{1}{2}n < k \leq n - 1$;
2. $t = \frac{1}{2}(n^2 + 1) + k$, n odd, k odd, $\frac{1}{2}(n - 1) \leq k \leq n - 1$.

It is conjectured in [11] that the above values of t do *not* belong to the spectrum of possible sizes of maximal partial Latin squares.

In this paper, we examine the analogous question for partial Latin cubes and define $ML(3, n)$ to be the set of integers t such that there exists a maximal partial Latin cube of order n with exactly t filled cells. It turns out that, in contrast to maximal partial Latin squares, there exist maximal partial Latin cubes with much less than half of the cells filled (cf. Theorem 15). Conversely, $ML(3, n)$ contains all values between $n^3/2$ and $n - 3$ when n is even and $n \geq 10$ and all values between $(n^3 + n)/2$ and $n - 3$ when n is odd and $n \geq 21$ (cf. Theorems 28 and 30).

Related to the problem of constructing maximal partial Latin cubes is the question of whether a partial Latin cube is completable. We briefly summarize the literature in this area. A *Latin cuboid* is a partial Latin cube with each layer either completely full or empty. Clearly a Latin cuboid cannot be maximal unless it is a Latin cube. In the 1980s, several authors [8, 10, 12] considered the problem of constructing non-completable $n \times n \times (n - 2)$ Latin cuboids. Subsequently, Kochol [13] proved that for any k and n satisfying $\frac{1}{2}n < k \leq n - 2$, there is a non-completable $n \times n \times k$ Latin cuboid. In [2], constructions for non-completable $n \times n \times k$ Latin cuboids are given in which $k = \lfloor n/2 \rfloor$, for any n not congruent to 1 mod 4 and greater than 6. Little is known about the embedding of a partial Latin cube into a complete Latin cube of higher order. Cruse [5] showed that any $n \times n \times r$ Latin cuboid with $r \leq n$ can be embedded in a Latin cube of order n^3 . Denley and Öhman [6] prove sufficient conditions for when a $m \times \ell \times k$ *Latin box*, i.e., a partial Latin cube whose filled cells form a $m \times \ell \times k$ rectangular box, can be extended to a $n \times \ell \times k$ Latin box and when it can be extended to a $n \times n \times k$ Latin cuboid. They also proved a cube analogue of Evan's Conjecture by proving that any partial Latin cube of order n with at most $n - 1$ filled cells is completable under certain conditions on the relative positions of the filled cells.

3 Lower bounds

We begin with an elementary bound.

Proposition 3. *If $t \in ML(3, n)$, then $t \geq n^4/(4n - 3)$.*

Proof. If L is a maximal partial Latin cube with t elements, then $[n]^4$ is the union of the t following unions of lines associated to each $(\ell, r, c, s) \in L$:

$$\{(\ell', r, c, s) \mid \ell' \in [n]\} \cup \{(\ell, r', c, s) \mid r' \in [n]\} \cup \{(\ell, r, c', s) \mid c' \in [n]\} \cup \{(\ell, r, c, s') \mid s' \in [n]\}.$$

Each of these line unions contains $4n - 3$ elements, so $t(4n - 3) \geq n^4$. □

The above bound is achieved exactly when $n = 3$. In larger orders where, for example, symbols may appear twice within a layer, the bound cannot be achieved. The following theorem offers an improvement for sufficiently large values of n .

Theorem 4. *If $t \in ML(3, n)$, then $t > (1 - 1/\sqrt{2})n^3 > 0.29289n^3$.*

Proof. Let L be a maximal partial Latin cube with t filled cells. Let m be the least number of filled cells in a layer; then $t \geq nm$. Let ℓ be a fixed layer of L with exactly m filled cells and note that $m \geq n$ since each layer must trivially contain at least n filled cells. Let r_i (resp., c_i) be the number of filled cells in row i (resp., column i) of layer ℓ . Observe that $\sum_{i=1}^n r_i = \sum_{i=1}^n c_i = m$.

For each $i \in [n]$, let \mathcal{C}_i be the set of columns c such that cell (ℓ, i, c) is empty. Then $|\mathcal{C}_i| = n - r_i$. For each i , define α_i to be the number of filled cells in the columns $[n] \setminus \mathcal{C}_i$ of layer ℓ .

For each $c \in \mathcal{C}_i$, the maximality of L implies that there must be at least $n - r_i$ filled cells in the union of column c of layer ℓ and the stack with row i and column c . Thus, there are at least $(n - r_i)^2$ filled cells which occur in some column $c \in \mathcal{C}_i$ either within layer ℓ or within row i in some layer $\ell' \neq \ell$.

Since ℓ has m filled cells, exactly $m - \alpha_i$ filled cells are in the former category, so at least $(n - r_i)^2 - (m - \alpha_i)$ are in the latter category. Thus there are at least

$$\sum_{i=1}^n [(n - r_i)^2 - (m - \alpha_i)] = \sum_{i=1}^n [n^2 - 2nr_i + r_i^2 - m + \alpha_i] = n^3 - 3nm + \sum_{i=1}^n r_i^2 + \sum_{i=1}^n \alpha_i$$

filled cells not in layer ℓ , so

$$t \geq n^3 - 3nm + \sum_{i=1}^n r_i^2 + \sum_{i=1}^n \alpha_i + m.$$

Now consider the term $\sum_{i=1}^n \alpha_i$ in the above expression. Within layer ℓ a fixed column j intersects a row i for exactly c_j choices of i . Each time such an intersection occurs, there is a contribution of c_j towards the sum $\sum_{i=1}^n \alpha_i$ due to filled cells within column j . Thus each column j contributes c_j^2 to the sum $\sum_{i=1}^n \alpha_i$, so

$$t \geq n^3 - 3nm + \sum_{i=1}^n r_i^2 + \sum_{j=1}^n c_j^2 + m.$$

Since $\sum_{i=1}^n r_i = \sum_{j=1}^n c_j = m$, by the Cauchy-Schwarz inequality,

$$t \geq n^3 - 3nm + 2m^2/n + m. \tag{1}$$

In particular,

$$t > n^3 - 3nm + 2m^2/n. \tag{2}$$

Suppose, for the sake of contradiction, that $t < n^3(1 - 1/\sqrt{2})$. Since $nm \leq t$, we must have $m < n^2(1 - 1/\sqrt{2})$. By rearranging terms, we see that

$$n^2 \left(\frac{1}{\sqrt{2}} - \frac{1}{4} \right) < \frac{3n^2}{4} - m$$

and squaring each side yields the inequality

$$n^4 \left(\frac{9}{16} - \frac{1}{2\sqrt{2}} \right) < \left(\frac{3n^2}{4} - m \right)^2 = \frac{9}{16}n^4 - \frac{3}{2}n^2m + m^2.$$

By subtracting $n^4/16$ and doubling each side, we see that

$$nt < n^4(1 - 1/\sqrt{2}) < n^4 - 3n^2m + 2m^2.$$

This contradicts (2). □

Note that it is possible to sharpen the bound in Theorem 4 just slightly, by applying a Taylor series to the inequality (1) instead of (2) as in the proof:

$$t > n^3(1 - 1/\sqrt{2}) + n^2(\sqrt{2} - 1)/4 + n/(16\sqrt{2}) + 1/(32\sqrt{2}).$$

This does not however provide any asymptotic improvement.

4 Bounds on the number of cells of a given symbol

We can also find a lower bound on the number of cells filled with a given symbol. Consider a maximal partial Latin cube L of order n and let χ_1, \dots, χ_n denote the number of cells in L filled with symbols $1, \dots, n$, respectively, and assume without loss of generality that $\chi_1 \geq \dots \geq \chi_n$.

Proposition 5. *If $t \in ML(3, n)$ and $j := \left\lceil \frac{n^3 - t}{3(n-1)} \right\rceil$, then for each $i \in [n]$,*

$$\max\{j, n\} \leq \chi_i \leq n^2.$$

Proof. Let $i \in [n]$ and consider a maximal partial Latin cube L of order n . No symbol can appear more than n times in any given cube layer, so $\chi_i \leq n^2$. Also, $\chi_i \geq n$, since if this were not true, then there would be some row and some column that did not contain symbol i ; but then their intersecting stack would contain at least one empty cell that could be filled with this symbol, thus contradicting the maximality of L .

Finally, each of the n^3 cells is either filled with one of the symbols $1, \dots, i-1, i+1, \dots, n$ or is in the same line as some cell that is filled by symbol i . Since each cell shares lines with $3n - 2$ cells,

$$(3n - 2)\chi_i + \sum_{i' \neq i} \chi_{i'} \geq n^3.$$

Subtracting $t = \chi_1 + \dots + \chi_n$ gives $3(n-1)\chi_i \geq n^3 - t$. Hence, $\chi_i \geq j$. \square

Although Proposition 3 and Theorem 4 provide better overall lower bounds on the numbers of filled cells than does Proposition 5, the latter provides more precise information on the required number of filled cells of each symbol. Proposition 5 can also be used to provide upper bounds on the number of cells filled with a particular symbol.

Theorem 6. *If $t \in ML(3, n)$ and $j := \left\lceil \frac{n^3 - t}{3(n-1)} \right\rceil$, then for each $i \in [n]$,*

$$\chi_i \leq \min\{\lfloor (t - (n-i) \max\{j, n\})/i \rfloor, n^2\}.$$

In particular, no symbol can appear in more than $t - (n-1) \max\{j, n\}$ cells.

Proof. By Proposition 5, $\chi_{i'} \geq \max\{j, n\}$ for all $i' \in [n]$, so

$$\chi_1 + \dots + \chi_i = t - (\chi_{i+1} + \dots + \chi_n) \leq t - (n-i) \max\{j, n\}.$$

The first part of the theorem follows from the inequalities $\chi_1 \geq \dots \geq \chi_i$ and from Proposition 5. The second part is obtained by setting $i = 1$. \square

The lower bounds in Proposition 5 may be tightened considerably by applying averaging arguments to the upper bounds in Theorem 6.

Theorem 7. If $t \in ML(3, n)$ and $j := \left\lceil \frac{n^3 - t}{3(n-1)} \right\rceil$, then for each $i \in [n]$,

$$\chi_i \geq \max \left\{ \max_{I \subseteq [n]: i = \min I} \left[\left(t - \sum_{i' \in [n]-I} \min \{ \lfloor (t - (n - i')) \max \{ j, n \} \rfloor / i' \rfloor, n^2 \} \right) / |I| \right], j, n \right\}.$$

Proof. For each set $I \subseteq [n]$,

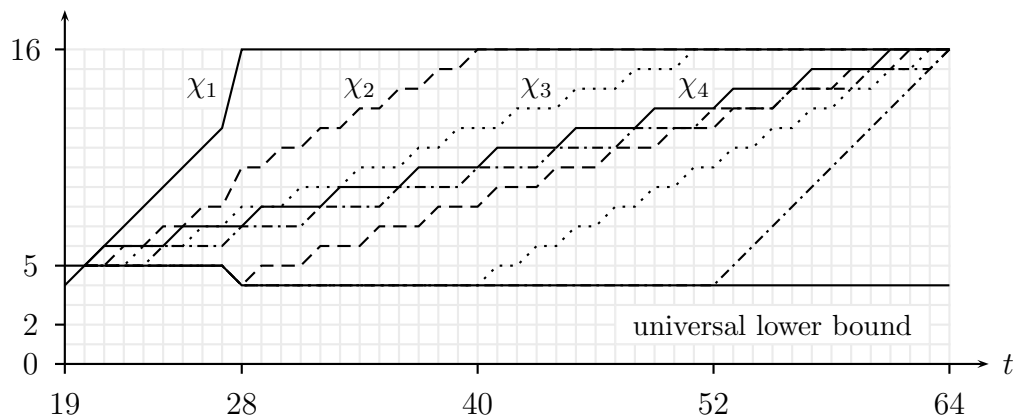
$$\sum_{i'' \in I} \chi_{i''} = t - \sum_{i' \in [n]-I} \chi_{i'}$$

and if $i = \min I$, then $\chi_i = \max \{ \chi_{i''} : i'' \in I \}$, so

$$\chi_i \geq \left\lceil \left(t - \sum_{i' \in [n]-I} \chi_{i'} \right) / |I| \right\rceil.$$

Applying Theorem 6 to $\chi_{i'}$, and Proposition 5, concludes the proof. \square

Example 8. In maximal partial Latin cubes of order $n = 4$, the least number of cells that each of the n symbols must fill is given by Proposition 5, illustrated as the lower black line in the grid below, where t is the number of filled cells. Theorem 7 generally gives far better individual lower bounds on χ_1, \dots, χ_4 . Meanwhile, Theorem 6 offers good individual upper bounds thereon. These individual upper and lower bounds are given by pairs of lines drawn in black (χ_1), dashed (χ_2), dotted (χ_3), and dash-dotted (χ_4).



It is immediately visible from the bounds above that $19 \notin ML(3, 4)$ since the lower bound is higher than one of the upper bounds at $t = 19$. Similarly, the bounds for $t = 20$ dictate that $\chi_1 = \dots = \chi_4 = 5$, and a simple counting argument shows that this is not possible; this shows that $20 \notin ML(3, 4)$.

Example 8 also illustrates the following open problem.

Open Problem. Are the true minimal values for χ_i non-decreasing for fixed i and increasing values of $t \in ML(3, n)$?

It does not appear to be possible to improve the upper or lower bounds on χ_i further by the methods that we have used above. However, let λ_i , ρ_i , and γ_i denote the number of filled cells in layer i , row i , and column i , respectively, and assume without loss of generality that $\lambda_1 \geq \dots \geq \lambda_n$, $\rho_1 \geq \dots \geq \rho_n$, and $\gamma_1 \geq \dots \geq \gamma_n$. Since the maximality of a partial Latin square is invariant under permutations of the layer, row, column, and symbol coordinates of the 4-tuples $(x, y, z, c) \in [n]^4$, we can significantly generalise the bounds from Theorems 6 and 7.

Theorem 9. *If $t \in ML(3, n)$ and $j := \left\lceil \frac{n^3 - t}{3(n-1)} \right\rceil$, then for each $i \in [n]$,*

$$f(i) \geq \lambda_i, \rho_i, \gamma_i, \chi_i \geq \max\left\{j, n, \max_{I \subseteq [n]: i = \min I} \left\lceil \left(t - \sum_{i' \in [n]-I} f(i')\right) / |I| \right\rceil\right\}$$

where $f(i) := \min\{\lfloor (t - (n - i) \max\{j, n\}) / i \rfloor, n^2\}$.

We may generalise these results and their proofs further. Define a partial Latin *hypercube* L of dimension d and order n to be a set of $(d + 1)$ -tuples from $[n]^{d+1}$ such that if (v_1, \dots, v_{d+1}) and (v'_1, \dots, v'_{d+1}) are distinct elements of L , then $v_i = v'_i$ for at most $d - 1$ values of i . Let $ML(d, n)$ denote the set of numbers t for which some *maximal* partial Latin hypercube of dimension d and order n contains exactly t elements. Let L be such a maximal partial Latin hypercube and let χ_i^ℓ denote the number of elements of L whose ℓ th coordinate has value i . Also, assume that $\chi_1^\ell \geq \dots \geq \chi_n^\ell$. The following result is a hypercube generalisation of (most of) Theorem 9.

Theorem 10. *If $t \in ML(d, n)$ for $d \geq 2$ and $j := \left\lceil \frac{n^d - t}{d(n-1)} \right\rceil$, then for each $\ell \in [d + 1]$ and each $i \in [n]$,*

$$f(i) \geq \chi_i^\ell \geq \max\left\{j, \max_{I \subseteq [n]: i = \min I} \left\lceil \left(t - \sum_{i' \in [n]-I} f(i')\right) / |I| \right\rceil\right\}$$

where $f(i) := \lfloor (t - (n - i)j) / i \rfloor$.

5 Spectra for small values of n

By inspection and by extensive computer search, we found the following (partial) spectra for the number of filled entries in maximal partial Latin cubes of orders $n = 2, 3, 4$, respectively.

Theorem 11.

$$\begin{aligned} ML(3, 2) &= \{4, 5, 8\} \\ ML(3, 3) &= \{9, 12, 14, \dots, 24, 27\} \\ ML(3, 4) &= \{31, \dots, 61, 64\} \cup S \quad \text{where } S \subseteq \{28, 29, 30\}. \end{aligned}$$

The Appendix presents examples of maximal partial Latin cubes for each of the spectrum values given in Theorem 11 above. The parallelised computer search used the results of Section 4, most particularly Theorem 9 and Example 8, to harshly prune the search tree. The cube of order 4 lent itself nicely to fast 64 bit representation, and precalculated permutation groups and subgroups were used together with shortcuts and optimisations to further prune the search tree and speed up the search.

We leave it as a challenging computational problem to determine whether each of the values 28, 29 and 30 is a member of $ML(3, 4)$.

6 Constructions with less than half of cells filled

In this section we show the existence of maximal partial Latin cubes with a small number of filled cells for any order n . Small in this case will be $n^3/3 + O(n^2)$.

First consider the case in which n is divisible by 3 and write $n = 3N$. There exists a maximal partial Latin cube L of order 3 with 9 filled cells (see Theorem 11). Replace each filled cell of L containing symbols $i \in \{1, 2, 3\}$ by a Latin cube of order N on element set $\{iN + 1, iN + 2, \dots, (i + 1)N\}$. The resultant Latin cube L_{3N} is clearly a maximal partial Latin cube of order $n = 3N$ with $n^3/3 = 9N^3$ filled cells.

Now suppose that n is not divisible by 3. For the sake of brevity, we use an existence proof to give an upper bound, rather than a completely explicit construction. The idea is based on the following simple observation.

Lemma 12. *Every partial Latin cube of order n is contained in a maximal partial Latin cube of order n .*

Suppose that $n = 3N + 1$ where $N \geq 1$. Take the partial Latin cube L_{3N} from above and add an empty row, an empty column and an empty layer to create a partial Latin cube of order $n = 3N + 1$. We then greedily add elements to filled cells until the resultant partial Latin cube is maximal. Each symbol occurs at most once per line, so the symbol $3N + 1$ can be added to at most n^2 empty cells, and each symbol $s \in [3N]$ can be added to fewer than $3n$ empty cells, each appearing in the final row, column and layer. The resultant maximal partial Latin cube thus has fewer than $9N^3 + n^2 + (3n)(3N)$ filled cells, and we therefore obtain the following lemma.

Lemma 13. *Let $n - 1$ be divisible by 3 and $n \geq 4$. Then there exists a maximal partial Latin cube of order n with no more than $(n^3 + 9n^2 - 6n - 4)/3$ filled cells.*

The argument for the case in which $n = 3N + 2$ is similar, adding an extra empty row, column and layer. The resultant maximal partial Latin cube has at most $9N^3 + 18nN + 2n^2$ filled cells.

Lemma 14. *Let $n - 2$ be divisible by 3 and $n \geq 5$. Then there exists a maximal partial Latin cube of order n with at most $(n^3 + 21n^2 - 3n - 4)/3$ filled cells.*

The linear terms above may be improved; for simplicity however, we summarize the results from this section as follows:

Theorem 15. *For each order $n \geq 3$ there exists a maximal partial Latin cube of order n with $n^3/3 + O(n^2)$ filled cells.*

7 Spectra for at least half-full partial Latin cubes

In this section, we show that maximal partial Latin cubes of any size from approximately half-full to completely full exist (for orders $n \geq 21$) unless either 1 or 2 cells are empty. We first show that neither $n^3 - 1$ nor $n^3 - 2$ belong to $ML(3, n)$. To this end, we need the following lemma.

Lemma 16. *If L is a maximal partial Latin cube with $n^3 - t'$ filled cells where $t' > 0$, then there are two empty cells (ℓ, r, c) , (ℓ', r', c') for which $r \neq r'$, $c \neq c'$ and $\ell \neq \ell'$.*

Proof. For a contradiction and without loss of generality, suppose that all empty cells in L belong to a single layer ℓ . Then every other layer of L is a Latin square.

Suppose that cell (ℓ, r, c) is empty. There is precisely one symbol s which does not occur in the stack defined by (r, c) . If $(\ell, r', c, s) \in L$ for some $r \neq r'$, then symbol s occurs n times within the Latin square L' defined by fixing c but 0 times within row r of L' , a contradiction. Similarly, $(\ell, r, c', s) \notin L$ for any $c' \neq c$. Thus, we may place symbol s in cell (ℓ, r, c) , contradicting the maximality of L . \square

Corollary 17. $n^3 - 1 \notin ML(3, n)$.

Lemma 18. $n^3 - 2 \notin ML(3, n)$.

Proof. Suppose for the sake of contradiction that L is a maximal partial Latin cube of order n with only 2 empty cells; let these be (ℓ, r, c) and (ℓ', r', c') . By the previous lemma, $\ell \neq \ell'$, $r \neq r'$ and $c \neq c'$. Thus the partial Latin squares defined by fixing ℓ , r or c each have exactly one empty cell. By Theorem 1, there is no maximal partial Latin square of order n with exactly one missing cell. Thus, the three partial Latin squares that include (ℓ, r, c) are each completable to a Latin square. In particular, there is a unique symbol s which occurs in none of the lines (ℓ, r) , (r, c) and (ℓ, c) . Thus L is not maximal, a contradiction. \square

Next, we focus on the upper half of the spectrum $ML(3, n)$. We will make use of the following construction for a partial Latin cube from two partial Latin squares; see, for example [14, Section 5].

Lemma 19. *Let L_1 and L_2 be two partial Latin squares of order n . Then*

$$L_1 \circ L_2 := \{(\ell, r, c, s) \mid (r, c, x) \in L_1, (\ell, s, x) \in L_2 \text{ for some } x\}$$

is a partial Latin cube of order n .

Note that $L_1 \circ L_2$ is a Latin cube whenever L_1 and L_2 are Latin squares.

If a partial Latin square L of order n has a subset of m rows and m columns such that the m^2 cells defined by these rows and columns form a Latin square of order m , then these cells are said to form a *Latin subsquare* of L . We define a *Latin subcube* similarly.

Corollary 20. *Let L_1 and L_2 be partial Latin squares sharing the same subsquare M of order m (on the same set of m^2 cells). Then $L_1 \circ L_2$ contains a Latin subcube of order m . Moreover, if L_2 is itself a Latin square, then the stacks of $L_1 \circ L_2$ corresponding to the subsquare M are full.*

Lemma 21. *Let L_1 be a maximal partial Latin square of order n and size t and let L_2 be any Latin square of order n . Then $L_1 \circ L_2$ is a maximal partial Latin cube of order n and size nt .*

Proof. Each layer of $L_1 \circ L_2$ is the maximal partial Latin square L_1 with the symbols relabeled. Hence, $L_1 \circ L_2$ is a maximal partial Latin cube. \square

We can determine certain subintervals of $ML(3, n)$ by modifying the construction in Lemma 21.

Lemma 22. *Suppose that some maximal partial Latin square L of order n and size t contains a Latin subsquare M of order $\lfloor n/2 \rfloor$ and suppose there exists a set S of β distinct elements (r_i, c_i, s_i) , $i \in [\beta]$, each in M and 2β distinct empty cells $(r_i, c'_i), (r'_i, c_i)$, $i \in [\beta]$, of L such that*

$$(L \setminus S) \cup \{(r_i, c'_i, s_i), (r'_i, c_i, s_i) \mid i \in [\beta]\}$$

is a partial Latin square of order n . Then there exists a maximal partial Latin cube of order n and any size t' , where $tn \leq t' \leq tn + \beta \lfloor n/2 \rfloor$.

Proof. Let $N = \lfloor n/2 \rfloor$. Without loss of generality, M is a subsquare on the first N rows, columns and symbols. We construct a partial Latin cube L' of order n as follows. Let L_1 be the partial Latin square L containing the Latin subsquare M . Since M has order $N \leq n/2$, Theorem 1.42 of [3] implies that M is contained in some Latin square L_2 of order n . We now set $L' = L_1 \circ L_2$.

By Lemma 21 and Corollary 20, the partial Latin cube L' is maximal and contains a Latin subcube of order N . Now, let $i \in [\beta]$ and let (r_i, c_i, s_i) be one of the β elements of M satisfying the conditions of the lemma. Let $\ell \in [N]$ and suppose that $(\ell, r_i, c_i, s') \in L'$. By construction, $s' \in [N]$.

Next, modify L' by removing s' from cell (ℓ, r_i, c_i) and placing s' in cells (ℓ, r_i, c'_i) and (ℓ, r'_i, c_i) of layer ℓ . The empty cell created cannot be filled with a symbol from $[N]$ (due to filled cells within layer ℓ); nor can it be filled with a symbol from $[n] \setminus [N]$ (due to filled cells in layers from $[n] \setminus [N]$). Thus, the resultant partial Latin cube is maximal and has size one more than L' . Moreover we may repeat this operation for any $i \in [\beta]$ and any layer $\ell \in [N]$. The result follows. \square

We illustrate the previous lemma in the case $n = 5$. The partial Latin square L_1 is given below with the elements of S in italics; for each $(r_i, c_i, s_i) \in S$, we let $r'_i = r_i + 2$ and $c'_i = c_i + 2$. The partial Latin square formed by replacing every element of S by two elements is given by L'_1 . By the above lemma, this guarantees the existence of maximal

partial Latin cubes of order 5 and sizes 65, . . . , 73.

| | | | | |
|---|---|---|---|---|
| 1 | 2 | | | |
| 2 | 1 | | | |
| | | 3 | 4 | 5 |
| | | 4 | 5 | 3 |
| | | 5 | 3 | 4 |

L_1

| | | | | |
|---|---|---|---|---|
| | | 1 | 2 | |
| | | 2 | 1 | |
| 1 | 2 | 3 | 4 | 5 |
| 2 | 1 | 4 | 5 | 3 |
| | | 5 | 3 | 4 |

L'_1

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |
| 2 | 1 | 4 | 5 | 3 |
| 3 | 4 | 5 | 1 | 2 |
| 4 | 5 | 2 | 3 | 1 |
| 5 | 3 | 1 | 2 | 4 |

L_2

A maximal partial Latin cube of order 5 and size 70 is now constructed as above.

| | | | | |
|---|---|---|---|---|
| | | 1 | 2 | |
| | | 2 | 1 | |
| 1 | 2 | 3 | 4 | 5 |
| 2 | 1 | 4 | 5 | 3 |
| | | 5 | 3 | 4 |

| | | | | |
|---|---|---|---|---|
| | 1 | 2 | | |
| 1 | 2 | | | |
| 2 | | 5 | 3 | 4 |
| | | 3 | 4 | 5 |
| | | 4 | 5 | 3 |

| | | | | |
|---|---|---|---|---|
| 4 | 5 | | | |
| 5 | 4 | | | |
| | | 1 | 2 | 3 |
| | | 2 | 3 | 1 |
| | | 3 | 1 | 2 |

| | | | | |
|---|---|---|---|---|
| 5 | 3 | | | |
| 3 | 5 | | | |
| | | 4 | 1 | 2 |
| | | 1 | 2 | 4 |
| | | 2 | 4 | 1 |

| | | | | |
|---|---|---|---|---|
| 3 | 4 | | | |
| 4 | 3 | | | |
| | | 2 | 5 | 1 |
| | | 5 | 1 | 2 |
| | | 1 | 2 | 5 |

The following lemma uses constructions very similar to those in Lemmas 9 and 10 of [11]. In what follows, an *idempotent* Latin square of order n contains symbol i in cell (i, i) for each $i \in [n]$.

Lemma 23. *If n is even, then there exists a maximal partial Latin square of order n and size $n^2/2 + 2k$ satisfying the conditions of Lemma 22, where $0 \leq k \leq n^2/4 - \beta$ and $\beta \in \{3, 4\}$. If n is odd and $n \geq 9$, then there exists a maximal partial Latin square of order n and size $(n^2+1)/2+2k$ satisfying the conditions of Lemma 22, where $\beta = (n-1)/2$ and $0 \leq k \leq (n-1)^2/4$.*

Proof. Let $N = \lfloor n/2 \rfloor$. Let Q be a Latin square of order N on symbol set $[N]$; this occupies the first N rows and columns of L . If n is odd, then we furthermore require that Q is idempotent; such a Latin square exists for $N \geq 3$ (see, for example, Lemma 1.2 of [4]).

Let P be a Latin square of order $n - N$ on symbol set $[n] \setminus [N]$. Next, let P' be a partial Latin square of size k which is a subset of P , subject to the following restrictions. If k is even, then P' can be any subset such that $0 \leq k \leq n^2/4 - 4$. If k is odd, then P' must not include the final row or column of P , so that $0 \leq k \leq (n-1)^2/4$.

If cell (i, j, s) is in P' , then place symbol s in cells $(i, j + N)$ and $(i + N, j)$ of L and place symbol $s - N$ in cell $(i + N, j + N)$ of L . Otherwise if $(i, j, s) \in P \setminus P'$, then place symbol s in cell $(i + N, j + N)$ of L . The resultant partial Latin square has the desired properties.

In particular, suppose that n is even. Whenever $(i, j, s) \in P \setminus P'$, the element $(i - n/2, j - n/2, s - n/2) \in Q$ can be one of the four elements of S chosen as in Lemma 22, since, by construction, cells $(i, j - n/2)$ and $(i - n/2, j)$ are empty and neither row i nor column j contain symbol $s - n/2$. For n odd, the β elements of S may be any $(i, i, i) \in Q$ where $i \in [N]$; cells (i, n) and (n, i) are always empty and neither row n nor column n contains symbol i . \square

Corollary 24. *If n is even and $n^3/2 \leq t \leq n^3 - 9n/2$, then there exists a maximal partial Latin cube of order n with t filled cells. If n is odd, $n \geq 9$ and $n(n^2 + 1)/2 \leq t \leq n^3 - (3n^2 - 2n - 1)/4$, then there exists a maximal partial Latin cube of order n with t filled cells.*

We now give a maximal partial Latin cube of order $n = 6$ and size 26×6 constructed as in the above proof. The symbols in italics can each be used to increase the size of the maximal partial Latin cube by 1 as described above, yielding maximal partial Latin cubes of sizes $26 \times 6, \dots, 26 \times 6 + 12 = 28 \times 6$.

| | | | | | | | | | | | | | | | | | |
|---|----------|----------|---|---|---|---|----------|----------|---|---|---|---|----------|----------|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 2 | 3 | 1 | 5 | 6 | 4 | 3 | 1 | 2 | 6 | 4 | 5 |
| 2 | <i>3</i> | <i>1</i> | 5 | | | 3 | <i>1</i> | <i>2</i> | 6 | | | 1 | <i>2</i> | <i>3</i> | 4 | | |
| 3 | <i>1</i> | <i>2</i> | | | | 1 | <i>2</i> | <i>3</i> | | | | 2 | <i>3</i> | <i>1</i> | | | |
| 4 | 5 | 6 | 1 | 2 | 3 | 5 | 6 | 4 | 2 | 3 | 1 | 6 | 4 | 5 | 3 | 1 | 2 |
| 5 | | | 2 | 6 | 4 | 6 | | | 3 | 4 | 5 | 4 | | | 1 | 5 | 6 |
| | | | 6 | 4 | 5 | | | | 4 | 5 | 6 | | | | 5 | 6 | 4 |
| 4 | 5 | 6 | 1 | 2 | 3 | 5 | 6 | 4 | 2 | 3 | 1 | 6 | 4 | 5 | 3 | 1 | 2 |
| 5 | 6 | 4 | 2 | | | 6 | 4 | 5 | 3 | | | 4 | 5 | 6 | 1 | | |
| 6 | 4 | 5 | | | | 4 | 5 | 6 | | | | 5 | 6 | 4 | | | |
| 1 | 2 | 3 | 4 | 5 | 6 | 2 | 3 | 1 | 5 | 6 | 4 | 3 | 1 | 2 | 6 | 4 | 5 |
| 2 | | | 5 | 3 | 1 | 3 | | | 6 | 1 | 2 | 1 | | | 4 | 2 | 3 |
| | | | 3 | 1 | 2 | | | | 1 | 2 | 3 | | | | 2 | 3 | 1 |

Finally, we consider the part of the spectrum $ML(3, n)$ for which there are relatively few ($O(n^2)$) empty cells. We first focus on even orders. A *partial incomplete Latin square* $PILS(n; t_1^{u_1} t_2^{u_2} \dots t_k^{u_k})$ is a partial Latin square of order n such that there exists a partition \mathcal{P} of $[n]$ of type $t_1^{u_1} t_2^{u_2} \dots t_k^{u_k}$ so that for each $P \in \mathcal{P}$, the subarray indexed by the rows and columns of P is empty and each row (column) of P and each column (row) of $[n] \setminus P$ contain each symbol from $[n] \setminus P$ exactly once. A PILS can always be completed to a Latin square of the same order by replacing each empty subarray indexed by P by a Latin square on the symbol set P .

Lemma 25. *If $a \geq 3$ and $0 \leq b \leq 2$, then there exists a $PILS(3^a b^1)$.*

Proof. We give a rough sketch of the construction. For $b = 0$, take an idempotent Latin square of order 3, replace each entry with a Latin square of order a and remove the Latin squares on the main diagonal. For $b \in \{1, 2\}$, take the construction for $b = 0$, making sure that the Latin squares of order a each possess at least two disjoint transversals. (Such Latin squares exist even for $a = 6$; see for example Latin square 6.2 from III.1.3 in [3].) Choose a set of transversals from the subsquares of order a so that each row or column contains b entries from a transversal. Add b extra rows and columns. Replace the transversals with entries $3a + 1$ (and $3a + 2$ if $b = 2$), shifting the original entries to the final b rows and columns. \square

Let L_1 be a PILS($3^a b^1$) (with the one holes of size b filled in) and let L_2 be a completion of L_1 to a Latin square. Observe that the partial Latin cube $L_1 \circ L_2$ contains a empty $3 \times 3 \times 3$ subarrays which may be replaced by any a partial Latin cubes of order 3 (on prescribed symbol sets). Moreover, if each of the inserted partial Latin cubes are maximal, then the overall partial Latin cube will also be maximal. These observations yield the following corollary to Lemma 25.

Corollary 26. *Let $27 - k_1, 27 - k_2, \dots, 27 - k_a \in ML(3, 3)$ for $a \geq 3$, and let $b \in \{0, 1, 2\}$. Then there exists a maximal partial Latin cube of order $3a + b$ with exactly $k_1 + k_2 + \dots + k_a$ empty cells.*

Theorem 27. *Let n and t be integers such that $n = 10$ or $n \geq 12$, $3 \leq t \leq 9n/2$. Then there exists a maximal partial Latin cube of order n and exactly $n^3 - t$ filled cells.*

Proof. Let $a = \lfloor n/3 \rfloor$ and $b = n - 3a$. Now, $ML(3, 3) = \{9, 12, 14, \dots, 24, 27\}$ by Theorem 11 (see Section 5). The theorem holds for $t = 18(a - 1) + 13$ which can be seen by setting $k_1 = k_2 = \dots = k_{a-1} = 18$ and $k_a = 13$; for lower values of t , k_a can be replaced by 12 or 11 and each other k_i can be replaced by 15 and so forth. Thus by the previous corollary, it suffices to show that $18(a - 1) + 13 \geq 9n/2$. This is clearly true for $n = 10$; also $18(n - 2)/3 - 5 = 6n - 17 \geq 9n/2$ for $n \geq 12$. \square

From Corollary 24 and the previous theorem we obtain the following.

Theorem 28. *Let n and t be integers such that n is even, $n \geq 10$ and $3 \leq t \leq n^3/2$. Then there exists a maximal partial Latin cube of order n and exactly $n^3 - t$ filled cells.*

We next focus on the case in which n is odd. It is well known that there exists a Latin square L_1 of order n with a subsquare of order m for any n and m such that $m \leq \lfloor n/2 \rfloor$. Let L'_1 be the partial Latin square created by removing this subsquare from L_1 . Then $L'_1 \circ L_1$ is a partial Latin cube with every entry filled except for a three-dimensional subarray of dimensions $m \times m \times m$. Moreover, we may insert a maximal partial Latin cube of order m into this ‘‘hole’’ to create a maximal partial Latin cube of order n . This implies the following lemma.

Lemma 29. *If $n \geq 2m$ and some maximal partial Latin cube of order m has t empty cells, then there is a maximal partial Latin cube of order n with t empty cells.*

Theorem 30. *Let n and t be integers such that n is odd, $n \geq 21$ and $3 \leq t \leq n(n^2 - 1)/2$. Then there exists a maximal partial Latin cube of order n and exactly $n^3 - t$ filled cells.*

Proof. Let $M = \lfloor n/4 \rfloor$. By Theorem 28, there exists a maximal partial Latin cube of order $2M$ with t empty cells, for any t with $3 \leq t \leq 4M^3$. Thus by the previous lemma, there exists a maximal partial Latin cube of order n with t empty cells. Since $M \geq (n - 3)/4$, Corollary 24 implies that it suffices to show that

$$4 \left(\frac{n - 3}{4} \right)^3 \geq \frac{3n^2 - 2n - 1}{4}$$

which is certainly true for $n \geq 21$. \square

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| 1 2 3 | 4 1 2 | 3 1 | 4 1 2 3 4 | 4 3 1 | 1 2 | 4 2 3 | 3 2 1 4 |
| 4 1 2 | 3 2 1 | 2 1 3 | 4 1 2 3 4 | 1 2 | 4 3 | 2 3 1 | 1 2 3 |
| 3 1 | 2 3 | 3 1 2 | 4 | 3 4 | 2 1 | 1 2 | 2 3 |
| 2 1 3 | 4 2 | 3 1 | 2 1 3 | 3 4 2 | 1 3 2 | 1 4 1 | 2 1 |
| 1 4 | 1 3 | 4 2 | 3 2 | 4 3 | 2 3 1 | 1 2 2 | 4 1 |
| 2 1 | 3 2 | 1 4 | 4 | 1 2 | 4 3 | 2 1 | 3 4 |
| 3 2 | 4 1 | 2 3 | 3 1 | 2 1 | 1 4 | 3 1 | 1 2 3 |
| 3 4 1 | 4 3 2 | 2 1 | 1 2 | 4 2 1 | 4 2 3 | 1 3 | 3 1 |
| 1 3 2 | 3 4 | 3 4 | 2 1 | 1 4 3 | 4 2 | 3 2 1 | 2 1 |
| 2 1 | 1 2 | 3 4 | 4 3 | 3 1 | 1 3 | 2 4 | 4 2 |
| 4 1 | 2 1 | 1 2 3 | 3 4 | 2 1 | 3 1 | 4 2 | 1 3 4 |
| 3 2 1 | 2 4 1 | 1 3 | 1 2 | 2 4 | 4 1 3 | 1 3 2 | 3 1 |
| 4 1 2 | 1 3 | 3 1 4 | 3 2 | 4 3 1 | 1 4 | 3 2 | 2 3 |
| 1 3 | 2 4 | 4 2 | 3 1 | 1 2 3 | 3 2 1 | 4 1 | 4 2 |
| 2 4 3 | 1 2 3 | 4 2 | 3 1 2 | 3 1 2 | 1 2 | 2 4 | 1 3 |
| 2 3 1 | 3 1 2 | 1 2 4 | 4 3 | 3 2 1 | 2 1 3 | 1 3 4 | 4 2 |
| 3 1 4 | 2 4 1 | 4 3 | 1 2 | 2 1 4 | 1 3 2 | 3 2 | 4 1 |
| 4 3 | 1 2 | 3 4 2 | 2 1 3 | 1 4 2 | 3 2 4 | 2 1 | 3 2 |
| 1 4 2 | 1 1 4 | 3 2 1 | 3 | 4 3 | 2 2 1 | 1 3 | 3 4 1 |
| 3 2 1 | 2 1 3 4 | 1 3 4 | 4 2 | 3 1 2 | 1 2 3 4 | 2 3 4 | 4 1 |
| 2 1 4 | 1 3 2 | 3 2 | 4 2 1 | 1 2 4 | 2 3 1 | 3 1 | 4 1 2 |
| 1 4 2 | 3 4 | 2 1 | 3 4 | 2 4 1 | 3 1 | 1 2 | 3 4 |
| 4 3 | 2 1 | 1 2 3 | 3 1 | 4 3 | 4 2 | 2 1 3 | 3 1 2 |
| 3 1 4 2 | 1 2 3 | 2 3 4 | 4 1 1 | 3 2 4 1 | 1 3 2 | 2 1 3 | 4 1 2 |
| 1 2 4 | 2 3 4 1 | 3 1 2 | 4 1 2 | 2 3 1 | 3 1 4 | 1 2 | 4 3 |
| 2 4 1 | 3 1 2 | 1 2 | 3 4 | 1 4 3 2 | 2 1 | 3 4 | 3 2 4 |
| 4 3 | 4 2 | 2 1 3 | 3 1 2 | 1 2 4 | 4 1 3 | 4 3 1 | 2 |
| 2 1 4 3 | 3 2 1 | 1 3 2 | 4 1 2 | 3 1 2 | 2 3 4 1 | 1 2 3 | 4 1 3 |
| 1 2 | 2 3 1 | 3 1 4 | 4 4 3 | 1 3 2 | 3 2 1 | 2 1 4 | 4 2 |
| 3 1 4 | 1 4 2 | 2 3 1 | 3 2 | 2 4 1 | 1 3 | 3 2 1 | 1 3 4 |
| 4 3 1 | 1 2 4 | 4 1 3 | 2 4 | 4 2 1 | 1 3 4 | 4 1 2 | 3 2 |
| 2 1 3 4 | 3 2 4 1 | 1 3 2 | 4 1 2 | 1 2 3 4 | 4 3 1 | 2 4 3 | 3 1 2 |
| 1 2 4 | 2 3 1 | 3 1 4 | 4 3 | 2 1 4 | 3 4 2 | 4 3 1 | 1 2 3 |
| 3 4 1 | 1 3 2 | 2 4 1 | 2 3 4 | 4 3 2 | 1 2 3 | 1 2 4 | 2 3 4 |
| 4 3 1 | 1 2 | 4 1 3 | 2 3 4 | 2 1 4 | 2 1 4 | 1 3 2 | 4 1 |
| 2 3 4 1 | 4 2 3 | 1 4 2 | 3 1 4 | 3 4 1 2 | 1 3 4 | 2 1 3 | 4 2 1 |
| 3 2 1 4 | 2 3 4 1 | 4 1 3 | 1 4 2 | 4 3 2 1 | 3 4 1 2 | 1 2 4 | 2 1 3 |
| 4 1 2 | 3 4 1 2 | 2 3 1 | 2 3 | 1 2 3 | 4 1 2 | 3 4 1 | 3 4 2 |
| 1 4 3 | 1 1 2 | 3 2 4 | 2 1 3 | 2 1 4 | 2 1 3 | 4 3 2 | 3 1 4 |
| 3 4 1 2 | 2 1 3 | 1 3 4 | 4 2 1 | 2 1 3 4 | 3 2 4 1 | 1 3 2 | 4 1 2 |
| 4 3 2 1 | 1 2 4 | 3 4 1 2 | 2 1 3 | 1 2 4 3 | 2 1 | 3 4 1 | 3 2 4 |
| 1 2 3 | 3 4 1 2 | 4 1 2 | 3 4 | 3 4 2 | 4 3 1 2 | 2 1 3 4 | 1 2 4 3 |
| 2 1 4 | 4 3 2 1 | 2 3 | 3 1 4 | 4 3 1 | 1 4 3 | 2 1 3 | 2 1 |
| 2 1 3 4 | 3 2 4 1 | 1 3 2 | 4 1 2 | 3 1 4 2 | 4 3 2 1 | 1 4 3 | 2 1 3 |
| 1 2 4 3 | 2 1 4 | 3 4 1 | 3 2 | 1 3 2 4 | 3 1 2 | 4 2 1 | 4 3 |
| 3 4 2 | 4 3 1 2 | 2 1 3 4 | 1 2 4 3 | 4 2 3 | 2 4 1 3 | 3 1 2 4 | 1 3 4 2 |
| 4 3 1 | 1 4 3 | 2 1 3 | 2 1 4 | 2 4 1 | 1 2 4 | 3 1 2 | 3 1 2 4 |
| 3 1 4 2 | 4 3 2 1 | 2 4 1 3 | 1 2 3 4 | 1 2 3 4 | 4 3 2 1 | 2 1 4 3 | 3 4 1 2 |
| 1 3 2 | 3 1 4 | 2 3 1 | 4 1 2 | 2 1 4 | 3 4 1 | 1 3 2 | 4 2 3 |
| 4 2 3 1 | 2 4 1 3 | 1 3 2 4 | 3 1 4 | 3 4 2 1 | 2 1 3 4 | 4 2 1 | 1 3 4 |
| 2 1 4 | 1 2 4 | 3 1 2 | 3 2 1 | 4 3 1 2 | 1 2 4 3 | 3 2 4 | 2 1 3 |
| 2 3 4 1 | 3 2 1 | 4 1 3 2 | 1 4 2 3 | 2 3 4 1 | 4 1 3 2 | 3 2 1 | 1 4 2 3 |
| 3 2 1 4 | 2 1 4 3 | 1 4 2 | 4 3 1 | 3 2 1 4 | 1 4 2 | 2 1 4 3 | 4 3 1 |
| 1 4 3 2 | 4 3 2 1 | 3 2 1 | 2 1 4 | 1 4 3 2 | 3 2 1 4 | 4 3 2 1 | 2 1 4 |
| 4 1 2 3 | 1 3 4 | 2 3 4 1 | 3 2 1 | 4 1 2 3 | 2 3 4 1 | 1 3 4 | 3 2 1 |
| 4 3 2 1 | 1 2 3 4 | 2 1 4 3 | 3 4 1 2 | 2 3 1 4 | 1 4 2 3 | 3 2 4 1 | 4 1 3 2 |
| 3 4 1 2 | 2 1 3 | 4 2 3 1 | 1 3 4 | 3 2 4 1 | 4 1 3 2 | 2 3 1 | 1 2 4 |
| 2 1 4 | 3 4 1 2 | 1 3 2 | 4 2 3 1 | 1 4 3 2 | 2 3 1 4 | 4 1 2 3 | 3 2 4 1 |
| 1 2 4 3 | 4 3 2 1 | 3 4 1 2 | 2 1 4 | 4 1 2 3 | 3 2 4 1 | 1 3 4 | 2 3 1 |
| 3 1 2 4 | 2 4 3 1 | 4 2 1 3 | 1 3 4 2 | 1 2 3 4 | 2 1 4 3 | 3 4 1 2 | 4 3 2 1 |
| 1 3 4 2 | 4 2 1 3 | 2 4 3 1 | 3 1 2 | 2 1 4 3 | 1 2 3 4 | 4 3 2 1 | 3 4 1 2 |
| 2 4 1 3 | 3 1 2 4 | 1 3 4 2 | 4 2 3 1 | 3 4 2 1 | 4 3 1 2 | 1 2 3 4 | 2 1 4 3 |
| 4 2 3 1 | 1 3 4 2 | 3 1 2 | 2 1 4 | 4 3 1 2 | 3 4 2 1 | 2 1 4 3 | 1 2 3 4 |