Theoretical Basis of assembling structures in the Rayleigh Ritz Method

S. Ilanko*

*The School of Engineering
The University of Waikato
Gate 1 Knighton Road
Private Bag 3105
Hamilton 3240
New Zealand
ilanko@waikato.ac.nz

Abstract

The use of negative structures in modelling cavities was discussed in previous symposia [1-3]. A formal proof that the use of negative structures to remove corresponding positive structural components in calculating natural frequencies has been established for discrete systems. This approach introduces extra spurious modes and methods of identifying and eliminating these modes have also been studied for the same. However, a corresponding proof for continuous systems, and its application poses some challenges. The focus of this paper is on these challenges.

To start with, let us consider a discrete n degree of freedom vibratory system A, which when connected to another m dof discrete system C^+ subject to r constraints that connect A and C^+ results in an $n+m-r$ dof system B. See Figure 1.

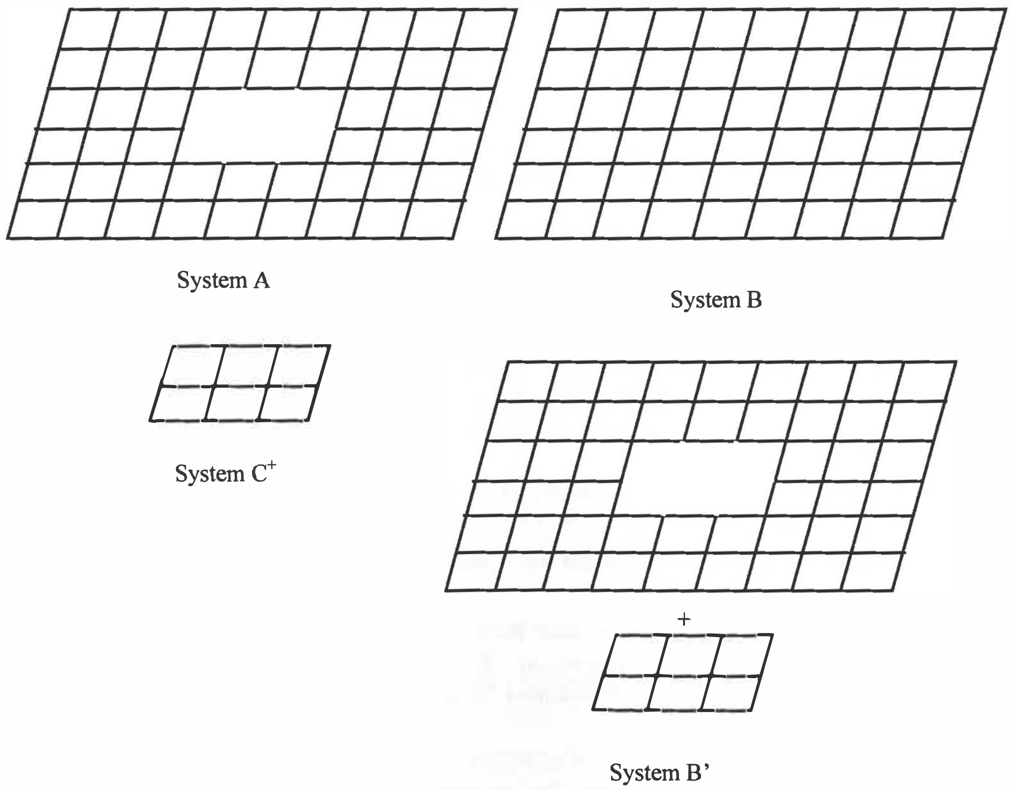

Figure 1

Here, the connection constraints are such that each of the corresponding pairs of r degrees of freedom of A and C^+ vibrate together. The constraint conditions may be written as

$d_{A_i^C} - d_{C_i^C} = 0$, for $i = 1,2..r$,

(1)

Where $d_{A_i^C}, d_{C_i^C}$ are the common degrees of freedom in A and C+.

There is an implication that $r < n$, and $r < m$.

Let $[\chi_A], [\chi_B], [\chi_C]$ be the normalised modal matrices of A, B, C respectively.

Then the ith modes of A, B, C are given by the ith columns of the corresponding matrices and are denoted by vectors $\{\chi_{A,i}\}, \{\chi_{B,i}\}\{\chi_{C,i}\}$ respectively.

Let $V_A(\bar{f}_A), V_B(\bar{f}_B), V_C(\bar{f}_C)$ etc, be the potential energy of A, B, C+ expressed in terms of assumed displacement vector forms$\{\bar{f}_A\}$, $\{\bar{f}_B\}$, $\{\bar{f}_C\}$ respectively. The displacement forms are obtained from a liner combination of sets of admissible vectors $\{\bar{\phi}_A\}, \{\bar{\phi}_B\}, \{\bar{\phi}_C\}$ which have the same length as the corresponding number of dofs.

i.e. $\bar{f}_{A,j} = \sum_i^n a_i \bar{\phi}_{A,j,i}, \ \bar{f}_B = \sum_i^{n+m-r} a_i \bar{\phi}_{B,j,i}, \ \bar{f}_C = \sum_i^m a_i \bar{\phi}_{C,j,i}$ (2a,b,c)

Similarly the kinetic energy functions of A, B, C^+ are $\psi_A(\bar{f}_A), \psi_B(\bar{f}_B), \psi_C(\bar{f}_C)$.

The energy terms of a negative structure C^- are simply negative of their positive counterparts.

i.e. $V_{C^-}(\bar{f}_C) = -V_C(\bar{f}_C); \psi_{C^-}(\bar{f}_C) = -\psi_C(\bar{f}_C)$ (3a,b)

For discrete systems, the exact frequencies may be readily obtained by minimising the Rayleigh coefficients in terms of an assumed form. For example, for A, an assumed form of displacement $\bar{f}_A$ formed by the admissible functions $[\bar{\phi}_A]$, will give the natural frequencies and modes of A using the Rayleigh-Ritz Method (RRM), if the functions$\{\bar{\phi}_A\}$ are independent, and the number of admissible functions is equal to the number of degrees of freedom.

In terms of the exact modes, the ith eigenvalue (the square of the natural frequency) of A is [4]

$$\omega_{A,i}^2 = \frac{V(\chi_A(i))}{\psi(\chi_A(i))} \quad (4a)$$

But from Rayleigh's Principle, the ith eigenvalue of A is also given by

$$\omega_{A,i}^2 = \min \frac{V(\bar{f}_A)}{\psi(\bar{f}_A)} | < \bar{f}_A, \chi_{A,j} > , \text{ for } j = 1,2..i-1 \quad (4b)$$

Now consider the combination of A and C^+, without the connections (see Figure 1), which for convenience may be labelled B′.

$B' = A \cup C^+$

(5)

In B′, it is possible for one of the constituting structures to vibrate in a natural mode while the other remains stationary. Therefore, the eigensolutions of B′ are combination of the eigensolutions of A and C^+.

$$\omega_{B',i}^2 = \min \frac{V(\bar{f}_A)+V(\bar{f}_C)}{\psi(\bar{f}_A)+\psi(\bar{f}_C)} | < \bar{f}_{B'}, \chi_{B',j} > , \text{ for } j = 1,2..i-1 \quad (6)$$

But $B = A \cup C^+ |$ eq.(1)

(7)

$$\omega_{B,i}^2 = \min \frac{V(\bar{f}_A)+V(\bar{f}_C)}{\psi(\bar{f}_A)+\psi(\bar{f}_C)} | < \bar{f}_{B'}, \chi_{B',j} > , \text{ for } j = 1,2..i-1 \text{ and eq.(1)} \quad (8)$$

Here, $\bar{f}_{B'}$denote the union of admissible forms for A and C^+ and , $\chi_{B',j}$ denotes the jth mode from the union of the sets of modes of A and C^+.

Since all possible degrees of freedom have been included in the chosen set of admissible forms and the constraint conditions (1) are enforced, eq. (8) gives the exact natural frequencies of B. Thus the natural frequencies of B may be obtained by using the admissible displacement forms for A and C^+ in the RRM.

Now let us consider systems A, B and C as discretised models of continuous systems $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$ with numbers of dofs that are of interest (let us refer to these as "significant degrees of freedom") being n, m and $n+m-r$ respectively. It is noted here that in order to get accurate modal

values for these displacements, additional degrees of freedom may have to be introduced in the models, such that the required natural frequencies and modes of all these systems are correct to the desired degree of accuracy. Therefore, for all practical purposes, the discretised models with the significant degrees of freedom can be used instead of the actual system. The significant dofs correspond to displacements (translations or rotations) at specified locations. The question then arises as to how to relate this to a typical Rayleigh-Ritz analysis of a continuous systems.
Let the actual number of degrees of freedom needed to obtain results with any desired level of convergence be n_B and n_C for systems B and C, where $n_B >> n+m-r$ and $n_C >> m$.
Then a linear combination of admissible functions $\varphi_{A,i}$ for i=1,2.. n_B and $\varphi_{C,j}$ for j=1,2.. n_C will yield independent coordinate values for all of the required number of independent degrees of freedom. This proves that the application of the Rayleigh Ritz Method to an assembly of structures based on the admissible functions for the component structures with the application of relevant connection constraints will yield the natural frequencies and modes of the assembled structure.
The proof above is applicable for any Rayleigh-Ritz model of continuous systems consisting of positive structures. If this argument is valid for combining positive and negative structures, we could then also infer that the natural frequencies of A could be obtained by combining the modes of B and C^- in a Rayleigh-Ritz procedure subject to continuity constraints at the interface/boundary. This would prove to be useful because the natural frequencies and modes of negative structures are identical to those of their positive counterparts because both kinetic energy function and potential energy for the negative structures are equal and opposite to their positive counterparts as given in eq. (3). However, it is not clear whether the Rayleigh's theorem of separation is applicable for negative structures. Currently work is in progress to find a proof that is applicable for assembling negative and positive structures.

References

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